Ideas and Results in Model Theory: 
Reference, Realism, Structure and Categoricity

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1 Introduction

Model theory is invoked by diverse branches of philosophy, including philosophy of mathematics, philosophy of science, philosophy of language and metaphysics. Such wide-ranging appeals to model theory have led to a highly fragmented philosophical literature. Very similar arguments and dialectical situations germinate in diverse areas, but with only limited cross-pollination. Our main aim in this paper is to rectify this situation and bring some unity to the philosophy of model theory.

We have not attempted a comprehensive survey of the recent literature in philosophy and model theory. For one thing, there is too much to consider. But, more important, there is not yet any such subject as philosophy of model theory (contrast this with philosophy of set theory). So we have had to decide upon our subject matter as we explore it.

We have focused on those topics mentioned in our subtitle: reference, realism, structure and categoricity. The title itself alludes to Prawitz’s ‘Ideas and Results in Proof Theory’ [1971]. In Prawitz’s writings, and those of Dummett, we find a careful survey of various philosophical ends to which proof-theoretic results might be put. We have aimed to do the same for key model-theoretic results, especially those relating to categoricity. Despite their centrality to the philosophy of mathematics, we know of no similar attempt to survey them.

After sketching some basic notions (§2), we begin by considering model theory as a tool for exploring reference (§3). Questions about the determinacy of reference force us to consider various versions of realism: in metaphysics, in the philosophy of mathematics, and in the philosophy of science (§4). We then consider model theory as a tool for exploring the intuitive idea of a mathematical structure; and this leads us to consider the interplay between categoricity results and a variety of philosophical-cum-mathematical positions (§§5–7). Continuing on this path, we explore various versions of structuralism (§§8–9), where

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all four topics of our subtitle connect. We close by examining the notion of uncountable categoricity, as used in recent work within model theory (§10).

By focusing on the topics of reference, realism, structure and categoricity, we have aimed to beat a fairly straight path through a fascinating but tangled landscape. Sadly, this means that many interesting topics have been omitted from this paper. We particularly regret that we have not had more to say about topics relating to nonstandard analysis, infinitary logic, definability, o-minimality, and the semantics of non-classical logic. Further, due to the presence of many recent surveys on logical consequence and the logical constants ([Blanchette 2001], [Shapiro 2005], [Bonay 2014], [Gomez-Torrente 2002]), we have opted to omit these topics from our treatment, despite their obvious centrality to the philosophy of model theory.

2 Basic notions

We start by introducing some of the most basic ideas from model theory. Those familiar with model theory can happily skim this section, extracting only our notational conventions.

2.1 Signatures, structures and satisfaction

A signature, $\mathcal{L}$, is something like a formal vocabulary. More precisely, it is a set of constant symbols $c_1, c_2, \ldots$, relation symbols $R_1, R_2, \ldots$ (also known as predicates), and function symbols $f_1, f_2, \ldots$. As is usual in model theory, we are fairly sloppy concerning quotation conventions, largely allowing context to distinguish between use and mention.

An $\mathcal{L}$-structure, $\mathcal{M}$, is then an underlying domain, $M$, together with an assignment of $\mathcal{L}$’s constant symbols to elements of $M$, of $\mathcal{L}$’s relation symbols to relations on $M$, and of $\mathcal{L}$’s function symbols to functions over $M$. In this paper, we always use $\mathcal{M}, \mathcal{N}, \ldots$ for structures, and $M, N, \ldots$ for their underlying domains. Where $s$ is any $\mathcal{L}$-symbol, we say that $s^\mathcal{M}$ is the object, relation or function (as appropriate) assigned to $s$ in the structure $\mathcal{M}$.

The preceding definition is always given a set-theoretic implementation. So: $\mathcal{M}$’s underlying domain, $M$, is just a set. Each $n$-place relation symbol, $R$, is interpreted by an $n$-ary relation, i.e. a subset of $M^n$. As is usual in set theory: $M^1 = M$ and $M^{n+1} = M^n \times M$, where $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$. Similarly, each $n$-place function symbol $f$ is interpreted by some subset of $M^{n+1}$. Consequently, $\mathcal{L}$-structures are individuated extensionally: they are identical iff they have exactly the same underlying domain and make exactly the same assignments to every $\mathcal{L}$-symbol.

We now know what ($\mathcal{L}$-)structures are. To move to the idea of a model, we need to think of a structure as making certain sentences true or false. Model theorists (typically) restrict their attention to the first-order sentences which we obtain by adding a signature’s symbols to a basic starter-pack of logical vocabulary (sentential connectives, quantifiers, and parentheses). As such, the sentences under consideration have a well-defined, recursive syntax. Since Tarski, we have known how to offer a precise, recursive definition of the idea that $\mathcal{M}$ satisfies the sentence $\varphi$, or (equivalently) that $\varphi$ is true in $\mathcal{M}$. We write this more briefly thus: $\mathcal{M} \models \varphi$. (For a full definition of satisfaction, pick up your favourite logic
textbook.) When $T$ is a theory, i.e. a set of sentences, and every sentence in $T$ is true in $\mathcal{M}$, we say that $\mathcal{M}$ is a *model of $T$*, or just $\mathcal{M} \models T$. Where every model of $T$ satisfies $\varphi$, we also write $T \models \varphi$.

### 2.2 Isomorphism and elementary equivalence

After satisfaction, one of the most fundamental ideas in model theory is *isomorphism*. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A bijection $\sigma : M \rightarrow N$ is an isomorphism iff it ‘preserves the structure’; i.e. iff for any atomic $\mathcal{L}$-formula $\varphi$ and any $a_1, \ldots, a_n \in M$ we have:

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \text{ iff } \mathcal{N} \models \varphi(\sigma(a_1), \ldots, \sigma(a_n))$$

It turns out that, if the condition holds for all atomic formulas, then it holds for all formulas of any complexity.\(^1\) When there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$, we say that they are *isomorphic*, and write $\mathcal{M} \cong \mathcal{N}$.

Another important notion is *elementary equivalence*. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent, written $\mathcal{M} \equiv \mathcal{N}$, iff they satisfy exactly the same sentences. Since sentences are just formulas with no free variables, we see immediately that isomorphism implies elementary equivalence, i.e. if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

Isomorphic—and hence elementary equivalent—structures are easy to construct. Indeed, given any structure and any bijection whose domain is the domain of that structure, we can treat that bijection as an isomorphism:

**The Push-Through Construction.** Let $\mathcal{L}$ be any signature, let $\mathcal{M}$ be any $\mathcal{L}$-structure, and let $\pi$ be any bijection whose domain is $M$. We can use $\pi$ to induce another $\mathcal{L}$-structure, $\mathcal{N}$, whose underlying domain is $N = \text{range}(\pi)$, just by ‘pushing through’ the assignments in $\mathcal{M}$, i.e. by stipulating that $s^N = \pi(s^M)$ for each $\mathcal{L}$-symbol $s$.\(^2\) It is easy to check that $\pi : \mathcal{M} \rightarrow \mathcal{N}$ is an isomorphism.

We invoke this Construction several times in the next few sections.

### 3 Referential indeterminacy

In the previous section, we sketched some technical notions. In this section, we shall apply these ideas by considering the *permutation argument*. Though made famous by Putnam [1981, pp. 33–5], the argument has a long history, running through Frege [1893, §10]; Carnap [1928, §§153–5]; Newman [1928, pp. 145–6]; Jeffrey [1964, pp. 82–4]; Winnie [1967, p. 226]; Field [1975, pp. 376–7]; Wallace [1979, p. 307]; Davidson [1979, pp. 9–10]. (We discuss the connection with Newman in §4.6.)

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\(^1\)For a proof, see [Marker 2002, Theorem 1.1.10]. Indeed, this facilitates a more long-winded (but in practice easier to use) definition of isomorphism; see [Marker 2002, Definition 1.1.3].

\(^2\)The set-theoretic implementation of relations and functions comes into its own here. For any set $X \subseteq M^n$, $\pi(X) = \{\pi(a_1, \ldots, a_n) \mid \langle a_1, \ldots, a_n \rangle \in X\}$. So in particular, if $R$ is a two-place relation symbol, then $R^N = \pi(R^M) = \{\langle \pi(a), \pi(b) \rangle \mid \langle a, b \rangle \in R^M\}$; and if $f$ is a one-place function symbol, then $f^N = \{\langle \pi(a), \pi(b) \rangle \mid \langle a, b \rangle \in f^M\}$, so that $f^N(\pi(a)) = \pi(f^M(a))$. 

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3.1 Intuitive ideas and the permutation argument

Model theory might seem to provide us with a rigorous way to think about truth and reference. In particular, we might employ the following heuristic:

Treat models as representing semantic possibilities. If $c^M = b$, then treat $M$ as representing a possible situation where the name $c$ refers to the object $b$. If $M \models \varphi$, treat $M$ as representing a possible situation where $\varphi$ is true.

To be clear: this treatment is not compulsory. It is a substantial philosophical claim that certain intuitive notions—semantic possibility, reference and truth—can be fruitfully explicated using mathematically precise notions. Nonetheless, the treatment is fairly natural, and pursuing it will allow us to tackle intuitive questions using model theory.

Consider, for instance, the following question: How many possible ways are there to make a given theory true? If structures represent semantic possibilities, then the question becomes: How many models does a given theory have? Here is a very naïve (partial) answer:

The Naïve Supervenience Thesis. Whenever two structures make exactly the same sentences true, they are really the same structure.

We call this a ‘supervenience’ thesis, because it holds that structures supervene upon a basis of (true) sentences. But the Naïve Supervenience Thesis is obviously false. Let $M$ be any model. Let $\pi$ be any bijection whose domain is $M$. Then, using the Push-Through Construction of §2.2, we can build an isomorphic and hence elementary equivalent structure $N$. So $N$ and $M$ make exactly the same sentences true. But it is very easy to choose $\pi$ so that $N \neq M$. So, we will have distinct models of exactly the same sentences, thereby refuting the Naïve Supervenience Thesis.

We said that it is easy to choose $\pi$ so that $N \neq M$. One easy way to do this is to start by selecting a set, $N$, which is distinct from $M$ but which has the same cardinality as $M$: because $M$ and $N$ have the same cardinality, there will be a suitable bijection $\pi$; and because $M \neq N$, the models built upon them will be different. An alternative approach is to allow that $N = M$, but to choose $\pi$ so that it ‘moves’ at least one of the constant, relation, or function symbol of $M$; in this case, $\pi$ is called a (non-trivial) permutation.

These thoughts underpin Putnam’s permutation argument [1981, pp. 33–5, 217–18]. (Putnam specifically focuses on the case where $N = M$, which gives the argument its name.) The main interest in the permutation argument is not, though, in its immediate refutation of the Naïve Supervenience Thesis; rather, it is in what Putnam went on to say.

Given the idea that models represent semantic possibilities, the Push-Through Construction shows that there are many different ways to make a given theory true and, in particular, many different ways to set up language–object reference relations whilst preserving the truth values of any sentences. Observing this, Putnam went on to argue that certain philosophical positions are forced to concede that reference is radically indeterminate: radically, because for any name and any object, we can use the Push-Through Construction to generate a

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3Some such bijection exists whenever $M$ is remotely interesting; for details, see [Button 2013, pp. 229–30].

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structure where that name refers to that object; and \textit{indeterminate}, because none of the reference candidates is to be preferred over any other. But, Putnam continued, the radical indeterminacy of reference is absurd. And so he concluded that those philosophical positions must themselves be rejected as absurd.

\section*{3.2 Just more theory}

We shall discuss some of the positions that Putnam rejected as absurd in a moment. Framed in this way, however, there is an obvious reaction to Putnam’s argument: Putnam has not yet shown that there is no preferred reference candidate. Consequently, we can respond to Putnam’s argument by invoking something like the following:

\textit{The Preferable Supervenience Thesis.} Whenever two preferable structures make exactly the same sentences true, they are really the same structure.

Whatever exactly one says about \textit{preferability}, the general idea is just that the models generated by an arbitrary permutation will not be (sufficiently) \textit{preferable} (see e.g. Merrill [1980, p. 80]; Lewis [1984, pp. 227–8]).

Putnam [1977, pp. 486–7, 1980, p. 477, 1981, pp. 45–8, 1983, p. ix] anticipated this line of response and suggested a perfectly general reply (aspects of which were anticipated by Winnie [1967, pp. 228–9]; Field [1975, pp. 383–4]; Wallace [1979, pp. 309–11]). Putnam insisted that an advocate of the Preferable Supervenience Thesis must explain her notion of \textit{preferability}, and tell us why preferable structures give us a better handle on reference than non-preferable ones. That is, she must advance a \textit{theory} of preferability. Putnam then noted that the Push-Through Construction establishes that \textit{any} theory has multiple models. This includes the theory of preferability itself, coupled with anything else we might want to say. In which case, the theory of preferability is \textit{just more theory}—more grist for the permutation-mill—and cannot constrain reference. For obvious reasons, this general argumentative strategy is called the \textit{just more theory manoeuvre}.

The just more theory manoeuvre has been widely criticised (see e.g. Lewis [1984, p. 225]; Bays [2001, pp. 342–8, 2008, pp. 197–207]; Button [2013, p. 29 n. 8] provides references to further criticisms of this sort). It is easy to explain why. To use one of Putnam’s own examples, suppose there are exactly as many cats as cherries, and that we are considering a permutation which sends every cat to a cherry. This permutation is supposed to make us worry that our predicate ‘... is a cat’ picks out the cherries, rather than the cats. But that permutation surely fails to respect the causal relationships that link our use of the word ‘cat’ to cats, rather than cherries. So, we are here suggesting that a \textit{preferable} model must respect certain causal constraints on language–object relations. In this context, the just more theory manoeuvre amounts to noting that we can reinterpret the word ‘causation’. But if causation does, in fact, fix reference, then this will just be a \textit{mis}interpretation, and of no significance.

This response is good, as far as it goes. But, in delivering a final verdict on the just more theory manoeuvre, we should remember that Putnam was not arguing \textit{for} radical semantic indeterminacy. Rather, Putnam was arguing only that \textit{certain} philosophical positions must
(absurdly) embrace it. The salient question is not, then, whether the Preferable Supervenience Thesis is true. The question is whether the positions that Putnam was concerned to attack can themselves invoke the Preferable Supervenience Thesis, or are instead bound, by their own lights, to treat the Preferable Supervenience Thesis as just more theory.

It would take us much too far afield to address that question in general. However, if we confine our attention to the philosophy of mathematics, then the just more theory manoeuvre can start to seem rather more compelling.

3.3 Moderate realism in philosophy of mathematics

Suppose that we adopt a position which Putnam [1980] called moderate realism. We shall discuss this position several times in what follows, so just to be clear: this is just one version of mathematical realism (and whether it is accurately characterised as moderate is besides the point).

Moderate realism is a combination of mathematical platonism with naturalism. The platonism dictates that mathematical entities are abstract, and not of our own creation, so that we cannot cannot pick them out by interacting with them. Or we cannot do so, unless we have some special, magical powers. But the naturalism rules out the existence of any such magical powers. Consequently, the only remaining hope for picking out mathematical entities is by laying down certain (formal) theories concerning how those entities behave. In a slogan: for the moderate realist, there can be no ‘reference by acquaintance’ to mathematical objects; ‘reference by description’ is our only hope.

So now: suppose that the moderate realist responds to the permutation argument by invoking the Preferable Supervenience Thesis. Given her naturalism, the preferability of one model over another cannot consist in the fact that our ‘mathematical gaze’ falls upon certain objects rather than others, for example. But it then becomes extremely unclear what preferability could consist in. It might be more reasonable to say, for example, that a preferable model of arithmetic is a countable model, i.e. one whose domain is countable. But such a claim faces two problems. First, it will not address the permutation argument, since the Push-Through Construction allows us to construct many isomorphic-but-distinct countable models of arithmetic. Second, the notion of being a ‘countable’ model is itself a mathematical notion. Consequently, the moderate realist herself holds that our only grasp of that notion comes from setting down some formal mathematical theory. And, as such, her attempt to constrain reference is, indeed, just more (mathematical) theory. It is yet more grist for Putnam’s permutation-mill; for to presuppose a grasp of it is to presuppose precisely what was at issue.

Simply put: a moderate realist who invokes the Preferable Supervenience Thesis faces a dilemma (see Hodes [1984, pp. 127, 133–5]; McGee [1997, pp. 35–8]; Button [2013, chs. 3–7] argues that the same dilemma applies in a rather more general context). If she invokes a notion of preferability which could fix reference, that notion will look just like the magic which she claims to reject. If she invokes a notion of preferability which is compatible with her moderate realism, then it cannot fix reference.
4 Ramsey sentences and the Newman objection

We shall return to the case of philosophy of mathematics in later sections. First, we shall explore some related topics in the philosophy of science. For the foregoing discussion puts us in an excellent position to make sense of some recent discussions of Ramsey sentences and the so-called Newman objection.

The motivating philosophical thought behind those discussions is that our scientific theorising is split into two parts. The first part is in good standing and poses no particular puzzles; the second is rather more problematic. For the sake of simplicity, we shall call the former part the observational part, and the latter the theoretical part. Note, though, that nothing much hangs on these labels, and we might equally call them the okay and troublesome parts, respectively. Indeed, in Lewis’s hands [1970], the observational/theoretical distinction became the (relativised) distinction between ‘old’ terms, with which we are familiar, and ‘theoretical’ terms, which we are seeking to define in terms of the ‘old’ terms. And within the philosophy of mathematics, there is Hilbert’s [e.g. 1925, p. 179] analogous real/ideal distinction.

In what follows, we shall track the distinction as follows. Where $L$ is a signature, $L_o \subseteq L$ is a privileged subset which is treated as $L$’s observational vocabulary, and $L_t = L \setminus L_o$ is its theoretical vocabulary.

4.1 Ramsey sentences

We shall work with a theory $T$ whose signature is $L$. For simplicity, we assume that $L$ is relational, i.e. that it contains only relation symbols. This is no real loss, as constants and functions can be simulated using relations and identity. However, from now until §4.7, we shall also make the substantive assumption that $T$ is finitely axiomatizable. Consequently, we shall regard $T$ as a single sentence (by conjoining its axioms if necessary). Since $T$ is finite, we can also regard $L$ as finite. So in particular, $L$’s theoretical vocabulary, $L_t$, is just a finite set of relation symbols $R_1, \ldots, R_n$. We can now define $T$’s Ramsey sentence: $\text{ram} T := \exists R_1 \ldots \exists R_n T$. That is, all of $T$’s theoretical vocabulary are treated as variables, and then bound by existential quantifiers. (Note that Ramsey sentences are only definable for finitely axiomatizable theories.)

By construction, $\text{ram} T$ is a second-order $L_o$-sentence. That is, its only non-logical vocabulary is all observational, but it quantifies into predicate position. In entertaining such second-order sentences, we go beyond the discussion of §2, which was restricted to first-order sentences. There are two distinct ‘classic’ semantics for second-order logic—full semantics, and Henkin semantics with all Comprehension instances—and they behave somewhat differently. Fortunately, all of the technical results we mention up until §4.6 (i.e. Propositions 4.1–4.6 inclusive) go through with either of these ‘classic’ semantics; so we shall suppress this point until §5.

The Ramsey sentence first formulated in Ramsey’s posthumously published paper “Theo-
ries” [1931]. It came to prominence in Carnap’s writings in the late 1950s and 1960s, where he related it to his long-standing concerns about the meaning of theoretical terms [1958, p. 245, 1959, pp. 160–5, 1966, 248, 252, 265ff.]. We shall present a contemporary (and mildly anachronistic) motivation for such concerns.

As noted in §3.2, one might be able to argue that causation guarantees that our predicate ‘. . . is a cat’ picks out cats rather than cherries. In this case, however, we are likely to treat ‘. . . is a cat’ as an observational predicate. And, even if causation can pin down the reference of observational predicates, it is much less clear what could pin down the reference of theoretical predicates: crudely put, we cannot point at electrons in the same way as we point at cats. So, to put matters rather roughly, it may seem that our knowledge of theoretical predicates will have to come via description—in terms of their impact on our observations—rather than by acquaintance. (Cf. Maxwell [1971], Zahar and Worrall [2001, pp. 239, 243], Hodesdon [forthcoming, §6], and the discussion of moderate realism in §3.3.)

If we buy this line of thought, then ramT will start to recommend itself to us. For ramT contains no theoretical vocabulary, and so is not vulnerable to the concerns just mentioned (cf. Carnap [1958, pp. 242, 245, 1966, pp. 251, 269]). And, crucially, nothing much is lost by considering ramT rather than T itself, for it is easy to show that the two are observationally equivalent in the following sense [Carnap 1958, p. 245, 1959, pp. 162–4, 1963, p. 965, 1966, p. 252; Psillos 1999, p. 292 n. 6; Zahar and Worrall 2001, pp. 242–3; Worrall 2007, p. 150]:

**Proposition 4.1.** Let T be a finitely axiomatizable L-theory. Then $T \models \varphi$ iff $\text{ram} T \models \varphi$, for all $\mathcal{L}_o$-sentences $\varphi$.

This led Carnap [1959, pp. 164–5] to claim that a theory’s Ramsey sentence expresses that theory’s ‘factual content’. 5

Suppose we follow Carnap’s initial motivations: there are difficulties with the theoretical vocabulary; but ramT does not share those difficulties; and indeed ramT has the same observational content as T, in the sense of Proposition 4.1. But suppose that we also maintain that ramT expresses some additional content beyond its observational consequences, namely, that there are some theoretical objects and relations. Moreover, we might hold that ramT specifies the behaviour of theoretical entities as tightly as we could hope; giving as full a statement of realism as one could hope to maintain, concerning a realm within which determinate reference is bound to be problematic (cf. Worrall [2007, p. 148]; Demopoulos [2011, p. 200]).

This line of thought culminates in what we call ramsified realism. This holds that (i) a

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5Define T’s Carnap Sentence as follows: $\text{car} T := (\text{ram} T \to T)$. There is a simple connection between Ramsey and Carnap sentences:

Let T be a finitely axiomatizable L-theory. Then $\text{ram} T \land \text{car} T \models T$ and $T \models \text{ram} T \land \text{car} T$.

Consequently, Carnap maintained that physical theories can be exclusively and exhaustively factored into a purely factual part, ramT, and a purely analytic part, carT, which records the theory’s ‘meaning postulates’ [1958, p. 246, 1959, pp. 163–4, 1963, p. 965, 1966, pp. 270–2]. So Carnap’s own interest in Ramsey sentences was tied to his long-standing interest in analyticity.
physical theory’s Ramsey sentence expresses its genuine content, but that (ii) the content that is thereby expressed goes significantly beyond its observational consequences.\textsuperscript{6}

4.2 Newman’s criticism of Russell

In what follows, we consider various problems for ramsified realism. These problems came to contemporary prominence when Demopoulos and Friedman [1985] related Ramsey sentences to Newman’s [1928] critique of Russell’s *The Analysis of Matter* [1927].

As Newman understood Russell’s 1927-position, Russell was committed to the doctrine that we could only know the structure of the external world. Here, structure was understood very much along the model-theoretic lines outlined in §2, and Newman’s objection was straightforward. For any structure $\mathcal{W}$,

Any collection of things can be organised so as to have the structure $\mathcal{W}$, provided there are the right number of them. Hence the doctrine that only structure is known involves the doctrine that nothing can be known that is not logically deducible from the mere fact of existence, except (‘theoretically’) the number of constituting objects. (Newman [1928, p. 144], with a slight change to typography)

Note that Newman’s initial claim is just a restatement of our Push-Through Construction from §2.2; indeed Newman [1928, pp. 145–6] gestures at the Construction himself.

Newman’s objection raises serious problems for anyone who thinks that we can know only the structure of the external world. But as Zahar and Worrall [2001, pp. 238–9, 244–5]; Worrall [2007, p. 150]; Ainsworth [2009, pp. 143–4] note, this seems to be a rather restricted target. Even if we grant that we can only know the structure of the external theoretical realm, we might think that we can know rather more about the external observational realm. Indeed, ramsified realism seems to embrace precisely this attitude. So it is natural to ask whether Newman’s objection against Russell can be turned into an objection against ramsified realism.

Over the next few subsections, we shall attempt to formulate the strongest possible Newman-style objection against ramsified realism. We think it arises by combining two simple thoughts: one relating to the Push-Through Construction, and the other relating to various model-theoretic notions of conservation.

4.3 The Conservation-Based Newman objection

We start by developing the notion of conservation. We shall show that Ramsey sentences are object-language statements of conservation, and we shall use this to present a Newman-style objection.

\textsuperscript{6}Psillos [1999, ch. 3] seems to suggest that Carnap himself was a ramsified realist. For a contrasting view, see Demopoulos [2003, pp. 384–90, 2011, pp. 195–7]. We cannot pursue this suggestion in detail, but we note that Carnap [esp. 1966, p. 256] may well have simply embraced the conclusions of the various Newman-style objections which we against present against ramsified realism in the remainder of this section.
Suppose theory \( S \) is formulated in a purely observational language. Imagine, too, that we have a guarantee that no more observational consequences follow from some theory \( T \) than follow from \( S \) alone. Obviously, the notions of ‘guarantee’ and ‘following from’ will need to be made more precise. But if they are suitably rich, this guarantee will allow us to become instrumentalists (or fictionalists) about everything in \( T \) which goes beyond \( S \). That is: we could in good conscience employ the full power of \( T \), without having to insist that its distinctively theoretical claims are true; they would simply be an expedient way to allow us to handle observable claims. (Note, in passing, that this line of thought seems to pull us away from ramsified realism, since ramsified realists want to incur substantive theoretical commitments.)

To tidy up the line of thought that points us towards instrumentalism, we can employ the following definition:

**Definition 4.2.** \( T \) is consequence-conservative over \( S \) iff if \( T \models \varphi \) then \( S \models \varphi \) for all sentences \( \varphi \) in the signature of \( S \)

The immediate interest in this definition is as follows. Suppose that \( S \) is the set of all true observation sentences (expressed in some suitably rich vocabulary). If \( T \) is consequence-conservative over \( S \), then we have a guarantee of the ‘observational-reliability’ of \( T \).

Here, though, is a second way to think about ‘observational-reliability’. Imagine that we can turn any model of \( S \) into a model of \( T \), without postulating any new entities, but simply by interpreting a few more symbols in \( T \)’s vocabulary. Then any configuration of observational matters that makes \( S \) true is compatible with the truth of \( T \). Intuitively, then, \( T \) cannot make any false judgements about observational matters; so, once again, \( T \) will be ‘observationally reliable’.

To formalise this second notion, we need some new notation. \( A \upharpoonright L \) is the \( L \)-structure one obtains by starting with \( A \), and then forgetting about the interpretation of any symbols in \( A \)’s signature but not in \( L \). We call \( A \upharpoonright L \) the \( L \)-reduct of \( A \) (for definitions, see Hodges [1993, 9ff.]; Marker [2002, p. 31]). To be clear: in obtaining the reduct, we do not remove any elements from \( A \)’s domain; in Quinean terms [1951, p. 14], we reduce the ideology and not the ontology. This affords a second notion of conservation:

**Definition 4.3.** Let \( T \) be an \( L \)-theory and \( S \) be an \( L_o \)-theory. \( T \) is expansion-conservative over \( S \) iff for any \( L_o \)-structure \( M \models S \), there is an \( L \)-structure \( N \models T \) such that \( M = N \upharpoonright L_o \).

It is natural to ask how these two notions of conservation connect with one another. In fact, the Ramsey sentence provides us with a simple demonstration that they align perfectly:

**Proposition 4.4.** Let \( T \) be any finitely axiomatizable \( L \)-theory, and let \( S \) be any \( L_o \)-theory. The following are equivalent:

1. \( T \) is consequence-conservative over \( S \)
2. \( T \) is expansion-conservative over \( S \)
3. \( S \models \text{ram} T \)
Proof. (3) \(\Rightarrow\) (2). Suppose \(S \models \text{ram}T\). Let \(\mathcal{M} \models S\). Since \(\mathcal{M} \models \text{ram}T\), we can choose suitable witnesses for \(\text{ram}T\)'s initial existential quantifiers, and then take these witnesses as the interpretations of the \(\mathcal{L}_t\)-predicates, giving us an \(\mathcal{L}\)-structure \(\mathcal{N} \models T\) such that \(\mathcal{M} = \mathcal{N}\rceil_{\mathcal{L}_o}\).

(2) \(\Rightarrow\) (1). Let \(\varphi\) be any \(\mathcal{L}_o\)-sentence such that \(T \models \varphi\). Suppose \(\mathcal{M} \models S\). If \(T\) is expansion-conservative over \(S\), then there is a structure \(\mathcal{N}\) such that \(\mathcal{N} \models T\) and \(\mathcal{M} = \mathcal{N}\rceil_{\mathcal{L}_o}\). Since \(\mathcal{N} \models T\), we have \(\mathcal{N} \models \varphi\). But since \(\mathcal{M} = \mathcal{N}\rceil_{\mathcal{L}_o}\), \(\mathcal{M} \models \varphi\). Since \(\mathcal{M}\) was arbitrary, \(S \models \varphi\).

(1) \(\Rightarrow\) (3). \(T \models \text{ram}T\) by rules for existential quantification; and \(\text{ram}T\) is an \(\mathcal{L}_o\)-sentence. So if \(T\) is consequence-conservative over \(S\), then \(S \models \text{ram}T\).

Informally glossed, Proposition 4.4 tells us that Ramsey sentences are object-language statements of conservation (in both senses).

We have not found Proposition 4.4 precisely stated in the literature. However, it allows for a neat Newman-style objection against ramsified realism:

**Conservation-Based Newman Objection.** Instrumentalists and realists can agree that theories should only entail true \(\mathcal{L}_o\)-sentences. Call this constraint \(\mathcal{L}_o\)-soundness. Let \(S\) be the set of all true \(\mathcal{L}_o\)-sentences. To say that \(T\) is \(\mathcal{L}_o\)-sound is now to say that \(T\) is consequence-conservative over \(S\). By Proposition 4.4, this is equivalent to the claim that \(S \models \text{ram}T\). In short: \(\text{ram}T\) is true iff \(T\) is \(\mathcal{L}_o\)-sound. So \(\text{ram}T\) expresses no content beyond \(T\)'s observational consequences, and ramsified realism is refuted.

### 4.4 Observational vocabulary versus observable objects

The ramsified realist has only one possible response against the Conservation-Based Newman Objection: she must deny that instrumentalists are committed to the \(\mathcal{L}_o\)-soundness of physical theories.

This is less strange than it might initially sound, as Worrall [2007, p. 152] has explained. Suppose \(T\) entails the existence of objects which fall under no \(\mathcal{L}_o\)-predicate. If we believe that our \(\mathcal{L}_o\)-vocabulary is adequate, we can characterise this by saying that \(T\) entails an \(\mathcal{L}_o\)-sentence which says, in effect, ‘there are unobservable objects’. So, in order for \(T\) to be \(\mathcal{L}_o\)-sound, there must be unobservable objects. But no instrumentalist should be committed to the existence of unobservables. So instrumentalists cannot be committed to the \(\mathcal{L}_o\)-soundness of physical theories (in general).

To accommodate this point, we shall assume that our observational vocabulary, \(\mathcal{L}_o\), includes a primitive one-place predicate, \(O\), which is intuitively to be read as ‘…is an observable object’. In this new setting, not just any \(\mathcal{L}_o\)-sentence will count as an observation.

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3. This predicate allows us to simulate Ketland’s [2004, pp. 289ff, 2009, pp. 38ff] use of a two-sorted language. Again, if the \(\mathcal{L}_o\)-vocabulary is adequate, then \(O\) can be defined as a disjunction of the other (finitely many) \(\mathcal{L}_o\)-predicates.
sentence. Rather, the observation sentences will be just those sentences which, intuitively,
are restricted to telling us about the observable objects. More precisely, the observation
sentences are the $\mathcal{L}_o$-sentences whose quantifiers are restricted to $O$. Call these the $\mathcal{L}_oO$
-sentences.

We can now easily accommodate Worrall’s point: ‘there are unobservable objects’, i.e.
$\exists x\neg Ox$, uses an unrestricted existential quantifier, and so is an $\mathcal{L}_o$-sentence but not an $\mathcal{L}_oO$-sentence. Moreover, in these terms, instrumentalists and realists will both insist
only upon $\mathcal{L}_oO$-soundness, i.e. that $T$ should entail only true $\mathcal{L}_oO$-sentences.11 But it is
clear that $\mathcal{L}_oO$-soundness is strictly weaker than $\mathcal{L}_o$-soundness, and it is only obvious that
realists have any motivation for insisting on the latter. Consequently, the Conservation-
Based Newman Objection seems to fail.12

4.5 The Strong Newman objection

In the previous two subsections, we focused on observation sentences, and regarded a theory
as inadequate if it entails any false observation sentence. As Ketland [2004, 2009] notes,
though, we can instead consider observable facts (in some intuitive sense). In this setting,
we will regard a theory as inadequate if it is incompatible with the actual observable facts.

We can use model theory to tidy up this idea of incompatibility. Let $\mathcal{I}$ be the intended model of the observable facts. So $\mathcal{I}$’s underlying domain consists of all and only the actually
observable entities in the physical universe. Moreover, any given observable entities stand in
a certain observational relation to each other iff $\mathcal{I}$ represents them as so doing. (Note, then,
that $\mathcal{I}$ is an $\mathcal{L}_o$-structure whose domain $I = O^\mathcal{I}$.) The question of whether $T$ is compatible
with the observable facts now becomes the question: Is there a model of $T$ whose ‘observable
part’ is just $\mathcal{I}$ itself? To make this more precise, we invoke the following definition:

**Definition 4.5.** For any $\mathcal{L}$-structure $\mathcal{M}$, its observable part, written $\text{Ob}(\mathcal{M})$, is the follow-
ing $\mathcal{L}_o$-structure:

1. $\text{Ob}(\mathcal{M})$’s domain is $OM$, i.e. exactly the ‘observables’ in $\mathcal{M}$
2. $R^{\text{Ob}(\mathcal{M})} = R^\mathcal{M} \cap (OM)^n$, for each $n$-place $\mathcal{L}_o$-predicate $R$
3. $\text{Ob}(\mathcal{M})$’s $n$-place second-order quantifiers range over $\{X \in M^n_{\text{rel}} \mid X \subseteq (OM)^n\}$,
   where $M^n_{\text{rel}}$ is the range of $\mathcal{M}$’s $n$-place second-order quantifiers.

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10We define this recursively: let $\varphi^O := \varphi$ if $\varphi$ is atomic, let $(\varphi \land \psi)^O := (\varphi^O \land \psi^O), let \neg(\varphi)^O := \neg(\varphi^O)$,
let $(\exists x \varphi)^O := \exists x(\exists x \land \varphi^O)$ and let $(\exists x \varphi)^O := \exists x(\exists x \land \varphi^O) \land (\exists x \varphi)^O := \exists x(\exists x \land \varphi^O) \land \varphi^O)$. The set of $\mathcal{L}_oO$-formulas is then $\{\varphi^O \mid \varphi$ is a $\mathcal{L}_o$-formula\}. This definition ignores complexities surrounding ‘mixed’ (i.e. part observational, part theoretical) predicates; we revisit this in footnote 14, below.

11This is essentially Ketland’s [2004, Definition D] notion of ‘weak empirical adequacy’.

12One might hope to keep the spirit of the Conservation-Based Newman Objection alive, by finding a notion of conservation appropriate to $\mathcal{L}_oO$-sentences. A notion is readily available: say that $T$ is $O$-conservative over $S$ iff if $T \models \varphi$ then $S \models \varphi$ for all $\mathcal{L}_oO$-sentences $\varphi$. (This resembles a notion which comes out of Field’s [1980] nominalist programme; see Shapiro and Weir [1999, p. 297]; Weir [2003, p. 21].) To reanimate the Conservation-Based Newman Objection, we would now want a result of the form: on minimal assumptions, $T$ is $O$-conservative over $S$ iff $S \cup \{‘\text{there are sufficiently many unobservables'}\} \models \text{raw}T$. Unfortunately, we cannot see how to obtain a result with suitably minimal assumptions.
We can now say that $T$ is compatible with the observable facts iff there is a $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $\text{Ob}(\mathcal{M}) = \mathcal{I}$ (cf. Ketland [2004, Definition F]). Note that, as in §3.1, we are using model theory to simulate possibilities.

The aim is to run a Newman-style argument in this setting. We shall need a suitable model-theoretic result, and one quickly suggests itself, as a combination of two ideas we have already seen. The first idea comes from the Push-Through Construction of §2.2, which was also the basis for Newman’s objection to Russell (see §4.2). Fairly trivially, if we consider a model in isolation, it does not matter which objects are in its domain; and in the present setting, the important point is that any object will do as an ‘unobservable’. The second idea relates to a trivial corollary of Proposition 4.4, namely that $T$ is expansion-conservative over $\text{ram}T$. Combining the two ideas, we obtain the desired result:

**Proposition 4.6.** Let $T$ be any $\mathcal{L}$-theory, with $O$ in $\mathcal{L}_o$. Let $\mathcal{M}$ be any $\mathcal{L}_o$-structure whose domain is $O^\mathcal{M}$. The following are equivalent:

1. There is an $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{N} \models T$ and $\mathcal{M} = \text{Ob}(\mathcal{N})$
2. There is some cardinal $\kappa$ such that, for any set $U$ of cardinality $\kappa$ with $M \cap U = \emptyset$, there is an $\mathcal{L}_o$-structure $\mathcal{P}$ with domain $M \cup U$, with $\mathcal{M} = \text{Ob}(\mathcal{P})$ and with $\mathcal{P} \models \text{ram}T$

**Proof.** (1) $\Rightarrow$ (2). Let $\mathcal{N}$ be as in (1), so $O^\mathcal{M} = O^\mathcal{N} = M$. Let $\kappa$ be the cardinality of the unobservables in $\mathcal{N}$, i.e. $\kappa = |N \setminus M|$. Let $U$ be any set of cardinality $\kappa$ with $M \cap U = \emptyset$; so there is a bijection $\sigma : (N \setminus M) \rightarrow U$. Define a further bijection $\pi : N \rightarrow M \cup U$ by setting: $\pi(x) := x$ if $x \in M$, and $\pi(x) := \sigma(x)$ otherwise. Now use $\pi$ to define an $\mathcal{L}$-structure $\mathcal{P}^*$ from $\mathcal{N}$ as in the Push-Through Construction, letting the range of the second-order variables in $\mathcal{P}^*$ be given by the image of the second-order variables in $\mathcal{N}$ under the action of $\pi$. Taking $\mathcal{P} = \mathcal{P}^*|_{\mathcal{L}_o}$, this has the required properties.

(2) $\Rightarrow$ (1). Let $\mathcal{P}$ be as in (2). Since $T$ is expansion-conservative over $\text{ram}T$, there is an $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{N} \models T$ and $\mathcal{P} = \mathcal{N}|_{\mathcal{L}_o}$, and hence $\mathcal{M} = \text{Ob}(\mathcal{P}) = \text{Ob}(\mathcal{N})$.

We can now very easily offer an objection against ramsified realism. (This is close to Ketland’s preferred version of the Newman objection; cf. also Ainsworth [2009, pp. 144–7]):

**Strong Newman Objection.** Instrumentalists and realists can agree that, at a minimum, $T$ must be compatible with the actual empirical facts. But Proposition 4.6 tells us that $T$ is compatible with the actual empirical facts iff there is some cardinal $\kappa$ such that we can obtain a model of $\text{ram}T$ simply by adding $\kappa$-many ‘unobservables’ to $\mathcal{I}$. Otherwise put: $\text{ram}T$ is true provided (i) it gets everything right about the observables and (ii) there are ‘sufficiently many’ unobservables. In which case, a commitment to $\text{ram}T$ scarcely qualifies as a version of realism.

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13Ketland’s objection falls out of his Theorem 6 [2004, pp. 298–9]. This is close to our Proposition 4.6, but it invokes full second-order logic (whereas, as noted in §4.1, Proposition 4.6) holds with both full and Henkin semantics.
4.6 Natural properties and just more theory

Ramsified realists might offer several responses against the Strong Newman Objection (see Ainsworth [2009, pp. 148ff.] for a survey). However, we shall pursue one response in detail.\(^{14}\)

Proposition 4.6 uses model theory to generate a ’trivialising’ structure, \(\mathcal{P}\), from a given structure, \(\mathcal{N}\). The existence of \(\mathcal{P}\) is guaranteed by the standard set-theoretic axioms which we assume whenever we do model theory. But, perhaps unsurprisingly, model theory is rather profligate when it comes to the existence of models and of second-order entities. The ramsified realist may, then, maintain that certain properties and relations are natural, whilst others are mere artefacts of a model theory that is suitable for mathematics but not for physical theorising. If she can uphold this point, then she can reject the significance of Proposition 4.6 for physical theorising, and so answer the Strong Newman Objection (cf. Psillos [1999, pp. 64–5]; Ketland [2009, p. 44]; Ainsworth [2009, pp. 167–9]).

The obvious question is whether belief in such natural kinds is consistent with ramsified realism. To believe that some properties or relations are natural is to postulate a third-order property of second-order entities, namely, naturalness. This third-order property, however, does not look like it could plausibly belong with the observational vocabulary; so it must belong with the theoretical vocabulary. But then the ramsified realist is duty-bound to ramsify away the property of naturalness; that is, she must treat it as an existentially bound third-order variable. And now, given even remotely permissive comprehension principles for third-order logic, the invocation of (a ramsified property of) naturalness will impose no constraint whatsoever upon our structures. (For a similar argument, see Ainsworth [2009, pp. 160–2, 169]; also Demopoulos and Friedman [1985, p. 629]; Psillos [1999, pp. 64–6]; Ketland [2009, p. 44].)

Doubtless, there is more to say on this point. However, we think it is illuminating to compare the preceding few paragraphs explicitly with the dialectic surrounding Putnam’s permutation argument.

\(^{14}\)A further point merits comment. We have made no particular assumptions about how the observable entities interact with the observational relations. Two perhaps plausible assumptions are:

(i) observable objects never fall under theoretical relations, i.e. for any \(\mathcal{L}\)-structure \(\mathcal{M}\) and any \(n\)-place \(\mathcal{L}_t\)-predicate \(R\), we have \(R^\mathcal{M} \cap (O^\mathcal{M})^n = \emptyset\).

(ii) unobservable objects never fall under observational relations, i.e. for any \(\mathcal{L}\)-structure \(\mathcal{M}\) and any \(n\)-place \(\mathcal{L}_o\)-predicate \(R\), we have \(R^\mathcal{M} \subseteq (O^\mathcal{M})^n\).

These kinds of assumptions are discussed in the literature, sometimes in terms of what to say about ‘mixed’—i.e. part-observational, part-theoretical—predicates. (See Cruse [2005, pp. 561ff.]; Ainsworth [2009, pp. 145–6, 155–60]; Ketland [2004, p. 292, 2009, pp. 38–9.]) However, we think we can largely sidestep this discussion, for the following reason.

A philosopher who rejects (ii) might insist that we enlarge our intended model of the observable facts, \(\mathcal{I}\), so that it includes those unobservable entities which stand in observation relations to observable entities. If we concede this point, then we shall similarly want to modify the definition of \(\text{Ob}(\mathcal{M})\). (It is unlikely that rejecting (i) would force us to change anything in our setup.) But making these changes this will not affect the two fundamental ideas behind Proposition 4.6, and hence behind the Strong Newman Objection. We can always arbitrarily permute away any entities we like, and so, in particular, those which our target philosopher designates as ’not observable’; and we can always ramsify away any vocabulary we like, and so, in particular, those predicates which our target philosopher designates as ’not observational’.

14
Our ramsified realist wanted to maintain that a Ramsey sentence is made true (if at all) only by the existence of a structure whose second-order entities are *natural*. This is just a version of the Preferable Supervenience Thesis, as considered in §3.2 as a response to Putnam’s permutation argument (cf. [Ainsworth 2009, pp. 162–3]). And, as in §3.2, any defender of that Thesis must face up to Putnam’s just more theory manoeuvre. In the present context, the allegation will be that to invoke ‘naturalness’ is just more theory.

Now, in §3.2, we noted that there are positions according to which the just more theory manoeuvre simply looks bizarre. However, we also noted that the just more theory manoeuvre poses real difficulties for certain other positions, such as moderate realism in the philosophy of mathematics. So here too: belief in intrinsically natural kinds may be a defensible thesis; but the allegation is that the ramsified realist is bound by her own lights to treat ‘naturalness’ as just more theory, in the precise sense that it belongs to theoretical-vocabulary rather than the observational-vocabulary, and hence must be ramsified away.

In sum, there are deep connections between the permutation argument and the Newman objection. First: ramsified realism itself can be motivated by permutation-style worries about the reference of theoretical vocabulary (see §4.1). Second: the Push-Through Construction is used in Putnam’s permutation argument, in Newman’s original argument, and in the Strong Newman Objection (cf. Demopoulos and Friedman [1985, pp. 629–30]; Lewis [1984, p. 224 n. 9]; Ketland [2004, pp. 294–5, 298–9]; Hodesdon [forthcoming, §5]). Third: similar responses are available to both arguments, and such responses must confront the just more theory manoeuvre head-on (cf. Hodesdon [forthcoming, §6]).

### 4.7 Conservation in first-order theories

Proposition 4.4 and Proposition 4.6 were presented in the context of finitely axiomatizable second-order theories. If we shift to first-order logic, and remove the restriction to finitely axiomatizable theories, we obtain a rather different result. We first need some new notation. For any \( \mathcal{L} \)-structures \( \mathcal{A} \) and \( \mathcal{B} \), we say that \( \mathcal{A} \) is an *elementary substructure* of \( \mathcal{B} \) (and \( \mathcal{B} \) is an *elementary extension* of \( \mathcal{A} \)), written \( \mathcal{A} \preceq \mathcal{B} \), iff for any \( n \)-tuple \( \bar{a} \in A^n \) and any \( n \)-place \( \mathcal{L} \)-formula \( \varphi \) we have \( \mathcal{A} \models \varphi(\bar{a}) \) iff \( \mathcal{B} \models \varphi(\bar{a}) \) (cf. Marker [2002, p. 44], Hodges [1993, p. 54]). We now obtain:

**Proposition 4.7.** Let \( S \) be a first-order \( \mathcal{L} \)-theory; let \( T \) be a first-order \( \mathcal{L}^+ \) theory, with \( \mathcal{L} \subseteq \mathcal{L}^+ \). The following are equivalent:

1. \( T \) is consequence-conservative over \( S \).

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15Putnam’s *infallibilism argument* is also closely related to the Newman objection. (The name ‘infallibilism argument’ is from Button [2013, pp. 17–18, 41–2]; Putnam himself does not sharply separate this from his permutation argument.) The infallibilism argument begins by noting that, by the Completeness Theorem, any consistent (essentially first-order) theory \( T \) has a model; this raises the question of why *that* model does not itself make \( T \) true. As Newman’s original objection seeks to trivialise \( \text{ram}T \) by reducing it to an expression of cardinality, the infallibilism argument seeks to trivialise \( T \) by reducing it to an expression of consistency. This connection is explored by Demopoulos and Friedman [1985, pp. 632–5]; Demopoulos [2003, pp. 385–8].
(2) For any $L$-structure $M$ such that $M \models S$, there is an $L^+$-structure $N$ such that $M \preceq N|_{L}$.

Demopoulos [2011, pp. 186–90] uses this result to present another Newman-style objection. There are other essentially first-order results that one could use in pursuit of a Newman-style objection (cf. Winnie [1967, p. 226]; Zahar and Worrall [2001, pp. 246–8]; Button [2013, pp. 41–2]). However, we think that the discussion of §§4.1–4.6 highlights the key philosophical contours that surround the Newman objection, and so we shall set these further variants to one side.

We are interested in Proposition 4.7 because it highlights that there are independently interesting cases where (1) and (2) of Proposition 4.7 obtain, but where $M$ must be distinct from $N|_{L}$.

Let $L$ be the signature of ordered fields, $\{0, 1, +, \times, <\}$. Let $M$ be the $L$-structure whose domain is the real numbers, with the $L$-symbols interpreted in the natural way. Let $S$ be the complete theory of $M$, which it turns out may be completely axiomatized by the recursive (but not finite) theory of real-closed fields [Marker 2002, §3.3]. Expand $L$ to $L^+$ by adding a single new constant, $c$. By the Compactness Theorem for first-order logic—i.e. the fact that, if every finite subset of a theory has a model, then the theory itself has a model—the following theory $T$ has a model, wherein $n > 0$ ranges over natural numbers:

$$T = S \cup \{0 < c < \frac{1}{n} \mid n > 0\}$$

Since $S$ is a complete theory, condition (1) of Proposition 4.7 holds trivially, and so (2) must hold too. But clearly we cannot turn $M$ into a model of $T$ by assigning the new constant $c$ to any real number. So if $N \models T$ and $M \preceq N|_{L}$, then $N$ must add some genuinely new elements to $M$. Such elements will be bigger than 0, but smaller than $\frac{1}{n}$ for any $n > 0$. That is, these elements will be infinitesimals.

This kind of case was central to Robinson’s development of nonstandard analysis in the 1960s. The fundamental point is that, if due care is taken, one can show rigorously that postulating infinitesimals is consequence-conservative over non-infinitesimals. As in §4.3, then, this seems to open the way to an instrumentalist or fictionalist attitude towards infinitesimals. So we see here a glimmer of the Leibnizian idea that ‘the infinitesimal is a well-founded fiction’ [Jesseph 2008, p. 233]. (Robinson [1966, p. 2] went further, and claimed to have ‘fully vindicated’ ‘Leibniz’s ideas’; for criticism, see Bos [1974, pp. 81–6].)

5 Informal structure-talk and categoricity

We shall now set aside ramsified realism, and philosophy of science more generally, and return to the case of philosophy of mathematics.

\[16^{\text{th}}\] In the literature on the Newman objection, such cases were first emphasized by Ketland [2004, p. 297, 2009, p. 43 n. 10]. Ketland raises these examples to emphasize the difference between (what we have called) $L_{o}$-soundness and compatibility with the observable facts.
In §3.3, we suggested that the moderate realist will have to accept that the reference of her mathematical language is radically indeterminate. It is worth being clear that, for plenty of mathematical theories, this is of no concern at all. To take one example: group theory is not about any single mathematical structure, and its vocabulary is not meant to refer to any particular entities. The whole point of group theory is that it can be applied to many different mathematical areas, such as modular arithmetic, certain classes of permutations, etc. Similar points hold for theories governing rings, fields, etc. Following Shapiro [1997, pp. 40–1], we call these kinds of theories algebraic.

This shows that, if referential indeterminacy is to raise any problems for mathematical theories, the theories in question will have to be nonalgebraic. That is, they must be theories which aim to ‘describ[e] a certain definite mathematical domain’ [Grzegorczyk 1962, p. 39] or ‘specify one particular interpretation’ [Kline 1980, p. 273]. But in fact, it is not clear that referential determinacy is a problem even for nonalgebraic theories.

To probe this point, we need to have a better idea of what it means to call a theory nonalgebraic. In this section, we shall see how model theory might allow us to simulate this, using isomorphism types. This will take us into the initially technical, and ultimately philosophical, question of whether mathematical theories can pin down a unique isomorphism type.

5.1 Treating informal-structures as isomorphism types

Let Mel be a fairly typical mathematician who, following the preceding suggestions, tells us: Arithmetic, unlike group theory, is not an algebraic theory. Group theory applies to many structures, whereas arithmetic describes a single, particular structure; the natural numbers.

If Mel uses the word ‘structure’ here in the technical, model-theoretic sense defined in §2, then we have already have good reasons to doubt her claim that arithmetic describes just one structure. After all, the Push-Through Construction yields an abundance of isomorphic models; so, unless one of them is somehow preferable (see §3.2), arithmetic cannot ‘describe’ one of these rather than any other.

But this observation should not, by itself, make us despondent about the possibility of using model theory to understand claims like Mel’s. Rather, we need to be sensitive to two points. First, Mel’s invocation of the word ‘structure’ may be rather informal; or, at the very least, not precisely the same word as it appears in the mouth of the model theorist. Second, mathematicians often care much more about isomorphism types than they do about particular \( \mathcal{L} \)-structures.

This second point merits development. For many mathematical purposes, it seems not to matter which of two isomorphic structures one works with. To use Benacerraf’s [1965, §II] famous example: when doing arithmetic, it makes no difference whether we think of the natural numbers as von Neumann’s finite ordinals \( \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots \), or as Zermelo’s finite ordinals \( \emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\emptyset\}\}, \ldots \).

Now, we must be careful not to over-generalise this point. It simply is not true that mathematicians only study properties of structures that are invariant under isomorphism. For instance, the computability of a structure is not preserved under isomorphism. Nevertheless,
one might respond that computability merely concerns the presentation of the structure and not the structure itself. And indeed it is fairly common for mathematicians to maintain (in a context) that they care about the identity of a structure only ‘up to isomorphism’.

Various philosophical theories of structuralism have been developed, which attempt to build upon this point. We shall explore two variants of structuralism in §§8–9. For now, though, we shall make a simple-minded suggestion. In §3.1, we noted that model-theoretic notions might be used to explicate the intuitive ideas of reference and truth. We now propose, instead, that model theoretic notions might be used to explicate the informal idea of a mathematical structure. In particular, we shall offer the following heuristic suggestion:

*Treat informal-structures as isomorphism types; that is, treat them as equivalence classes of isomorphic \( L \)-structures.*

To be clear: this is not an ontological hypothesis. (Similarly: when we thought of \( L \)-structures as describing semantic possibilities, we were not advancing a hypothesis about modal ontology.) The main point of treating informal-structures as isomorphism types is that it will allow us to deploy model theory when discussing informal issues that arise within mathematics. This is, then, a project in applied mathematics: the mathematical theory which is being applied is model theory, and it is being applied, not to some branch of the physical sciences, but to mathematical discourse and mathematical practice.

5.2 Signature-sensitivity and sameness of structure

A model theorist’s structures are always relative to a specific signature (see §2). Consequently, the isomorphism type of the natural numbers under successor, and the isomorphism type of the natural numbers under less-than, are different entities. So, if Mel thinks there is just one natural number structure (informally construed), there is a mismatch between her way of speaking and the suggestion that we should treat informal-structures as isomorphism types. (And this is just the tip of the iceberg, for Mel’s informal ways of speaking; see Baldwin [2013, pp. 91ff.].)

The essential problem is that *isomorphism* is a rather fine-grained equivalence relation between structures. If we want to simulate Mel’s way of speaking, we need a coarser, signature-insensitive, relation. An obvious candidate arises by considering definitional equivalence. Roughly, two structures which have the same domain, but perhaps different signatures, are *definitionally equivalent* iff each can offer an explicit definition of the primitives of the other (for details, see Hodges [1993, pp. 59–63]; Visser [2006, §3]). Such a notion would capture the intuitive idea that two structures might be merely ‘notational variants’ of one another [Corcoran 1980b, p. 232], or ‘descriptions amounting to the same thing’ [Bouve`re 1965b, p. 622; Bouve`re 1965a; cf. Walsh 2014, pp. 96–7]. So we might instead, perhaps, treat informal-structures as definitional-equivalence-types.\(^\text{17}\)

\[^{17}\text{i.e. as equivalence classes given by the following relation between structures }\mathcal{A}\text{ and }\mathcal{B}: \text{there is some }\mathcal{A}'\text{ such that both }\mathcal{A} \cong \mathcal{A}'\text{ and also }\mathcal{A}'\text{ and }\mathcal{B}\text{ are definitionally equivalent. Both Shapiro and Resnik invoke this relation whilst developing their structuralism(s); we revisit this in §8.1.}\]
Definitional equivalence is, of course, relative to a choice of background logic: if we increase our logical resources, then more things become explicitly definable, and so the equivalence relation coarsens (cf. [Resnik 1997, p. 251]). Indeed, there are a host of related notions of ‘sameness of structure’ which are studied by model theorists, including mutual interpretability and biinterpretability. Given this plurality of notions of sameness of structure, it is not clear that any particular one of them will mesh perfectly with our informal structure-talk.

Despite this, we shall continue to pursue the idea of treating informal-structures as isomorphism types (and simply be ‘forgetful’ about choice of signature). The idea of an isomorphism type provides an easily understood bridge between informal and formal notions of structure, which will allow us to bring model theory to bear on philosophical questions.\(^{18}\)

### 5.3 Categoricity and semantic (in)determinacy

Consider, now, the following claim:

*The Categoricity Thesis.* Certain key mathematical theories pick out a single isomorphism type (though not a single \(\mathcal{L}\)-structure).

If we can defend the Thesis, and also agree to treat informal-structures as isomorphism types, then we will wholly vindicate Mel’s way of speaking (modulo the comments of §5.2).

To explain how the Categoricity Thesis might be defended, it will help to explain its name. We say that a theory is *categorical* iff all of its models are isomorphic. So, if we can demonstrate that certain key mathematical theories are categorical, we have a way to defend the Categoricity Thesis. In what follows, we shall discuss the prospects of such a defence.

First, it is worth noting that the Categoricity Thesis involves entirely surrendering the determinacy of mathematical reference. We cannot, now, maintain that any particular constant symbol, such as ‘0’, picks out exactly one entity. Of course, within each of the many (isomorphic) models of arithmetic, ‘0’ will pick out exactly one entity. But, by a simple application of the Push-Through Construction, for any object \(x\) there is some model \(M\) such that ‘0’ picks out \(x\) in \(M\).

Nonetheless, such referential indeterminacy need not lead to indeterminacy of truth value.\(^{19}\) Given a theory \(T\), consider the following *supervaluational* semantics for the sentences in \(T\)’s language:

\[
\text{(i) } \varphi \text{ is true } (T \models \varphi) \text{ iff every model of } T \text{ satisfies } \varphi.
\]

\(^{18}\)Walsh [2014, §4.2, pp. 102ff.] argues that a very specific epistemic variety of logicism might run into problems regarding the specific choice of signature and of ‘sameness’ relation. The difficulty arises because that type of logicism sought to use prior knowledge of mathematical structures to obtain knowledge of certain basic mathematical principles. But this difficulty need not affect all positions. For example, someone who assumes that they already know a great deal of mathematics might want to treat informal-structures as isomorphism types (cf. the *Mathematics-First Attitude* of §5.6).

\(^{19}\)McGee [1997, p. 43 n. 11] emphasises this point. Awodey and Reck [2002, p. 18] make a plausible case that the first author to have this point clearly in mind is Veblen in 1904–6. The observation is also a key part of the exchange between Kreisel [1967, pp. 147–52]; Weston [1976, p. 286].
(ii) \( \varphi \) is false \( (T \models \neg \varphi) \) iff every model of \( T \) satisfies \( \neg \varphi \).

(iii) \( \varphi \) is indeterminate otherwise.

If \( T \) is categorical, then exactly one of (i) and (ii) must obtain, for any given sentence \( \varphi \) in \( T \)'s language. For, let \( M \) be any model of \( T \). Either \( M \models \varphi \) or \( M \models \neg \varphi \). Now suppose also that \( N \models T \). Since \( T \) is categorical, \( M \cong N \) and hence, by the observations made in §2.2, \( M \equiv N \). So if \( M \models \varphi \) then \( N \models \varphi \); and if \( M \models \neg \varphi \) then \( N \models \neg \varphi \). Generalising: either every model of \( T \) satisfies \( \varphi \), or every model of \( T \) satisfies \( \neg \varphi \). Consequently, for a categorical theory with at least one model, the supervaluational semantics obeys standard (classical) rules. So, if we can defend the Categoricity Thesis, then we will also be able to defend:

The Truth-Determinacy Thesis. Certain key mathematical theories are such that every sentence in their language has a determinate truth value.

We can therefore summarise the impact of the Categoricity Thesis on semantic determinacy as follows. The Categoricity Thesis involves abandoning the idea that a mathematical theory is true of any particular entities. But we can still maintain that the theory is simply true. Provided, of course, that we can establish that theory’s categoricity.

5.4 The Löwenheim–Skolem Theorem

Unfortunately, an elementary result imposes an immediate barrier on establishing the categoricity of a mathematical theory (cf. [Marker 2002, §2.3 pp. 44 ff], [Rautenberg 2010, p. 112], [Shapiro 1991, p. 80], [Mancosu et al. 2009, §4 pp. 352 ff]):

Löwenheim–Skolem Theorem. Every countable first-order theory with an infinite model has a model of every infinite size.

As an immediate corollary, no countable first-order theory with an infinite model is categorical (since isomorphic models must be of the same size). But the theories we are likely to be most interested in here—arithmetic, analysis, and set theory—all have an infinite model, and so cannot be categorical. Would-be defenders of the Categoricity Thesis, then, will have to look beyond first-order logic. And so we turn to second-order logic.

5.5 Dedekind’s Categoricity Theorem

Second-order logic expands on the syntax of first-order logic by allowing quantification into predicate-position and function-position. We then need to specify the semantics of such formulas. To illustrate, here is the satisfaction-clause for full second-order universal quantification:

\[ (5.1) \quad M \models \forall P \phi \iff M \models \phi[X/P] \text{ for every } X \in \varphi(M) \]

\(^{20}\)Notation: \( M \models \phi[X/P] \) means that, when we augment \( M \)'s signature with a new predicate ‘\( S \)’ which is interpreted as standing for \( X \), \( \phi(S) \) is true in the augmented model.
In this, the powerset \( \wp(X) = \{ Y \mid Y \subseteq X \} \) is the set of all subsets of \( X \), which is also frequently denoted as \( P(X) \). Crucially, the Löwenheim–Skolem Theorem does not hold for full second-order logic [Shapiro 1991, p. 93; Manzano 1996, pp. 256–7]. Indeed, in this setting, there are some landmark categoricity results. For now, though, we shall consider only the categoricity of arithmetic, turning to set theory in \( \S 7.2 \).

When arithmetic is formulated as a first-order theory, known as PA, we require an induction scheme: \((\varphi_0 \land \forall x(\varphi x \to \varphi sx)) \to \forall x \varphi x\). That is to say, PA involves all of the (infinitely many) formulas which result by replacing the schematic variable ‘\( \varphi \)’ with a formula in the language of arithmetic. However, we can instead formulate arithmetic as a second-order theory, known as PA\(^2\). In that case, we replace the induction scheme with an induction axiom; a single second-order sentence: \( \forall P((P_0 \land \forall x(Px \to Psx)) \to \forall x Px) \). We must also include all of the instances of the Comprehension Scheme, \( \exists X \forall x(Xx \leftrightarrow \Phi x) \) for any formula \( \Phi \) in the signature not containing \( X \). Having done so, we can invoke Dedekind’s classic result, which establishes the categoricity of PA\(^2\):

\textbf{Dedekind’s Categoricity Theorem.} All full models of PA\(^2\) are isomorphic.

(For Dedekind’s original proof, see [Dedekind 1888, ¶132, 1930-1932, vol. 3 p. 376; Ewald 1996, vol. 2 p. 821]. For a contemporary proof, see [Shapiro 1991, pp. 82–3; Enderton 2001, p. 287].) So, if it is legitimate to insist on full second-order logic, then this Theorem will give us a way to defend the Categoricity Thesis, when it comes to arithmetic.

The ‘second-order’ component of ‘full second-order logic’ is surely unobjectionable. No one can prevent mathematicians from speaking a certain way, or from formalising their theories using any symbolism they like. The qualifying expression ‘full’, however, is rather more delicate.

In this context, the word ‘full’ describes a particular semantics for second-order logic; one in which the second-order quantifiers must range over the entire powerset of the first-order domain of the structure. This was shown vividly in the semantic clause (5.1). But there are alternative semantics for second-order logic. In particular, Henkin semantics is much more permissive than full semantics, concerning the range of the second-order quantifiers. When we specify a Henkin model, \( \mathcal{M} \), we must also specify some \( M_{\text{rel}} \subseteq \wp(\mathcal{M}) \) for the quantifiers to range over;\(^{21}\) and our satisfaction clause (5.1) is replaced with the following:

\[
\mathcal{M} \models \forall P \phi \iff \mathcal{M} \models \phi[X/P] \text{ for every } X \in M_{\text{rel}}
\]

Crucially, the Löwenheim–Skolem Theorem holds in second-order logic with Henkin semantics. As an immediate corollary, no second-order theory with an infinite model is categorical.

\(^{21}\)We may also want to insist that \( M_{\text{rel}} \) vindicates Choice and all instances of the Comprehension Scheme; for details, see [Shapiro 1991, pp. 88–9].
in Henkin semantics. Specifically, then, Dedekind’s Categoricity Theorem fails if we replace ‘full’ with ‘Henkin’.

Everything, then, turns on the question of whether it is legitimate for us to restrict our attention to full models of second-order logic.

5.6 Flexibility in the treatment

We shall probe that question in a moment. First, we wish to clarify an important dialectical point, building upon Shapiro [2012].

So far in this section, we have mentioned several purely technical points concerning categoricity. We have also made a more philosophical (but fairly commonplace) suggestion, that we can fruitfully treat informal-structures as isomorphism types. We want to be clear that this treatment will not itself settle many philosophical debates. Rather, its primary purpose is to furnish those debates with sharper tools. To emphasise this point, we want to note that four distinct attitudes towards Dedekind’s Categoricity Theorem are compatible with treating informal-structures as isomorphism types.

The Sceptical Attitude. There are various positions within the philosophy of mathematics that one might characterise as ‘sceptical’. A fairly extreme sceptic might deny that there is a single natural number structure (intuitively construed). Such a sceptic need not repudiate the idea that we can treat informal-structures as isomorphism types, and she should certainly not doubt the perfect mathematical rigour of Dedekind’s Categoricity Theorem. Rather, she—and all parties—should conclude that any indeterminacy in the natural numbers will be mirrored by indeterminacy in second-order quantification (see Shapiro [2012, p. 308] and cf. Mostowski [1967, p. 107]; Read [1997, p. 92]).

The Mathematics-First Attitude. One might treat it as a datum, given to us by mathematical practice, that there is exactly one natural number structure (informally construed). If we treat informal-structures as isomorphism types, then one can use the Löwenheim–Skolem Theorem to argue that we must have access to resources beyond first-order logic. For those who are not caught up in the Sceptical Attitude, this pattern of argumentation will look like nothing more than an inference to the best explanation. This is part of Shapiro’s approach to second-order logic [1991, pp. xii–xiv, 100, 207, 217–8, 2012, p. 306], and Shapiro cites Church [1956, p. 326 n. 535] as a precedent.

The Logic-First Attitude. Suppose we begin with the idea that our grasp on full second-order logic is completely unproblematic. Suppose we also believe that there is just one arithmetical structure (informally construed). We can then use our antecedently given grasp of second-order logic, coupled with our treatment of informal-structures as isomorphism types, to explain how we can pin down that single informal-structure. Equally, by invoking the supervaluational semantics of §5.3, we can explain how finite creatures like us could come to speak a language, all of whose infinitely many sentences have determinate truth values (cf. Read [1997, p. 89]).

The Holistic Attitude. The previous two attitudes seem to presuppose a clean separation between full second-order logic and mathematics. However, one might well think that the boundary between the two is rather artificial (see Shapiro [2012, pp. 311–22]; Väänänen
The final alternative, then, is to embrace a more holistic attitude, according to which our grasp of full second-order logic and our grasp of determinate informal-structures come together, with each helping to illuminate the other.

On any of the last three attitudes, the existence of a categorical axiomatization is likely to be regarded as some kind of hallmark of success in the project of providing axiomatisations (cf. Read [1997, p. 92]; Meadows [2013, pp. 525–7, 536–40]). Whether that is correct or not, our point here is fairly simple. By themselves, the categoricity results are simply results within pure mathematics. If they are to be deployed in philosophical discussion, we need some bridging principle which connects informal structure-talk with the technical notion supplied by model theory. But the existence of such bridging principles is compatible with many different attitudes concerning what (if anything) the categoricity results show.

6 Categoricity and moderate realism

We have not yet attempted to address whether the Categoricity Thesis is correct. And we shall not address that question directly in this section either. Instead, we shall ask whether the moderate realist, of §3.3, can sustain the Categoricity Thesis. In fact, we shall see very quickly that the just more theory manoeuvre prevents her from embracing the Categoricity Thesis, just as it prevents her from embracing the Preferable Supervenience Thesis.

6.1 Full models as preferable models

In §5.5, we saw that any defender of the Categoricity Thesis must argue that we can restrict our attention to full structures rather than Henkin structures. Otherwise put: a moderate realist who wants to defend the Categoricity Thesis must explain why full models are preferable models. This is an apt description of the case, since comparison of (5.1) and (5.2) immediately shows that full structures can naturally be regarded as a privileged kind of Henkin structure; namely, they are just the Henkin structures where \( M_{\text{rel}} = \wp(M) \).

In the light of the Löwenheim–Skolem Theorem, then, a moderate realist who wants to defend the Categoricity Thesis must invoke a very specific version of the Preferable Supervenience Thesis, from §3.2. Consequently, the Categoricity Thesis essentially inherits the latter’s vulnerability to Putnam’s just more theory manoeuvre. In particular, any attempt to defend the Categoricity Thesis, via the use of full second-order logic, will presuppose just what was at issue, namely, whether we can grasp full second-order logic.\(^{22}\)


\(^{22}\)One might worry that the moderate realist is being asked to prove too much here. We revisit this concern in §§6.4–7.1.
the ‘intended’ interpretation of the second-order formalism is not fixed by the use of the formalism (the formalism itself admits so-called ‘Henkin models’), and so it becomes necessary to attribute to the mind special powers of ‘grasping second-order notions’. [1980, p. 481]

Recall that the moderate realist explicitly denies that we have special (magical) powers which enable us to grasp mathematical objects directly (see §3.3). But the notions invoked in spelling out the distinction between full semantics and Henkin semantics are themselves mathematical notions; indeed, they are straightforwardly set-theoretic notions. So they are surely as hard to grasp—indeed, harder—than the arithmetical notions which the moderate realist had hoped to secure by appealing to full second-order logic. In short: invoking full second-order logic seems to be invoking just more (set) theory.23

At this point, the moderate realist might insist that her grasp of the second-order quantifiers is simply logical, or purely combinatorial, and so simply is not up for reinterpretation via Henkin semantics.24 But this seems only to label the problem, and not to solve it. Whatever honorific we give these notions—logical, combinatorial, mathematical, or whatever—the question remains how the moderate realist can claim to have grasped them.

6.2 Categoricity for arithmetic in weaker logics

The moderate realist evidently has difficulties, if she wants to invoke full second-order logic. In this section, we consider whether she might appeal to some logic which is intermediate between first-order logic and full second-order logic.

Over the past couple of decades, there has been interest in what we might call second-order-real-variable logic, or, more briefly, sorv logic.25 Like first-order logic, sorv logic does not allow second-order quantification. Consequently, PA2’s induction axiom cannot be formulated within sorv logic. However, sorv logic augments first-order logic with a new kind of symbol, predicate-variables. Where P is one of these predicate-variables, we can deal with arithmetical induction by using a single formula of sorv logic: \((P0 \land \forall x (Px \rightarrow Psx)) \rightarrow \forall x Px\). Call the resulting theory of arithmetic PA_{solv}.

23One might ask: what exactly does ‘grasping’ amount to in this context and what exactly is ‘grasped’? While this is Putnam’s language (see previous quotation), perhaps the best way to think about it is in terms of how moderate realism was formulated in §3.3, in terms of a preference for reference by description rather than reference by acquaintance, at least in the case of mathematics. Thus to grasp some part of mathematics is to refer determinately to it, which given the moderate realist’s commitments will have to be done via a theory of that part of mathematics.

24Some people seem to read Shapiro [1991] this way. However, as remarked earlier, Shapiro [2012, p. 308 n. 1] does not think that categoricity can be used in the present context to secure the Categoricity Thesis. It is perhaps a plausible reading of McGee [1997, pp. 45–6; for criticism, see Field [2001, p. 352].

25The technical point here goes back to Corcoran [1980a, pp. 192–3], and is also discussed by Shapiro [1991, pp. 247–8]. McGee [1997, pp. 56ff.;] Lavine [1994, pp. 224–40, 1999]; Parsons [2008, pp. 262–93] have all invoked sorv logic in defence of philosophical arguments based upon categoricity results. Importantly, though, none of them were appealing to Dedekind’s Categoricity Theorem as stated in §5.5. The significance of this is explored in §9, especially footnote 41.
Having introduced some new syntax, we need to lay down its semantics. The semantics of sorv logic is exactly the same as for first-order logic, but with an additional clause to handle our new symbols. Where $P$ is a one-place predicate-variable, we offer the following semantic clause:

\[(6.1) \quad M \models \phi(P) \iff M \models \phi[X/P] \text{ for every } X \in \phi(M)\]

It is now easy to show that PA$^\text{sorv}$ is categorical, just as PA$^2$ is. But this should neither come as a surprise nor occasion much interest (as noted by Field [2001, p. 354]; Walmsley [2002, p. 253]; Pedersen and Rossberg [2010, pp. 333–4]; Shapiro [2012, pp. 309–10]). Comparing (6.1) with (5.1), it is immediately clear that sorv logic is just a notational variant for the $\Pi^1_1$-fragment of full second-order logic.\(^{26}\) The notational variant amounts to the trivial point that, since we are restricting our interest to $\Pi^1_1$-sentences, we can always omit the initial second-order quantifiers without risk of ambiguity. In the present context, this notational variant is surely no more philosophically significant than the fact that, in propositional logic, we can omit the outermost pairs of brackets in a sentence without risk of ambiguity.

The metaphysical realist might consider various other intermediate logics. These three augmentations of first-order logic allow for a categorical axiomatization of arithmetic:\(^{27}\)

1. Treat ‘0’ and ‘s’ as logical constants.
2. Add a new quantifier, ‘$\forall$’, with the clause that $M \models \forall x \phi x$ iff there are only finitely many $\phi$s in $M$.
3. Introduce a single one-place predicate, $P$, whose fixed interpretation is given by $P^M = \{ x \in M \mid \text{there is an } n \geq 0 \text{ such that } M \models x = s^{(n)}0 \}$, where $s^{(n)}0$ is defined recursively by $s^{(0)}0 = 0$ and $s^{(n+1)}0 = ss^{(n)}0$.

But, as Read [1997, p. 91] notes, invoking either (1) or (2) would simply ‘shift’ the problem from the identification of postulates characterizing [the natural numbers] categorically... into the semantics and model theory of the logic used to state the postulates’. For obvious reasons, exactly the same problem affects options (3).

A marginally more interesting approach is:

4. Add Härtig’s binary quantifier, ‘$\exists$’, with the clause that $M \models \exists xy(\phi(x); \psi(y))$ iff there are exactly as many $\phi$s as $\psi$s in $M$.

To produce a categorical theory of arithmetic, we now add to PA an axiom stating ‘if there are exactly as many entities less than $x$ as there are entities less than $y$, then $x = y$’; such a claim would be false of certain nonstandard numbers. But to grasp this point seems to require a grasp of the behaviour of cardinality in general, which again seems to presuppose in the semantics precisely the notions we were seeking to secure. Similar, we could consider the following

\(^{26}\)A $\Pi^1_1$-sentence is any sentence whose second-order quantifiers are all universal quantifiers occurring at the very beginning of the sentence.

(5) Allow sentences containing (countably) infinitely many conjunctions and disjunctions (i.e. move from $L_{\omega,\omega}$ to $L_{\omega_1,\omega}$)

We get categoricity by adding to PA the axiom $\forall x (x = 0 \lor x = s^{(1)}0 \lor \ldots \lor s^{(n)}0 \lor \ldots)$. But exactly the same worry arises when we ask how we can grasp the idea of a countable infinity of disjunctions in the metatheory.

Mathematically the most interesting alternative is:

(6) Stipulate that the arithmetical function symbols ‘+’ and ‘×’, even if not logical constants, must always stand for computable functions.

To obtain categoricity, we then invoke Tennenbaum’s Theorem [1959], that all computable models of first-order PA are isomorphic (for proofs, see Kaye [2011]; Ash and Knight [2000, p. 59]). As a way for the moderate realist to defend either the Categoricity or the Truth-Determinacy Theses, however, option (6) faces similar problems. It invokes a grasp of a computable function, which is surely at least as problematic as the notion of a natural number. (For more on the philosophical interpretation of Tennenbaum’s Theorem, see McCarty [1987, pp. 561–3]; Dean [2002]; Halbach and Horsten [2005]; Quinon [2010]; Button and Smith [2012]; Horsten [2012]; Dean [2014].)

Time and again, the moderate realist reencounters the same problem. The moderate realist was asked to explain how we grasped certain mathematical concepts. She attempted to secure a grasp of those concepts by appealing to a categoricity theorem. But to prove categoricity, she has had to invoke resources beyond those supplied by first-order logic; resources which invoked precisely the kinds of concepts that were at issue in the first place, and which she was hoping to secure by appeal to a categoricity theorem. In short: the moderate realist’s attempts to go beyond first-order logic all seem to be just more theory.

### 6.3 Troubles with Truth-Determinacy

In response to this line of argument, the moderate realist may propose the following gambit. Even the notions employed in first-order semantics (or second-order Henkin semantics) are deeply mathematical. So if the just more theory manoeuvre succeeds at all, then you should not even allow me access to first-order model theory. But this just shows that something has obviously gone wrong in your argumentative strategy. (This argument is implicit in a remark by Kreisel [Mostowski 1967, p. 101] and explicitly developed by Bays [2008, pp. 197–207].)

There is no doubt that we are in dangerously ‘unstable’ territory here, and we shall probe that instability in the next section. For now, though, we can respond to this gambit as follows. There are sound and complete inference rules for first-order logic and for second-order logic with a Henkin semantics. So the moderate realist can plausibly demonstrate her grasp of those logical ideas, just by demonstrating her mastery of the inference rules. By contrast, full second-order logic has no sound and complete deductive system. So we...

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28That said, there is a substantial question of whether she can demonstrate her grasp of the notion of an arbitrary finite number of applications of such rules.
can expand on Putnam’s observation, quoted in §6.1: it is not just that the ‘formalism’ of second-order logic fails to fix our attention on full models; additionally, no (finitary) system of valid inference rules can so fix our attention.

We can develop this point by considering Gödel’s First Incompleteness Theorem. Let $T$ be a consistent and categorical theory of arithmetic. Since $T$ is categorical, $T$ is negation-complete, i.e. for any sentence $\gamma$ in the language of $T$, either $T \models \gamma$ or $T \models \neg \gamma$ (via the argument given in §5.3). So, by the Incompleteness Theorem, either (i) $T$ is not recursively axiomatisable; or (ii) the notion of $T$-provability is not primitive recursive; or (iii) $T$-provability is not complete for the semantics according to which $T$ is categorical. Consequently, to claim a grasp of this supposedly categorical theory, we must claim to grasp either a theory or a semantics which cannot be laid down in any finitary fashion. Given the moderate realist’s self-conscious naturalism, it is hard to see how she could sustain such a claim (although cf. [Weir 2010]).

The preceding argument raises a difficulty for the Categoricity Thesis which, in effect, goes via a difficulty for the Truth-Determinacy Thesis. In fact, the First Incompleteness Theorem is just the beginning of the moderate realist’s difficulties concerning Truth-Determinacy. Consider the following result from Hamkins and Yang [2014, Theorem 1]:

**Non-Absoluteness Theorem.** Given any consistent extension $\text{ZFC}^+$ of ZFC, there are $M_1 \models \text{ZFC}^+$ and $M_2 \models \text{ZFC}^+$ which share exactly the same natural number substructure, but which disagree in their theories of arithmetical truth. Specifically: there is (a code for) an arithmetical sentence, $\gamma$, such that $M_1 \models (\mathbb{N} \models \gamma)$ and $M_2 \models (\mathbb{N} \not\models \gamma)$.

So, even granting that you and I somehow manage to pick out exactly the same model of arithmetic with our set-theoretic language, there is no guarantee that you and I agree on arithmetical truth.

To be clear: the Non-Absoluteness Theorem does not conflict with the simple observation that isomorphism implies elementary equivalence (see §2.2). That observation holds good within $M_1$, and it also holds good within $M_2$. However, once we start modelling theories of truth, there are settings where we can make coherent but divergent assignments of truth for one and the same underlying model. The result highlights just how ‘pathological’ the interpretations of first-order theories can be.

In this regard, it must be mentioned that every object mentioned in the Non-Absoluteness Theorem—$M_1$, $M_2$, their shared natural number substructure, and the (code of) the troublesome sentence, $\gamma$—must be nonstandard. Indeed, we can easily prove that ‘any two truth predicates on a[ny] given model of arithmetic[, standard or otherwise,] must agree on their judgments for standard-finite formulas’ [Hamkins and Yang 2014, p. 27]. So the moderate realist might attempt to reject the philosophical significance of the Non-Absoluteness Theorem, on the grounds that it is shot through with appeal to nonstandard entities. Such a rebuttal succeeds, though, if and only if the moderate realist can already claim to have distinguished between standard and nonstandard entities (cf. Hamkins and Yang [2014, §6]). And that is just what the moderate realist has failed to do so far, either by appeal to Dedekind’s

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29They [2014, p. 4] describe it as a ‘folklore’ result.
Categoricity Theorem, or by other means. Otherwise put: so far, the distinction between standard and nonstandard entities itself looks like just more theory.

6.4 Wrapping up moderate realism

As a last-ditch response, the moderate realist might push back, as follows: *You have not shown that I cannot embrace either the Categoricity or the Truth-Determinacy Theses. You have shown only that I cannot demonstrate, to the satisfaction of some radical sceptic, that either Thesis is true. But my present inability to answer this sceptic is no more embarrassing than our general inability to answer a radical Cartesian sceptic about the external world. And we should not allow our philosophy to be dictated entirely by sceptics!*

The moderate realist is right, here, in two ways. First, she has, indeed, failed to answer the sceptic. Second, we should not allow the sceptic to dictate all of our philosophising.

But such observations do not let moderate realism off the hook. It is not as if some sceptical character has loomed out of the shadows and foisted some unreasonable standards of justification upon the moderate realist. Rather, as Putnam repeatedly emphasised [Putnam 1980, 1994, pp. 284–5; see also Button forthcoming], the standards in play have always been those of the moderate realist herself. The moderate realist has a distinctive philosophical position, which brings with it distinctive commitments concerning our grasp of mathematics; in turn, those commitments naturally generate sceptical challenges, which are unanswerable by the moderate realist’s own lights. So, if we want to move beyond these supposedly ‘unreasonable’ sceptical challenges, then we must move beyond moderate realism itself.

7 Probing the Sceptical Attitude

The last section showed that the moderate realist cannot defend either the Categoricity or Truth-Determinacy Theses. It did not establish that either of those Theses were false. In this section, then, we shall consider these Theses a bit more directly, by probing the Sceptical Attitude of §5.6; or rather, by probing a variety of different sceptical attitudes.

7.1 Transcendental arguments

We begin with an extreme version of the Sceptical Attitude: the *model-theoretic sceptic*, who was the moderate realist’s sparring partner in the last section (or perhaps her own shadow). On the basis of the now-familiar limiting results from first-order logic, the model-theoretic sceptic denies that there is a unique natural number structure (informally construed).

Several authors have suggested that there is something deeply *unstable* about model-theoretic scepticism. There are two main lines of argument to this effect. The first suggests that the model-theoretic sceptic cannot even state her position without undermining it.\(^{30}\)

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\(^{30}\)Button [2013, p. 60, forthcoming, §4] presents a similar transcendental argument against a permutation-induced scepticism about reference. Briefly put: if my word ‘cat’ does not refer to cats, as permutation might suggest, then I cannot even formulate the sceptical scenario by saying ‘my word “cat” does not refer
First Transcendental Argument. The model-theoretic sceptic seems to proceed as follows: she presents us with a nonstandard model, and asks us how we can be sure that this was not the target of our mathematical language, given that it is elementary equivalent to the supposedly standard model. But, insofar as we understand what she has presented us with, we can see that it is nonstandard, and so not our target. For example: suppose the sceptic has given us a nonstandard model of arithmetic. If our grasp on induction is sufficiently open-ended to license induction in the language in which the nonstandard model is presented, then we can immediately see that the nonstandard model violates induction (see Parsons [1990b, pp. 37–9, 2008, pp. 287–8, 292–3]; McGee [1997, p. 59]; Lavine [1999, p. 65]; Pollard [2007, p. 89]; and for criticism, Field [2001, p. 355]). Alternatively: suppose the sceptic has presented us with a model of set theory which is merely countable (using the Löwenheim–Skolem Theorem). If that were the target of our mathematical discourse, our word ‘countable’ would pick out only those sets which are countable-in-the-model; but since the model itself is not countable-in-the-model, it could not now be described to us as a ‘countable’ model (see Tymoczko [1989, pp. 287–90]; Moore [2001, pp. 165–7, 2011, §3]; Button [forthcoming, §3], all of whom draw inspiration from Putnam’s [1977, p. 487, 1981, ch. 1] brain-in-vat argument).

The second line of argument suggests that the model-theoretic sceptic is not entitled, by her own lights, to employ the first-order model theory with which she raises a sceptical challenge:

Second Transcendental Argument. The model-theoretic sceptic is happy to employ the full power of first-order model theory. However, as Bays [2001, p. 345] notes, ‘the notions of finitude and recursion are needed to describe first-order model theory’, since first-order formulas ‘can be of arbitrary finite length, but they cannot be infinite’ and first-order satisfaction is defined via recursion (cf. also Field [1994, pp. 397, 410–11, 2001, pp. 318, 338, 343]). Thus, the use of first-order model theory itself presupposes a grasp of the natural numbers (at least up to isomorphism). So if the model-theoretic sceptic employs first-order model theory in an attempt to argue that the natural numbers are not unique up to isomorphism, she saws off the branch on which she sits.

As our names suggest, these are both transcendental arguments against model-theoretic scepticism. To pursue them any further would take us well beyond the confines of this paper. (In particular, we would need to consider whether there is any sense to the idea of an ineffable sceptical challenge.) But it is worth emphasising that, whatever the merits of these arguments, they cannot be wielded in defence of moderate realism. As noted in §6.4, moderate realism inevitably leads to a model-theoretic sceptical challenge which it finds unanswerable. If some transcendental argument subsequently shows the sceptical challenge to be incoherent, then that merely shows the incoherence of moderate realism (cf. Button [2013, ch. 7, forthcoming].)

(Note that someone who embraces the Second Transcendental Argument will believe that we can grasp the natural numbers. As such, they may feel able to invoke some of the
intermediate logics discussed as options (1)–(6) of §6.2, and so believe that they can present a categorical axiomatisation of arithmetic. However, it is not clear whether they should want to go on to embrace the Logic-First Attitude of §6.2; that is, it is not clear that they should attempt to explain how we grasp the arbitrary-but-finite by invoking (as unproblematic) a logic which itself presupposes a grasp of the arbitrary-but-finite, for there is a threat of circularity. To discuss this point any further would, though, take us into the territory of Poincaré’s argument against logicism, and so we set it to one side.

7.2 Categoricity and Set Theory

Since §5.5, we have considered the (putative) categoricity of arithmetic. We now turn to the case of set theory. Here, we do not have a categoricity result per se but rather Zermelo’s classic quasi-categoricity result ([Ewald 1996, vol. 2 pp. 1219 ff. Zermelo 2010, pp. 400ff. Kanamori 2003, p. 19]):

**Zermelo’s Quasi-Categoricity Theorem.** Given any two full models of $ZFC^2$, either they are isomorphic, or one is isomorphic to a proper initial segment of the other.

Here, $ZFC^2$ is the axiomatic theory of second-order Zermelo-Fraenkel set theory with Choice, and the initial segments refer to initial segments of the cumulative hierarchy:

$$V_0 = \emptyset, \quad V_{\alpha+1} = P(V_\alpha), \quad V_\alpha = \bigcup_{\beta < \alpha} V_\beta, \quad \alpha \text{ limit}$$

If $\langle M, E_M \rangle$ is a full model of $ZFC^2$ with its first-order set-sized domain $M$ and membership relation $E$, then $V^M_\alpha$ denotes the cumulative hierarchy relative to $M$ up to level $\alpha$. So if $\langle M, E_M \rangle$ and $\langle N, E_N \rangle$ are two full models, then Zermelo’s Quasi-Categoricity Theorem says that either $\langle M, E_M \rangle \cong \langle N, E_N \rangle$, or there is $\alpha$ in $N$ such that $\langle M, E_M \rangle \cong \langle V^N_\alpha, E_N \rangle$ or there is $\alpha$ in $M$ such that $\langle N, E_N \rangle \cong \langle V^M_\alpha, E_M \rangle$. Moreover, the proof of this result tells us that the level $\alpha$ is a strongly inaccessible cardinal, so that we obtain some very specific information about the initial segments.

The contemporary interest in this result is due to Kreisel [1967, p. 150], who noted that it offered a sense in which the continuum hypothesis (CH) was determined by the axioms. For, if as in §5.3, one stipulates $ZFC^2 \models \varphi$ to mean that all full models of $ZFC^2$ satisfy $\varphi$, then since the continuum hypothesis concerns sets at very low levels of the cumulative hierarchy, Zermelo’s Quasi-Categoricity Theorem implies that $ZFC^2 \models \text{CH}$ or $ZFC^2 \models \neg\text{CH}$. By the same token, however, both $ZFC^2 \not\models \varphi$ and $ZFC^2 \not\models \neg\varphi$, where $\varphi$ states ‘there is an inaccessible cardinal’. For if we assume in the background metatheory that there are at least two inaccessibles and we let $\kappa < \lambda$ be the smallest two inaccessibles, then $\langle V_\kappa, \in \rangle$ and $\langle V_\lambda, \in \rangle$ both model $ZFC^2$, but only the second models $\varphi$.

Of course, if we expand $ZFC^2$ with the sentence ‘there are no inaccessible cardinals’, we obtain full categoricity. It is non-trivial to say why this is impermissible; perhaps the idea is that postulating larger and larger axioms of infinity is a part of set-theoretic practice, and we want broad consonance between this practice and the truth-values predicted by our formalism.

31
full second-order logic, decides the truth value of some but not all set-theoretic statements; Zermelo’s Quasi-Categoricity Theorem can only go so far towards securing a set-theoretic Truth-Determinacy Thesis.

This limitation has been recently discussed by Isaacson. According to Isaacson, when we have a categorical theory, we are thereby presented with a ‘particular structure’ (this is his phrase for our notion of an informal-structure). However, Isaacson also wants to treat the set hierarchy as a particular structure, even though as just noted our set theory is not categorical. So Isaacson confronts a problem: ‘How is it that for set theory quasi-categoricity suffices to establish a particular structure whereas in the case of other particular structures categoricity is required?’ [2011, p. 53]. Isaacson’s response to this problem is the following:

[...] what is undecided in virtue of this degree of non-categoricity is genuinely undecided, in the same way that the fifth postulate of Euclid’s geometry is genuinely undecided by the axioms. This includes GCH (or some version that survives the refutation of CH), and the existence of large cardinals[...]. [2011, p. 53; see also pp. 4, 50]

So on Isaacson’s view, set theory constitutes a particular structure; however, that structure is somehow partial or incomplete, because it may be realized in incompatible ways, with one realization modeling ‘there is an inaccessible cardinal’ and another not. There is obviously much to say here; however, we shall confine ourselves to a single observation. If we want to treat questions such as ‘is there an inaccessible cardinal?’ as undecided for these kinds of reasons, then it will be highly confusing if we continue to talk about the set hierarchy.

In fact, when Kreisel [1967] first appealed to Zermelo’s Quasi-Categoricity Theorem, he was concerned to rebut an extreme version of the view that there is not just one single set hierarchy. Following Cohen’s development of forcing to prove the independence of CH from ZFC, the 1960s saw an explosion of set-theoretic independence results. Reflecting upon these results, Mostowski articulated the view that was contrary to Kreisel’s:

[...] the incompleteness of set-theory[...] is comparable[...] to the incompleteness of group theory or of similar algebraic theories. These theories are incomplete because we formulated their axioms with the intention that they admit many non-isomorphic models. In [the] case of set-theory we did not have this intention but the results are just the same. [1967, p. 94]

So, Mostowski was arguing that set theory has turned out to be algebraic. Mostowski went on to suggest that the algebraic nature of set theory may threaten set theory’s purported foundational role:

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32 McGee also discusses this limitation. However, since McGee (unlike Isaacson) is concerned with what we shall call internal categoricity, we postpone discussion of McGee until §9.6.

33 He writes [2011, p. 32]: ‘The requirement of consistency is sufficient for structure, with categoricity determining whether that structure is general or particular.’ In order to use formal categoricity results to tell us about informal-structures (i.e. ‘particular structures’), Isaacson requires a bridging principle between informal-structures and isomorphism types, like our treatment of informal-structures as isomorphism types. Isaacson [2011, p. 26] has explicit metaphysical qualms with that treatment; but as we mentioned in §5.1, the treatment is heuristic rather than ontological.
Of course if there are a multitude of set-theories then none of them can claim the central place in mathematics. Only their common part could claim such a position; but it is debatable whether this common part will contain all the axioms needed for a reduction of mathematics to set-theory. [1967, pp. 94-5]

Inverting this reasoning process might give us cause to maintain that there is a single determinate concept of set, for that would safeguard its foundational role. In turn, this would give us a reason to value the quasi-categoricity of ZFC\(^2\); for quasi-categoricity will leave us with essentially only one notion of set.

Any such argument, however, would have serious limitations. We shall note two.

First: the foundational role which Mostowski isolates is the ability to reduce mathematics to set theory. Putting aside the fact that set theory might play other foundational roles, there is a lacuna in Mostowski’s reasoning. As Kreisel noted, Mostowski has not ruled out the possibility that there is one (and only one) ‘fundamental’ set theory, within which it is possible to define all of the other notions of set [Mostowski 1967, p. 100]. To pursue this idea any further, we would need to get much clearer on the relevant notion of reduction.

But one well-understood notion is that of interpretation. Roughly, the idea here is that the primitives of the reduced theory can be explicitly defined in terms of the reducing theory, in a way that systematically preserves theoremhood (Walsh [2014, §2] provides an overview of this notion, with further references). As Koellner [2009, p. 99] emphasises, though, ZFC + CH, and ZFC + ¬CH, and ZFC are all mutually interpretable. This seems to undermine Kreisel’s hope that there is a single ‘fundamental theory’. Equally, though, it undercuts our purported reason for valuing (quasi-)categoricity. For it seems that plain, vanilla, first-order ZFC itself serves as a decent ‘common part’ of the equally acceptable set theories, and no one doubts that classical mathematics can all be reduced to—i.e. interpreted within—ZFC.

Second: even if adopting an algebraic conception of set theory forces us to abandon something which we hold valuable, such as some reductive project, the algebraic conception may make it possible to secure something else which is also valuable. If so, then we will have to engage in some kind of cost-benefit analysis. And, to make the case for the algebraic side, note that set theory has thrived as a mathematical discipline, precisely because the independence results have led to an examination of the rich relations between the multitude of models of first-order ZFC. Thus we find Hamkins writing:

Set theory appears to have discovered an entire cosmos of set-theoretic universes, revealing a category-theoretic nature for the subject, in which the universes are connected by the forcing relation or by large cardinal embeddings in complex commutative diagrams [...] [2012, p. 418]

For Hamkins, then, the categoricity arguments serve only to highlight the essentially algebraic nature of second-order quantification [2012, pp. 427-8]. So Hamkins embraces what we might call set-theoretic scepticism—and hence, of course, a similar attitude towards second-order logic—precisely because he assigns a high value to the mathematical fertility of an algebraic conception of set.
8 Shapiro’s structuralism

Since §5, we have been exploring the consequences of treating informal-structures as isomorphism types. In fact, various structuralist approaches in the philosophy of mathematics offer rather more detailed accounts of ‘mathematical structures’. One particular example is Shapiro. His ‘structures’ are not the model-theorist’s $\mathcal{L}$-structures; rather, they are philosophical posits. In this section, we shall use our model-theoretic resources to outline and assess Shapiro’s structuralism.

8.1 Shapiro’s ontology

To make sense of informal-structure talk, Shapiro postulates certain fundamental ontological categories. He begins with structures, which we shall call ante-structures, for reasons that will become clear in a moment. These are abstract entities, consisting of places with certain intra-structural relations holding between them.34

These abstract entities should be considered along the lines of (platonistic) universals, which can be multiply realised. Shapiro calls such realisations systems, and a place-holder is an object which, on that realisation, instantiates a particular place in the realised ante-structure [1997, pp. 73–4]. Shapiro’s ante rem structuralism is then characterized as the contention that the ante-structures exist independently from the systems that realise them [1997, pp. 9, 84–5, 109].

These ontological categories allow Shapiro to make straightforward sense of the idea that there are many different objects—i.e. different systems—that mathematicians might consider when studying some single ante-structure. To reuse the example from §5.1: both the systems of Zermelo’s ordinals, and of von Neumann’s finite ordinals, instantiate the natural number ante-structure, and so either are suitable objects of study. But we can also see why mathematicians may profess uninterest in the precise differences between the two systems, if their aim is to study the ante-structure itself.

Shapiro goes on to maintain that ‘two structures are identical if they are isomorphic’ [Shapiro 1997, p. 93]. But we should take a little care with this claim: he mentions ‘structures’, but these are his ante-structures, rather than the model theorist’s $\mathcal{L}$-structures; and so his ‘isomorphisms’ cannot quite be those of a model theorist. However, we can explain the idea by building a toy model of Shapiro’s structure-theory (and we really do mean a model, in the sense of §2).

Fix an initial segment of the cumulative hierarchy, and fix a signature $\mathcal{L}$. Let Systems consist of all the $\mathcal{L}$-structures in this initial segment. By fixing a well-order of this initial segment, we can let Structure contain, for each $\mathcal{L}$-structure in Systems, its least isomorphic copy. Then one can consider the formal model $(Systems, \in, Structure)$, whose domain consists of Systems, with a primitive membership relation symbol interpreted as normal, and with a single one-place predicate whose extension is given by Structure. If we identify the elements

34Shapiro’s ante-structures can be compared with fruitfully with Resnik’s [1981, 1997] patterns; his places with Resnik’s positions.
of Systems with Shapiro’s systems, and identify the elements Structure with Shapiro’s ante-
structures, then it is trivially true on this formal model that two structures are identical iff
they are isomorphic. This kind of formal model is part of the idea behind Shapiro’s thought
that his structure theory and set theory are ‘notational variants of each other’ [1997, p. 96].

The preceding model treats Shapiro’s systems just as the model-theorist’s \( \mathcal{L} \)-structures.
But this very natural treatment leads us back to an issue we first encountered in §5.2. We
will want to have a criterion for when two systems instantiate the same ante-structure.
An obvious answer is to say that they do so iff they are isomorphic. But now we have a
problem: suppose we consider an \( \omega \)-sequence of entities under less-than, or an \( \omega \)-sequence
of entities under a successor function; the difference in signature will mean that they are
not, strictly speaking, isomorphic systems. To circumvent this problem, Shapiro—exactly as
in §5.2—suggests that two systems with the same places instantiate the same structure iff
they are definitionally equivalent. (Here Shapiro [1997, p. 91] explicitly follows Resnik [1981,
pp. 535–6, 1997, pp. 207–8].)

Having set out Shapiro’s structuralism, we now want to consider three issues that have
been raised against it. Doing so will help us better understand what structuralism might
amount to, and might achieve.

8.2 Referential determinacy

Shapiro claimed that his structuralism helped to resolve the threat of referential indetermi-
nacy that was raised in §3:

if we construe a language of mathematics as about a[n ante-]structure or a class
of models, then there is no inscrutability. In effect, the insights behind the
inscrutability of reference are shared by structuralism, but different conclusions
on the nature of objects and reference ensue. [1997, p. 55 n. 15]

In the first sentence of this quote, Shapiro is essentially treating his ante-structures as on
a par with a model-theorist’s isomorphism types. (This differs from the model of Shapiro’s
structuralism which we consired in the previous subsection; we shall revisit this difference in
a moment.) But the idea that this will alleviate referential indeterminacy is surprising. As
we saw back in §5.3, we cannot avoid the indeterminacy (or inscrutability) of reference by
construing mathematical language as about isomorphism types. If we grant full second-order
logic, for example, then a mathematical theory such as \( \text{PA}^2 \) can itself be said to pick out
a single entity, namely, an isomorphism type; but if we think of the singular terms of our
arithmetical vocabulary, such as ‘27’, as referring at all, then they will indeterminately refer
to different entities in different models of \( \text{PA}^2 \).

There is, however, an infelicity in treating Shapiro’s ante-structures as isomorphism types.
Ante-structures are made up of distinct places, so the natural number ante-structure has
positions 0, 1, \ldots, 27, \ldots. By contrast, there is no immediate analogue of the 0-place, or
the 27-place, in an isomorphism type. (There was an analogue of these places in the model we
provided in §8.1: the places are just the domains of the structures.) So the idea might be that
posing places in ante-structures is what cures the structuralist of referential indeterminacy. Shapiro seems to suggest as much himself:

Once we accept the Peano axioms [i.e. PA\(^2\)] as an implicit definition of the natural-number [ante-]structure, the numeral ‘27,’ for example, refers to a place in this [ante-]structure. There is no room for doubt or inscrutability concerning just which place this is. [1997, p. 14; cf. p. 141 n. 8]

Postulating places, in addition to ante-structures, therefore at least renders it intelligible to maintain that singular mathematical terms refer determinately.

Nonetheless, it is doubtful that structuralism removes the threat of referential indeterminacy (Balaguer [1998, pp. 80–4]; Hellman [2001, pp. 193–6, 2005, p. 546] raise similar objections). Ante rem structuralism is a version of platonism; so there is presumably a moderate-realist version of ante rem structuralism. Such a moderate realist structuralist will maintain that our only handle on mathematical entities—now considered as abstract structures and their positions—comes via our theories. So the question arises of why the expression ‘27’ refers to the 27-position in the natural number structure, rather than any other abstract entity. To make the question tricky, observe that a simple Push-Through Construction (of §2.2) allows us to create an arithmetical structure (in the model theorist’s sense) or system (in Shapiro’s terminology) according to which ‘27’ refers to any other entity we like. In response, the moderate realist structuralist might emphasise that our Push-Through Construction generates a mere system, rather than an ante-structure. She could then argue that ante-structures are the preferred reference candidates of our mathematical language. This, however, is evidently just a version of the Preferable Supervenience Thesis of §3.2, and it is unclear why it sounds any better coming from a moderate realist who happens to be a structuralist, than it sounded back in §3.2 coming from a pre-structural moderate realist.

The situation is therefore as follows. Since Shapiro’s ante-structures are meant to have distinct places, it might be possible to argue for the determinacy of reference. However, to alleviate the threat of referential indeterminacy completely, the ante rem structuralism must (at a minimum) eschew moderate realism.

### 8.3 The identity of indiscernibles

Recent discussions of structuralism have focussed on its purported commitment to the Identity of Indiscernibles. In particular, Burgess [1999, pp. 287–8]; Keränen [2001, 2006]; Hellman [2001, p. 193] alleged that Shapiro’s structuralism is committed to the claim that no ante-structure can contain two distinct places which possess exactly the same (one-place, intrastructural) properties.\(^{35}\) Model-theoretic tools will help us understand what is at stake here.

Given an ℒ-structure \(\mathcal{M}\) and a set of parameters \(A\) from \(M\), one says that two elements \(a, b\) are \(A\)-indiscernible iff for any formula \(\varphi(x)\) with parameters from \(A\), one has that \(\mathcal{M} \models \varphi(a) \iff \varphi(b)\). (Model theorists express this by saying that \(a\) and \(b\) have the

\(^{35}\) See also Field [2001, p. 328].
same complete 1-type over $A$; we return to this in §10.3.) To illustrate the idea: in the rational numbers under $<$, all the elements are $\emptyset$-indiscernible. But of course given any $\mathcal{L}$-structure $\mathcal{M}$ and distinct elements $a, b$, it is always the case that $a, b$ are not $\{a\}$-indiscernible, since $a$ satisfies $x = a$ in $\mathcal{M}$, whereas $b$ does not.

Keränen’s claim, that positions are identical iff they have the same one-place properties, now comes out as the view that $\emptyset$-indiscernible elements are identical [2001, p. 316]. This would prevent structuralists from being able to consider a whole host of ante-structures, including perhaps the integers, the rationals, the Euclidean plane, etc.

A number of responses suggest themselves. First: the structuralist could fix upon certain structures within which all elements are $\emptyset$-discernible, and argue that these correspond to the core, non-algebraic, theories which she wanted to pick out a unique ante-structure (cf. Parsons [2004, §IV, 2008, p. 108]). Second: the structuralist might argue for some more permissive versions of the identity of indiscernibles, thereby arguing that the problem is less widespread than it might have seemed (cf. Ladyman [2005]; MacBride [2006]). These two responses can be combined (cf. Button [2006, pp. 220–1]; Shapiro [2006b, pp. 169–70]). But the most straightforward option for the structuralist is simply to reject any form of the identity of indiscernibles. And indeed, most contemporary structuralists seem to embrace this line (see Parsons [2004, pp. 67, 71]; Shapiro [2006a, p. 140]; Ketland [2006, pp. 311–2]; Leitgeb and Ladyman [2008]).

That said, this again suggests that ante rem structuralism may be less distinctive than it initially seemed; for another apparently distinctive doctrine has fallen to one side.

### 8.4 Indeterminacy of identity

Shapiro also wanted to embrace a thesis which we might put as follows:

**Indeterminacy of Identity.** Identities between some mathematical objects are indeterminate.

The motivation for this idea can be traced back to Benacerraf [1965]. As remarked already (§§ 5.1 and 8.1), we can think of the natural numbers as either Zermelo’s ordinals or von Neumann’s ordinals. Zermelo would tell us that $2 = \{\emptyset\}$, von Neumann would tell us that $2 = \{\emptyset, \{\emptyset\}\}$, and set theory alone tells us that $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$. But it seems it would be arbitrary to say that Zermelo is right, rather than von Neumann, or vice versa. And saying that *both* are flat-out wrong would seem to threaten the status of 2 altogether (though cf. Kalderon [1996, pp. 249–50]; Field [2001, pp. 326–7]). So we might hope to say that it is in some sense *indeterminate* whether $2 = \{\emptyset\}$ or $2 = \{\emptyset, \{\emptyset\}\}$. We would thereby flout, in an intriguing way, the Quinean doctrine that there is no entity without identity (see Shapiro [2006a, p. 127]; though cf. his [1997, pp. 92–3]).

Shapiro [1997, pp. 79–82] went beyond the Indeterminacy of Identity, and stated that there is no fact as to whether $2 = \{\emptyset\}$, unless and until we decide to stipulate one. However, Shapiro [2006a, p. 124] soon realised that this contradicted his platonism. His

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36Ladyman et al. [2012]; Caulton and Butterfield [2012]; Button [MS] have recently explored various notions of indiscernibility.
platonism entails that positions in ante-structures exist and are not of our own creation; consequently, their identity or non-identity cannot be down to us.

Simply insisting that it is indeterminate whether \(2 = \{\emptyset\}\) would not run into that problem. However, there is a famous and general argument against indeterminate identity, formulated by Evans 1978, and pushed by Chihara [2004, pp. 81–83] in this context. From the supposition that it is indeterminate whether \(2 = \{\emptyset\}\) and the fact that it is determine that \(2 = 2\), we obtain, by classical logic and Leibniz’s Law, that \(2 \neq \{\emptyset\}\). Such considerations seem to have led Shapiro [2006, pp. 128–31] to abandon the Indeterminacy of Identity. What indeterminacy remains, if any, is relocated to a linguistic level. (For further discussion, see MacBride [2005, pp. 570–1].)

We pursue the idea of linguistic indeterminacy in §9.1. However, over the past three subsections, we have seen ante rem structuralism give up on many of its distinctive commitments. It contributes little to issues concerning referential indeterminacy (§8.2); it has no particularly distinctive take on the Identity of Indiscernibles (§8.3); and it is not obviously able to sustain the Indeterminacy of Identity (§8.4). This may lead us to ask two questions. First: to what extent is ante rem structuralism a distinctive version of platonism (cf. MacBride 2005, pp. 582–6)? Second: does ante rem structuralism contribute much more to our understanding of informal structure-talk, than we gain just by treating informal-structures as isomorphism types? We cannot hope to settle either question here, but we would note that Shapiro consistently adopts the Mathematics-First Attitude of §5.6 [esp. 2006a, pp. 110–2]. In deciding between competing views, then, Shapiro will be less concerned to defend the claim that his is a novel version of platonism, and much more concerned to find the view which best meshes with informal mathematical practice.

9 Parsons’ structuralism and internal categoricity

There are many alternative versions of structuralism; too many, indeed, for us to consider. So in this section, we shall focus simply on Parsons’ version of structuralism. The main reason for this is that Parsons’ notion of ‘structure’ is very different from any of the notions considered so far in this paper. Pursuing this point will also lead us to consider a very different kind of categoricity theorem from that considered in §§5–8.

9.1 Contextualism about identity statements

Parsons defines his structuralism as:

\[
\text{[\ldots] the view that reference to mathematical objects is always in the context of some background structure, and that the objects involved have no more by way of a ‘nature’ than is given by the basic relations of the structure [2008, p. 40, 1990a, pp. 303, 333]}
\]

In short, Parsons is a contextualist about mathematical reference, and the relevant contexts are his structures. We shall say more about what Parsons takes structures to be soon. But
his contextualism immediately allows Parsons to distinguish between two different kinds of ‘Indeterminacy of Identity’. On the one hand: since the referents of terms varies from context to context, so identity statements will shift truth-value from context to context. On the other hand, in a specific context, some identity statements might have an indeterminate truth-value, because they are not explicable in terms of the resources of the background structure itself. To illustrate: ‘2 = \{\emptyset\}’ might be true in a Zermelo-style set-theoretic context, but false in a von Neumann-style set-theoretic context [2008, p. 103; see also p. 77], and simply indeterminate in a purely number-theoretic context.

Neither kind of indeterminacy do not obviously involve a violation of Quine’s dictum that there is no entity without identity (though we return to this in §9.5). Really, what we have here is the context-sensitivity (and potential indeterminacy) of identity statements.

9.2 Structures as predicates

The contextualist approach to indeterminacy is compatible with many different attitudes towards ‘structures’. As mentioned in §8.4, Shapiro [2006a, pp. 128–31] now embraces something like this contextualism, whilst coupling it with realism about ante-structures and their places. Parsons, however, has a very distinctive take on structure. He essentially treats structures as simple predicates along with various relations on them (Parsons [2008, p. 112, 1990a, p. 335]).

This is very different from the model-theorist’s notion of an \( L \)-structure, which we now briefly recall by reference to arithmetic. If we speak of a ‘model of arithmetic’ as an \( L \)-structure, we might write \( \langle N, s, z \rangle \models \text{PA}^2 \), and explain its meaning following §2. First, \( \langle N, s, z \rangle \) is a structure whose domain is \( N \), where \( s : N \rightarrow N \) implements the successor function, and where \( z \in N \) implements zero. Second, \( \text{PA}^2 \) is a (set of) sentence(s). Third and finally, \( \models \) is a relation between the structure and the sentence(s): a recursively defined bona fide language–object relation.

But when Parsons talks about ‘models of arithmetic’, he has something deeply different in mind. For him, a ‘model of arithmetic’ is (with a grain of salt): a one-place predicate \( N \), a one-place function symbol \( s \), and a first-order variable \( z \), such that:

\[
Nz \land \forall x(Nx \rightarrow Nsx) \land \forall x(Nx \rightarrow z \neq sx) \land \\
\forall x \forall y([Nx \land Ny \land sx = sy] \rightarrow x = y) \land \\
\forall X([Xz \land \forall x((Xx \rightarrow Nx) \land (Xx \rightarrow Xsx))] \rightarrow \forall x[Xx \leftrightarrow Nx])
\]

This is just a formula in pure second-order logic, which axiomatises arithmetic relative to \( N, s, z \). In what follows, we abbreviate this formula by \( \text{PA}(N, s, z) \).

We cannot emphasise enough the difference between the expressions \( \langle N, s, z \rangle \models \text{PA}^2 \) and \( \text{PA}(N, s, z) \). The latter formula does not mention any (sets of) sentences, and all suggestion of a satisfaction relation has vanished. So, when Parsons speaks of ‘models of arithmetic’, there is no longer any hint of any language–object relation.

\[37\] Note that this approach is possible only because \( \text{PA}^2 \) is finitely axiomatisable. As we described \( \text{PA}^2 \) in §5.5, \( \text{PA}^2 \) also required infinitely many instances of the Comprehension Scheme. In the present context, however, these are assumed to be relegated to the background (second-order) logic.
(At this point, one might ask whether Parsons is still doing model theory at all. Hodges [MS, p. 1] suggests that the very many brands of model theory all ‘rest on one fundamental notion, and that is the notion of a formula \( \varphi \) being true under an interpretation \( I \).’ Parsons’ notion of structure does not involve that idea, so one might think that we have moved beyond the confines of model theory altogether. That said, Parsons’ notion does connect with the idea of a class model (see e.g. Kunen [1980, p. 112]), so it is not unprecedented.)

9.3 The internal categoricity of arithmetic

In common with most versions of structuralism, Parsons [1990a, p. 13, 2008, p. 272] wants to pin down a unique natural number structure. To do this, he invokes the following categoricity result:

**Parsons’ Categoricity Theorem.** All models of PA\(^2\) are isomorphic.

This looks a lot like Dedekind’s Categoricity Theorem. Oddly, though, we have dropped the qualifier ‘full’, whose significance we emphasised at such great length in previous sections. To make sense of this omission, we must note that ‘models of PA\(^2\)’ is meant, here, in Parsons’ sense, rather than the language–object sense outlined in §2. Indeed, Parsons’ Categoricity Theorem is just a sentence of pure second-order logic, of the following shape:

\[
\forall N_1 \forall S_1 \forall z_1 \forall N_2 \forall S_2 \forall z_2 (\text{PA}(N_1, S_1, z_1) \land \text{PA}(N_2, S_2, z_2)) \rightarrow \\
\exists F \text{Iso}(F, N_1, S_1, z_1, N_2, S_2, z_2)
\]

where ‘Iso(\(F, \ldots\))’ indicates that \( F \) is a (second-order) isomorphism.\(^{38}\) At the risk of repetition then: whilst we might offer similar informal glosses on Parsons’ and Dedekind’s Theorems, they are very different statements. Dedekind’s Theorem concerns structures, satisfaction and sentences; Parsons’ Theorem is just a sentence of pure second-order logic.

Crucially, Parsons’ Categoricity Theorem can be proved within a deductive system for second-order logic (for a simple proof, see Väänänen and Wang [forthcoming]). Consequently, we do not need to say anything about the semantics for second-order quantification in order to secure the result.\(^{39}\)

We call Parsons’ result an internal categoricity result,\(^{40}\) and contrast this with Dedekind’s external categoricity result. These labels are appropriately suggestive.Crudely: an external

\(^{38}\)So Iso(\(F, N_1, S_1, z_1, N_2, S_2, z_2\)) just abbreviates the conjunction of: \(\forall x \forall y (Fx = y \rightarrow (N_1 x \land N_2 y))\), i.e. ‘\( F \)’s domain is \( N_1 \) and range is \( N_2 \); and \(\forall x \forall y (Fx = Fy \rightarrow x = y) \land \forall y (N_2 y \rightarrow \exists x Fx = y)\), i.e. ‘\( F \) is a bijection’; and \( Fz_1 = z_2 \land \forall x (N_1 x \rightarrow FS_1 x = S_2 Fx)\), i.e. ‘\( F \) preserves structure’.

\(^{39}\)This is why Parsons [1990b, p. 34] says that ‘Dedekind’s theorem...is essentially first-order’. His point is that we can prove categoricity (in an internal sense) in a logic for which there is a sound and complete deductive system.

Note that sorv logic (see §6.2) can also be given a suitable deductive system; unsurprisingly, the system in question is much like that for second-order logic. Lavine [1994, pp. 224–40, 1999]; Parsons [2008, pp. 262–93] then invoke internal categoricity results for sorv logic, rather than second-order logic.

\(^{40}\)Walmsley [2002, pp. 249–51] seems to be the first author to use the phrase ‘internal categoricity’, but Parsons [1990b, pp. 31–9] seems to be the first author to invoke such a result. Sometimes internal categoricity results are called relative categoricity results.

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result requires that we stand back from the object language we were using, and consider its semantics in some metalanguage. By contrast, an internal result can be proved within the object language.

9.4 What might an internal categoricity result show?

We here have a kind of categoricity which does not require any discussion of the semantics for full second-order logic. We now need to consider what such a result might show.

To be very clear: internal categoricity results cannot possibly help the moderate realist. The challenge facing the moderate realist, first raised in §3.3, is to explain how the semantics for mathematical language(s) get fixed. Parsons’ Categoricity Theorem is a sentence of pure second-order logic, which tell us nothing (directly) about any language–object relation, or indeed about any semantic facts. Of course, the moderate realist might stand back from Parsons’ Theorem, as it were, and start to explore the semantics for the language in which she proved that Theorem. In so doing, she will see that Parsons’ Theorem entails Dedekind’s Theorem [Lavine 1999, p. 64; Väänänen 2012, p. 99]. But now Parsons’ Theorem has simply taken us on a long-winded detour back to §6, where moderate realism went to die.

Simply put, internal categoricity results are just the wrong shape for use by moderate realists. But Parsons is not a moderate realist; so let us now consider how he attempts to use internal categoricity.

Following Parsons [1990b, pp. 35–8, 2008, pp. 283–8], imagine two characters, Kurt and Michael, who are both ‘doing arithmetic’. So, Kurt has a predicate ‘. . . is a natural number’, which we can symbolise as $N_{kurt}$ or $N_k$, he has a notion of function ‘the successor of. . .’, which we can symbolise as $s_k$, and he has a name ‘zero’, which we can symbolise as $z_k$. We similarly symbolise Michael’s arithmetical vocabulary with $N_m$, $s_m$ and $z_m$. Assuming reasonable levels of communication between Kurt and Michael, it will be obvious to them that they are engaged in somewhat similar practices. But what they might want to show is that their languages are essentially identical; that, for arithmetical purposes, they differ only in the subscripts we have imposed; that they are ‘syntactically isomorphic’, to use Lavine’s [1999, p. 47] phrase. Parsons suggests that Kurt and Michael can establish this via the following route. Let us suppose that Kurt and Michael are in communication with one another to the point that both are able to take the other’s vocabulary into his own language (and both know this). Both can now prove Parsons’ Categoricity Theorem; and so, since they both have access to each other’s vocabulary, both can prove:

$$[\text{PA}(N_k, s_k, z_k) \land \text{PA}(N_m, s_m, z_m)] \rightarrow \exists F \text{Iso}(F, N_k, s_k, z_k, N_m, s_m, z_m)$$

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41In footnote 25, we noted that Parsons, Lavine and McGee all invoke sorv logic. We can now see why this is so. Parsons, Lavine and McGee want the predicate-variables of sorv logic to be open ended over any future expansion of the object language. This is why Kurt is able to establish a result which uses a predicate which originally belonged to Michael’s vocabulary (and vice versa). However, see Field [2001, pp. 358–60] for discussion of Kurt’s warrant for assuming that Michael may countenance and perform induction on predicates which Kurt introduces.
Furthermore, both can presumably see that the antecedent obtains: they affirm one of the conjuncts themselves, and their interlocutor happily affirms the other. They therefore obtain the consequent. And this guarantees that, for arithmetical purposes, their languages differ only in the subscripts we have imposed.

9.5 Internal categoricity and acquiescence in the mother tongue

To be clear, the foregoing argument will not decide whether \( z_k = z_m \). This is no surprise, given what we said at the start of this section. Indeed, context may even leave it indeterminate whether \( z_k = z_m \) is true. But this does seem to generate a slight tension for Parsons, for a natural kind of disquotation principle would then tell us that it may be indeterminate whether \( z_k = z_m \); and yet it is a theorem that \( z_k = z_m \lor z_k \neq z_m \). The more general tension here is as follows: Parsons' notion of 'structure' echews semantic notions; however, his contextualism is essentially semantic; and we are not given much guidance as to how the non-semantic and the semantic relate to one another.

We can illustrate the same point by considering what impact Parsons’ result might have on someone who wanted to defend the Truth-Determinacy Thesis. Once Kurt and Michael have established the existence of their second-order isomorphism, they can see that if they ever disagree (modulo subscripts) about any arithmetical sentence, only one of them is right. But does this enable them to say that every sentence in the language of arithmetic is either true or false? Presumably we shall want to say so; but this evidently requires the use of semantic principles which do not themselves appear anywhere within the categoricity result.

If the semantic principles that we need to invoke here are, in the end, just the notions provided by model theory, then Parsons’ treatment of structures as predicates, and his invocation of internal categoricity results, would have been a needless detour. In fact, Parsons seems to have a rather different view in mind. Roughly put, he holds that, once we consider semantic questions for our mother tongue, we can only answer them by ‘acquiescence in our mother tongue’ [1990b, p. 39, 2008, p. 288]. Whatever exactly this amounts to, it seems to involve the suggestion that there is no need to look to model theory (for example) to supply a semantic theory for the mother tongue. Rather, when I reach the mother tongue, I am just being asked to use my own words to say what my own words refer to, at which point, all I can provide is disquotational platitudes.

To probe this idea would evidently take us even further beyond the discussion of model theory than we have already gone. So let us simply suppose that the notion of acquiescence is coherent, and briefly consider how a Parsons-style structuralist will respond to the challenge of referential indeterminacy, posed in §3. Given his contextualism, Parsons will not expect context-insensitive referential determinacy for arithmetical vocabulary: ‘2’ may refer to one set in one context, and to a different set in another context. So, if the challenge of §3 is to pose any problems, they will have to concern referential principles which plausibly should be

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42In the next section, we shall see that McGee appeals to the internal categoricity of set theory; and his stated aim is to arrive at Truth-Determinacy [1997, pp. 40–2].

43Continuing with the preceding footnote: McGee does not discuss ‘acquiescence in our mother tongue’; however, he is a fairly thorough-going disquotationalist and so is likely to reach a similar conclusion.
context-insensitive, e.g. that ‘2’ refers to 2. But for Parsons, this will be guaranteed just by acquiescing in the mother tongue.

To be clear, then: Parsons’ response to sceptical challenges does not really invoke his notion of structure, or his internal categoricity result. His answer involves repudiating moderate realism and then acquiescing in the mother tongue. His notion of structure, and his categoricity result, come in to play further downstream, in showing that different parties are engaged in the same practice. And note that Kurt cannot demonstrate the existence of a shared practice, just by acquiescing in his mother tongue; for what needs to be shown is that Kurt and Michael indeed share a tongue.

The situation might, then, be put as follows. I can deal with the sceptic by acquiescing in my mother tongue. But now the threat of mathematical solipsism arises. This latter threat can be addressed by invoking internal categoricity. We thereby arrive at the possibility of mathematical intersubjectivity. However, this falls short of establishing mathematical objectivity, in that nothing so far establishes that we are all talking about the same objects.

This is ultimately how we understand Parsons’ [1990b, pp. 38–9, 2008, pp. 287–8] invocation of internal categoricity. We suggest a fairly similar reading of Pollard [2007, p. 89]. It is not clear whether this is quite the right reading of McGee’s [1997] invocation of internal categoricity (more on McGee below). And it may well be in some tension with Lavine’s [1994, 1999] view, since Lavine seems to hold that internal categoricity plays an active role in answering the sceptic. Ultimately, we must confess that it is not always clear to us, what is being claimed of an internal categoricity result.

9.6 Internal categoricity of set theory

We have focussed on internal categoricity for arithmetic. Unsurprisingly, there is an internal quasi-categoricity result for set theory (cf. Lavine [1999, pp. 55–66, 89–95]; Väänänen and Wang [forthcoming, §4]). But of course, since it is an internal categoricity result, it simply abbreviates a second-order formula of roughly the following shape:¹⁴

$$\forall V_1 \forall E_1 \forall V_2 \forall E_2 ([\text{ZFC}(V_1, E_1) \land \text{ZFC}(V_2, E_2)] \to \exists F \text{Seg}(F, V_1, E_1, V_2, E_2))$$

Crucially, and as in the case of Parsons’ Categoricity Theorem, no satisfaction relation appears anywhere in this sentence. Consequently, it tells us nothing directly about any language-object relation. Nonetheless, a Parsons-style structuralist could invoke this result in order to generate the intersubjectivity of set theory, ‘so far as it goes’.

There is, though, a stronger result to which we can appeal. Typical formulations of set theory only involve quantification over sets. But if set theory is to be applicable, then we shall want to be able to form sets from entities that are not themselves sets, also known as urelements (cf. McGee [1997, p. 49]; Potter [2004, pp. vi, 24, 50–1]). We can then distinguish pure sets—those which, intuitively, involve no urelements anywhere in their construction—from impure sets. Let $\text{ZFCU}^2$ be second-order Zermelo–Fraenkel set theory with Choice and

¹⁴So ‘$\text{ZFC}(V, E)$’ abbreviates a fairly long conjunction, axiomatizing second-order set theory relative to ‘$V$’ (intuitively, the sets) and ‘$E$’ (intuitively, membership). Likewise ‘$\text{Seg}(F, \ldots)$’ indicates that $F$ is a (second-order) isomorphism between an ‘initial segment’ of one and the ‘whole’ of the other.
with urelements, coupled with the axiom that there is a set containing all the urelements. McGee [1997] has proved the following internal categoricity result:

**McGee’s Categoricity Theorem.** Any two models of $\text{ZFCU}^2$ with unrestricted first-order quantifiers have isomorphic pure sets.

Since this is an *internal* categoricity result, it again has the shape:

$$\forall U_1 \forall E_1 \forall U_2 \forall E_2 ((\text{ZFCU}(U_1, E_1) \land \text{ZFCU}(U_2, E_2)) \rightarrow \exists F \text{PureIso}(F, U_1, E_1, U_2, E_2))$$

Such a result could then be invoked by a Parsons-style structuralist to secure complete intersubjectivity for the pure sets.

At the risk of repetition, we wish to emphasise that McGee’s Theorem mentions no sentences and involves no satisfaction relation. McGee [1997, pp. 49–50] is explicit on this point; however, subsequent discussion of this result has been rather less clear. Indeed, since McGee’s Theorem is an internal categoricity result, we can prove the result with a suitable deductive system for second-order logic, ignoring all semantic questions. As such, its proof does not turn on whether a first-order domain can be ‘absolutely unrestricted’; we need only use the logic without imposing any restriction on our first-order quantifiers within our object language. Equally, the result does not invoke semantics for full second-order quantifiers; we only need access to the deductive system.

The crucial point is as follows. ‘Structuralists about sets’ may want to invoke McGee’s Theorem. However, only Parsons-style structuralists are obviously entitled to do so. Structuralists who treat structures as objects, and so who are interested in genuine language–object relations, should be aware that McGee’s Theorem tells us nothing directly about any genuine language–object relation.

### 10 Uncountable categoricity, geometry, and classification

Much of this paper has focussed on philosophical considerations surrounding categoricity. But considerations related to categoricity have animated much of the work in model theory in the last decades. This might initially seem puzzling: the theories considered in this

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45Here ‘$\text{ZFCU}(U, E)$’ abbreviates a fairly long conjunction axiomatizing second-order set theory with urelements relative to ‘$U$’ (intuitively, the urelements) and ‘$E$’ (intuitively, membership), with *no* explicit relativisation on the hierarchy itself (as it were). Likewise ‘$\text{PureIso}(F, \ldots)$’ indicates that there is a (second-order) isomorphism between ‘the pure sets’.

46McGee is pushed in this direction by Tarski’s Indefinibility Theorem (roughly, that no sufficiently rich, consistent theory can define a truth predicate which provably has the properties we would want it to have). As a consequence, he faces a choice: either offer a *third-order* definition of satisfaction suitable for *second-order* theories, or relocate to internal categoricity. Since the former approach seems to embark on a futile regress, McGee embraces the latter approach, describing it as ‘devious’ [1997, pp. 47–8, 50, 62].

47In fact, just as Parsons’ Categoricity Theorem can be proved for sorv logic (see footnote 39, above), so McGee [1997, pp. 56ff.] also proves his result in sorv logic.
model-theoretic work are always first-order theories, so that the Löwenheim–Skolem Theorem immediately precludes categoricity for any ‘interesting’ theories. But, as Zilber retrospectively put the point, the obvious unavailability of categoricity simply ‘entailed a rethinking of the concept of categoricity’ [1993, p. 1].

In this section, we wish to explain the basic ideas behind this ‘rethinking’ of categoricity, to give some sense of its mathematical and foundational significance.

### 10.1 Uncountably categoricity

For the sake of simplicity, in this section we restrict attention to complete first-order theories in countable signatures. In this context, the Löwenheim-Skolem Theorem implies that any theory with an infinite model is not categorical.

However, suppose that we restrict attention to structures of a given infinite cardinality, \( \kappa \). For a fixed infinite \( \kappa \), there are many natural examples of theories that have exactly one model up to isomorphism of cardinality \( \kappa \). For example, Cantor proved that the complete first-order theory of the rationals as a linear order, known as DLO, has exactly one model of cardinality \( \aleph_0 \) up to isomorphism (Marker [2002, p. 48]; Hodges [1993, p. 100] present neat proofs). However, DLO has many non-isomorphic models of higher cardinalities: the real numbers as a linear order satisfy DLO, as do the real numbers minus a single given real number (e.g. 0), but these two models clearly cannot be isomorphic. As another example, take \( \text{Th}(\mathbb{Z}, s) \), the first-order theory of the integers where our only non-logical primitive is a one-place function symbol, \( s \), for successor. Models of \( \text{Th}(\mathbb{Z}, s) \) consist of one or more copies of the integers, with no relations ‘between’ any of these copies. So \( \text{Th}(\mathbb{Z}, s) \) has, up to isomorphism, \( \aleph_0 \)-many models of cardinality \( \aleph_0 \); but \( \text{Th}(\mathbb{Z}, s) \) has exactly one model up to isomorphism of cardinality \( \kappa \), for any uncountable \( \kappa \).

Given such examples, Loś [1954, p. 62] asked whether there were theories with exactly one model up to isomorphism in one uncountable cardinality, but not in others. Morley [1965] showed that this could not happen:

**Morley’s Theorem.** A theory \( T \) has exactly one model up to isomorphism of some uncountable cardinality \( \kappa \) iff it has exactly one model up to isomorphism of every uncountable cardinality \( \lambda \).

Theories which satisfy either side of the biconditional in Morley’s Theorem are called *uncountably categorical*. Uncountable categoricity is the ‘rethought’ notion of categoricity, mentioned in the earlier quote from Zilber.

It is worth briefly noting that uncountable categoricity is as good as ordinary categoricity, if all we want to do is to aim for Truth-Determinacy, and pin down the truth value of every sentence in the signature. For, if \( T \) has an infinite model and is uncountably categorical, then \( T \) is deductively-complete, since any incompleteness would be registered on distinct uncountable models by the Löwenheim–Skolem Theorem.
10.2 Pregeometries and dimension

The proof of Morley’s Theorem has been refined subsequently. These refinements have made transparent a connection between uncountably categoricity, and a very natural notion of dimension.

It transpires that the notion of a pregeometry is sufficient to provide us with an abstract, algebraic, axiomatic treatment of certain basic ideas related to dimension (cf. Marker [2002, p. 289]; Hodges [1993, pp. 170–1]; Buechler [1996, p. 52]; Baldwin [2014, §4.2]):

Definition 10.1. Suppose \( G \) is a set and \( \text{cl} : \varphi(G) \to \varphi(G) \). Then \((G, \text{cl})\) is a pre-geometry iff it satisfies the following four axioms:

1. \( A \subseteq \text{cl}(A) \) and \( \text{cl}(\text{cl}(A)) = \text{cl}(A) \).
2. If \( A \subseteq B \) then \( \text{cl}(A) \subseteq \text{cl}(B) \).
3. If \( a \in \text{cl}(A \cup \{b\}) \setminus \text{cl}(A) \) then \( b \in \text{cl}(A \cup \{a\}) \).
4. If \( a \in \text{cl}(A) \) then there is a finite \( A_0 \subseteq A \) such that \( a \in \text{cl}(A_0) \).

Where \((G, \text{cl})\) is any pregeometry:

1. A set \( B \subseteq G \) is independent iff \( c \notin \text{cl}(B \setminus \{c\}) \) for all \( c \in B \).
2. A set \( A \subseteq G \) is closed iff \( A = \text{cl}(A) \).
3. A subset \( B \) of a closed set \( A \) is a basis of \( A \) iff \( B \) is independent and \( \text{cl}(B) = A \).
4. The dimension of a closed set \( A \) is the cardinality of any basis for \( A \).

These definitions straightforwardly generalise the notion of dimension that we encounter when we deal with Euclidean space. In more detail: the elements of \( n \)-dimensional Euclidean space, \( \mathbb{R}^n \), are vectors \( \overline{v} = (v_1, \ldots, v_n) \), each of whose entries \( v_1, \ldots, v_n \) are real numbers. Along with the operation of pointwise addition \( \overline{v} + \overline{u} = (v_1 + u_1, \ldots, v_n + u_n) \), we have the operation of scalar multiplication \( c \cdot \overline{v} = (c \cdot v_1, \ldots, c \cdot v_n) \), where \( c \) is any real number. Given a subset \( A \subseteq \mathbb{R}^n \), the linear span of \( A \) is \( \text{span}(A) = \{c_1 \cdot \overline{a}_1 + \ldots + c_k \cdot \overline{a}_k \mid c_i \in \mathbb{R}, \overline{a}_i \in A\} \). If we now define \( \text{cl}(A) = \text{span}(A) \), then it is easy to check that have a pregeometry. Indeed, the ensuing definitions of a closed set, a basis, and a dimension are exactly the standard ones. So, suppose we are working in \( \mathbb{R}^3 \), and let \( A = \{(0, 0, 1)\}, B = \{(0, 0, 1), (0, 1, 0)\} \) and \( C = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} \); then we have \( \dim(\text{span}(A)) = 1 \), \( \dim(\text{span}(B)) = 2 \), and \( \dim(\text{span}(C)) = 3 \).

For model-theoretic purposes, an important example of pregeometry comes from the idea of a strongly minimal set. The definition of this notion employs the definition of elementary extension from the outset of §4.7. Let \( G = \{\overline{a} \in M^n \mid M \models \varphi(\overline{a})\} \) be any infinite definable subset in the structure \( M \). Then \( G \) is strongly minimal iff, for every elementary extension \( N \) of \( M \), the set \( G(N) = \{\overline{a} \in N^n \mid N \models \varphi(\overline{a})\} \) has only finite or cofinite definable subsets (i.e. if \( Y \) is a definable subset of \( G(N) \), then either \( Y \) is finite or \( G(N) \setminus Y \) is finite). Given a strongly minimal set \( G \), defined by a formula with parameters from some finite set \( A_0 \), we define a closure operation by \( \text{cl}(A) = \text{acl}(A \cup A_0) \cap G \). (Here \( \text{acl}(B) \) is the set of elements \( b \in M \) such that there is some formula \( \psi \) with parameters from \( B \) such that \( M \models \psi(b) \) and only finitely many elements in \( M \) satisfy \( \psi \).) So defined, \((G, \text{cl})\) is a
pregeometry. (For further details, see Marker [2002, pp. 208, 290]; Hodges [1993, pp. 134, 164, 171]; Buechler [1996, pp. 15, 51–3].)

This can seem rather technical at first, but it describes some very classical situations and examples. We list three which are particularly important (cf. Marker [2002, p. 291]; Hodges [1993, pp. 164, 167]; Buechler [1996, pp. 51–2]):

(i) *The integers under successor*; i.e. the structure \( \langle \mathbb{Z}, s \rangle \). Here, \( G = \mathbb{Z} \) is itself strongly minimal, and \( \text{acl}(A) \) is the set of points which are ‘finitely far away’ from some element of \( A \).

(ii) *The rationals as a vector space*; i.e. the structure \( \langle \mathbb{Q}^n, 0, + \rangle \), augmented with linear maps \( f_p \) for each \( p \in \mathbb{Q} \) such that \( f_p(\bar{a}) = p \cdot \bar{a} \). Here, \( G = \mathbb{Q}^n \) is itself strongly minimal and \( \text{acl}(A) = \text{span}(A) \), in the Euclidean sense of ‘linear span’ described above.

(iii) *The complex field*; i.e. the structure \( \langle \mathbb{C}, +, \times \rangle \). Here, \( G = \mathbb{C} \) is itself strongly minimal and \( \text{acl}(A) \) is the smallest subfield of \( \mathbb{C} \) containing \( A \) such that every non-zero polynomial with coefficients in the field has a root in the field.

With all of these definitions in place, Morley’s Theorem is a direct consequence of the following result (see Marker [2002, p. 214]; Buechler [1996, p. 68]):

**Theorem 10.2.** Suppose that \( T \) has only one model up to isomorphism for some uncountable cardinality. Then \( T \) has a model \( \mathcal{M} \) with strongly minimal set \( G \) such that

1. \( \mathcal{M} \) is an elementary substructure of any model of \( T \),
2. Any model \( \mathcal{N} \) of \( T \) of cardinality \( \lambda > \omega \) satisfies \( \dim(G(\mathcal{N})) = \lambda \),
3. Models \( \mathcal{N}, \mathcal{N'} \) of \( T \) with \( \dim(G(\mathcal{N})) = \dim(G(\mathcal{N'})) \) are isomorphic.

A few comments are required. The notion of dimension in the statement of the Theorem is given via the pregeometry \((G, \text{cl})\) where \( \text{cl}(A) = \text{acl}(A \cup A_0) \cap G \) and \( A_0 \) is the finite set of parameters used to define \( G \). Further, the statement of the Theorem retains the convention (flagged at the start of this section) that all theories are complete first-order theories in countable signatures. Finally, it should be mentioned that the above Theorem is typically regarded as implicit in the proof of the Baldwin–Lachlan Theorem (Baldwin and Lachlan [1971]; Marker [2002, pp. 213–4]; Buechler [1996, p. 68]).

The idea arising from the refinements of the proof of Morley’s Theorem can be summarised as follows: uncountable categoricity requires the presence of geometrical resources like *dimension*. Zilber expresses this idea as follows:

[... ] the main logical problem after answering the question of J. Los was what properties of \( \mathcal{M} \) make it \( \kappa \)-categorical for uncountable \( \kappa \)? [¶] The answer is now reasonably clear: the key factor is that we can measure definable sets by a rank-function (dimension) and the whole construction is highly homogeneous. [2010, p. 200]

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48Baldwin–Lachlan states that \( T \) is uncountably categorical iff both (i) \( T \) has no Vaughtian pairs (for definition, see Marker [2002, p. 151]; Buechler [1996, p. 58]) and (ii) \( T \) is \( \omega \)-stable (for definition, see below). A related theorem of Baldwin–Lachlan states that if \( T \) is an uncountably categorical theory (which is complete and in a countable language) then \( T \) has either 1 or \( \aleph_0 \)-many countable models (cf. Marker [2002, p. 215]; Buechler [1996, p. 92]).
We have just explained the notion of a ‘dimension’ to which Zilber is referring, and we see it occurring explicitly in conditions (2) and (3) of Theorem 10.2. The idea that the ‘construction is highly homogeneous’ follows from condition (1) of that Theorem (for details, see [Buechler 1996, pp. 11–12, 17, 21], [Marker 2002, pp. 129–130, 133–4]).

10.3 Research programs and classification

Zilber is also well-known for advancing an ambitious research program, aimed at enhancing our understanding of uncountable categoricity. In particular, he has articulated different versions of a so-called Trichotomy Conjecture. Loosely put, these conjecture that the pregeometries associated to uncountably categorical theories are similar to the three examples of pregeometries which we gave in the previous subsection, namely: (i) the integers under successor; (ii) the rationals as a vector space; and (iii) the complex field.

It would take us too far afield to offer precise statements of the various versions of the Trichotomy Conjecture ([Zilber 1984, p. 362; Marker 2002, p. 292; Hodges 1993, p. 199]) and Hrushovski’s counterexamples to the original versions of the conjecture ([Hrushovski 1993; Ziegler 2013]). However, the motivation for the conjecture was that it would indicate that all of the examples of pregeometries associated to uncountably categorical theories are both (a) essentially already known and (b) highly classical. The conjecture therefore expresses ‘[…] a belief in a strong logical predetermination of basic mathematical structures’ [Zilber 2010, p. 201].

Moreover, the most interesting of the three cases is that of the complex field, since the model theory of the complex field can be viewed as a part of algebraic geometry. This led Hrushovski to say, of a proven special case of the Trichotomy Conjecture, that ‘this was originally conceived as a foundational result, showing that algebraic geometry is sui generis’ [1998, p. 288]. Hodges cites the work of Hrushovski as part of the motivation for viewing model theory as ‘algebraic geometry minus fields’ [1997, p. vii].

Other model theorists suggest that recent model theory is significant, at least in part, because it may give us a means by which to organize parts of mathematics outside of mathematical logic proper; or, perhaps, a means by which to conceptualize the organization of these other parts of mathematics. For example:

In effect one aspect of the theory of models is to find the hidden relations between the different mathematical disciplines, most often the algebraic ones. [Lascar 1998, p. 238]

A useful contribution of post-Morley model theory is to explain these extreme classifications in terms of a geometrical independence theory […] From these explanations one does understand why there are so few extreme classifications in algebra, and one understands some absolutely new things, for example that there are no such extreme classifications in ordered algebra. [Macintyre 2003, p. 199]

The stability hierarchy is a collection of properties of theories […] that organize complete first order theories (that is structures) into families with similar mathematically important properties [Baldwin 2014, §4.2].
Some definitions are needed in order to appreciate the more specific aspects of the last two quotes from Macintyre and Baldwin.

Baldwin mentions stability, which requires a prior definition of the type space, which is also known as the Stone space. In detail: given an $\mathcal{L}$-structure $\mathcal{M}$ and a set of parameters $A \subseteq M$, we let $\mathcal{L}(A)$ be the expanded signature with new constant symbols for the elements of $A$, and we view $\mathcal{M}$ as an $\mathcal{L}(A)$-structure in the natural way. Further, we say that an $n$-type is a set $p(x_1, \ldots, x_n)$ in $n$-free variables of $\mathcal{L}(A)$-formulas which is complete and consistent with the $\mathcal{L}(A)$-theory of $\mathcal{M}$. The Stone space, $S_n(A)$, is then the set of $n$-types.

A theory $T$ is then said to be $\lambda$-stable iff $S_1(A)$ has cardinality $\leq \lambda$ whenever $A$ is a set of parameters with cardinality $\leq \lambda$ from a model of $T$; a theory $T$ is said to be stable iff it is $\lambda$-stable for some infinite $\lambda$. It turns out that uncountably categorical theories are $\lambda$-stable for all infinite $\lambda$. Further, structures which admit a linear order are not $\lambda$-stable, for any infinite $\lambda$, and so are not uncountably categorical. So in Macintyre’s quotation, the technical observation that infinite linearly ordered structures are not uncountably categorical is the basis of the ‘absolutely new’ realisation that there are no ‘extreme classifications in ordered algebra.’

To our knowledge, there is no extant work within philosophy of mathematics specifically on the topic of classification.\footnote{Though one can perhaps regard Lakatos’s *Proofs and Refutations* [1976] as touching upon the topic.} This stands in contrast to the large amounts written on classification within the philosophy of biology (see e.g. Ereshefsky [2007]), which one might think to be broadly analogous, in terms of its basic questions and distinctions. Perhaps this absence is related to the fact that classification is slightly orthogonal to proof, which has dominated most discussion within philosophy of mathematics. For instance: classification programs are usually begun well before the statement of the final classification theorem is articulated, and the plurality of solutions to a given classification problem might lead one to resist identifying its successful solution with any one theorem. But viewing categorical axiomatization as an extreme case of classification is at least one way of seeing how categoricity connects to the broader foundational interest of contemporary work in model theory.

## 11 Conclusions

Model theory is, of course, a branch of mathematical logic with its own goals and metrics for assessing what counts as progress. In this last section, we have hinted at some of these goals and metrics, and gestured at the way in which they have animated some of the more recent work in model theory.

But philosophers have traditionally used models to formalize disparate concepts. On the one hand, we have seen how models have been used by philosophers to approximate the notion of reference. This is the role that models play in Putnam’s permutation argument. On the other hand, models have been used to capture the informal structures that seem to be the subject-matter of some parts of mathematics. This is the role that models have come to occupy within some structuralist projects.
We have found a similar plurality of perspectives concerning the concept of categoricity. Simply put, a categoricity proof shows that a theory describes a single model up to isomorphism. But this survey had made vivid that one’s view of a given categoricity proof—such as Dedekind’s or Zermelo’s—depends both upon the logic or formalism in which the proof is embedded, and on the philosophical or foundational ends to which the proof is put. As described in the last section, one might view categoricity results just as extreme cases of classification. But we have also seen philosophers who wish to appeal to categoricity in order to arrive at much more distinctively philosophical theses, such as the determinacy of truth-value, or the possibility of mathematical intersubjectivity. Even structuralists who share certain goals—such as Shapiro and Parsons—approach the notion of categoricity via very different formal frameworks, allowing us to distinguish external from internal categoricity theorems. This underscores the importance of Dedekind’s and Zermelo’s classic results; however, it also shows that we must take great care to respect the sheer variety of philosophical perspectives on categoricity and other key ideas of model theory.

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References


Dedekind, R. (1888). *Was sind und was sollen die Zahlen?* Braunschweig.


Infinitesimal Magnitudes”. In: *Infinitesimal Differences: Controversies between Leibniz
and his Contemporaries*. Ed. by U. Goldenbaum and D. Jesseph. de Gruyter, pp. 215–
233.

*Philosophia Mathematica* 4, pp. 238–55.


Kaye, R. (2011). “Tennenbaum’s Theorem for Models of Arithmetic”. In: *Set Theory, Arith-
in Logic. Cambridge, pp. 66–79.

Keränen, J. (2001). “The Identity Problem for Realist Structuralism”. In: *Philosophia Math-

— (2006). “The Identity Problem for Realist Structuralism II: A Reply to Shapiro”. In:

Philosophy of Science*, pp. 287–300.


the Philosophy of Mathematics*. Ed. by O. Bueno and O. Linnebo. New York: Palmgrave,
pp. 80–116.

Kreisel, G. (1967). “Informal Rigour and Completeness Proofs [with Discussion]”. In: *Pro-
blems in the Philosophy of Mathematics*. Ed. by I. Lakatos. Amsterdam: North-Holland,
pp. 138–186.

ics. Amsterdam: North-Holland.

Ladyman, J. (2005). “Mathematical Structuralism and the Identity of Indiscernibles”. In:
*Analysis* 65.3, pp. 218–21.

Ladyman, J. et al. (2012). “Identity and Discernibility in Philosophy and Logic”. In: *The


l’algèbre. Un point de vue tendancieux”. In: *Revue d’Histoire des Mathématiques* 4.2,


Leitgeb, H. and J. Ladyman (2008). “Criteria of Identity and Structuralist Ontology”. In:
*Philosophia Mathematica* 16, pp. 388–96.

