Online Appendix to "Optimal Monetary Policy during a Cost-of-Living Crisis"

Alan Olivi, Vincent Sterk, and Dajana Xhani

March 6, 2024

Contents

Guidance: This Appendix contains technical, supplemental material. The derivations underlying the model and analytical results involve mostly straightforward algebra, though some equations are quite long given the richness of the model. Appendix A contains the main derivations for the model. Appendix B simply summarizes all the equations. Proofs for the analytical results (under simplifying assumptions) are provided in Appendix D. Derivations underlying the optimal policy simulations and analytical results are given in Appendix E. Finally, additional analytical result for extended cases are presented in Appendix F.

A Model derivations

Households

In this section, we derive the optimal response of households' consumption and labor supply decisions to changes in prices (subvariety prices, wage and interest rate) near a steady state where subvariety prices are equal within sectors and the real interest rate satisfies $R_t = (1-\delta)\beta$. Preferences are weakly separable for subvarieties across sectors, additively separable in consumption and leisure and additively separable across time. This allows us to characterize households' decisions in three steps. We first study the inner intratemporal consumption problem which determines individual demand for subvarieties conditional on subvarity prices and sectoral expenditure. Second, we determine individual expenditure across sectors and labor supply conditional on subvariety prices, wage and total (intratemporal) consumption expenditure (outer intratemporal problem). These first two problems are the same for both unconstrained and Hand-to-Mouth households. Finally, we determine individual expenditure across time by solving the intertemporal problem of unconstrained households and the decision rule of Hand-to-Mouth households.

Inner intratemporal consumption problem (valid for unconstrained and HtM households)

We start with the allocation of a household's expenditures on varieties within a sector. Note that this is an intratemporal problem. For any such problem, we omit time subscripts in this appendix, unless stated otherwise.

For any sector k, let $v_k(\boldsymbol{p}_k, e_k)$ be the indirect subutility function for a given vector of prices $\boldsymbol{p}_k$ and total expenditure e_k, defined as:

$$v_k(\boldsymbol{p}_k, e_k) = \max_{\{\boldsymbol{c}_k\}} \mathcal{U}_k(\boldsymbol{c}_k) \quad s.t. \quad \int p_k(j)c_k(j)dj \leq e_k.$$

Let $d_k(p_k(j^*), \boldsymbol{p}_k, e_k)$ be the household's demand for variety j^* and note that this function is C^2 and symmetric in $\boldsymbol{p}_k$.[1] As noted in the main text, we consider a steady state with identical prices within sectors, i.e. $p_k(j) = P_k$ for all j. Let $\partial_p d_k$ denote the own-price derivative and $\partial_j d_k$ be the Gateaux derivative of d_k with respect to the price of variety j. By symmetry of the subutility function $\mathcal{U}_k$, and the fact that prices are the same in equilibrium, it holds in the steady state that$d_k(p_k(j^*), \boldsymbol{p}_k, e_k) = e_k/P_k$ for any e_k and $\partial_j d_k = \partial_{j'} d_k$ for any two subvarieties. Using the fact that the demand function is homogeneous of degree zero we can apply Euler's theorem to obtain:

$$\left(\partial_p d_k\right) p_k(j^*) + \int \left(\partial_j d_k\right) p_k(j)dj + \left(\partial_{e_k} d_k\right) e_k = 0.$$

Applying the symmetry property noted above then gives:

$$\left(\partial_p d_k\right) P_k + P_k \left(\partial_j d_k\right) + \left(\partial_{e_k} d_k\right) e_k = 0.$$

After rearranging, we obtain the following expression for the derivative of d_k with respect to the price of variety j:

$$\partial_j d_k = -\partial_p d_k - \frac{1}{P_k^2} e_k.$$

Note that this equation is simply a decomposition of demand for j^* to a change in the price of j into substitution and income effects. This result allows us to derive the first-order change in consumption as:[2]

$$\begin{aligned}
dc_k(j^*) &= \left(\partial_p d_k\right) dp_k(j^*) + \int \left(\partial_j d_k\right) dp_k(j)dj + \partial_{e_k} d_k de_k, \\
&= \left(\partial_p d_k\right) dp_k(j^*) - \left(\left(\partial_p d_k\right) + \frac{1}{P_k}\partial_e d_k e_k\right) \int dp_k(j)dj + \partial_{e_k} d_k de_k, \\
&= \left(\partial_p d_k\right) \left(dp_k(j^*) - dP_k\right) + \frac{1}{P_k}\left(de_k - \frac{e_k}{P_k} dP_k\right).
\end{aligned}$$

This equation relates changes in subvariety consumption with respect to its own relative price $(dp_k(j^*) - dP_k)$ to the inner elasticity of substitution $\epsilon_k = -P_k \partial_p d_k / d_k$which is the standard statistic of the firm pricing problem in steady state. Furthermore, exploiting the fact that $\partial_p d_k$is homogeneous of degree -1, symmetric in $\boldsymbol{p}_k$ one can again apply Euler's theorem to obtain:

$$\left(\partial_{pp} d_k\right) p(j^*) + \int \left(\partial_{pj} d_k\right) p_k(j)dj + \left(\partial_{pe_k} d_k\right) e_k = -\partial_p d_k,$$

$\Leftrightarrow$

$$P_k \left(\partial_{pp} d_k + \partial_{pj} d_k\right) + \left(\partial_{pe_k} d_k\right) e_k = -\partial_p d_k,$$

$\Leftrightarrow$

$$\partial_{pj} d_k = -\frac{\partial_p d_k}{P_k} - c_k \left(\partial_{pe_k} d_k\right) - \partial_{pp} d_k.$$

Using this result: we can derive the following expression for the first-order change in the own-price derivative of sector-k demand:

[1] Note that $\boldsymbol{c}_k$ lives in L^1, since $\mathcal{U}$ is (strictly) concave the problem has a unique solution which satisfies the set of first order conditions. Applying the implicit function theorem – for Banach spaces – shows that $\boldsymbol{c}_k$ is a C^2 function of $\{\boldsymbol{p}_k, e_k\}$.

[2] Recall that, by definition, $c_k(j^*) = d_k(p_k(j^*), \boldsymbol{p}_k, e_k)$.

$$
\begin{aligned}
d\partial_p d_k &= \left(\partial_{pp} d_k\right) dp_k\left(j^*\right) + \int \left(\partial_{pj} d_k\right) dp_k(j)dj + \left(\partial_{pe_k} d_k\right) de_k, \\
&= \left(\partial_{pp} d_k\right) dp_k\left(j^*\right) + \int \left(-\frac{\partial_p d_k}{P_k} - c_k \partial_{pe} d_k - \partial_{pp} d_k\right) dp_k(j)dj + \left(\partial_{pe_k} d_k\right) de_k, \\
&= \left(\partial_{pp} d_k\right) \left(dp_k\left(j^*\right) - dP_k\right) - \partial_p d_k \frac{dP_k}{P_k} + \left(\partial_{pe_k} d_k\right) \left(de_k - c_k dP_k\right).
\end{aligned}
$$

This expression will allow us to characterize the changes in elasticities of substitution away from steady state and their impact on firms' pricing decisions – through changes in endogenous markups.

Outer intratemporal consumption problem (valid for unconstrained and HtM households)

We now turn to the allocation of expenditures over different sectors. Let $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots \boldsymbol{p}_K)$ be the full vector of prices and let $v_i(\boldsymbol{P}, e)$ the indirect utility function of the outer problem which can be household-specific, hence we momentarily re-introduce the subscript i. The problem is to choose expenditure levels across different sectors, conditional on optimally choosing the bundle of varieties $\boldsymbol{c}_k$, which we solved for in the previous section. Recall that we assume that U_i is increasing, strictly concave and C^3. The problem can be expressed as:

$$
v_i\left(\mathbf{P}, e\right) = \max_{\{e_1, e_2, \ldots e_K\}} U_i\left(\nu_1(\boldsymbol{p}_1, e_1), \nu_2(\boldsymbol{p}_2, e_2), \ldots, \nu_K(\boldsymbol{p}_K, e_K)\right), \quad s.t. \quad \sum_{k=1}^{K} e_k = e.
$$

The associated first-order optimality condition is given by $U^{'}_{i,k} \partial_{e_k} \nu_k = \iota$, where $\iota = \partial_e v_i$ is the Lagrange multiplier. The problem defines a spending function $e_{k,i}\left(e, \boldsymbol{P}\right)$ which is C^2. Note that, by symmetry and since subvariety prices are equal within sectors, it holds in steady state that $\partial_{p_k(j)} \nu_k = \partial_{p_k(j')} \nu_k$ for any j, j' and e_k, so we have $\partial_{p_k(j)} e_k\left(e, \boldsymbol{P}\right) = \partial_{p_k(j')} e_k\left(e, \boldsymbol{P}\right) \equiv \partial_{P_k} e_k\left(e, \boldsymbol{P}\right)$. The derivative of the indirect utility function with respect to the price of a variety j in sector k is given by:

$$
\partial_{p_k(j)} v_i = -\partial_e v_i c_k(j),
$$

which follows by Roy's identity, where $c_k(j)$ is a shorthand for $d_k(p_k(j), \boldsymbol{p}_k, e_{k,i}\left(e, \boldsymbol{P}\right))$.The expression for the mixed derivative (which we will employ later on) is given by:

$$
\begin{aligned}
P_k \partial_{e p_k(j)} v_i &= -P_k \left(\partial_{ee} v_i c_k(j) + \partial_e v_i \partial_e e_{k,i} \partial_{e_k} c_k(j)\right), \\
&= -\left(\partial_{ee} v_i e_{k,i} + \partial_e v_i \partial_e e_{k,i}\right).
\end{aligned}
$$

Given $\partial_{p_k(j)} e_{k,i}\left(e, \boldsymbol{P}\right) = \partial_{P_k} e_{k,i}\left(e, \boldsymbol{P}\right)$ we can now write the change in sector-k expenditures in terms of the change in the sectoral prices, $\frac{dP_k}{p_k} = \hat{P}_k = \int \hat{p}_k(j)dj$:

$$
\begin{aligned}
de_{k,i} - e_{k,i}\hat{P}_k &= \sum_{l=1}^{K} P_l \partial_{P_l} e_{k,i} \hat{P}_l - e_{k,i}\hat{P}_k + \partial_e e_{k,i} de, \\
&= \left(P_k \partial_{P_k} e_{k,i} + \partial_e e_{k,i} e_{k,i} - e_{k,i}\right)\hat{P}_k + \sum_{l \neq k}\left(P_l \partial_{P_l} e_{k,i} + \partial_e e_{k,i} e_l\right)\hat{P}_l - dP_l + \partial_e e_{k,i}\left(de - \sum_l e_{l,i}\hat{P}_l\right), \\
&\equiv \partial_e e_{k,i}\left(de - \sum_l e_{l,i}\hat{P}_l\right) + e_{k,i}\sum_l \rho_{k,l}\left(i\right)\hat{P}_l.
\end{aligned}
$$

Note that we have $\sum_l \rho_{k,l} = 0$, as $e_k\left(e, \boldsymbol{P}\right)$ is homogeneous of degree one. In addition, consider the spending responses to a compensated change in the price of sector k:$\hat{P}_k = 1$, $de = e_k$. Inspecting the budget constraint gives $\sum_{l=0}^{K}\left(P_k \partial_{P_k} e_l + \partial_e e_l e_k\right) = e_k$ so we have $\sum_l e_l \rho_{l,k} = 0$.

Labor Supply (valid for unconstrained and HtM households) We start by solving for the labor supply response for an agent of type i in period t, which we derive from the first-order optimality condition for labor supply, which is given by $\chi'\left(\frac{n(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)} = \partial_e v_i W$. Taking a first order approximation of this condition, we obtain:

$$
\chi''\left(\frac{n(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)}\frac{dn(i)}{\vartheta(i)} = \left(\partial_{ee} v_i de(i) + \sum_k \int \left(\partial_{e p_k(j)} v\right) dp_k(j)dj\right) W + \partial_e v_i dW,
$$

$\Leftrightarrow$

$$
\frac{\chi''\left(n(i)/\vartheta(i)\right)}{\chi'\left(n(i)/\vartheta(i)\right)}\frac{dn(i)}{\vartheta(i)} = \left(\frac{\partial_{ee} v_i}{\partial_e v_i} de(i) - \sum_k \left(\frac{\partial_{ee} v_i}{\partial_e v_i} e_k(i) + \partial_e e_k(i)\right)\hat{P}_k\right) + \frac{dW}{W},
$$

$\Leftrightarrow$

$$
\hat{n}(i) = \psi\left\{\hat{W} - \sum_k \partial_e e_k(i)\hat{P}_k\right\} - \frac{\psi}{\sigma}\left(\hat{e}\left(i\right) - \sum_l s_l\left(i\right)\hat{P}_l\right).
$$

Intertemporal Decision (valid for non-HtM households only)

A household of type i born in t_0 has initial bond holdings $b_{t_0}(i) = b(i)\left(1 + \sum_l \bar{s}_l \frac{P_{l,t_0} - P_l^*}{P_l^*}\right)$ with P_l^* the steady state price of l and $P_{l,t_0} = \int p_{l,t_0}(j)\, dj$. Using the definition of the indirect utility function $v_i(\boldsymbol{P}, e)$, one can write the Lagrangian of the non-HtM households intertemporal problem as:

$$V(i) = \max_{\{e_{t+s}, n_{t+s}, b_{t+s+1}\}_{s=0}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta(1-\delta))^{t+s} \left(v_i\left(\mathbf{P}_{t+s}, e_{t+s}(i)\right) - \chi\left(\frac{n_{t+s}(i)}{\vartheta(i)}\right)\right)$$
$$+ \theta_{t+s}(i)\left\{ b_{t+s}(i) + n_{t+s}(i) W_{t+s} + \sum_k \varsigma_k(i) Div_{k,t+s} - e_{t+s}(i) - \frac{b_{t+s+1}(i)}{R_{t+s}} \right\},$$

with the first-order conditions given by

$$\frac{\partial V(i)}{\partial e_{t+s}(i)} = \mathbb{E}_t\left[(\beta(1-\delta))^{t+s} \partial_e v_i\left(\mathbf{P}_{t+s}, e_{t+s}(i)\right) - \theta_{t+s}(i)\right] = 0,$$
$$\frac{\partial V(i)}{\partial n_{t+s}(i)} = \mathbb{E}_t\left[-\chi'\left(\frac{n_{t+s}(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)} + \theta_{t+s}(i) W_{t+s}\right] = 0,$$
$$\frac{\partial V(i)}{\partial b_{t+s+1}(i)} = \mathbb{E}_t\left[-\frac{\theta_{t+s}(i)}{R_{t+s}} + \theta_{t+s+1}(i)\right] = 0.$$

We now linearize the consumption Euler Equation, $\partial_e v_{t,i} = \beta(1-\delta) R_t \mathbb{E}_t\left[\partial_e v_{t+1,i}\right]$, around a stationary steady state with no uncertainty:

$$\partial_{ee} v_i de_t(i) + \sum_k \int \left(\partial_{e p_k(j)} v_i\right) dp_{k,t}(j) dj = \beta(1-\delta) dR_t \partial_e v_i$$
$$+ \beta(1-\delta) R \left(\partial_{ee} v_i de_{t+1}(i) + \sum_k \int \left(\partial_{e p_k(j)} v_i\right) dp_{k,t+1}(j) dj\right),$$
$$\Leftrightarrow$$
$$\frac{\partial_{ee} v_i}{\partial_e v_i} de_t(i) + \sum_k \int \left(\frac{\partial_{e p_k(j)} v_i}{\partial_e v_i}\right) dp_{k,t}(j) dj = \frac{dR_t}{R} + \frac{\partial_{ee} v_i}{\partial_e v_i} de_{t+1}(i) + \sum_k \int \left(\frac{\partial_{e p_k(j)} v_i}{\partial_e v_i}\right) dp_{k,t+1}(j) dj,$$
$$\Leftrightarrow$$
$$\frac{\partial_{ee} v_i}{\partial_e v_i}\left(de_t(i) - \sum_k e_k \hat{P}_{k,t}\right) - \sum_k \partial_e e_k(i) \hat{P}_{k,t} = \hat{R}_t + \frac{\partial_{ee} v_i}{\partial_e v_i}\left(de_{t+1}(i) - \sum_k e_k \hat{P}_{k,t+1}\right) - \sum_k \partial_e e_k(i) \hat{P}_{k,t+1},$$
$$\Leftrightarrow$$
$$\frac{e\partial_{ee} v_i}{\partial_e v_i}\left(\hat{e}_t - \sum_k s_k \hat{P}_{k,t}\right) - \sum_k \partial_e e_k(i) \hat{P}_{k,t} = \hat{R}_t + \frac{e\partial_{ee} v_i}{\partial_e v_i}\left(\hat{e}_{t+1} - \sum_k s_k \hat{P}_{k,t+1}\right) - \sum_k \partial_e e_k(i) \hat{P}_{k,t+1},$$
$$\Leftrightarrow$$
$$\left(\hat{e}_t - \sum_k s_k \hat{P}_{k,t}\right) = \left(\hat{e}_{t+1} - \sum_k s_k \hat{P}_{k,t+1}\right) - \sigma\left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1}\right),$$

where $s_l = e_l(i)/e(i)$ and the third line uses the fact that $P_k \frac{\partial_{e p_k(j)} v_i}{\partial_e v_i} = -\left(\partial_{ee} v_i e_k + \partial_e v_i \partial_e e_k\right)$. We define $\sigma \equiv -\partial_e v_i / e\partial_{ee} v_i$ as the elasticity of intertemporal substitution.

Note: In the formula above and in the labor supply decision problem, we assumed that the EIS $\sigma = -\partial_e v_i / e\partial_{ee} v_i$ is equal across households. It is always possible to renormalize the intratemporal indirect utility of consumption v_i to obtain an arbitrary EIS without affecting the allocation of expenditure (at given $e_t(i)$) across markets and subvarieties. Indeed, if the utility of the households is renormalized to $\Upsilon_i\left(U_i\left(\mathcal{U}_1\left(\boldsymbol{c}_1\right), \ldots, \mathcal{U}_K\left(\boldsymbol{c}_K\right)\right)\right)$, demand for subvarieties $d_k(p_k(j^*), \boldsymbol{p}_k, e_k)$ and the sectoral expenditure functions $e_{k,i}\left(e, \boldsymbol{P}\right)$remains the same while indirect utility of consumption becomes $\Upsilon_i\left(v_i(e, \boldsymbol{P})\right)$. Defining $\Upsilon_i(\cdot) = \left(v_i^{-1}(\cdot, \boldsymbol{P})\right)^{1-\frac{1}{\sigma}} / \left(1 - \frac{1}{\sigma}\right)$ with $\boldsymbol{P}$ fixed at its steady state value allows us to parametrize the EIS to any value σ.

Expenditure of Hand-to-Mouth households.

HtM households consume all their current income, i.e. they never adjust their bond holdings. This allows one to directly solve for the real consumption change in period t from the budget constraint in period t only. In addition, a HtM household of type i born in t_0 has initial bond holdings $b_{t_0}(i) = b(i)\left(1 + \sum_l \bar{s}_l \frac{P_{l,t_0} - P_l^*}{P_l^*}\right)$. Differentiating

$$b_{t+1}(i) = R_t\left(b_t(i) + n_t(i) W_t + \sum_k \varsigma_k(i) Div_{k,t} - e_t(i)\right)$$

gives:

$$dR_t\left(b(i)+n(i)W-e(i)\right) \;+\; R\left(dn_t(i)W+n(i)dW_t+\sum_k \varsigma_k(i)dDiv_{k,t}(i)-de_t(i)+b(i)\sum_l \bar{s}_l\hat{P}_{l,t_0}\right) \;=\; b(i)\sum_l \bar{s}_l\hat{P}_{l,t_0},$$

$\Leftrightarrow$

$$R\left(Wn(i)\left(\left(1+\psi\right)\hat{W}_t-\frac{\psi}{\sigma}\left(\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}\right)-\sum_k \psi\partial_e e_k(i)\hat{P}_{k,t}\right)+\sum_k \varsigma_k(i)dDiv_{k,t}(i)-e\left(i\right)\left(\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}+\sum_l s_l(i)\hat{P}_{l,t}\right)\right)$$

$$=\left(1-R\right)b(i)\sum_l \bar{s}_l\hat{P}_{l,t_0}-\hat{R}_t b\left(i\right),$$

$\Leftrightarrow$

$$\hat{R}_t b\left(i\right)+R\left(\psi Wn(i)\hat{W}_t+Wn(i)\sum_k\left(\bar{s}_k-\psi\partial_e e_k(i)\right)\hat{P}_{k,t}-\sum_k e_k(i)\hat{P}_{k,t}+Wn(i)\sum \bar{s}_k\tilde{A}_{k,t}\right)+\left(R-1\right)b(i)\sum_l \bar{s}_l\hat{P}_{l,t_0}$$

$$=R\left(e\left(i\right)+\frac{\psi}{\sigma}Wn(i)\right)\left(\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}\right),$$

$\Leftrightarrow$

$$\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(\hat{R}_t\frac{b}{R}+Wn(i)\left(\psi\hat{W}_t-\psi\sum_k \overline{\partial_e e_k}\hat{P}_{k,t}+\sum \bar{s}_k\tilde{A}_{k,t}\right)-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)$$

$$+\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(1-\frac{1}{R}\right)b(i)\sum_l \bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)=\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}.$$

Using the definition of $\hat{\mathcal{Y}}_t \equiv \frac{\sigma}{\sigma+\psi}\left(\psi\hat{W}_t-\psi\sum_k \overline{\partial_e e_k}\hat{P}_{k,t}+\sum_k \bar{s}_k\tilde{A}_{k,t}\right)$, we obtain:

$$\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}=\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(\hat{R}_t\frac{b(i)}{R}+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_t-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)$$

$$+\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(\left(1-\frac{1}{R}\right)b(i)\sum_l \bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right)$$

where we have used the fact that the equity share of agent i in sector k is the same as the income share and the change in aggregate profits is $d\Pi_t=\sum_k P_k Y_k\left(\hat{P}_{k,t}+\hat{A}_{k,t}-\Omega_{N,k}\hat{W}_t-\sum_l \Omega_{k,l}\hat{P}_{l,t}\right)=\sum_k E_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}-\hat{W}_t\right)$, with $\tilde{\mathbf{A}}_t=\left(Id-\Omega\right)^{-1}\hat{\mathbf{A}}_t$ so that $dDiv_t(i)=\frac{Wn(i)}{WN}\sum_k E_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}-\hat{W}_t\right)$ (See subsection on Firm's Input choice for a definition of Ω).[3]

Firms

In this section, we derive the sectoral New Keynesian Phillips Curves. In each sector, identical firms with constant return to scale technology produce subvarieties of good k using labor and a bundle of sector l goods, aggregated by a representative intermediary as inputs. We first derive the firm's pricing equation away from steady state as a function of the change in unit marginal cost. We then study the firm's intratemporal problem to derive changes in demand for intermediate inputs and labor. Finally, using market clearing conditions for goods and labor, we derive the sectoral NKPCs in terms of sectoral prices, the output gap and changes in endogenous markups.

Intermediate inputs producers

We start with competitive intermediaries producing intermediate inputs. They aggregate differentiated varieties into $\tilde{Y}_k$ using a symmetric and CRS technology, and sell them to firms at a price P_k:

$$P_k=\inf_{y_k[j]}\int p_k(j)y_k(j)di$$
$$s.t. 1=\mathcal{F}^{\mathcal{I}}{}_k(\boldsymbol{y_k})$$

where $\mathcal{F}^{\mathcal{I}}{}_k$ is symmetric, increasing, strictly concave, C^3 and with $\mathcal{F}^{\mathcal{I}}{}_k(\boldsymbol{y_k})=1$ if $y_k\left(j\right)=1$ for all j.[4] The intermediary problem defines a unit demand function for subvarieties (indexed by j):

$$D^{\mathcal{I}}_k(p_k[j],\boldsymbol{p_k}).$$

[3] Note that the real consumption change for HtM agents is given by their MPC times the real income change in a given period that comes from three channels: interest rate changes, output gap and relative prices.

[4] The assumption $\mathcal{F}^{\mathcal{I}}{}_k(\boldsymbol{y_k})=1 if y_k\left(j\right)=1$ is simply a normalization ensuring that when all prices are equal with $p_k(j)=p_k\,\forall j$, $P_k=p_k$.

Goods varieties firms: price setting

We now turn to the firms producing individual goods varieties. We can re-write the present value of firm profits given in Equation \ref in terms of the reset price and using the fact that production of firms in k has constant returns to scale:[5]

$$\max_{p_{k,t}(j^*)} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(p_{k,t}(j^*)D_k\left(p_{k,t}(j^*),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right) - (1-\tau_k)MC_{k,t+s}(j^*)D_k\left(p_{k,t}(j^*),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right) - T_{k,t}\right)$$

with $D_k\left(p_{k,t}(j^*),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right) = \int d_k(p_k(j^*),\boldsymbol{p}_k,e_k(i))di + D_k^{\mathcal{I}}(p_k[j^*],\{p_k\})\tilde{Y}_{k,t+s}$ and where MC_k is the marginal cost, to be specified below. The first-order optimality condition is given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(D_{k,t+s}(j^*) + (p_{k,t}(j^*) - (1-\tau_k)MC_{k,t+s}(j^*))\,\partial_p D_{k,t+s}(j^*)\right) = 0.$$

Using the derivations in section **\ref** and aggregating over the distribution of agents, we can express the change in demand, to a first-order approximation, as:

$$\begin{aligned}
dD_{k,t+s}(j^*) = &\int \partial_p d_k(i,j^*)\left(dp_{k,t}(j^*) - dP_{k,t+s}\right) + \partial_{e_k} d_k(i,j^*)\left(de_{k,t+s}(i) - e_k(i)\hat{P}_{k,t+s}\right)di \\
&+ \partial_p D_k^{\mathcal{I}}\left(dp_{k,t}(j^*) - dP_{k,t+s}\right)\tilde{Y}_{k,t+s} + D_k^{\mathcal{I}}(p_k[j^*],\{p_k\})d\tilde{Y}_{k,t+s}, \\
= &\left(\frac{P_k\partial_p D_k^C}{D_k^C}C_k + \frac{P_k\partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}}\tilde{Y}_{k,t+s}\right)\left(\hat{p}_{k,t}(j^*) - \hat{P}_{k,t+s}\right) \\
&+ \frac{1}{P_k}\int\left(de_{k,t+s}(i) - c_k(i)dP_{k,t+s}\right)di + D_k^{\mathcal{I}}(p_k[j^*],\{p_k\})d\tilde{Y}_{k,t+s},
\end{aligned}$$

where $\partial_p D_k^C = \int \partial_p d_k(i,j^*)\,di$, $D_k^C = \int d_k(i,j^*)\,di$ and we have used that $D_k^{\mathcal{I}}(p_k[j^*],\boldsymbol{p_k}\}) = 1$ in the steady state. Similarly, for the second term:

$$\begin{aligned}
d\left(\partial_p D_{k,t+s}\right) = &\int\left(\partial_{pp}d_k(i,j^*)\right)\left(dp_{k,t}(j^*) - dP_{k,t+s}\right) - \partial_p d_k(i,j^*)\hat{P}_{k,t+s} \\
&+ \partial_{pe}d_k(i,j^*)\left(de_{k,t+s}(i) - e_k(i)\hat{P}_{k,t+s}\right)di + d\left(\partial_p D_k^{\mathcal{I}}\tilde{Y}_{k,t+s}\right) \\
= &\left(P_k\partial_{pp}D_k^C + P_k\partial_{pp}D_k^{\mathcal{I}}\tilde{Y}_{k,t+s}\right)\left(\hat{p}_{k,t}(j^*) - \hat{P}_{k,t+s}\right) - \left(\frac{P_k\partial_p D_k^C}{D_k^C}C_k + \frac{P_k\partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}}\tilde{Y}_{k,t+s}\right)\hat{P}_{k,t+s} \\
&+ \int \partial_{pe_k}d_k(i,j^*)\left(de_{k,t+s}(i) - c_k(i)dP_{k,t+s}\right)di + \partial_p D_k^{\mathcal{I}}d\tilde{Y}_{k,t+s}
\end{aligned}$$

Taking a first-order approximation of the first-order optimality condition and using the expressions above, we obtain:

$$\begin{aligned}
0 = &\,\mathbb{E}_t\sum_{s=0}^{\infty}\Lambda_{t,t+s}\theta_k^s\left\{\left(\hat{p}_{k,t}(j^*) - \hat{P}_{k,t+s}\right)P_k\partial_p D_{k,t+s} + \frac{1}{P_k}\int\left(de_{k,t+s}(i) - e_k(i)\hat{P}_{k,t+s}\right)di + D_k^{\mathcal{I}}(p_k[j^*],\{p_k\})d\tilde{Y}_{k,t+s}\right\} \\
&+ \mathbb{E}_t\sum_{s=0}^{\infty}\Lambda_{t,t+s}\theta_k^s\left(p_k(j^*) - (1-\tau_k)MC_{k,t+s}(j^*)\right)\left\{\left(P_k\partial_{pp}D_k^C + P_k\partial_{pp}D_k^{\mathcal{I}}\tilde{Y}_{k,t+s}\right)\left(\hat{p}_{k,t}(j^*) - \hat{P}_{k,t+s}\right) - P_k\partial_p D_{k,t+s}\hat{P}_{k,t+s}\right\} \\
&+ \mathbb{E}_t\sum_{s=0}^{\infty}\Lambda_{t,t+s}\theta_k^s\left(p_k(j^*) - (1-\tau_k)MC_{k,t+s}(j^*)\right)\left\{\int\partial_{pe_k}d_k(i,j^*)\left(de_{k,t+s}(i) - e_k(i)\hat{P}_{k,t+s}\right)di + \partial_p D_k^{\mathcal{I}}d\tilde{Y}_{k,t+s}\right\} \\
&+ \mathbb{E}_t\sum_{s=0}^{\infty}\Lambda_{t,t+s}\theta_k^s\left(dp_{k,t}(j^*) - (1-\tau_k)dMC_{k,t+s}(j^*)\right)\partial_p D_{k,t+s}
\end{aligned}$$

Grouping the terms together and using the fact that in steady state $p_k(j^*) - (1-\tau_k)MC(j^*) = \frac{P_k}{\bar{\epsilon}_k}$, $D_k + (P_k - (1+\tau_k)MC_k)\,\partial_p D_k = 0$, $\frac{P_k\partial_p D_k^C}{D_k^C} = \frac{P_k\partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}} = -\bar{\epsilon}_k$ and, in the steady state, $\Lambda_{t,t+s} = \tilde{\beta}^s$, where we assume that $\tilde{\beta} = (1-\delta)\beta = 1/R$, we obtain:

$$\begin{aligned}
0 = &\left(1-\tilde{\beta}\theta_k\right)^{-1}\left(2P_k\partial_p D_{k,t+s} + \frac{P_k}{\bar{\epsilon}_k}\left(P_k\partial_{pp}D_k^C + P_k\partial_{pp}D_k^{\mathcal{I}}\tilde{Y}_{k,t+s}\right)\right)\hat{p}_{k,t}(j^*) \\
&- \mathbb{E}_t\sum_{s=0}^{\infty}\left(\tilde{\beta}\theta_k\right)^s\left(2P_k\partial_p D_{k,t+s} + \frac{P_k}{\bar{\epsilon}_k}\left(P_k\partial_{pp}D_k^C + P_k\partial_{pp}D_k^{\mathcal{I}}\tilde{Y}_{k,t+s}\right)\right)\hat{P}_{k,t+s} \\
&+ \mathbb{E}_t\sum_{s=0}^{\infty}\left(\tilde{\beta}\theta_k\right)^s\int\left(\frac{1}{P_k} + \frac{P_k}{\bar{\epsilon}_k}\partial_{pe_k}d_k(i,j^*)\right)\left(de_{k,t+s}(i) - e_k(i)\hat{P}_{k,t+s}\right)di + \mathbb{E}_t\sum_{s=0}^{\infty}\left(\tilde{\beta}\theta_k\right)^s\left(\bar{\epsilon}_k - 1\right)D_k\hat{mc}_{k,t+s}
\end{aligned}$$

[5]This implies that total costs TC can be written as $TC_{k,t}(j) = MC\left(W_t,\boldsymbol{P}_t^I\right)D_{k,t}(j)$.

where $\hat{mc}_{k,t+s} \equiv \hat{MC}_{k,t+s} - \hat{P}_{k,t+s}$ is common across firms. Rewriting this expression recursively gives:

$$\hat{p}_{k,t}(j^*) = \left(1-\tilde{\beta}\theta_k\right)\hat{P}_{k,t} - \frac{\left(1-\tilde{\beta}\theta_k\right)\left(\bar{\epsilon}_k-1\right)}{2P_k\partial_p D_{k,t+s} + \frac{P_k}{\bar{\epsilon}_k}\left(P_k\partial_{pp}D_k^C + P_k\partial_{pp}D_k^{\mathcal{I}}\tilde{Y}_{k,t+s}\right)}\left\{\int\left(\frac{1}{P_k\left(\bar{\epsilon}_k-1\right)} + \frac{P_k}{\bar{\epsilon}_k\left(\bar{\epsilon}_k-1\right)}\partial_{pe_k}d_k\left(i,j^*\right)\right)\left(de_{k,t}\left(i\right) - e_k(i)\hat{P}_{k,t}\right)di + D_k\hat{mc}_{k,t}\right\} + \tilde{\beta}\theta_k\mathbb{E}_t\hat{p}_{k,t+1}(j^*)$$

Next recall the following definitions:

$$\bar{\epsilon}_k^s \equiv P_k\partial_p\ln(\bar{\epsilon}_k^s) = \left(-\int\left(\epsilon_k(j)-\bar{\epsilon}_k\right)^2\frac{e_k(j)}{E_k}dj + \int P_k\partial_p\epsilon_k(j)\frac{e_k(j)}{E_k}dj\right)/\bar{\epsilon}_k,$$

$$\bar{\epsilon}_k^{s,I} \equiv P_k\partial_p\ln(\epsilon_k^{\mathcal{I}}),$$

$$\gamma_{e,k}(i) \equiv \left(1-\frac{\epsilon_k(i)}{\bar{\epsilon}_k}\left(1+\frac{\partial ln(\epsilon_k(i))}{\partial ln(e_k(i))}\right)\right)/\left(\bar{\epsilon}_k-1\right).$$

Plugging these definition into the optimal price equation, we obtain:

$$\hat{p}_{k,t}(j^*) = \left(1-\tilde{\beta}\theta_k\right)\hat{P}_{k,t} + \frac{\left(1-\tilde{\beta}\theta_k\right)\left(\bar{\epsilon}_k-1\right)}{\bar{\epsilon}_k-1+s_k^C\bar{\epsilon}_k^s+\left(1-s_k^C\right)\bar{\epsilon}_k^{s,I}}\left\{s_k^C\int\gamma_{e,k}(i)\frac{de_{k,t}\left(i\right)-e_k(i)\hat{P}_{k,t}}{E_k}di + \hat{mc}_{k,t}\right\} + \tilde{\beta}\theta_k\mathbb{E}_t\hat{p}_{k,t+1}(j^*).$$

Note that all firms that can reset their prices choose the same $\hat{p}_{k,t}^*$ and $\hat{P}_{k,t} = \left(1-\theta_k\right)\hat{P}_{k,t}^* + \theta_k\hat{P}_{k,t-1}$. It follows that:

$$\pi_{k,t} = \frac{\left(1-\tilde{\beta}\theta_k\right)\left(1-\theta_k\right)}{\theta_k}\frac{\left(\bar{\epsilon}_k-1\right)}{\bar{\epsilon}_k-1+s_k^C\bar{\epsilon}_k^s+\left(1-s_k^C\right)\bar{\epsilon}_k^{s,I}}\left\{s_k^C\int\gamma_{e,k}(i)\frac{de_{k,t}\left(i\right)-e_k(i)\hat{P}_{k,t}}{E_k}di + \hat{mc}_{k,t}\right\} + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1}.$$

Defining

$$\lambda_k \equiv \frac{\left(1-\tilde{\beta}\theta_k\right)\left(1-\theta_k\right)}{\theta_k}\frac{\left(\bar{\epsilon}_k-1\right)}{\bar{\epsilon}_k-1+s_k^C\bar{\epsilon}_k^s+\left(1-s_k^C\right)\bar{\epsilon}_k^{s,I}},$$

we can write the sectoral NKPC as:

$$\pi_{k,t} = \lambda_k\left\{s_k^C\int\gamma_{e,k}(i)\frac{de_{k,t}\left(i\right)-e_k(i)\hat{P}_{k,t}}{E_k}di + \hat{mc}_{k,t}\right\} + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1}.$$

Goods varieties firms: intermediate input choice

The cost-minimization problem of the firm is given by: $\min WL_k(j) + \sum_l P_l\tilde{Y}_{l,k}(j) \quad s.t. \quad A_kF_k(n_k(j), \tilde{Y}_{1,k}(j), \tilde{Y}_{2,k}(j), \ldots, \tilde{Y}_{K,k}(j)) \geq y_k(j)$. Since F_k has constant return to scale we can express the change in the marginal cost as:

$$dMC_k = \frac{Wn_k}{Y_k}\hat{W} + \sum_l\frac{P_l\tilde{Y}_{l,k}}{Y_k}\hat{P}_l - MC_k\hat{A}_k,$$

$$\Leftrightarrow$$

$$\hat{MC}_k = \left(1+\mu_k\right)\left(1-\tau_k\right)\left(\Omega_{N,k}\hat{W} + \sum_l\Omega_{k,l}\hat{P}_l\right) - \hat{A}_k,$$

$$= \left(\Omega_{N,k}\hat{W} + \sum_l\Omega_{k,l}\hat{P}_l\right) - \hat{A}_k.$$

The subsidy is chosen to eliminate markup distortions in the steady state, i.e. $\left(1+\mu_k\right)\left(1-\tau_k\right) = 1$. Ω is the matrix of intermediate input shares ($\Omega_{k,l} = \frac{P_l\tilde{Y}_{l,k}}{P_kY_k}$), Ω_N a column vector of length K of labor shares ($\Omega_{N,k} = 1 - \sum_{l=1}^K\Omega_{k,l}$). Since F_k has CRS, we can write demand for input l has $\tilde{Y}_{l,k}(j) = \mathcal{Y}_{l,k}(\boldsymbol{P}, W)\frac{y_k(j)}{A_k}$ (where $\mathcal{Y}_{l,k}(\boldsymbol{P}, W)$ the unit demand for input l by firms in k is common to all firms in k) and derive change in aggregate demand for input bundle l as:

$$\frac{d\tilde{Y}_l}{\tilde{Y}_l} = \sum_k\mathcal{Q}_{l,k}\left(\hat{Y}_k - \hat{A}_k\right) + \tilde{\mathcal{T}}_{l,W}\hat{W} + \sum_k\tilde{\mathcal{T}}_{l,k}\hat{P}_k.$$

Let $\tilde{\mathcal{T}}$ be the matrix of aggregate input price elasticities such that $\tilde{\mathcal{T}}_{l,k} = \sum_m\frac{\tilde{Y}_{l,m}}{\tilde{Y}_l}\frac{\partial\mathcal{Y}_{l,m}}{\partial P_k}\frac{P_k}{\mathcal{Y}_{l,m}}$, $\tilde{\mathcal{T}}_{l,W} = \sum_m\frac{\tilde{Y}_{l,m}}{\tilde{Y}_l}\frac{\partial\mathcal{Y}_{l,m}}{\partial W}\frac{W}{\mathcal{Y}_{l,m}}$ be the column vector of wage elasticities and $\mathcal{Q}_{l,k} = \mathcal{Y}_{l,k}$ be the matrix of intermediate shares. Since intermediary input producers have a CRS technology we can write the (aggregated) market clearing equation for subvariety k as $\hat{Y}_k = s_k^C\hat{C}_k + (1-s_k^C)\tilde{Y}_k$.We have, denoting $\mathcal{D}\left[s^c\right]$ and $\mathcal{D}\left[PY\right]$ as the diagonal matrices with share of consumption demand and sectoral revenue on the diagonal ($s_k^C = \frac{E_k}{P_kY_k}$and P_kY_k), $\hat{\boldsymbol{Y}}$,$\hat{\boldsymbol{C}}$, $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{P}}$ the column vectors of sectoral output, consumption, TFP shocks and prices:

$$\hat{\boldsymbol{Y}} = \mathcal{D}\left[s^c\right]\hat{\boldsymbol{C}} + \left(Id - \mathcal{D}\left[s^c\right]\right)\left(\mathcal{Q}\left(\hat{\boldsymbol{Y}} - \hat{\boldsymbol{A}}\right) + \tilde{\mathcal{T}}_W\hat{W} + \tilde{\mathcal{T}}\hat{\boldsymbol{P}}\right),$$
$$\Leftrightarrow$$
$$\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)\hat{\boldsymbol{Y}} = \mathcal{D}\left[s^c\right]\hat{\boldsymbol{C}} - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\hat{\boldsymbol{A}} + \mathcal{T}_W\hat{W} + \mathcal{T}\hat{\boldsymbol{P}}.$$

where we use the fact that $\left[\left(Id - \mathcal{D}\left[s^c\right]\right)\mathcal{Q}\right]_{k,l} = \frac{P_k\tilde{Y}_k}{P_kY_k}\frac{\tilde{Y}_{k,l}}{\tilde{Y}_k} = \frac{P_kY_k}{P_lY_l}\Omega_{l,k}$. Note that $\mathcal{T}_W = \left(Id - \mathcal{D}\left[s^c\right]\right)\tilde{\mathcal{T}}_W$ and similarly $\mathcal{T} = \left(Id - \mathcal{D}\left[s^c\right]\right)\tilde{\mathcal{T}}$.

Labour Demand Response

We can similarly write demand for labor for a firm j in sector k as $n_k(j) = \mathcal{N}_k(\boldsymbol{P}, W)\frac{y_k(j)}{A_k}$. Differentiating and aggregating this function, we can express the percentage change in aggregate labor demand as:

$$\hat{N} = \boldsymbol{s^N}\left(\left(\hat{Y} - \hat{A}\right) + \tilde{\mathcal{T}}_W^N\hat{W} + \tilde{\mathcal{T}}^N\hat{P}\right),$$

where $\boldsymbol{s^N} = \left[\frac{W\int n_1(j)dj}{WN}, ..., \frac{W\int n_K(j)dj}{WN}\right]$, $\tilde{\mathcal{T}}_W^N = \left[\partial_{\ln(W)}\ln\left(\mathcal{N}_1\right), ..., \partial_{\ln(W)}\ln\left(\mathcal{N}_K\right)\right]$, $\tilde{\mathcal{T}}_{k,l}^N = \partial_{\ln(P_l)}\ln\left(\mathcal{N}_k\right)$. One can show that the change in labor demand will only depend on the change in consumption and productivities as follows:

$$\hat{N} = \boldsymbol{s^N}\left(\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left(\mathcal{D}\left[s^c\right]\hat{C} - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\hat{A} + \mathcal{T}_W\hat{W} + \mathcal{T}\hat{P}\right) - \hat{A} + \tilde{\mathcal{T}}_W^N\hat{W} + \tilde{\mathcal{T}}^N\hat{P}\right),$$
$$= \boldsymbol{s^N}\left(\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left(\mathcal{D}\left[s^c\right] - \hat{A}\right)\right) + \boldsymbol{s^N}\left(\mathcal{T}_W^N\hat{W} + \mathcal{T}^N\hat{P} + \left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left(\mathcal{T}_W\hat{W} + \mathcal{T}\hat{P}\right)\right).$$

Note that, as $\left(1 + \mu_k\right)\left(1 - \tau_k\right) = 1$, we have

$$\left[WN_1, ..., WN_K\right]\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1} = \left[PY_1, ..., PY_K\right],$$
$$\partial_{ln(W)}\mathcal{N}_k\hat{W} + \sum_l \partial_{ln(P_l)}\mathcal{Y}_{l,k}\hat{W} = 0,$$
$$\partial_{ln(P_l)}\mathcal{N}_k\hat{P}_l + \sum_m \partial_{ln(P_l)}\mathcal{Y}_{m,k}\hat{P}_l = 0,$$

where Id denotes the diagonal matrix. We thus obtain:

$$\hat{N} = \boldsymbol{s^N}\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left(\mathcal{D}\left[s^c\right]\hat{C} - \hat{A}\right)$$
$$= \sum_k \bar{s}_k\left(\hat{E}_k - \hat{P}_k\right) - \sum_k \frac{P_kY_k}{E}\hat{A}_k.$$

Aggregate Consumption Response

We can derive aggregate spending in sector k by simply aggregating individual decisions:

$$\hat{E}_k - \hat{P}_k = \frac{1}{E_k}\int de_k(i) - e_k(i)\hat{P}_k di,$$
$$= \int \frac{e(i)}{E_k}\partial_e e_k(i)\left(\hat{e} - \sum_l s_l\hat{P}_l\right)di + \sum_l S_{k,l}\hat{P}_l,$$

where $S_{k,l} = \int \rho_{k,l}(i)\frac{e_k(i)}{E_k}di$ is the aggregate compensated price elasticity of sector k with respect to P_l.

Labour Market Clearing

Let us re-introduce time subscripts. Recall that:

$$\hat{n}_t(i) = \psi\left\{\hat{W}_t - \sum_k \partial_e e_k(i)\hat{P}_{k,t}\right\} - \frac{\psi}{\sigma}\left(\hat{e}_t\left(i\right) - \sum_l s_l\left(i\right)\hat{P}_{l,t}\right).$$

Aggregating over all households we obtain:

$$\hat{N}_t = \psi\left(\hat{W}_t - \sum_k \int \frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}\right) - \frac{\psi}{\sigma}\int \frac{Wn(i)}{WN}\left(\hat{e}_t\left(i\right) - \sum_l s_l\left(i\right)\hat{P}_{l,t}\right)di.$$

So labor market clearing becomes:

$$\psi\left(\hat{W}_t-\sum_k\int\frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}\right)-\frac{\psi}{\sigma}\int\frac{Wn(i)}{WN}\left(\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}\right)di=\sum_k\left(\bar{s}_k\left(\hat{E}_{k,t}-\hat{P}_{k,t}\right)-\frac{P_kY_k}{E}\hat{A}_{k,t}\right)$$

$$=\sum_k\bar{s}_k\left(\int\frac{e(i)}{E_k}\partial_e e_k(i)\left(\hat{e}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)di+\sum_l S_{k,l}\hat{P}_{l,t}\right)-\sum_k\frac{P_kY_k}{E}\hat{A}_{k,t},$$

$\Leftrightarrow$

$$\psi\hat{W}_t-\psi\sum_k\int\frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}+\sum_k\frac{P_kY_k}{E}\hat{A}_{k,t}=\left(\int\frac{e(i)}{E}\left(\sum_k\partial_e e_k(i)+\frac{\psi Wn(i)}{\sigma e(i)}\right)\left(\hat{e}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)di+\sum_l\left(\sum_k\bar{s}_k\mathcal{S}_{k,l}\right)\hat{P}_{l,t}\right),$$

$$=\int\frac{e(i)}{E}\left(1+\frac{\psi Wn(i)}{e(i)\sigma}\right)\left(\hat{e}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)di,$$

where we have used the fact that $\sum_k e_k\rho_{k,l}(i)=0$ for all l,i so $\sum_k\bar{s}_k\mathcal{S}_{k,l}=0$. Finally, recall the definitions:

$$\tilde{\mathbf{A}}_t\equiv(Id-\Omega)^{-1}\hat{\mathbf{A}}_t,$$

$$\hat{\mathcal{Y}}_t\equiv\frac{\sigma}{\sigma+\psi}\left(\psi\hat{W}_t-\psi\sum_k\overline{\partial_e e_k}\hat{P}_{k,t}+\sum_k\bar{s}_k\tilde{A}_{k,t}\right),$$

where the last line uses the fact that $[E_1,...,E_k]\,(Id-\Omega)^{-1}=[P_1Y_1,...,P_kY_k]$.So we have:

$$\hat{\mathcal{Y}}_t=\frac{\sigma}{\sigma+\psi}\int\frac{e(i)}{E}\left(1+\frac{\psi Wn(i)}{e(i)\sigma}\right)\left(\hat{e}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)di+\frac{\sigma\psi}{\sigma+\psi}\sum_k\int\frac{wn(i)-e(i)}{E}\partial_e e_k(i)di\hat{P}_{k,t}.$$

Defining the natural level of aggregate demand as the level that prevails in the absence of markups distortions we obtain our formula for the output gap:

$$\hat{\mathcal{Y}}^*{}_t\equiv\frac{\sigma}{\sigma+\psi}\left(\left(\sum_k\psi\overline{\partial_e e_k}+\bar{s}_k\right)\tilde{A}_{k,t}\right),$$

$$\tilde{\mathcal{Y}}_t=\frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t-\sum_k\overline{\partial_e e_k}\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right).$$

see the optimal policy section for a justification of the efficiency of $\hat{\mathcal{Y}}^*{}_t$. Note that in the absence of markup distortions it holds that $\hat{P}_{k,t}=\hat{W}_t-\tilde{A}_{k,t}$. We will show later, that the output gap shows up in the social welfare function.

Production Efficiency (Detour)

In this section we briefly show that our set of steady state subsidies ($(1+\mu_k)\,(1-\tau_k)=1$) renders production efficient in the steady state. Production is efficient if the steady state consumption bundle $\{C_1,...,C_K\}$is produced at minimum labor cost.

$$\hat{L}=\sum_k s_k^N\left(\hat{Y}_k+\partial_{ln(W)}ln\left(\mathcal{N}_k\right)\hat{W}+\sum_l\partial_{ln(P_l)}ln\left(\mathcal{N}_k\right)\hat{P}_l\right),$$

$$\left(Id-\mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)\hat{Y}=\left\{\sum_k\frac{W\partial\mathcal{Y}^h_{j,k}}{Y_j\partial W}\right\}_j\hat{W}+\left\{\sum_k\frac{P_l\partial\mathcal{Y}^h_{j,k}}{Y_j\partial P_l}\right\}_{j,l}\hat{P}_l.$$

Therefore:

$$\sum_k s_k^N\partial_{ln(W)}ln\left(\mathcal{N}_k\right)+\boldsymbol{s}^{\boldsymbol{N}}\left(Id-\mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left\{\sum_k\frac{W\partial\mathcal{Y}^h_{j,k}}{Y_j\partial W}\right\}_j=0,$$

$$\sum_k s_k^N\partial_{ln(P_l)}ln\left(\mathcal{N}_k\right)+\boldsymbol{s}^{\boldsymbol{N}}\left(Id-\mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left\{\sum_k\frac{P_l\partial\mathcal{Y}^h_{j,k}}{Y_j\partial P_l}\right\}_j=0,$$

So $\boldsymbol{s}^{\boldsymbol{N}}\left(Id-\mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}=[P_jY_j/WN]$, which gives $WN_k=P_kY_k-\sum P_l\tilde{Y}_{l,k}$, or $(1+\mu_k)\,(1-\tau_k)=1$ for all k.

Sectoral NKPC

Recall that

$$\pi_{k,t} = \lambda_k \left\{ s_k^C \int \gamma_{e,k}(i) \frac{de_{k,t}(i) - e_k(i)\hat{P}_{k,t}}{E_k} di + \hat{mc}_{k,t} \right\} + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1},$$

$$\hat{mc}_{k,t} = \left(\Omega_{N,k} \hat{W}_t + \sum_l \Omega_{k,l} \hat{P}_{l,t} \right) - \hat{A}_{k,t} - \hat{P}_{k,t},$$

$$\tilde{\mathcal{Y}}_t = \frac{\sigma\psi}{\sigma+\psi} \left(\hat{W}_t - \sum_k \overline{\partial_e e_k} \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right),$$

$$de_{k,t}(i) - e_k(i)\hat{P}_{k,t} = \partial_e e_k(i) \left(de_t(i) - \sum_l e_l(i)\hat{P}_{l,t} \right) + e_k(i) \sum_l \rho_{k,l}(i) \hat{P}_{l,t}.$$

Combining these equations, we obtain:

$$\begin{aligned}
\pi_{k,t} &= \lambda_k \left\{ s_k^C \tilde{\mathcal{M}}_{k,t} + \Omega_{N,k} \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t + \Omega_{N,k} \sum_l \overline{\partial_e e_l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} \right) + \sum_l \Omega_{k,l} \hat{P}_{l,t} - \hat{A}_{k,t} - \hat{P}_{k,t} \right\} + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1}, \\
&= \lambda_k \left\{ s_k^C \tilde{\mathcal{M}}_{k,t} + \Omega_{N,k} \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t + \Omega_{N,k} \sum_l \overline{\partial_e e_l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right) + \sum_l \Omega_{k,l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right) \right\} + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1},
\end{aligned}$$

with

$$\tilde{\mathcal{M}}_{k,t} = \mathcal{M}_{k,t}^P + \mathcal{M}_{k,t}^E,$$

$$\mathcal{M}_{k,t}^P = \sum_l \int \gamma_{e,k}(i) \frac{e_k(i)}{E_k} \rho_{k,l}(i) di \hat{P}_{l,t},$$

$$\mathcal{M}_{k,t}^E = \int \gamma_{e,k}(i) \partial_e e_k \frac{e(i)}{E_k} \left(\hat{e}(i) - \sum_l s_l(i) \hat{P}_{l,t} \right) di$$

Finally, defining

$$\kappa_k \equiv \lambda_k \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \left(1 + \frac{\sigma\psi}{\sigma+\psi} \Gamma_k \right),$$

$$\Gamma_k \equiv \frac{R}{R-1} \frac{\sigma+\psi}{\sigma} \int \gamma_{b,k}(i) \frac{Wn(i)}{WN} di,$$

$$\mathcal{M}_{k,t}^D \equiv \mathcal{M}_{k,t}^E - \frac{1+\frac{\bar{\psi}}{\bar{\sigma}}}{1-\frac{1}{R}} \int \gamma_{b,k}(i) \frac{Wn(i)}{WN} di \hat{\mathcal{Y}}_t,$$

$$\mathcal{M}_{k,t} \equiv \mathcal{M}_{k,t}^P + \mathcal{M}_{k,t}^D + \frac{1+\frac{\bar{\psi}}{\bar{\sigma}}}{1-\frac{1}{R}} \int \gamma_{b,k}(i) \frac{Wn(i)}{WN} di \hat{\mathcal{Y}}_t^*,$$

$$\mathcal{NH}_t \equiv \sum_{l=1}^K (\overline{\partial_e e_l} - \bar{s}_l)(\hat{P}_{l,t} + \tilde{A}_{l,t}),$$

$$\bar{\mathcal{P}}_{k,t} \equiv (\hat{P}_{k,t} - \sum_{l=1}^K \bar{s}_l \hat{P}_{l,t}) + (\tilde{A}_{k,t} - \sum_{l=1}^K \bar{s}_l \tilde{A}_{l,t}),$$

$$\mathcal{P}_{k,t} \equiv \Omega_{N,k} \bar{\mathcal{P}}_{k,t} - \sum_{l=1}^K \Omega_{k,l} \left(\bar{\mathcal{P}}_{l,t} - \bar{\mathcal{P}}_{k,t} \right),$$

and noting that $\Omega_{N,k} \sum_{l=1}^K \overline{\partial_e e_l} + \sum_{l=1}^K \Omega_{k,l} = 1$, gives the formula in the model equation appendix. To obtain the equations of the main text without the Input-Output structure, we simply set $\Omega_{N,k} = 1$, $s_k^C = 1$ and $\mathcal{P}_{k,t} = \bar{\mathcal{P}}_{k,t}$, and obtain:

$$\pi_{k,t} = \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\mathcal{NH}_t + \mathcal{M}_{k,t} - \mathcal{P}_{k,t} \right) + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1}.$$

Evolution of arbitrary demand indices

In this section, we derive the dynamic equations characterizing the evolution of averages of individual households expenditures for arbitrary weights, taking into account the death/birth process. These equations can be used to compute the full distribution of consumption expenditures. In the next two subsection, we also use these equations to derive the dynamic equation for the output gap and for the endogenous markup wedge.

Denote by $C_t(\omega) = \int \omega(i) \left(\hat{e}_t(i) - \sum_l s_l(i) \hat{P}_{l,t} \right) di$ an arbitrary demand index with weight ω. Moreover, denote by $C_{t+1}^u(\omega) = \int (1-\varphi(i)) \omega(i) \left(\hat{e}_t(i) - \sum_l s_l(i) \hat{P}_{l,t} \right) di$ the contribution of unconstrained (=non-HtM) households to the demand index. We have:

$$\mathbb{E}_t C^u_{t+1}(\boldsymbol{\omega}) = (1-\delta)\, C^u_t(\boldsymbol{\omega}) + (1-\delta)\,\sigma \int (1-\varphi(i))\,\omega(i)\left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right) di + \delta \tilde{C}^{u,0}_{t+1}(\boldsymbol{\omega})$$

Here, we use the individual Euler equation, as derived above:

$$\left(\hat{e}_t - \sum_l s_l \hat{P}_{l,t}\right) = \mathbb{E}_t\left(\hat{e}_{t+1} - \sum_l s_l \hat{P}_{l,t+1}\right) - \sigma \mathbb{E}_t\left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right)$$

for households "born" before $t+1$. $\tilde{C}^{u,0}_{t+1}(\boldsymbol{\omega})$ is the consumption of the households born at $t+1$. Note that the lifetime budget constraint of the households born at t with wealth $b(i)\left(1+\sum_l \bar{s}_l \hat{P}_{l,t}\right)$ is

$$-b(i)\sum_l \bar{s}_l \hat{P}_{l,t} = \mathbb{E}_t \sum_{s=0}^{\infty} \frac{1}{R^s}\left(\frac{b(i)}{R}\hat{R}_{t+s} + Wn(i)\left(\hat{W}_{t+s} - \hat{n}_{t+s}\right) + dDiv_{t+s}(i) - e(i)\sum_k s_k(i)\hat{P}_{k,t+s}\right) - e(i)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{e}_{t+s} - \sum_k s_k(i)\hat{P}_{k,t+s}\right)$$

Using labor supply decisions and $dDiv_t(i) = \frac{Wn(i)}{WN}\sum_k E_k\left(\hat{P}_k + \tilde{A}_{k,t} - \hat{W}\right)$ we obtain:

$$-b(i)\sum_l \bar{s}_l \hat{P}_{l,t} + \left(e(i) + \frac{\psi}{\sigma}Wn(i)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{e}_{t+s} - \sum_k s_k(i)\hat{P}_{k,t+s}\right) =$$

$$\mathbb{E}_t \sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\hat{R}_{t+s} + \left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_{t+s} - \sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s} - \left(1-\frac{1}{R}\right)b(i)\sum_l \bar{s}_l \hat{P}_{l,t+s}\right)$$

$\Leftrightarrow$

$$\left(e(i) + \frac{\psi}{\sigma}Wn(i)\right)\sum \frac{1}{R^s}\left(\hat{e}_{t+s} - \sum_k s_k(i)\hat{P}_{k,t+s}\right) =$$

$$\mathbb{E}_t \sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\left(\hat{R}_{t+s} - \pi_{cpi,t+s+1}\right) + (1+\frac{\psi}{\sigma})Wn(i)\hat{\mathcal{Y}}_{t+s} - \sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s}\right).$$

Using the Euler equation, $\hat{e}_{t+u} - \sum_k s_k(i)\hat{P}_{k,t+u} = \hat{e}_t - \sum_k s_k(i)\hat{P}_{k,t} + \sigma\mathbb{E}_t\sum_{s=0}^{u-1}\left(\hat{R}_{t+s} - \sum_k \partial_e e_k(i)\pi_{k,t+s+1}\right)$ so we obtain:

$$\hat{e}_t - \sum_k s_k(i)\hat{P}_{k,t} =$$

$$\frac{1-\frac{1}{R}}{e(i)+\frac{\psi}{\sigma}Wn(i)}\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\left(\hat{R}_{t+s} - \pi_{cpi,t+s+1}\right) + (1+\frac{\psi}{\sigma})Wn(i)\hat{\mathcal{Y}}_{t+s} - \sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s}\right)$$

$$-\sigma\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^{s+1}}\left(\hat{R}_{t+s} - \sum_k \partial_e e_k(i)\pi_{k,t+s+1}\right).$$

Averaging across households with the arbitrary weights $\boldsymbol{\omega}$, we have:

$$\tilde{C}^{u,0}_t(\boldsymbol{\omega}) - \frac{1}{R}\mathbb{E}_t N\tilde{C}^{u,0}_{t+1}(\boldsymbol{\omega}) = -\sigma\frac{1}{R}\mathbb{E}_t\int(1-\varphi(i))\,\omega(i)\left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right)di +$$

$$\int \frac{(1-\varphi(i))\,\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{b(i)}{R}\left(\hat{R}_t - \pi_{cpi,t+1}\right) + (1+\frac{\psi}{\sigma})Wn(i)\hat{\mathcal{Y}}_{t+s} - \sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)di.$$

Defining $C^{u,0}_t(\boldsymbol{\omega}) \equiv \tilde{C}^{u,0}_t(\boldsymbol{\omega}) - C^u_t(\boldsymbol{\omega})$, we have:

$$\mathbb{E}_t C^u_{t+1}(\boldsymbol{\omega}) = C^u_t(\boldsymbol{\omega}) + \sigma\mathbb{E}_t\int(1-\varphi(i))\,\omega(i)\left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right)di + \frac{\delta}{1-\delta}C^{u,0}_{t+1}(\boldsymbol{\omega}),$$

$$C_t^{u,0}(\omega) - \frac{1}{R}\mathbb{E}_t C_{t+1}^{u,0}(\omega) + C_t(\omega) - \frac{1}{R}\mathbb{E}_t C_{t+1}^{u}(\omega) =$$

$$\mathbb{E}_t \int \frac{(1-\varphi(i))\,\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left(\frac{b(i)}{R}\left(\hat{R}_t - \pi_{cpi,t+1}\right) + \left(1+\frac{\psi}{\sigma}\right) Wn(i)\hat{\mathcal{Y}}_t - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right) di$$

$$-\sigma\frac{1}{R}\mathbb{E}_t \int (1-\varphi(i))\,\omega(i)\left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right) di,$$

$$C_t^{u,0}(\omega) - \frac{1}{R(1-\delta)}\mathbb{E}_t C_{t+1}^{u,0}(\omega) + \left(1-\frac{1}{R}\right) C_t^{u}(\omega) =$$

$$\mathbb{E}_t \int \frac{(1-\varphi(i))\,\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left(\frac{b(i)}{R}\left(\hat{R}_t - \pi_{cpi,t+1}\right) + \left(1+\frac{\psi}{\sigma}\right) Wn(i)\hat{\mathcal{Y}}_t - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right) di.$$

Now we consider the contribution of the HtMs, we have:

$$\mathbb{E}_t C_{t+1}^{HtM}(\omega) = (1-\delta)\, C_t^{HtM}(\omega) + \delta \tilde{C}_{t+1}^{HtM,0}(\omega)$$

$$+ (1-\delta)\,\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{\Delta\hat{R}_{t+1}\frac{b}{R} + \left(1+\frac{\psi}{\sigma}\right) Wn(i)\Delta\hat{\mathcal{Y}}_{t+1} - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\} di$$

$$-(1-\delta)\,\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{\left(1-\frac{1}{R}\right) b(i)\pi_{cpi,t+1}\right\} di,$$

$$\tilde{C}_t^{HtM,0}(\omega) - \frac{1}{R}\mathbb{E}_t \tilde{C}_{t+1}^{HtM,0}(\omega) =$$

$$-\frac{1}{R}\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{\Delta\hat{R}_{t+1}\frac{b}{R} + (1+\frac{\psi}{\sigma})Wn(i)\Delta\hat{\mathcal{Y}}_{t+1} - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\} di$$

$$-\frac{1}{R}\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{-\left(1-\frac{1}{R}\right) b(i)\pi_{cpi,t+1}\right\} di$$

$$+\left(1-\frac{1}{R}\right)\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{\hat{R}_t\frac{b}{R} + (1+\frac{\psi}{\sigma})Wn(i)\hat{\mathcal{Y}}_t - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right\} di$$

$$+\left(1-\frac{1}{R}\right)\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{-\left(1-\frac{1}{R}\right) b(i)\hat{P}_{cpi,t}\right\} d$$

$$+\mathbb{E}_t \int \varphi(i)\,\omega(i)\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(1-\frac{1}{R}\right) b(i) di \left(\hat{P}_{cpi,t} - \frac{1}{R}\hat{P}_{cpi,t+1}\right)$$

Defining $C_t^{HtM,0}(\omega) \equiv \tilde{C}_t^{HtM,0}(\omega) - C_t^{HtM}(\omega)$, we have:

$$\mathbb{E}_t C_{t+1}^{HtM}(\omega) = C_t^{HtM}(\omega) + \frac{\delta}{1-\delta}\mathbb{E}_t \tilde{C}_{t+1}^{HtM,0}(\omega) + \mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{-(1-\frac{1}{R})b(i)\pi_{cpi,t+1}\right\} di$$

$$\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{\Delta\hat{R}_{t+1}\frac{b}{R} + (1+\frac{\psi}{\sigma})Wn(i)\Delta\hat{\mathcal{Y}}_{t+1} - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\} di,$$

$$\tilde{C}_t^{HtM,0}(\omega) - \frac{1}{R(1-\delta)}\mathbb{E}_t \tilde{C}_{t+1}^{HtM,0}(\omega) + \left(1-\frac{1}{R}\right) C_t^{HtM}(\omega) =$$

$$\left(1-\frac{1}{R}\right)\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)} \left\{\frac{b}{R}\left(\hat{R}_t - \pi_{cpi,t+1}\right) + (1+\frac{\psi}{\sigma})Wn(i)\hat{\mathcal{Y}}_t - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right\} di.$$

Putting everything together, we obtain:

$$\mathbb{E}_t C_{t+1}(\boldsymbol{\omega}) = C_t(\boldsymbol{\omega}) + \sigma \mathbb{E}_t \int (1-\varphi(i))\,\omega(i) \left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1} \right) di + \frac{\delta}{1-\delta} \mathbb{E}_t C^0_{t+1}(\boldsymbol{\omega}) +$$

$$\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i) + \frac{\psi}{\sigma} Wn(i)} \left(\Delta \hat{R}_{t+1} \frac{b}{R} + (1+\frac{\sigma}{\psi}) Wn(i) \Delta \hat{\mathcal{Y}}_{t+1} - \sum_k \left(e(i)\left(s_k(i) - \bar{s}_k\right) + \psi Wn(i) \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \right) \pi_{k,t+1} \right) di$$

$$- \mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i) + \frac{\psi}{\sigma} Wn(i)} \left((1-\frac{1}{R}) b(i) \pi_{cpi,t+1} \right) di,$$

$$C^0_t(\boldsymbol{\omega}) - \frac{1}{R(1-\delta)} \mathbb{E}_t C^0_{t+1}(\boldsymbol{\omega}) + \left(1 - \frac{1}{R}\right) C_t(\boldsymbol{\omega})$$
$$= \int \frac{\omega(i)(1-\frac{1}{R})}{e(i) + \frac{\psi}{\sigma} Wn(i)} \left(\frac{b(i)}{R} \left(\hat{R}_t - \pi_{cpi,t+1} \right) + \left(1 + \frac{\psi}{\sigma}\right) Wn(i) \hat{\mathcal{Y}}_t - \sum_k \left(e(i)\left(s_k(i) - \bar{s}_k\right) + \psi Wn(i) \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \right) \hat{P}_{k,t} \right) di$$

with

$$C^0_0(\boldsymbol{\omega}) = (1-\delta) \int \frac{\omega(i)(1-\frac{1}{R})}{\left(e(i) + \frac{\psi}{\sigma} Wn(i)\right)} b(i) di P_{cpi,0.}$$

Euler Equation for the output gap. We now derive the evolution of the output gap. Recall the definition $\tilde{\mathcal{Y}}_t = \hat{\mathcal{Y}}_t - \mathcal{Y}^*{}_t$, with $\hat{\mathcal{Y}}_t = \frac{\sigma}{\sigma+\psi} \left(\psi \hat{W}_t - \psi \sum_k \overline{\partial_e e_k} \hat{P}_{k,t} + \sum_k \bar{s}_k \tilde{A}_{k,t} \right)$. Using the labor market condition, $\hat{\mathcal{Y}}_t$ can be expressed in terms of a demand index $C_t(\boldsymbol{\omega})$, with $\omega(i) = \frac{e(i)}{E} + \frac{\psi}{\sigma}\frac{Wn(i)}{WN}$: $\hat{\mathcal{Y}}_t = \frac{\sigma}{\sigma+\psi} C_t \left(\frac{e}{E} + \frac{\psi}{\sigma}\frac{Wn}{WN} \right) - \frac{\sigma\psi}{\sigma+\psi} \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN} \right) \partial_e e_k(i) di \hat{P}_{k,t}$. Therefore, applying the formulas derived above, we have:

$$\mathbb{E}_t \hat{\mathcal{Y}}_{t+1} - \hat{\mathcal{Y}}_t = \sigma \int (1-\varphi(i)) \frac{\sigma \frac{e(i)}{E} + \psi \frac{Wn(i)}{WN}}{\sigma + \psi} \left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1} \right) di =$$
$$- \frac{\sigma\psi}{\sigma+\psi} \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN} \right) \partial_e e_k(i) di \pi_{k,t+1} + \frac{\delta}{1-\delta} \hat{\mathcal{Y}}^0_{t+1}$$
$$+ \frac{\sigma}{\sigma+\psi} \mathbb{E}_t \int \varphi(i) \left\{ \Delta \hat{R}_{t+1} \frac{b}{RE} + (1 + \frac{\psi}{\sigma}) \frac{Wn(i)}{WN} \Delta \hat{\mathcal{Y}}_{t+1} \right\} di$$
$$- \frac{\sigma}{\sigma+\psi} \mathbb{E}_t \int \varphi(i) \left\{ \sum_k \left(\frac{e(i)}{E} (s_k(i) - \bar{s}_k) + \psi \frac{Wn(i)}{WN} (\partial_e e_k(i) - \overline{\partial_e e_k}) \right) \pi_{k,t+1} + \left(1 - \frac{1}{R}\right) \frac{b(i)}{E} \pi_{cpi,t+1} \right\} di,$$

$$\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} + \left(1 - \frac{1}{R}\right) \hat{\mathcal{Y}}_t + \frac{\sigma\psi}{\sigma+\psi} \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN} \right) \partial_e e_k(i) di \hat{P}_{k,t} =$$
$$\frac{\sigma}{\sigma+\psi} \left(1 - \frac{1}{R}\right) \int \left(\frac{b(i)}{RE} \left(\hat{R}_t - \pi_{cpi,t+1} \right) + \left(1 + \frac{\psi}{\sigma}\right) \frac{Wn(i)}{WN} \hat{\mathcal{Y}}_t - \sum_k \left(\frac{e(i)}{E} \left(s_k(i) - \bar{s}_k\right) + \psi \frac{Wn(i)}{WN} \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \right) \hat{P}_{k,t} \right) di.$$

Using

$$\int \frac{b(i)}{RE} di = \int \sum_k \frac{e(i)}{E} \left(s_k(i) - \bar{s}_k\right) di = 0$$

we obtain:

$$\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} + \left(1 - \frac{1}{R}\right) \hat{\mathcal{Y}}_t + \frac{\sigma\psi}{\sigma+\psi} \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN} \right) \partial_e e_k(i) di \hat{P}_{k,t} =$$
$$\frac{\sigma}{\sigma+\psi} \left(1 - \frac{1}{R}\right) \int \left(\left(1 + \frac{\psi}{\sigma}\right) \frac{Wn(i)}{WN} \hat{\mathcal{Y}}_t - \sum_k \psi \frac{Wn(i)}{WN} \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \hat{P}_{k,t} \right) di,$$
$$\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} = 0.$$

Using $\hat{\mathcal{Y}}_0^0 = 0$ and $\frac{1}{R(1-\delta)} > 1$, we have $\hat{\mathcal{Y}}_t^0 = 0$ for all t. Defining $\varphi^E \equiv \int \varphi(i) \frac{e(i)}{E} di$, $\varphi^N \equiv \int \varphi(i) \frac{Wn(i)}{WN} di$, we obtain:

$$\left(1-\varphi^N\right)\left(\mathbb{E}_t \hat{\mathcal{Y}}_{t+1} - \hat{\mathcal{Y}}_t\right) = \left(1-\varphi^N\right)\sigma \mathbb{E}_t \left(\hat{R}_t - \sum_k \overline{\partial_e e}_k \pi_{k,t+1}\right)$$
$$+ \mathbb{E}_t \int \frac{\sigma\varphi(i)}{\sigma+\psi}\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1} - \sigma(R-1)\hat{R}_t\right) - \frac{e(i)}{E}\sum_k \left((s_k(i)-\bar{s}_k) - \sigma(\partial_e e_k(i) - \overline{\partial_e e}_k)\right)\pi_{k,t+1}\right\} di$$
$$- \mathbb{E}_t \int \frac{\sigma\varphi(i)}{\sigma+\psi}\left\{-\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\left(\pi_{cpi,t+1} - \sigma\pi_{mcpi,t+1}\right)\right\} di$$

By definition, we have $r_t^* = \frac{1}{\sigma}\left(\mathbb{E}_t \hat{\mathcal{Y}}^*{}_{t+1} - \hat{\mathcal{Y}}_t^*\right)$ so $r_t^* \equiv \mathbb{E}_t \frac{1}{\sigma+\psi}\left(\left(\sum_k \psi\overline{\partial_e e}_k + \bar{s}_k\right)\left(\tilde{A}_{k,t+1} - \tilde{A}_{k,t}\right)\right)$. The evolution of the output gap $\tilde{\mathcal{Y}}_t = \hat{\mathcal{Y}}_t - \hat{\mathcal{Y}}_t^*$ is given by:

$$\left(1-\varphi^N\right)\left(\mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t\right) = \left(1-\varphi^N\right)\sigma \mathbb{E}_t \left(\hat{R}_t - \sum_k \overline{\partial_e e}_k \pi_{k,t+1} - r_t^*\right) +$$
$$\mathbb{E}_t \int \frac{\sigma\varphi(i)}{\sigma+\psi}\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1} - \sigma(R-1)\hat{R}_t\right) - \frac{e(i)}{E}\sum_k \left((s_k(i)-\bar{s}_k) - \sigma(\partial_e e_k(i) - \overline{\partial_e e}_k)\right)\pi_{k,t+1}\right\} di$$
$$- \mathbb{E}_t \int \frac{\sigma\varphi(i)}{\sigma+\psi}\left\{\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\left(\pi_{cpi,t+1} - \sigma\pi_{mcpi,t+1}\right)\right\} di,$$

which gives the equation of the main text.

Euler Equation for $\mathcal{M}_{k,t}^D$. Using $\omega(i) = \gamma_{e,k}(i)\partial_e e_k(i)\frac{e(i)}{E_k}$, we obtain:

$$\mathbb{E}_t \mathcal{M}_{k,t+1}^E - \mathcal{M}_{k,t}^E = \sum_l \sigma_{k,l}^{\mathcal{M}^E,u}\left(\hat{R}_t - \pi_{l,t+1}\right) + \frac{\delta}{1-\delta}\mathcal{M}_{k,t+1}^0$$
$$\int \left(\varphi(j)\frac{\gamma_{e,k}(i)\,\partial_e e_k(i)}{\left(1+\frac{Wn(i)\psi}{e(i)\sigma}\right)}\frac{b(i)}{RE_k}\right) di\Delta\hat{R}_{t+1} + \left(1+\frac{\bar{\psi}}{\bar{\sigma}}\right)\int\left(\varphi(i)\frac{\gamma_{e,k}(i)\,\partial_e e_k(i)}{\left(1+\frac{Wn(i)\psi}{e(i)\sigma}\right)}\frac{Wn(i)}{E_k}\right) di\Delta\hat{\mathcal{Y}}_{t+1}$$
$$+\sum_l \int \left(\varphi(i)\frac{\gamma_{e,k}(i)\,\partial_e e_k(i)}{\left(1+\frac{Wn(i)\psi}{e(i)\sigma}\right)}\left(-\frac{(R-1)\,b(i)}{RE_k}\bar{s}_l - \frac{e(i)}{E_k}\left(s_l(i)-\bar{s}_l\right) + \frac{Wn(i)}{E_k}\psi\left(\overline{\partial_e e}_l - \partial_e e_l(i)\right)\right)\right) di\pi_{l,t+1}$$

$$\mathcal{M}_{k,t}^0 - \frac{1}{(1-\delta)R}\mathbb{E}_t\mathcal{M}_{k,t+1}^0 = \int \gamma_{b,k}(i)\frac{b(i)}{RE}di\left(\hat{R}_t - \sum_l \bar{s}_l\pi_{l,t+1}\right)$$
$$+\sum_l \int \gamma_{b,k}(i)\left(\frac{e(i)}{E}\left(\bar{s}_l - s_l(i)\right) + \frac{wn(i)}{WL}\left(\bar{\psi}\overline{\partial_e e}_l - \psi(i)\partial_e e_l(i)\right)\right) di\hat{P}_{l,t}$$
$$+\left(1+\frac{\bar{\psi}}{\bar{\sigma}}\right)\int \gamma_{b,k}(i)\frac{Wn(i)}{WN}di\hat{\mathcal{Y}}_t - \frac{R-1}{R}\mathcal{M}_{k,t}^E$$

with

$$\sigma_{k,l}^{\mathcal{M}^E,u} = \int \gamma_{e,k}(j)\frac{(1-\varphi(i))\,e(i)}{E_k}\partial_e e_k(i)\partial_e e_l(i)di$$

Note that $\mathcal{M}_{k,t}^D = \mathcal{M}_{k,t}^E - \frac{1+\frac{\bar{\psi}}{\bar{\sigma}}}{1-\frac{1}{R}}\int \gamma_{b,k}(i)\frac{Wn(i)}{WN}di\hat{\mathcal{Y}}_t$, so using the equation for the output gap, we have:

$$\mathbb{E}_t\mathcal{M}_{k,t+1}^D - \mathcal{M}_{k,t}^D = \sum_l \sigma_{k,l}^{\mathcal{M},u}\left(\hat{R}_t - \pi_{l,t+1}\right) + \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}_{k,t+1}^0$$
$$+\frac{R}{R-1}\int\left(\gamma_{b,k}^u(i)\left(\varphi(i)\frac{b(i)}{RE} - \frac{(1-\varphi(i))\,Wn(i)}{(1-\varphi^L)\,WN}\int\left(\varphi(i)\frac{b(i)}{RE}\right)di\right)\right) di\mathbb{E}_t\Delta\hat{R}_{t+1}$$
$$-\frac{R}{R-1}\sum_l\int\gamma_{b,k}^u(i)\left\{(1-\frac{1}{R})\left(\varphi(i)\frac{b(i)}{E} - \frac{(1-\varphi(i))\,Wn(i)}{(1-\varphi^L)\,WN}\int\left(\varphi(i)\frac{b(i)}{E}\right)di\right)\bar{s}_l\right\} di\mathbb{E}_t\pi_{l,t+1}$$
$$-\frac{R}{R-1}\sum_l\int\gamma_{b,k}^u(i)\left\{\left(\varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) - \frac{(1-\varphi(i))\,Wn(i)}{(1-\varphi^L)\,WN}\int\varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)di\right)\right\} di\mathbb{E}_t\pi_{l,t+1}$$
$$-\frac{R}{R-1}\sum_l\int\gamma_{b,k}^u(i)\left\{\frac{Wn(i)}{WN}\psi\left(\varphi(i)\left(\partial_e e_l(i) - \overline{\partial_e e}_l\right) - \frac{1-\varphi(i)}{1-\varphi^E}\int\varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i) - \overline{\partial_e e}_l\right)di\right)\right\} di\mathbb{E}_t\pi_{l,t+1},$$

$$\mathcal{M}^0_{k,t} - \frac{1}{(1-\delta)R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} = \int \gamma^u_{b,k}(i)\frac{b(i)}{RE}di\left(\hat{R}_t - \sum_l \bar{s}_l \mathbb{E}_t \pi_{l,t+1}\right)$$
$$- \sum_l \int \gamma^u_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \psi\frac{Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\hat{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}$$

with

$$\sigma^{\mathcal{M},u}_{k,l} = \sigma\int \gamma_{e,k}(i)\frac{(1-\varphi(i))\,e(i)}{E_k}\partial_e e_k(i)\partial_e e_l(i)di - \overline{\partial_e e_l}^u\frac{R}{R-1}\int\frac{(1-\varphi(i))\,Wn}{(1-\varphi^L)\,WN}\gamma^u_{b,k}(i)di\left(\sigma\left(1-\varphi^E\right)+\psi\left(1-\varphi^L\right)\right).$$

B Model equations

Below we present the equations of the full linearized model with an interest rate rule. Derivations are provided in the previous appendix.

Coefficients - households Individual coefficients:

$$
\begin{aligned}
s_k(i) &= \frac{e_k(i)}{e(i)} \\
\partial_e e_k(i) &= \frac{\partial e_k(i)}{\partial e(i)} \overset{(NHCES)}{=} s_k(i)\left(\eta + (1-\eta)\frac{\zeta_k}{\bar{\zeta}(i)}\right) \quad \text{where} \quad \bar{\zeta}(i) = \sum_l s_l(i)\zeta_l \\
\rho_{k,l}(i) &= \partial_{P_l} e_k(i)/P_k + e_l(i)/P_l \partial_e e_k(i)/P_k \overset{(NHCES)}{=} (s_l(i) - 1 \cdot \mathbb{I}[k=l])\,\eta \\
\epsilon_k(i) &= -\frac{\partial e_k(i,j)}{\partial p_k(j)}\frac{p_k(j)}{c_k(i,j)} \overset{(HARA)}{=} a_k + \frac{b_k}{e_k(i)} \\
\epsilon_k^s(i) &= \frac{\partial \epsilon_k(i)}{\partial p_k(j)}\frac{p_k(j)}{\epsilon_k(i)} \overset{(HARA)}{=} \frac{b_k}{c_k(i)} \\
\gamma_{e,k}(i) &= \left(1 - \frac{\epsilon_k(i)}{\bar{\epsilon}_k}\left(1 + \frac{\partial \epsilon_k(i)}{\partial e_k(i)}\frac{e_k(i)}{\epsilon_k(i)}\right)\right) / (\bar{\epsilon}_k - 1) \overset{(HARA)}{=} \left(1 - \frac{a_k}{\bar{\epsilon}_k}\right)\frac{1}{\bar{\epsilon}_k - 1} \\
MPC(i)^u &= \frac{R-1}{R} / \left(1 + \frac{Wn(i)\psi}{e(i)\sigma}\right) \\
MPC(i)^{HtM} &= 1/\left(1 + \frac{Wn(i)\psi}{e(i)\sigma}\right) \\
MPC(i) &= \varphi(i) MPC(i)^{HtM} + (1-\varphi(i)) MPC(i)^u \\
\gamma_{b,k}^u(i) &= MPC(i)^u \gamma_{e,k}(i) \partial_e e_k(i)/\bar{s}_k \\
\gamma_{b,k}^{HtM}(i) &= MPC(i)^{HtM} \gamma_{e,k}(i) \partial_e e_k(i)/\bar{s}_k \\
\gamma_{b,k}(i) &= \varphi(i)\gamma_{b,k}^{htm}(i) + (1-\varphi(i))\gamma_{b,k}^u(i)
\end{aligned}
$$

where the second equality sign imposes the assumed preferences in the calibration.
Aggregate coefficients:

$$
\begin{aligned}
\bar{s}_k &= \frac{E_k}{E} = \frac{\int e_k(i)di}{\int e(i)di} \\
\bar{s}_k^u &= \int \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^E)\, E}\frac{e_l(i)}{e(i)}di \\
\overline{\partial_e e_l} &= \int \frac{e(i)}{E}\partial_e e_k(i)di \\
\overline{\partial_e e_l}^u &= \int \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^E)\, E}\partial_e e_l(i)di \\
\bar{\epsilon}_k &= \int \frac{e_k(i)}{E_k}\epsilon_k(i)di \overset{(HARA)}{=} a_k + \frac{b_k}{E_k} \\
\bar{\epsilon}_k^s &= \left(-\int (\epsilon_k(i) - \bar{\epsilon}_k)^2 \frac{e_k(i)}{E_k}di + \int \epsilon_k(i)\epsilon_k^s(i)\frac{e_k(i)}{E_k}di\right)/\bar{\epsilon}_k \overset{(HARA)}{=} \frac{b_k}{E_k} \\
s_k^C &= \frac{E_k}{P_k Y_k} \\
\mathcal{S}_{k,l} &= \int \frac{e_k(i)}{E_k}\gamma_{e,k}(i)\rho_{k,l}(i)di
\end{aligned}
$$

$$\varphi^E = \frac{\int e(i)\varphi(i)di}{E}$$

$$\varphi^N = \frac{\int Wn(i)\varphi(i)di}{WN}$$

$$\sigma_{k,l}^{\mathcal{M},u} = \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \overline{\partial_e e_l}^u \frac{R}{R-1} \int \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^N)\, WN} \gamma_{b,k}^u(i) di \left(\sigma\left(1-\varphi^E\right) + \psi\left(1-\varphi^L\right)\right)$$

$$R = \frac{1}{\beta(1-\delta)}$$

Coefficients - firms

$$\Omega_{N,k} = \frac{WN_k}{P_k Y_k}$$

$$\Omega_{k,l} = \frac{P_l \mathcal{Y}_{l,k}}{P_k Y_k}$$

$$\tilde{\Omega} = (Id - \Omega)^{-1}$$

$$\overline{\epsilon}_k^{\mathcal{I}} = \overline{\epsilon}_k^s$$

where Id is the identity matrix.

Coefficients - equations NKPC:

$$\lambda_k = \frac{(1-\theta_k)(1-\beta\theta_k)}{\theta_k} \frac{\overline{\epsilon}_k - 1}{\overline{\epsilon}_k - 1 + s_k^C \overline{\epsilon}_k^s + \left(1 - s_k^C\right) \overline{\epsilon}_k^{\mathcal{I}}}$$

$$\kappa_k = \lambda_k \left(\frac{1}{\sigma} + \frac{1}{\psi}\right) \left(\Omega_{N,k} + s_k^C \frac{\sigma\psi}{\sigma+\psi} \Gamma_k\right)$$

$$\Gamma_k = \frac{R}{R-1} \frac{\sigma+\psi}{\sigma} \int \gamma_{b,k}^u(i) \frac{Wn(i)}{WN} di$$

Other:

$$dhtm\_R_k = \frac{R}{R-1} \int \left(\gamma_{b,k}^u(i) \left(\varphi(i)\frac{b(i)}{RE} - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^N)\, WN} \int \left(\varphi(i)\frac{b(i)}{RE}\right) di\right)\right) di$$

$$\begin{aligned} dhtm\_\pi_{k,l} = & -\frac{R}{R-1} \int \gamma_{b,k}^u(i) \left(1 - \frac{1}{R}\right) \left(\varphi(i)\frac{b(i)}{E} - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^N)\, WN} \int \left(\varphi(i)\frac{b(i)}{E}\right) di\right) \bar{s}_l di \\ & -\frac{R}{R-1} \int \gamma_{b,k}^u(i) \left(\varphi(i)\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^N)\, WN} \int \varphi(i)\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) di\right) di \\ & -\frac{R}{R-1} \int \gamma_{b,k}^u(i) \frac{Wn(i)}{WN} \psi \left(\varphi(i)\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) - \frac{1-\varphi(i)}{1-\varphi^E} \int \varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) di\right) di \end{aligned}$$

$$m0\_r_k = \int \gamma_{b,k}^u(i) \frac{b(i)}{RE} di$$

$$m0\_P_{k,l} = -\int \gamma_{b,k}^u(i) \left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \psi\frac{Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right) di$$

$$ygap\_htm\_\pi_l = -\left(\sigma\left(\overline{\partial_e e_l}^u - \overline{\partial_e e_l}\right)\left(1-\varphi^E\right) - \left(s_l^u - \bar{s}_l\right)\left(1-\varphi^E\right) - \left(\varphi^E - \varphi^N\right)\left(\sigma\overline{\partial_e e_l} - \bar{s}_l\right)\right)$$

Sectoral equations. For every sector $k = 1, ..., K$ we have:

$$
\begin{aligned}
\pi_{k,t} &= \hat{P}_{k,t} - \hat{P}_{k,t-1} \\
\pi_{k,t} &= \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\Omega_{N,k} \mathcal{N}\mathcal{H}_t + s_k^C \mathcal{M}_{k,t} - \mathcal{P}_{k,t} \right) + \beta(1-\delta)\mathbb{E}_t \pi_{k,t+1} \\
\mathcal{M}_{k,t} &= \Gamma_k \hat{\mathcal{Y}}_t^* + \mathcal{M}_{k,t}^P + \mathcal{M}_{k,t}^D \\
\mathcal{M}_{k,t}^P &= \sum_l \mathcal{S}_{k,l} \left(\hat{P}_{l,t} - \hat{P}_{k,t} \right) \\
\mathbb{E}_t \mathcal{M}_{k,t+1}^D - \mathcal{M}_{k,t}^D &= \sigma_k^{\mathcal{M},u} \hat{R}_t - \sum_l \sigma_{k,l}^{\mathcal{M},u} \pi_{l,t+1} + \frac{\delta}{1-\delta} \mathbb{E}_t \mathcal{M}_{k,t+1}^0 + dhtm\_R_k \left(\mathbb{E}_t \hat{R}_{t+1} - \hat{R}_t \right) \\
&\quad + \sum_l dhtm\_\pi_{k,l} \mathbb{E}_t \pi_{l,t+1} \\
\mathcal{M}_{k,t-1}^0 - \frac{1}{(1-\delta) R} \mathbb{E}_t \mathcal{M}_{k,t}^0 &= m0\_r_k \left(\hat{R}_{t-1} - \mathbb{E}_t \pi_{cpi,t} \right) + \sum_l m0\_P_{k,l} \hat{P}_{l,t-1} - \frac{R-1}{R} \mathcal{M}_{k,t-1}^D \\
\bar{\mathcal{P}}_{k,t} &= \left(\hat{P}_{k,t} - \hat{P}_{cpi,t} \right) - \left(\hat{P}_{k,t}^* - \hat{P}_{cpi,t}^* \right) \\
\mathcal{P}_{k,t} &= \Omega_{N,k} \bar{\mathcal{P}}_{k,t} - \sum_l \Omega_{k,l} \left(\bar{\mathcal{P}}_{l,t} - \bar{\mathcal{P}}_{k,t} \right) \\
\hat{P}_{k,t}^* &= -\tilde{A}_{k,t} \\
\hat{A}_{k,t} &= \rho \hat{A}_{k,t-1} + \varepsilon_{k,t} \\
\tilde{A}_{k,t} &= \sum_l \tilde{\Omega}_{k,l} \hat{A}_{l,t}
\end{aligned}
$$

Aggregate equations

$$
\begin{aligned}
\tilde{\mathcal{Y}}_t &= \mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \sigma \mathbb{E}_t \left(\hat{R}_t - \pi_{mcpi,t+1} - \hat{r}_t^* \right) \\
&\quad - \frac{1}{1-\varphi^N} \frac{\sigma}{\sigma+\psi} \left(\frac{\varphi^E - \varphi^N}{R-1} \left(\mathbb{E}_t \hat{R}_{t+1} - (1 + \sigma (R-1)) \hat{R}_t \right) + \sum_l ygap\_htm\_\pi_l \cdot \pi_{l,t+1} \right) \\
\hat{r}_t^* &= \frac{1}{\sigma+\psi} \sum_l \left(\psi \overline{\partial_e e}_l + \bar{s}_l \right) \left(\mathbb{E}_t \tilde{A}_{l,t+1} - \tilde{A}_{l,t} \right) \\
\hat{\mathcal{Y}}_t^* &= \frac{1}{1 + \frac{\psi}{\sigma}} \sum_l \left(\psi \overline{\partial_e e}_l + \bar{s}_l \right) \tilde{A}_{l,t} \\
\mathcal{N}\mathcal{H}_t &= \sum_l \left(\overline{\partial_e e}_l - \bar{s}_l \right) \left(\hat{P}_{l,t} - \hat{P}_{l,t}^* \right) \\
P_{cpi,t} &= \sum_l \bar{s}_l \hat{P}_{l,t} \\
P_{cpi,t}^* &= \sum_l \bar{s}_l \hat{P}_{l,t}^* \\
\pi_{cpi,t} &= \sum_l \bar{s}_l \pi_{l,t} \\
\pi_{mcpi,t} &= \sum_l \overline{\partial_e e}_l \pi_{l,t} \\
\hat{R}_t &= \phi \pi_{cpi,t} + u_t^R \\
u_t^R &= \rho^R u_{t-1}^R + \varepsilon_t^R
\end{aligned}
$$

Equations for demand indices. Coefficients:

$$
\begin{aligned}
fracu_{\omega} &= \int \left(1-\varphi(i)\right)\omega(i)di \\
msu_{\omega,l} &= \int \left(1-\varphi(i)\right)\omega(i)\partial_e e_l(i)di \\
chtm\_R_{\omega} &= \int \left(\varphi(i)\frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}\frac{b(i)}{R}\right)di \\
chtm\_Y_{\omega} &= \left(1+\frac{\psi}{\sigma}\right)\int \left(\varphi(i)\frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}Wn(i)\right)di \\
chtm\_\pi_{\omega,l} &= \int \left(\varphi(i)\frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}\left(\frac{(R-1)\,b(i)}{R}\bar{s}_l+e(i)\left(s_l(i)-\bar{s}_l\right)+Wn(i)\psi\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)\right)\right)di \\
c0\_r_{\omega} &= \frac{R-1}{R}\int \frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}\frac{b(i)}{R}di \\
c0\_P_{\omega,l} &= -\frac{R-1}{R}\int \frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}\left(e(i)\left(s_l(i)-\bar{s}_l\right)+\psi Wn(i)\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)\right)di \\
c0\_Y_{\omega} &= \frac{R-1}{R}\left(1+\frac{\psi}{\sigma}\right)\int \frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}Wn(i)di
\end{aligned}
$$

Equations:

$$
\begin{aligned}
\mathbb{E}_t\hat{C}_{t+1}\left(\boldsymbol{\omega}\right)-\hat{C}_t\left(\boldsymbol{\omega}\right) = \sigma\left(fracu_{\omega}\hat{R}_t-\sum_l msu_{\omega,l}\pi_{l,t+1}\right)+\frac{\delta}{1-\delta}\mathbb{E}_t\hat{C}^0_{t+1}\left(\boldsymbol{\omega}\right) \\
+chtm\_R_{\omega}\left(\mathbb{E}_t\hat{R}_{t+1}-\hat{R}_t\right)+chtm\_Y_{\omega}\left(\mathbb{E}_t\hat{\mathcal{Y}}_{t+1}-\hat{\mathcal{Y}}_t\right)-\sum_l chtm\_\pi_{\omega,l}\mathbb{E}_t\pi_{l,t+1}
\end{aligned}
$$

$$
\hat{C}^0_{t-1}\left(\boldsymbol{\omega}\right)-\frac{1}{\left(1-\delta\right)R}\mathbb{E}_t\hat{C}^0_t\left(\boldsymbol{\omega}\right) = c0\_r_{\omega}\left(\hat{R}_{t-1}-\mathbb{E}_t\pi_{cpi,t}\right)+\sum_l c0\_P_{\omega,l}\hat{P}_{l,t-1}+c0\_Y_{\omega}\hat{\mathcal{Y}}_{t-1}-\frac{R-1}{R}\hat{C}_{t-1}\left(\boldsymbol{\omega}\right)
$$

C Proofs Analytical results Section 3

Result 1

Denote $\tilde{P}_{k,t} = \hat{P}_{k,t} - \sum_l \frac{\lambda}{\lambda_l}\overline{\partial_e e_l}\hat{P}_{l,t}$ and $\tilde{\pi}_{k,t} = \pi_{k,t} - \sum_l \frac{\lambda}{\lambda_l}\overline{\partial_e e_l}\pi_{l,t}$ the sector price and inflation relative to the 'Divine Coincidence index' $\hat{P}_{d,t} = \sum_l \frac{\lambda}{\lambda_l}\overline{\partial_e e_l}\hat{P}_{l,t}$ with $\frac{1}{\lambda} = \sum_l \frac{\overline{\partial_e e_l}}{\lambda_l}$, define similarly $\tilde{P}^*_{k,t}$. Under (**A.1**), we can aggregate the sectoral NKPCs with the divine coincidence weights to obtain:

$$\pi_{d,t} = \kappa\tilde{\mathcal{Y}}_t + \lambda\sum_k \overline{\partial_e e_k}\mathcal{M}_{k,t} + \beta(1-\delta)\mathbb{E}_t\pi_{d,t+1},$$

$$\tilde{\pi}_{k,t} = \left(\lambda_k\left(\tilde{P}^*_{k,t} - \tilde{P}_{k,t}\right) - \lambda_k\sum_l \overline{\partial_e e_l}\left(\tilde{P}^*_{l,t} - \tilde{P}_{l,t}\right) + \lambda_k\mathcal{M}_{k,t} - \lambda\sum_l \overline{\partial_e e_l}\mathcal{M}_{l,t}\right) + \beta(1-\delta)\mathbb{E}_t\tilde{\pi}_{k,t+1}.$$
$$\tilde{P}_{k,t} = \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1}.$$

Next, assume $\int \gamma_{b,k}(i)b(i)di = 0$ for all k, which is a weaker version of assumption (**A.2**). Recall that:

$$\mathcal{M}^D_{k,t} = \mathbb{E}_t\mathcal{M}^D_{k,t+1} - \sum_l \sigma^{\mathcal{M}}_{k,l}(\hat{R}_t - \mathbb{E}_t\pi_{l,t+1}) - \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}^0_{k,t+1},$$

$$\mathcal{M}^0_{k,t} = \frac{1}{(1-\delta)R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} + \int \gamma_{b,k}(i)\frac{b(i)}{RE}di\left(\hat{R}_t - \pi_{cpi,t+1}\right)$$
$$-\sum_l\int \gamma_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \frac{\psi Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\hat{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t},$$

$$\sigma^{\mathcal{M}}_{k,l} = \sigma\int \gamma_{e,k}(i)\frac{e(i)}{E_k}\partial_e e_k(i)\partial_e e_l(i)di - \sigma\overline{\partial_e e_l}\frac{R}{R-1}\frac{\sigma+\psi}{\sigma}\int \gamma_{b,k}(i)\frac{Wn(i)}{WN}di.$$

Given $\int \gamma_{b,k}(i)b(i)di = 0$, we can write:

$$\begin{aligned}(\sigma+\psi)\int \gamma_{b,k}(i)\frac{Wn(i)}{WN}di &= \int \gamma_{b,k}(i)\left(\psi\frac{Wn(i)}{WN} + \sigma\left(\frac{Wn(i)}{WN} + \left(1-\frac{1}{R}\right)\frac{b(i)}{E}\right)\right)di,\\ &= \int \gamma_{b,k}(i)\left(\psi\frac{Wn(i)}{WN} + \sigma\frac{e(i)}{E}\right)di,\\ &= \int\left(1-\frac{1}{R}\right)\frac{\gamma_{e,k}(i)\partial_e e_k(i)}{1+\frac{Wn(i)\psi}{e(i)\sigma}}\frac{E}{E_k}\left(\psi\frac{Wn(i)}{WN} + \sigma\frac{e(i)}{E}\right)di,\\ &= \sigma\left(1-\frac{1}{R}\right)\int \gamma_{e,k}(i)\partial_e e_k(i)\frac{e(i)}{E_k}di,\end{aligned}$$

and therefore:

$$\sigma^{\mathcal{M}}_{k,l} = \sigma\int \gamma_{e,k}(i)\frac{e(i)}{E_k}\partial_e e_k(i)\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)di,$$
$$\sum_l \sigma^{\mathcal{M}}_{k,l} = 0,$$

and

$$\mathcal{M}^D_{k,t} = \mathbb{E}_t\mathcal{M}^D_{k,t+1} + \sum_l \sigma^{\mathcal{M}}_{k,l}\mathbb{E}_t\tilde{\pi}_{l,t+1} - \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}^0_{k,t+1},$$
$$\mathcal{M}^0_{k,t} = \frac{1}{(1-\delta)R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} - \sum_l\int \gamma_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \frac{\psi Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}.$$

Recall that we can decompose the endogenous markup wedge $\mathcal{M}_{k,t} = \Gamma_k\mathcal{Y}^*_t + \mathcal{M}^P_{k,t} + \mathcal{M}^D_{k,t}$, and note that the first component, $\Gamma_k\mathcal{Y}^*_t$, is exogenous and hence independent of monetary policy. To show that the other components are independent of monetary policy too, we proceed as follows. Since $\sum_l \rho_{k,l}(i) = 0$, we can write the sectoral substitution component of the endogenous markup wedge as:

$$\mathcal{M}^P_{k,t} = \sum_{l=1}^K\int \gamma_{e,k}(i)\frac{e_k}{E_k}\rho_{k,l}(i)di\tilde{P}_{l,t}.$$

Therefore, the relative price equations can be rewritten as:

$$\tilde{\pi}_{k,t} - \beta(1-\delta)\mathbb{E}_t\tilde{\pi}_{k,t+1} = -(\lambda_k - \lambda)\left(\frac{1}{\psi}+\frac{1}{\sigma}\right)\hat{\mathcal{Y}}_t^* + \lambda_k\left(\tilde{P}_{k,t}^* - \sum_l \overline{\partial_e e_l}\tilde{P}_{l,t}^*\right) + \sum \alpha_{k,l}\tilde{P}_{l,t} + \sum\left(\lambda_k\tilde{\mathcal{M}}_{k,t} - \lambda\sum_l \overline{\partial_e e_l}\tilde{\mathcal{M}}_{l,t}\right),$$

$$\tilde{P}_{k,t} = \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1},$$

$$\tilde{\mathcal{M}}_{k,t} = \mathbb{E}_t\tilde{\mathcal{M}}_{k,t+1} - \frac{\delta}{1-\delta}\mathbb{E}_t\tilde{\mathcal{M}}_{k,t+1}^0,$$

$$\tilde{\mathcal{M}}_{k,t}^0 = \frac{1}{(1-\delta)R}\mathbb{E}_t\tilde{\mathcal{M}}_{k,t+1}^0 - \sum\beta_{k,l}\tilde{P}_{l,t} - \left(1-\frac{1}{R}\right)\tilde{\mathcal{M}}_{k,t}^D,$$

with

$$\alpha_{k,l} = -\lambda_k\mathbb{1}_{k=l} + \overline{\partial_e e_l}\lambda_l + \lambda_k\int\gamma_{e,k}(i)\frac{e}{E_k}\rho_{k,l}(i) - \lambda\sum_n\overline{\partial_e e_n}\int\gamma_{e,n}(i)\frac{e}{E_n}\rho_{n,l}(i)di - \lambda_k\sigma_{k,l}^{\mathcal{M}} + \lambda\sum_n\overline{\partial_e e_n}\sigma_{n,l}^{\mathcal{M}},$$

$$\beta_{k,l} = \int\gamma_{b,k}(i)\frac{e(i)}{E}\left((s_l(i)-\bar{s}_l) + \sigma\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di + \left(1-\frac{1}{R}\right)\sigma_{k,l}^{\mathcal{M}}.$$

Since $\hat{\mathcal{Y}}_t^*$ and $\tilde{P}_{l,t}^*$ are exogenous, $\tilde{P}_{k,t}$, $\tilde{\pi}_{k,t}$, $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_{k,t}^0$ are pinned down by a system of $4(K-1)$ equations which does not involve $\hat{R}_t$. These variables are therefore independent of monetary policy. From the above equations we observe that $\mathcal{M}_{k,t}^D$ and $\mathcal{M}_{k,t}^P$ depend only on $\tilde{\pi}_{k,t}$ and $\tilde{P}_{k,t}$. Therefore, these wedges are independent of monetary policy as well. Finally, the non-homotheticity and relative price wedge can be written as:

$$\mathcal{NH}_t = \sum_{l=1}^K(\overline{\partial_e e_l} - \bar{s}_l)(\tilde{P}_{l,t} - \tilde{P}_{l,t}^*),$$

$$\mathcal{P}_{k,t} = (\tilde{P}_{k,t}^* - \sum_l\bar{s}_l\tilde{P}_{l,t}^*) - (\hat{P}_{k,t}^* - \hat{P}_{cpi,t}^*).$$

It now follows that all the wedges are independent of monetary policy.

Additions to Result 1. In Appendix F we present a number of additions to Result 1. Specifically, we derive an inflation index implementing the Divine Coinvidence. We also extend Result 1 to the case with HtM households and Input-Output linkages.

Result 2

Note that if $\mathcal{M}_t = 0$, then $\kappa_k = \lambda_k\left(\frac{1}{\sigma}+\frac{1}{\psi}\right)$, so (**A.1**) becomes $\lambda_k = \lambda$ for all k. We can now write the NKPC for the MCPI as :

$$\pi_{mcpi,t} = \kappa\tilde{\mathcal{Y}}_t + \beta(1-\delta)\mathbb{E}_t\pi_{mcpi,t+1}.$$

And the Euler equations remains:

$$\tilde{\mathcal{Y}}_t = \mathbb{E}_t\tilde{\mathcal{Y}}_{t+1} - \sigma\mathbb{E}_t\left(\hat{R}_t - \pi_{mcpi,t+1} - \hat{r}_t^*\right).$$

As in the standard model, implementing

$$\hat{R}_t = \hat{r}_t^* + \phi\pi_{mcpi,t}$$

therefore stabilizes jointly the output gap and MCPI inflation (when $\phi > 1$). Indeed we obtain:

$$\mathbb{E}_t\pi_{mcpi,t+2} - (1+R+R\kappa\sigma)\mathbb{E}_t\pi_{mcpi,t+1} + (R+R\kappa\sigma\phi)\pi_{mcpi,t} = 0.$$

For $\phi > 1$, the roots of the polynomial are strictly larger than 1, so the only non explosive solution is $\pi_{mcpi,t} = 0$ which implies $\tilde{\mathcal{Y}}_t = 0$, see e.g. Woodford (2003).

Result 3

Denote the gap between MCPI and CPI inflation by $\pi_{\Delta,t} = \sum\left(\overline{\partial_e e_l} - \bar{s}_l\right)\pi_{l,t}$, and analogously define $\hat{P}_{\Delta,t}$ and $\hat{A}_{\Delta,t}$. Recall that if if $\mathcal{M}_t = 0$ then (**A.1**) becomes $\lambda_k = \lambda$ for all k. We can write the NKPC for $\pi_{\Delta,t}$ as:

$$R\pi_{\Delta,t} = -\lambda R\left(\hat{P}_{\Delta,t} + \hat{A}_{\Delta,t}\right) + \pi_{\Delta,t+1}$$

$\Leftrightarrow$

$$\hat{P}_{\Delta,t+1} - (1+R+R\lambda)\hat{P}_{\Delta,t} + R\hat{P}_{\Delta,t-1} = \lambda R\hat{A}_{\Delta,t}$$

The eigenvalues of the system are:

$$\mu_\pm = \frac{R + R\lambda + 1 \pm \sqrt{(R + R\lambda - 1)^2 + 4R\lambda}}{2}$$

With $\mu_+ > R + R\lambda$, $\mu_- < 1$. We obtain:

$$\hat{P}_{\Delta,t} = -\lambda \sum_0^t \mu_-^{t-s+1} \sum \frac{1}{\mu_+^u} \hat{A}_{\Delta,u+s}.$$

Therefore, we have:

$$\mathcal{NH}_t = -\lambda \sum_0^t \mu_-^{t-s+1} \sum \frac{1}{\mu_+^u} \hat{A}_{\Delta,u+s} + \hat{A}_{\Delta,t}.$$

Now suppose that we have a negative shock in a necessity (luxury) sector, in that case $\hat{A}_{\Delta,t} \geq 0$ ($\hat{A}_{\Delta,t} \leq 0$). Assume in addition that $\left|\hat{A}_{\Delta,t}\right| \leq \left|\hat{A}_{\Delta,0}\right|$ (the shock is larger on impact), then we have for a shock in a necessity sector

$$\begin{aligned}
\mathcal{NH}_0 &\geq \left(1 - \lambda\mu_- \sum_{u \geq 0} \frac{1}{\mu_+^u}\right) \hat{A}_{\Delta,0}, \\
&\geq \left(1 - \frac{\lambda\mu_-\mu_+}{\mu_+ - 1}\right) \hat{A}_{\Delta,0}, \\
&\geq \left(1 - \frac{\lambda R}{R + R\lambda - 1}\right) \hat{A}_{\Delta,0} \geq 0.
\end{aligned}$$

Similarly for a shock in a luxury sector, we have:

$$\mathcal{NH}_0 \leq \left(1 - \frac{\lambda R}{R + R\lambda - 1}\right) \hat{A}_{\Delta,0} \leq 0.$$

Result 3A.0 Analytical formulas for AR(1) shocks

In this section, we assume that shocks vanish at a constant rate ρ_a and derive analytical formulas for $\pi_{cpi,t}$, $\pi_{mcpi,t}$ and $\tilde{\mathcal{Y}}_t$. We show the following:

i. There exists a time $t_{\mathcal{NH}}$ ($t_{\mathcal{NH}} = 0$ if $\rho_a = 0$, $t_{\mathcal{NH}} = \infty$if $\rho_a = 1$) such that for a negative shock in a necessity (luxury) sector and $t \leq t_{\mathcal{NH}}$ then $\mathcal{NH}_t \geq 0$ ($\mathcal{NH}_t \leq 0$) and for $t > t_{\mathcal{NH}}$ $\mathcal{NH}_t \leq 0$ ($\mathcal{NH}_t \geq 0$)

ii. The gap $\pi_{cpi,t} - \pi_{mcpi,t}$ evolves independently of the policy rule. There exists t^* ($t^* = 0$ if $\rho_a = 0$, $t^* = \infty$if $\rho_a = 1$) such that for a negative shock in a necessity (luxury) sector and $t \leq t^*$ then $\pi_{cpi,t} \geq \pi_{mcpi,t}$ ($\pi_{cpi,t} \leq \pi_{mcpi,t}$) and for $t > t^*$ $\pi_{cpi,t} \leq \pi_{mcpi,t}$ ($\pi_{cpi,t} \geq \pi_{mcpi,t}$)

iii. Under the MCPI rule $\hat{R}_t = \phi\pi_{mcpi,t} + \hat{r}_t^*$ (with $\phi > 1$), we have $\pi_{mcpi,t} = \tilde{\mathcal{Y}}_t = 0$ so for a negative shock in a necessity (luxury) sector and $t \leq t^*$ then $\pi_{cpi,t} \geq 0$ ($\pi_{cpi,t} \leq 0$) and for $t > t^*$ $\pi_{cpi,t} \leq 0$ ($\pi_{cpi,t} \geq 0$)

iv. Under the CPI rule $\hat{R}_t = \phi\pi_{cpi,t} + \hat{r}_t^*$, There exists a time $t_{\mathcal{Y}}$ ($t_{\mathcal{Y}} = 0$ if $\rho_a = 0$, $t_{\mathcal{Y}} = \infty$ if $\rho_a = 1$) such that for a negative shock in a necessity (luxury) sector and $t \leq t_{\mathcal{Y}}$ then $\tilde{\mathcal{Y}}_t \leq 0$ ($\tilde{\mathcal{Y}}_t \geq 0$) and for $t > t_{\mathcal{Y}}$.$\tilde{\mathcal{Y}}_t \geq 0$ ($\tilde{\mathcal{Y}}_t \leq 0$).

v. Under the CPI rule $\hat{R}_t = \phi\pi_{cpi,t} + \hat{r}_t^*$, there exists a level of persistence ρ^*such that for $\rho_a \leq \rho^*$, for negative shocks in a necessity (luxury) sector $\pi_{cpi,t} \geq 0$ ($\pi_{cpi,t} \leq 0$) for all t. For $\rho_a > \rho^*$, There exists t_{CPI} ($t_{CPI} = \infty$if $\rho_a = 1$) such that for a negative shock in a necessity (luxury) sector and $t \leq t_{CPI}$ then $\pi_{cpi,t} \leq 0$ ($\pi_{cpi,t} \geq 0$) and for $t > t_{CPI}$ $\pi_{cpi,t} \geq 0$ ($\pi_{cpi,t} \leq 0$).

vi. Under the alternative rule $\hat{R}_t = \phi\pi_{mcpi,t}$ or $\hat{R}_t = \phi\pi_{cpi,t}$, the response of the output gap and both inflation indices at t are simply shifted up proportionally to $\rho_a^t \hat{r}_0^*$. Normalizing shocks such that $\hat{r}_0^* = -1$ (equal impact of sectoral shocks on efficient output), we have that for $t \leq t_{\mathcal{Y}}$ ($t > t_{\mathcal{Y}}$) and CPI targeting the output gap will be higher (lower) following a shock in a luxury sector rather than in a necessity sector. In addition, for high enough persistence the output gap will be negative under CPI targeting following a shock in a necessity sector.

Dynamics of the $\mathcal{NH}$ wedge. Rewriting $\mathcal{NH}_t = -\lambda \sum_0^t \mu_-^{t-s+1} \sum \frac{1}{\mu_+^u} \hat{A}_{\Delta,u+s} + \hat{A}_{\Delta,t}$, with $\hat{A}_{\Delta,t} = \rho_a^t \hat{A}_{\Delta,0}$ we have:

$$\mathcal{NH}_t = \frac{1}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \left((R - \rho_a)(1 - \rho_a)\rho_a^t - (R - \mu_-)(1 - \mu_-)\mu_-^t\right) \hat{A}_{\Delta,0}$$

Define $t^* = ln\left(\frac{(R-\mu_-)(1-\mu_-)}{(R-\rho_a)(1-\rho_a)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$, for $t \le t^*$, $\mathcal{NH}_t$same sign as $A_{\Delta,0}$ and for $t > t^*$, $\mathcal{NH}_t$same sign as $-A_{\Delta,0}$. For transitory shock $t^* = 0$, for a permanent shock $t^* = \infty$.
We now derive the evolution of inflation (CPI and MCPI) and the output gap under some particular interest rules.

Case $\hat{R}_t = \phi\pi_{mcpi,t} + \hat{r}^*_t$. The system of equations becomes

$$R\pi_{mcpi,t} = R\kappa\tilde{\mathcal{Y}}_t + \mathbb{E}_t\pi_{mcpi,t+1}$$
$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma\left(\phi\pi_{mcpi,t} - \pi_{mcpi,t+1}\right)$$

The eigenvalues of the system are

$$\lambda_\pm = \frac{R + R\kappa\sigma + 1 \pm \sqrt{\left(R + R\kappa\sigma - 1\right)^2 - 4R\kappa\sigma\left(\phi - 1\right)}}{2}$$

For $\phi > 1$, the eigenvalues are larger than 1in modulus, we therefore have $\pi_{mcpi,t} = \tilde{\mathcal{Y}}_t = 0$ for all t. The evolution of CPI is then

$$R\pi_{cpi,t} = R\lambda\mathcal{NH}_t + \mathbb{E}_t\pi_{cpi,t+1}$$

$$\pi_{cpi,t} = \frac{R\lambda}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\left((1-\rho_a)\rho_a^t - (1-\mu_-)\mu_-^t\right)\hat{A}_{\Delta,0}$$

We have that $\pi_{cpi,0}$ has the same sign as $A_{\Delta,0}$ (positive for a shock in a necessity sector, negative for a shock in a luxury sector). In addition, define $t^* = ln\left(\frac{(1-\mu_-)}{(1-\rho_a)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$, for $t \le t^*$, $\pi_{cpi,t}$ has same sign as $\hat{A}_{\Delta,0}$ and for $t > t^*$, $\pi_{cpi,t}$ has the same sign as $-\hat{A}_{\Delta,0}$. For transitory shock $t^* = 0$, for a permanent shock $t^* = \infty$.

Case $\hat{R}_t = \phi\pi_{cpi,t} + \hat{r}^*_t$.

$$R\pi_{cpi,t} = R\kappa\tilde{\mathcal{Y}}_t + R\lambda\mathcal{NH}_t + \mathbb{E}_t\pi_{cpi,t+1}$$
$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma\left(\phi\pi_{cpi,t} - \pi_{mcpi,t+1}\right)$$

In that case, we have

$$\pi_{cpi,t} = \frac{R\lambda}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\left\{\left(1 - \frac{R\kappa\sigma\phi}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a)}\right)(1-\rho_a)\rho_a^t - \left(1 - \frac{R\kappa\sigma\phi}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-)}\right)(1-\mu_-)\mu_-^t\right\}\hat{A}_{\Delta,0}$$

$$\tilde{\mathcal{Y}}_t = -\frac{R\lambda\sigma\phi}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\left\{\frac{(1-\rho_a)(R-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a)}\rho_a^t - \frac{(1-\mu_-)(R-\mu_-)}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-)}\mu_-^t\right\}\hat{A}_{\Delta,0}$$

$$\pi_{cpi,t} - \pi_{mcpi,t} = \frac{R\lambda}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\left((1-\rho_a)\rho_a^t - (1-\mu_-)\mu_-^t\right)\hat{A}_{\Delta,0}$$

Note that the fraction $\frac{(1-x)(R-x)}{(\mu_+-x)(\mu_--x)}$ is decreasing in x. From this we deduce that $\tilde{\mathcal{Y}}_t$ initially has the same sign as $-\hat{A}_{\Delta,0}$ (for $t \le t^* = t^* = ln\left(\frac{(R-\mu_-)(1-\mu_-)}{(R-\rho_a)(1-\rho_a)}\frac{(\lambda_+-\rho_a)(\lambda_--\rho_a)}{(\lambda_+-\mu_-)(\lambda_--\mu_-)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$) then for $t > t^*$ has the same sign as $A_{\Delta,0}$ (for a transitory shock$\tilde{\mathcal{Y}}_t$ has the same sign as $\hat{A}_{\Delta,0}$ for $t > 0$, for a permanent shock, $\tilde{\mathcal{Y}}_t$ has the same sign of $-A_{\Delta,0}$ for all t). This implies that the output gap is always negative on impact in response to a negative shock in the necessity sector, positive for a shock in a luxury sector.

The response of CPI is more ambiguous and depends on the persistence of the shock. There exist a persistence $0 < \nu = \frac{R+R\kappa\sigma+1-\sqrt{(R+R\kappa\sigma-1)^2+4R\kappa\sigma}}{2} < \rho^* < \frac{R+R\lambda+1-\sqrt{(R+R\lambda-1)^2+4R\lambda}}{2} = \mu_-$ such that for $\rho_a \le \rho^*$, $\pi_{cpi,t}$ always has the same sign as $\hat{A}_{\Delta,0}$. In that case, $\pi_{cpi,t}$ and the output gap initially move in opposite direction. If $\rho_a > \rho^*$, initially cpi inflation has the same sign as $-A_{\Delta,0}$ and then switches sign (keeping the sign of $-\hat{A}_{\Delta,0}$ if $\rho_a = 1$). In that case, $\pi_{cpi,t}$ and the output gap initially co-move. To see this consider the polynomial $P(x) = ((1-x)(R-x) - R\kappa\sigma x)(1-x) - (\lambda_+ - x)(\lambda_- - x)\left(1 - \frac{R\kappa\sigma\phi}{(\lambda_+-\mu_-)(\lambda_--\mu_-)}\right)(1-\mu_-)$. It is a third order polynomial with a negative dominant term. It is direct to check that $P(x) \ge 0$ for $x \le \nu$, $P(\mu_-) = 0$, $P(1) = 0$ and $P'(\mu_-) = 0$. This implies $P(x) \ge 0$ for $x \in [0,\rho^*] \cup [\mu_-,1]$, $P(x) \le 0$ for $x \in [\rho^*,\mu_-]$ with $\nu < \rho^* < \mu_-$. Inspecting the formula for $\pi_{cpi,t}$ then gives the result. In the extreme case where $\phi \to \infty$, we have $\pi_{cpi,t} = 0$, $\tilde{\mathcal{Y}}_t = -\frac{\sigma\psi}{\sigma+\psi}\mathcal{NH}_t$: stabilizing CPI inflation comes at the cost of distorting the output gap. Finally, since by Result 1 $\pi_{cpi,t} - \pi_{mcpi,t}$ is independent of monetary policy, we have as in the previous case that for a negative shock in a necessity sector, $\pi_{cpi,t}$ is initially higher than $\pi_{mcpi,t}$ and then lower and the opposite is true for a negative shock in a luxury sector.

Case $\hat{R}_t = \phi \pi_{mcpi,t}$. The system of equations becomes

$$R\pi_{mcpi,t} = R\kappa \tilde{\mathcal{Y}}_t + \mathbb{E}_t \pi_{mcpi,t+1}$$
$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma\left(\phi \pi_{mcpi,t} - \pi_{mcpi,t+1} - \hat{r}_t^*\right)$$

In that case

$$\pi_{mcpi,t} = \frac{R\kappa\sigma}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \rho_a^t \hat{r}_0^*$$
$$\tilde{\mathcal{Y}}_t = \frac{\sigma(R - \rho_a)}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \rho_a^t \hat{r}_0^*$$

The response is as in the standard model with $\pi_{mcpi,t}$ and $\tilde{\mathcal{Y}}_t$ both increasing in response to a negative shock (sectoral or aggregate) and increase is smaller the stronger the Taylor rule. In addition

$$\pi_{cpi,t} = \frac{R\lambda}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \left((1-\rho_a)\rho_a^t - (1-\mu_-)\mu_-^t\right)\hat{A}_{\Delta,0} + \frac{R\kappa\sigma}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \rho_a^t \hat{r}_0^*$$

$\pi_{cpi,t}$ increases relatively more than $\pi_{mcpi,t}$ for $t \leq t^* = ln\left(\frac{(1-\mu_-)}{(1-\rho_a)}\right) / ln\left(\frac{\rho_a}{\mu_-}\right)$, (less for $t > t^*$) for a negative shock in a necessity sector, relatively less for a negative shock in a luxury sector.

Case $\hat{R}_t = \phi \pi_{cpi,t}$. The system of equations becomes

$$R\pi_{cpi,t} = R\kappa \tilde{\mathcal{Y}}_t + R\lambda \mathcal{N}\mathcal{H}_t + \mathbb{E}_t \pi_{cpi,t+1}$$
$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma\left(\phi \pi_{cpi,t} - \pi_{mcpi,t+1} - \hat{r}_t^*\right)$$

We have:

$$\pi_{cpi,t} = \frac{R\lambda}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \left\{\left(1 - \frac{R\kappa\sigma\phi}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a)}\right)(1-\rho_a)\rho_a^t - \left(1 - \frac{R\kappa\sigma\phi}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-)}\right)(1-\mu_-)\mu_-^t\right\}\hat{A}_{\Delta,0} + \frac{R\kappa\sigma}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a)} \rho_a^t \hat{r}_0^*,$$

$$\tilde{\mathcal{Y}}_t = -\frac{R\lambda\bar{\sigma}\phi}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \left\{\frac{(1-\rho_a)(R-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a)}\rho_a^t - \frac{(1-\mu_-)(R-\mu_-)}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-)}\mu_-^t\right\}\hat{A}_{\Delta,0} + \frac{\sigma(R-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a)} \rho_a^t \hat{r}_0^*.$$

Using the results of the previous cases, we can directly see that following a negative shock in a necessity sector, the output gap is lower under targeting than under MCPI targeting. In addition, if we compare the response of a negative shock in a luxury sector and a necessity sector which have the same impact on efficient output ($\mathcal{Y}_t^* = \frac{1}{1+\frac{\psi}{\sigma}} \sum_l \left(\psi \overline{\partial_e e_l} + \bar{s}_l\right) \hat{A}_{l,t}$), the output gap is relatively lower in response to the shock in the necessity sector. If the shock is sufficiently persistent the output gap is negative in response to a shock in a necessity sector (as $\hat{r}_0^* \to 0$ when $\rho_a \to 1$).

Additions to Result 3. In Appendix F we extend provide a number of additional analytical results for the case with Hand-to-Mouth households.

Result 4

We first give an example of a shock[6] that is such that there is no inflation index ($\pi_t = \sum_k \tilde{\omega}_k \pi_{k,t}$ with $\sum_k \tilde{\omega}_k \neq 0$) that can be stabilized alongside the output gap under A.1 and A.2. We then argue that for any persistence of the shock ρ_a, the set of shocks for which an inflation index can be jointly stabilized with the output gap is of measure 0. Finally we extend the argument without A.1 and A.2. We denote $\tilde{P}_{k,t}\left(\left\{\hat{\mathbf{A}}_t\right\}_{t\geq 0}\right)$ the solution of the relative price system (described in Result 1) for an arbitrary sequence of shocks $\left\{\hat{\mathbf{A}}_t\right\}_{t\geq 0}$ (and similarly $\tilde{\pi}_{k,t}\left(\left\{\hat{\mathbf{A}}_t\right\}_{t\geq 0}\right)$, $\tilde{\mathcal{M}}_{k,t}\left(\left\{\hat{\mathbf{A}}_t\right\}_{t\geq 0}\right)$ the implied relative price inflation and endogenous markups). Consider a shock $\hat{\mathbf{A}}_t = \left\{\hat{A}_{1,t}, ..., \hat{A}_{k,t}\right\}_{t\geq 0}$ such that for $k = 1, ..., K-1$ and all t:

$$-\left(\frac{1}{\psi} + \frac{1}{\sigma}\right)(\lambda_k - \lambda)\hat{\mathcal{Y}}_t^* + \lambda_k \tilde{P}_{k,t}^* - \lambda_k \sum_l \overline{\partial_e e_l} \tilde{P}_{l,t}^* = 0$$

Re-expressed in terms of $\hat{\mathbf{A}}_t$ this becomes:

[6] There are, of course, other examples as well.

$$-\left(\lambda_k-\lambda\right)\sum_l\left(\overline{\partial_e e_l}+\frac{\bar{s}_l}{\psi}\right)\hat{A}_{l,t}-\lambda_k\hat{A}_{k,t}+\lambda_k\sum_l\overline{\partial_e e_l}\hat{A}_{l,t}=0$$

Note that this is a system of $K-1$ equations in K unknowns, so it admits a non trivial solution $\hat{\mathbf{A}}^*\neq 0$. We necessarily have:

$$\sum_l\left(\overline{\partial_e e_l}+\frac{\bar{s}_l}{\psi}\right)\hat{A}_l^*\neq 0.$$

We reason by contradiction: if $\sum_l\left(\overline{\partial_e e_l}+\frac{\bar{s}_l}{\psi}\right)\hat{A}_l^*=0$, then $\lambda_k\left(\hat{A}_k^*-\sum_n\overline{\partial_e e_n}\hat{A}_n^*\right)=\lambda_l\left(\hat{A}_l^*-\sum_n\overline{\partial_e e_n}\hat{A}_n^*\right)$ for all l,k (note that the K^{th} sector equation is a linear combination of the other $K-1$ equations). Under (**A.1**), we have that $\lambda_k>0$ for all k, which implies $\hat{A}_k^*=0$ for all k. Indeed, noting $\underline{A}^*=min\left(\hat{A}_l^*\right),\overline{A}^*=max\left(\hat{A}_l^*\right)$, we have $0\leq\overline{\lambda}\left(\overline{A}^*-\sum_n\overline{\partial_e e_n}\hat{A}_n^*\right)=\underline{\lambda}\left(\underline{A}^*-\sum_n\overline{\partial_e e_n}\hat{A}_n^*\right)\leq 0$, so $\hat{A}_k^*$ is constant across sectors which implies $\hat{A}_k^*=0$ for all k. This contradicts the fact that $\hat{\mathbf{A}}^*$ is a non trivial solution of the system.

Next, define the shock $\hat{\mathbf{A}}^{*,\rho_a}$ such that $\hat{\mathbf{A}}_t^{*,\rho_a}=\rho_a^t\hat{\mathbf{A}}^*$ for $0\leq\rho_a\leq 1$, in that case, the system for relative prices is given by:

$$\begin{aligned}\tilde{\pi}_{k,t}-\beta\left(1-\delta\right)\mathbb{E}_t\tilde{\pi}_{k,t+1}&=\sum\alpha_{k,l}\tilde{P}_{l,t}+\sum\left(\lambda_k\tilde{\mathcal{M}}_{k,t}-\lambda\sum_l\overline{\partial_e e_l}\tilde{\mathcal{M}}_{l,t}\right)\\ \tilde{P}_{k,t}&=\tilde{\pi}_{k,t}+\tilde{P}_{k,t-1}\\ \tilde{\mathcal{M}}_{k,t}&=\mathbb{E}_t\tilde{\mathcal{M}}_{k,t+1}-\frac{\delta}{1-\delta}\mathbb{E}_t\tilde{\mathcal{M}}^0_{k,t+1}.\\ \tilde{\mathcal{M}}^0_{k,t}&=\frac{1}{\left(1-\delta\right)R}\mathbb{E}_t\tilde{\mathcal{M}}^0_{k,t+1}-\sum\beta_{k,l}\tilde{P}_{l,t}-\left(1-\frac{1}{R}\right)\tilde{\mathcal{M}}^D_{k,t}\end{aligned}$$

We therefore that have $\tilde{P}_{k,t}\left(\hat{\mathbf{A}}^{*,\rho_a}\right)=0$ for all k, t is a solution of the system. This implies that the NKPC for the index $\pi_{d,t}$ is:

$$\pi_{d,t}=\kappa\tilde{\mathcal{Y}}_t+\lambda\sum_k\overline{\partial_e e_k}\Gamma_k\sum_l\frac{\sigma\left(\psi\overline{\partial_e e_l}+\bar{s}_l\right)}{\sigma+\psi}\rho_a^t\hat{A}_l^*++\beta\mathbb{E}_t\pi_{d,t+1}$$

Since $\sum_l\frac{\sigma\left(\psi\overline{\partial_e e_l}+\bar{s}_l\right)}{\sigma+\psi}\hat{A}_l^*\neq 0$, the index $\pi_{d,t}$ cannot be stabilized jointly with the output gap for any $\hat{\mathbf{A}}^{*,\rho_a}$.

Now Take an arbitrary inflation index π_t, decomposing it in the basis of $\pi_{d,t}$ and relative prices, we have:

$$\pi_t=\omega_d\pi_{d,t}+\sum_{k=1}^{K-1}\omega_k\tilde{\pi}_{k,t}$$

Suppose $\omega_d\neq 0$. We have for an arbitrary shock persistence ρ_a, $\hat{\mathbf{A}}^{\rho_a}$ such that $\hat{\mathbf{A}}_t^{\rho_a}=\rho_a^t\hat{\mathbf{A}}$ that the NKPC for the index π_t is:

$$\pi_t-\beta\mathbb{E}_t\pi_{t+1}=\omega_d\kappa\tilde{\mathcal{Y}}_t+\mathcal{W}_t\left(\hat{\mathbf{A}}^{\rho_a}\right)$$

Where the wedge $\mathcal{W}_t\left(\hat{\mathbf{A}}^{\rho_a}\right)$ is given by

$$\begin{aligned}\mathcal{W}_t\left(\hat{\mathbf{A}}^{\rho_a}\right)=&\sum_l\left(\omega_d\lambda\overline{\partial_e e_l}+\sum_{m=1}^{K-1}\omega_m\left(\lambda_m-\lambda\overline{\partial_e e_m}\right)\right)\left\{\Gamma_l\sum_m\frac{\sigma\left(\psi\overline{\partial_e e_m}+\bar{s}_m\right)}{\sigma+\psi}\rho_a^t\hat{A}_m+\left(\mathcal{M}^D_{l,t}\left(\hat{\mathbf{A}}^{\text{æ}_\mathbf{a}}\right)+\mathcal{M}^P_{l,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)\right)\right\}\\&-\sum_{l=1}^{k-1}\omega_l\left(\lambda_k\left(\rho_a^t\hat{A}_m+\tilde{P}_{l,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)\right)-\lambda_k\sum_m\overline{\partial_e e_m}\left(\rho_a^t\hat{A}_m+\tilde{P}_{m,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)\right)\right)\end{aligned}$$

Since the system of relative prices (described in Result 1) is linear and that shock enters linearly, we have that the mapping $\hat{\mathbf{A}}\mapsto\tilde{P}_{k,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)$ is linear. Therefore we directly have that the mappings $\hat{\mathbf{A}}\mapsto\mathcal{M}^D_{k,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)$, $\hat{\mathbf{A}}\mapsto\mathcal{M}^P_{k,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)$ are linear (as $\mathcal{M}^D_{k,t}$ and $\mathcal{M}^P_{k,t}$ are linear functions of relative prices). This implies that the mapping $\hat{\mathbf{A}}\mapsto\mathcal{W}_t\left(\hat{\mathbf{A}}^{\rho_a}\right)$ is also linear. Note that since $\mathcal{W}_0\left(\hat{\mathbf{A}}^{*,\rho_a}\right)=\omega_d\lambda\left(\sum\overline{\partial_e e_k}\Gamma_k\right)\sum_l\frac{\sigma\left(\psi\overline{\partial_e e_l}+\bar{s}_l\right)}{\sigma+\psi}\hat{A}_l^*\neq 0$, we have that the kernel of $\hat{\mathbf{A}}\mapsto\mathcal{W}_0\left(\hat{\mathbf{A}}^{\rho_a}\right)$ is at most of dimension $K-1$. Since a subspace of $\mathbb{R}^K$ of dimension $K-1$ has Lebesgue measure 0, that implies that for a any ρ_a, $\mathcal{W}_0\left(\hat{\mathbf{A}}^{\rho_a}\right)\neq 0$ on a subset of measure 1. This implies that no index with $\omega_d\neq 0$ can be stabilized jointly with the output gap. Therefore only relative prices can be stabilized jointly with the output gap. However, as shown in Result 1, relative prices are independent from monetary policy. So the only inflation index that could be stabilized

jointly with inflation would be a trivial index which does not respond to any shock.
Note that previous argument remains valid if we relax A.1 and A.2. for permanent shocks. Consider a policy that stabilizes the output gap. We have $\tilde{\mathcal{Y}}_t = 0$, $\hat{R}_t = r_t^* + \pi_{mcpi,t+1}$. We first show that no inflation index $\pi_t = \sum_k \tilde{\omega}_k \pi_{k,t}$ with $\sum_k \tilde{\omega}_k \neq 0$ can be stabilized jointly with the output gap.

$$\tilde{\pi}_{k,t} = \sum_m (\lambda_k \Gamma_k - \lambda \Gamma_m) \overline{\partial_e e}_m \sum_l \frac{\sigma\left(\psi \overline{\partial_e e}_l + \bar{s}_l\right)}{\sigma + \psi} \hat{A}_{l,t} - \left(\lambda_k \hat{A}_{k,t} - \lambda_k \sum_l \overline{\partial_e e}_l \hat{A}_{l,t}\right)$$
$$- \left(\lambda_k \tilde{P}_{k,t} - \lambda_k \sum_l \overline{\partial_e e}_l \tilde{P}_{l,t}\right) + \lambda_k \left(\mathcal{M}^{\mathcal{D}}{}_{k,t} + \mathcal{M}^{\mathcal{P}}{}_{k,t}\right) - \lambda \sum_l \overline{\partial_e e}_l \left(\mathcal{M}^{\mathcal{D}}{}_{l,t} + \mathcal{M}^{\mathcal{P}}{}_{l,t}\right) + \beta (1-\delta) \mathbb{E}_t \tilde{\pi}_{k,t+1}.$$
$$\tilde{P}_{k,t} = \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1}.$$

Where the endogenous markup now solves:

$$\mathcal{M}^P_{k,t} = \sum_{l=1}^K \int \gamma_{e,k}(i) \frac{e_k}{E_k} \rho_{k,l}(i) di \tilde{P}_{l,t},$$
$$\mathcal{M}^D_{k,t} = \mathbb{E}_t \mathcal{M}^D_{k,t+1} - \sum_l \sigma^{\mathcal{M}}_{k,l} (r_t^* + \tilde{\pi}_{mcpi,t+1} - \mathbb{E}_t \tilde{\pi}_{l,t+1}) - \frac{\delta}{1-\delta} \mathbb{E}_t \mathcal{M}^0_{k,t+1},$$
$$\mathcal{M}^0_{k,t} = \frac{1}{(1-\delta) R} \mathbb{E}_t \mathcal{M}^0_{k,t+1} + \int \gamma_{b,k}(i) \frac{b(i)}{RE} di \left(r_t^* + \tilde{\pi}_{mcpi,t+1} - \tilde{\pi}_{cpi,t+1}\right)$$
$$- \sum_l \int \gamma_{b,k}(i) \left(\frac{e(i)}{E} (s_l(i) - \bar{s}_l) + \frac{\psi W n(i)}{WN} \left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right) di \tilde{P}_{l,t} - \frac{R-1}{R} \mathcal{M}^D_{k,t},$$
$$\sigma^{\mathcal{M}}_{k,l} = \sigma \int \gamma_{e,k}(i) \frac{e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l} \frac{R}{R-1} \frac{\sigma + \psi}{\sigma} \int \gamma_{b,k}(i) \frac{W n(i)}{WN} di.$$

Note that under a policy that stabilizes the output gap, the evolution of relative prices is independent from $\pi_{d,t}$: relative prices only depend on themselves. Consider a permanent shock ($\hat{A}_{l,t} = \hat{A}_{l,0}$ for all l, t) such that $\sum_m (\lambda_k \Gamma_k - \lambda \Gamma_m) \overline{\partial_e e}_m \sum_l \frac{\sigma\left(\psi \overline{\partial_e e}_l + \bar{s}_l\right)}{\sigma+\psi} \hat{A}_{l,t} - \left(\lambda_k \hat{A}_{k,t} - \lambda_k \sum_l \overline{\partial_e e}_l \hat{A}_{l,t}\right) = 0$ for all k and denote it $\hat{\mathbf{A}}^*$. This implies $r_t^* = 0$ for all t and therefore $\tilde{P}_{k,t}\left(\hat{\mathbf{A}}^*\right) = 0$ for all k, t. As before, we necessarily have $\sum_l \left(\overline{\partial_e e}_l + \frac{\bar{s}_l}{\psi}\right) \hat{A}^*_l \neq 0$. If we consider an inflation index $\pi_t = \sum_k \tilde{\omega}_k \pi_{k,t}$ with $\sum_k \tilde{\omega}_k \neq 0$ it can be rewritten $\pi_t = \omega_d \pi_{d,t} + \sum_{k=1}^{K-1} \omega_k \tilde{\pi}_{k,t}$ with $\omega_d \neq 0$. Consider the set of permanent shocks $\hat{\mathbf{A}}^1$, the NKPC for π_t is then $\pi_t - \beta \mathbb{E}_t \pi_{t+1} = \mathcal{W}_t\left(\hat{\mathbf{A}}^1\right)$ and note that $\mathcal{W}_t\left(\hat{\mathbf{A}}^*\right) = \omega_d \sum_m \lambda \Gamma_m \overline{\partial_e e}_m \sum_l \frac{\sigma\left(\psi \overline{\partial_e e}_l + \bar{s}_l\right)}{\sigma+\psi} \hat{A}^*_l \neq 0$. Since $\hat{\mathbf{A}} \mapsto \mathcal{W}_t\left(\hat{\mathbf{A}}^1\right)$ is again a linear map, this implies that the set of permanent shocks such that $\mathcal{W}_0\left(\hat{\mathbf{A}}^1\right) = 0$ as dimension at most $K-1$ and therefore has a Lebesgue measure of 0. We can extend the argument to the set of shocks $\hat{\mathbf{A}}(\boldsymbol{\alpha})$such that $\hat{A}_{k,t}(\boldsymbol{\alpha}) = \sum_{i=0}^I \alpha_{i,k} \rho_i^t$ where $0 = \rho_0 < ... < \rho_I = 1$ and $\{\alpha_{i,k}\}_{0 \leq i \leq I, 1 \leq k \leq K}$ are arbitrarily scalars ($\hat{\mathbf{A}}(\boldsymbol{\alpha})$ is an arbitrary combination of I shocks with persistence ρ_0,...,ρ_I). Indeed consider $\hat{\mathbf{A}}(\boldsymbol{\alpha}^*)$ such that $\alpha_{I,k} = \hat{A}^*_k$ and $\alpha_{i,k} = 0$ for $i \neq I$, we have $\mathcal{W}_t\left(\hat{\mathbf{A}}(\boldsymbol{\alpha}^*)\right) = \omega_d \sum_m \lambda \Gamma_m \overline{\partial_e e}_m \sum_l \frac{\sigma\left(\psi \overline{\partial_e e}_l + \bar{s}_l\right)}{\sigma+\psi} \hat{A}^*_l \neq 0$ so using the same logic, for a given t, the subset of shocks such that $\mathcal{W}_t\left(\hat{\mathbf{A}}(\boldsymbol{\alpha})\right) = 0$ is of measure 0 .Therefore no inflation index with $\sum_k \tilde{\omega}_k \neq 0$ can be stabilized jointly with the output gap on any set of combination of AR(1) shocks.

Result 5

Under the assumption that $\lambda = \lambda_k$ for all k, the equations for relative prices (defined with respect to MCPI) can be rewritten as:

$$\tilde{\pi}_{k,t} - \beta (1-\delta) \mathbb{E}_t \tilde{\pi}_{k,t+1} = \lambda \left(\tilde{P}^*_{k,t} + \sum \alpha_{k,l} \tilde{P}_{l,t} + \sum \left(\tilde{\mathcal{M}}_{k,t} - \sum_l \overline{\partial_e e}_l \tilde{\mathcal{M}}_{l,t}\right)\right),$$
$$\tilde{P}_{k,t} = \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1},$$
$$\tilde{\mathcal{M}}_{k,t} = \mathbb{E}_t \tilde{\mathcal{M}}_{k,t+1} - \frac{\delta}{1-\delta} \mathbb{E}_t \tilde{\mathcal{M}}^0_{k,t+1},$$
$$\tilde{\mathcal{M}}^0_{k,t} = \frac{1}{(1-\delta) R} \mathbb{E}_t \tilde{\mathcal{M}}^0_{k,t+1} - \sum \beta_{k,l} \tilde{P}_{l,t} - \left(1 - \frac{1}{R}\right) \tilde{\mathcal{M}}^D_{k,t}$$

with

$$\alpha_{k,l} = -\mathbb{1}_{k=l} + \overline{\partial_e e_l} + \int \gamma_{e,k}(i)\frac{e}{E_k}\rho_{k,l}(i) - \lambda\sum_n \overline{\partial_e e_n}\int \gamma_{e,n}(i)\frac{e}{E_n}\rho_{n,l}(i)di - \sigma^{\mathcal{M}}_{k,l} + \sum_n \overline{\partial_e e_n}\sigma^{\mathcal{M}}_{n,l}$$

$$\beta_{k,l} = \int \gamma_{b,k}(i)\frac{e(i)}{E}\left((s_l(i)-\bar{s}_l) + \sigma\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di + \left(1-\frac{1}{R}\right)\sigma^{\mathcal{M}}_{k,l}$$

For an aggregate shock, we have $\tilde{P}^*_{k,t} = 0$ for all k so $\tilde{P}_{k,t} = 0$ for all k,t. Since we have

$$\mathcal{M}^D_{k,t} = \mathbb{E}_t\mathcal{M}^D_{k,t+1} + \sum_l \sigma^{\mathcal{M}}_{k,l}\mathbb{E}_t\tilde{\pi}_{l,t+1} - \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}^0_{k,t+1}$$

$$\mathcal{M}^0_{k,t} = \frac{1}{(1-\delta)R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} - \sum_l\int\gamma_{b,k}(i)\left(\frac{e(i)}{E}(s_l(i)-\bar{s}_l) + \frac{\psi W n(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}.$$

and

$$\mathcal{M}^P_{k,t} = \sum_l\int\gamma_{e,k}(i)\frac{e_k}{E_k}\rho_{k,l}(i)di\tilde{P}_{l,t}$$

This implies $\mathcal{M}^D_{k,t} = \mathcal{M}^P_{k,t} = 0$ for all k,t. Therefore,

$$\mathcal{M}_{k,t} = \Gamma_k\hat{\mathcal{Y}}^*_t < 0$$

For all k,t if $\Gamma_k > 0$.

Result 6

Assume that the households' utility function associated with intratemporal sectoral consumption takes the form

$$u(c_k,...,c_K) = \frac{1}{1-\frac{1}{\sigma}}\left(\prod_{k=1}^K\left(c_k - \underline{c_k}\right)^{\alpha_k}\right)^{1-\frac{1}{\sigma}}.$$

With $c_k = e_k/P_k$ (recall that subvariety prices are equal in steady state) and $\sum\alpha_k = 1$. We have:

$$\alpha_k\left(e - \sum P_{k=l}\underline{c_l}\right) = P_k\left(c_k - \underline{c_k}\right).$$

Therefore

$$\partial_e e_k = \alpha_k$$

$$\partial_{P_l}c_k + \frac{\partial_e e_k}{P_k}c_l = -\frac{\alpha_k}{P_k}\underline{c_l} + \frac{\alpha_k}{P_k}\left(\frac{\alpha_l}{P_l}\left(e - \sum P_k\underline{c_k}\right) + \underline{c_l}\right) - \mathbb{1}_{k=l}\frac{\alpha_k}{P_k^2}\left(e - \sum P_k\underline{c_k}\right),$$

$$P_l\partial_{P_l}c_k + P_l\frac{\partial_e e_k}{P_k}c_l = \frac{\alpha_k}{P_k}\left(\alpha_l - \mathbb{1}_{k=l}\right)\left(e - \sum P_k\underline{c_k}\right),$$

and

$$\begin{aligned}\bar{s}_k &= \int\frac{1}{E}\left(\alpha_k\left(e(i) - \sum P_l\underline{c_l}\right) + P_k\underline{c_k}\right)di,\\ &= \overline{\partial_e e_k} + \frac{P_k\underline{c_k} - \overline{\partial_e e_k}\sum P_l\underline{c_l}}{E},\end{aligned}$$

$$\begin{aligned}\frac{e(i)}{E}\left(s_k(i) - \bar{s}_k\right) &= \frac{1}{E}\left(\overline{\partial_e e_k}\left(e(i) - \sum P_l\underline{c_l}\right) + P_k\underline{c_k} - e(i)\left(\overline{\partial_e e_k} + \frac{P_k\underline{c_k} - \overline{\partial_e e_k}\sum P_l\underline{c_l}}{E}\right)\right),\\ &= \frac{1}{E}\left(1 - \frac{e(i)}{E}\right)\left(P_k\underline{c_k} - \overline{\partial_e e_k}\sum P_l\underline{c_l}\right) = \left(1 - \frac{e(i)}{E}\right)\left(\bar{s}_k - \overline{\partial_e e_k}\right).\end{aligned}$$

Defining $\tilde{P}_{k,t} = P_{k,t} - \sum_l\overline{\partial_e e_l}P_{l,t}$ and $\tilde{\pi}_{k,t} = \pi_{k,t} - \sum_l\overline{\partial_e e_l}\pi_{l,t}$, we therefore have

$$\begin{aligned}\mathcal{M}^P_{k,t} &= \sum_l\int\gamma_{e,k}(i)\frac{\overline{\partial_e e_k}}{E_k}\left(e - \sum P_k\underline{c_k}\right)di\overline{\partial_e e_l}\tilde{P}_{l,t} - \int\gamma_{e,k}(i)\frac{\overline{\partial_e e_k}}{E_k}\left(e - \sum P_k\underline{c_k}\right)di\tilde{P}_{k,t}\\ &= -\int\gamma_{e,k}(i)\frac{\overline{\partial_e e_k}}{E_k}\left(e - \sum P_k\underline{c_k}\right)di\tilde{P}_{k,t}\end{aligned}$$

$$\mathcal{M}^D_{k,t} = \mathbb{E}_t \mathcal{M}^D_{k,t+1} - \frac{\delta}{1-\delta}\mathbb{E}_t \mathcal{M}^0_{k,t+1}$$

$$\mathcal{M}^0_{k,t} = \frac{1}{(1-\delta)R}\mathbb{E}_t \mathcal{M}^0_{k,t+1} - \sum_l \int \gamma_{b,k}(i)\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) di \tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}$$

Note that under A3 we have $\gamma_{e,k}(i)\frac{\overline{\partial_e e_k}}{E_k} = \gamma_e(i)\frac{1}{E}$ and $\gamma_{b,k}(i) = \gamma_{b,l}(i)$ for all k so $\mathcal{M}^D_{k,t} = \mathcal{M}^D_t$,$\mathcal{M}^0_{k,t} = \mathcal{M}^0_t$,

$$\mathcal{M}^P_{k,t} = -\int \gamma_e(i)\frac{e(i) - \sum P_k \underline{c_k}}{E} di \tilde{P}_{k,t}$$

$$\mathcal{M}^D_t = \mathbb{E}_t \mathcal{M}^D_{t+1} - \frac{\delta}{1-\delta}\mathbb{E}_t \mathcal{M}^0_{t+1}$$

$$\mathcal{M}^0_t = \frac{1}{(1-\delta)R}\mathbb{E}_t \mathcal{M}^0_{t+1} - \sum_l \int \gamma_b(i)\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) di \tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_t$$

Next under (**A.1**) and (**A.3**), we necessarily have $\Gamma_k = \Gamma$, $\lambda_k = \lambda$ for all k so the NKPC for $\tilde{\pi}_{k,t}$ is

$$\tilde{\pi}_{k,t} = \lambda\left(\left(\tilde{P}^*_{k,t} - \tilde{P}_{k,t}\right) - \int \gamma_e(i)\frac{\left(e - \sum P_k \underline{c_k}\right)}{E} di \tilde{P}_{k,t}\right) + \beta \mathbb{E}_t \tilde{\pi}_{k,t+1}$$

The evolution of relative price k only depends on itself. Denoting $\tilde{\lambda} = \lambda\left(1 + \int \gamma_e(i)\frac{\left(e - \sum P_k \underline{c_k}\right)}{E}\right)$ the eigenvalues of the system are:

$$\nu_{\pm} = \frac{R + R\tilde{\lambda} + 1 \pm \sqrt{\left(R + R\tilde{\lambda} - 1\right)^2 - 4R\tilde{\lambda}}}{2}$$

(Note that for $\int \gamma_e(i)\frac{\left(e - \sum P_k \underline{c_k}\right)}{E} > -1$, $0 < \nu_- < 1$, $R + R\tilde{\lambda} < \nu_+$) and the evolution of $\tilde{P}_{k,t}$ is given by:

$$\tilde{P}_{k,t} = \lambda \sum_0^t \nu_-^{t-s+1} \sum \frac{1}{\nu_+^u}\tilde{P}^*_{k,s+u}$$

For a negative sequence of shocks in k $\{\hat{P}^*_{k,t}\}_{t\geq 0} > 0$, we therefore have $\tilde{P}_{k,t} > 0$ for all t and $\overline{\partial_e e_k}\tilde{P}_{k,t} = -\left(1 - \overline{\partial_e e_k}\right)\tilde{P}_{l,t}$ for all $l \neq k$ so we have

$$\begin{aligned}
\mathcal{M}^P_{cpi,t} &= \sum_l \bar{s}_k \mathcal{M}^P_{k,t} = -\int \gamma_e(i)\frac{e(i) - \sum P_k \underline{c_k}}{E} di \sum \bar{s}_l \tilde{P}_{l,t} \\
&= -\int \gamma_e(i)\frac{e(i) - \sum P_k \underline{c_k}}{E} di \left(\bar{s}_k - \overline{\partial_e e_k}\right)\lambda \sum_0^t \nu_-^{t-s+1} \sum \frac{1}{\nu_+^u}\tilde{P}^*_{k,s+u}
\end{aligned}$$

So $\mathcal{M}^P_{cpi,t} < 0$ following a shock in a necessity sector, $\mathcal{M}^P_{cpi,t} > 0$ following a shock in a luxury sector. In addition we have for a shock in sector k

$$\begin{aligned}
\mathcal{M}^D_t = &-\int \gamma_b(i)\frac{1}{E}\left(1 - \frac{e(i)}{E}\right) di (1-\delta)^{t+1} \sum_{u=0}^{\infty} \frac{1}{R^u} \sum_l \left(P_l \underline{c_l} - \overline{\partial_e e_l} \sum P_n \underline{c_n}\right)\tilde{P}_{l,u} \\
&- \delta \int \gamma_b(i)\frac{1}{E}\left(1 - \frac{e(i)}{E}\right) di \sum_{s=0}^{t} (1-\delta)^{t-s} \sum_{u=0}^{\infty} \frac{1}{R^u} \sum_l \left(P_l \underline{c_l} - \overline{\partial_e e_l} \sum P_n \underline{c_n}\right)\tilde{P}_{l,s+u} \\
= &-\int \gamma_b(i)\left(1 - \frac{e(i)}{E}\right) di (1-\delta)^{t+1} \sum_{u=0}^{\infty} \frac{1}{R^u}\left(\bar{s}_k - \overline{\partial_e e_k}\right)\frac{\tilde{P}_{k,u}}{\left(1 - \overline{\partial_e e_k}\right)} \\
&- \delta \int \gamma_b(i)\left(1 - \frac{e(i)}{E}\right) di \sum_{s=0}^{t} (1-\delta)^{t-s} \sum_{u=0}^{\infty} \frac{1}{R^u}\left(\bar{s}_k - \overline{\partial_e e_k}\right)\frac{\tilde{P}_{k,s+u}}{\left(1 - \overline{\partial_e e_k}\right)}
\end{aligned}$$

So if $Cov\left(\gamma_b(i), \frac{e(i)}{E}\right) > 0$, $\mathcal{M}^D_t = \mathcal{M}^D_{cpi,t} > 0$ following a shock in a necessity sector. Note that under the stronger assumption that $b(i) = 0$ for all i, we have $\gamma_b(i) = \gamma_e(i)\sigma/(\sigma + \psi)$ so if $\gamma_e(i)$ is increasing in $e(i)$, $Cov\left(\gamma_b(i), \frac{e(i)}{E}\right) > 0$.

D Calibration procedure and numerical details

Outer Preferences

To calibrate the non-homothetic CES preferences we use the LCF survey, which is the most comprehensive survey on household spending in the UK. Each member of the household keeps a detailed spending diary for a period of two weeks, while expenditure information on bigger items (like cars, vacations, housing etc.) are collected during interviews with the household head. We map these highly disaggregated consumption data into the standard 3-digit COICOP categories using a mapping table provided by the ONS. Aggregating these to the COICOP division level, forms the basis of our definition of sectors for the UK economy as well as providing the data for estimating the household-specific marginal propensities to consume across different sectors.

We exclude housing costs from household expenditures by redefining the relevant consumption category (COICOP4) to only include expenditure on Electricity, Gas and Other Fuels.[7] Furthermore, we exclude the following four sectors from our model: Alcohol & Tobacco, Health, Communication and Education. Health and Education are largely publicly provided in the UK and hence only a very small fraction of households report any private spending in these sectors. The other two sectors account for a small budget share so overall we still capture the vast majority of private expenditure, with the notable exception of housing.[8]

We construct household-specific price indices using the observed consumption shares in the 3-digit subcategories of each COICOP group so that $\ln P_{k,t}(i) = \sum_{m \in M_k} s_{m,k,t}(i) \ln P_{m,k,t}$. Whenever indices of 3-digit COICOP categories are not available (only occurring before 2015 and for a small subset of categories), we use the 2-digit price index of the corresponding group. To guard against any potential endogeneity of prices (similarly to what is done in **?**) we construct Hausman-type price instruments by using the shares of all other households in the same region and for any given sector. To instrument for total expenditure we use log disposable income as well as the expenditure quintile of the household.

We impose that the individual parameter shifters take the following form:

$$\ln \mathcal{V}_{i,k} = x_i \beta_k + v_i^k,$$

where x_i are household demographic characteristics and v_i^k is an idiosyncratic and time invariant preference shifter that satisfies $\mathbb{E}\left[v_i^k | x_i\right] = 0$. The specific demographic controls include the size of the household $(1, 2+\text{adults})$, number of children $(0, 1+)$ and the age of the household head $(18-37, 38-50, 51-64, 65+)$. Note that since households are surveyed at different points during the year, we also include quarter dummies to allow for potential seasonal effects in the consumption of different goods. The table below shows the results across a different set of specifications, with the first column showing our baseline version.For all estimates in this table, we use Clothing as the base sector and assume an elasticity of substitution equal to 0.1. We conduct different robustness checks to show that our results do not qualitatively change with the specific assumptions made in the baseline. The other columns show in turn the GMM results from winsorizing the data, adding regional controls (there are 12 regions in the UK) and expanding the sample to include all years available. For the winsorized sample we mark the households that are in the bottom or top 2% of expenditure shares in each of the eight COICOP categories and then drop them from the estimation. The GMM results are quite robust to outliers so the exact cut-off does not matter much. Note also that in specification 4 we add year dummies on top of the quarter dummies that are present in all specifications. We have also run other robustness checks where we use different instruments or weight the observations by household expenditure and qualitatively the results are unchanged.

[7] Note that these are not the only direct expenditure on energy as households who own vehicles will also spend on diesel and petrol, included in the Transport category.

[8] The correlation between the three different measures of total expenditure (i.e. the original variable, excluding housing and excluding housing plus the four sectors) is always greater than 0.966.

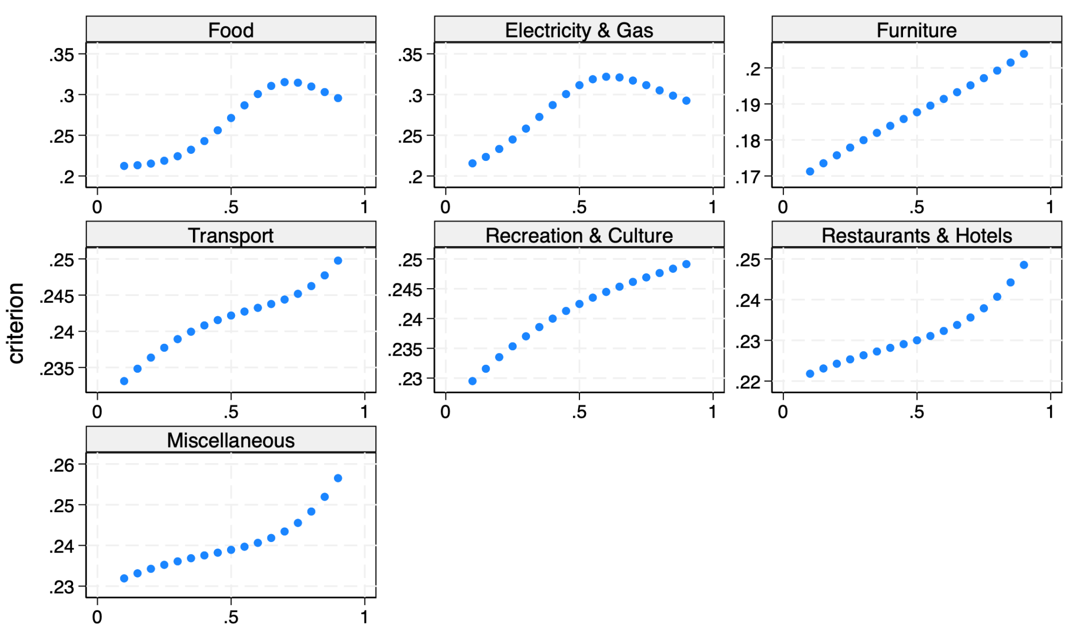
	(1)	(2)	(3)	(4)
Food	0.50	0.46	0.49	0.37
	(0.02)	(0.02)	(0.02)	(0.00)
Electricity & Gas	0.52	0.47	0.51	0.30
	(0.02)	(0.02)	(0.02)	(0.00)
Furniture	1.21	1.12	1.19	1.20
	(0.05)	(0.05)	(0.05)	(0.01)
Transport	0.90	0.89	0.88	1.10
	(0.04)	(0.04)	(0.04)	(0.01)
Recreation	1.23	1.15	1.20	1.11
	(0.05)	(0.04)	(0.04)	(0.01)
Restaurants & Hotels	0.98	0.97	0.96	0.99
	(0.04)	(0.04)	(0.03)	(0.01)
Miscellaneous	0.86	0.82	0.84	0.90
	(0.03)	(0.03)	(0.03)	(0.01)
N	3,164	2,815	3,164	56,538

These estimates allows us in turn to construct the marginal budget share $\partial_e e_k(i) = \eta + (1-\eta)\frac{\zeta_k}{\bar{\zeta}(i)}$, where $\bar{\zeta}(i)$ is the household specific 'average' non-homotheticity measure given by $\bar{\zeta}(i) = \sum_k s_k(i)\zeta_k$. This implies that richer households that spend more on luxury goods will have a higher $\bar{\zeta}(i)$. These preferences also imply that the compensated price elasticities take the following form:

$$\rho_{k,l}(i) = \begin{cases} \eta s_l(i) & \text{if } k \neq l, \\ -\eta\,(1 - s_l(i)) & \text{if } k = l. \end{cases}$$

Elasticity of Substitution Parameter. We set the elasticity of substitution parameter equal to 0.1, following the **?** estimation for their 10-sector model. Here we show that increasing the value of η worsens the fit of the model, as measured by the criterion function of the GMM procedure. Figure 8 plots the criterion value as we vary the value of the elasticity parameter between 0.05 and 0.9. Regardless which of the sectors we choose as the base, the fit of the model worsens with higher values of η. [9]

Figure 1. Criterion Value for different values of the elasticity parameter.



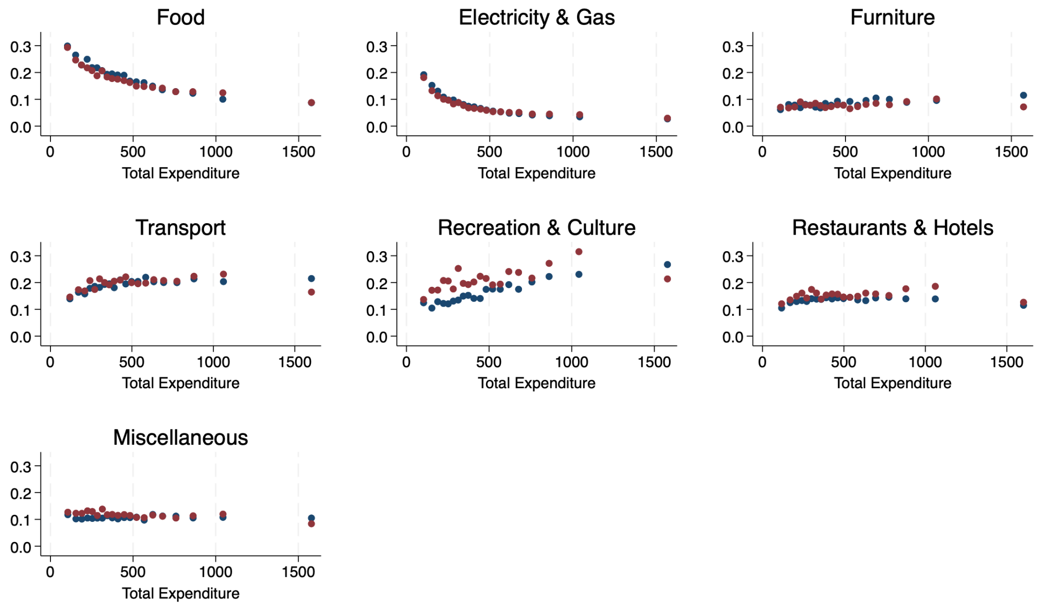
Notes: Each panel plots the minimised criterion function for the same GMM procedure for a given base sector.

Finally, figure 2 shows that the estimated preferences reproduce reasonably well the non-homotheticiets observed in the data. In particular, we plot each category's budget share against total household expenditure. The data are binned in 20 equally sized groups by total expenditure, and each dot represents the

[9]Note that we do not estimate the parameter η jointly with the ζ's because as the figure shows the estimation would demand an η that goes to zero and so the procedure is not well behaved.

average budget share in that bin. In blue we have the observed budget shares while the red dots represent the fitted budget shares, and they align quite well for almost all sectors.

Figure 2. Actual vs. Predicted budget shares by household total expenditure.



Notes: Each point represents the average expenditure share on a given sector by total expenditure bin. The data has been binned into 20 equally sized groups.

Inner Preferences

Our quantitative exercise assumes an inner aggregator that takes the HARA form and is sector specific. The sectoral bundle for household i in sector k is given by

$$\mathcal{U}_k\left(\boldsymbol{c}_k(i)\right) = \frac{1}{a_k - 1}\int \left(b_k + a_k c_k(i,j)\right)^{\frac{a_k-1}{a_k}} dj,$$

where $\{a_k, b_k\}$ are the two parameters that govern the HARA function. The optimal bundle of varieties given a total sectoral expenditure $e_k(i)$ is the solution to the following problem

$$\max_{\boldsymbol{c}_k(i)} \mathcal{U}_k\left(\boldsymbol{c}_k(i)\right) + \lambda_k(i)\left(e_k(i) - \int p_k(j)c_k(i,j)dj\right),$$

where $\lambda_k(i)$ is the Lagrange multiplier and is household-specific due to the fact that households have different expenditure levels. Taking the FOC of this problem and re-writing allows us to derive the HARA demand function as

$$c_k(i,j) = \frac{1}{a_k}\left(\left(\lambda_k(i)p_k(j)\right)^{-a_k} - b_k\right).$$

We can then use the definition of price elasticity $\epsilon_k(i) \equiv \frac{\partial \ln c_k(i,j)}{\partial \ln p_k(j)}$ and take the derivative of the previous expression to derive that the elasticity is equal to $a_k + \frac{b_k}{c_k(i)}$, as given in the main text. Since subvariety prices are all equal in equilibrium, the household will have the same elasticity of demand for all subvarieties and therefore we suppress the j in the notation. Nonetheless, if $b_k < 0$ households that spend more money on a given sector and therefore consume higher amounts will be less price elastic.

A few more lines of algebra allow us to derive the superelasticity for household i in sector k starting from its definition

$$\begin{aligned}
\epsilon^s_k(i) &\equiv \frac{\partial \ln \epsilon_k(i)}{\partial \ln p_k(j)},\\
&= -\frac{b_k}{c^2_k(i,j)} \frac{\partial c_k(i,j)}{\partial p_k(j)} \frac{p_k(j)}{\epsilon_k(i)},\\
&= \frac{b_k}{c_k(i,j)} \left(-\frac{\partial c_k(i,j)}{\partial p_k(j)} \frac{p_k(j)}{c_k(i,j)} \right) \frac{1}{\epsilon_k(i)},\\
&= \frac{b_k}{c_k(i,j)}.
\end{aligned}$$

Given the household level elasticity and super-elasticity, we can derive the aggregate counterpart of these objects which will in turn determine the sectoral markup and price passthrough. To recover the aggregate elasticity we take the the average household elasticity, weighted by the expenditure shares to get that $\bar{\epsilon}_k = a_k + \frac{b_k}{C_k}$. Finally, to get the expression for the aggregate super-elasticity, we plug in the expressions for $\epsilon_k(i)$ and $\epsilon^s_k(i)$ in the formula[10] $\bar{\epsilon}^s_k = \left(-\int (\epsilon_k(i) - \bar{\epsilon}_k)^2 \frac{e_k(i)}{E_k} di + \int \frac{e_k(i)}{E_k} \epsilon^s_k(i) \epsilon_k(i) di \right) / \bar{\epsilon}_k$ and we get that $\bar{\epsilon}^s_k = \frac{b_k}{C_k}$. Note that these formulas are slightly different than the ones given in the main text where we use expenditure rather than actual consumption levels. Normalising the price to one is innocuous since the elasticity and super-elasticity values that are recovered for each household are independent of the assumed price level. The reason for this is that while we can recover a_k for the other coefficient we can only identify $\frac{b_k}{C_k} = \frac{\frac{1}{\text{passthrough}} - 1}{\text{markup} - 1}$. This is sufficient to get the household objects since with a slight re-writing we have that $\epsilon_k(i) = a_k + \left(\frac{e_k(i)}{E_K} \right)^{-1} \frac{b_k}{C_k}$ and $\epsilon^s_k(i) = \left(\frac{e_k(i)}{E_K} \right)^{-1} \frac{b_k}{C_k}$. The same is true for the markup sensitivity parameter which an application of the formula shows to be equal to $\gamma_{e,k}(i) = \left(1 - \frac{a_k}{\bar{\epsilon}_k} \right) \frac{1}{\bar{\epsilon}_k - 1}$.

Input-Output. To calibrate the parameters relating to the IO part of the model, we use the tables of intermediate input consumption provided by the ONS. These tables of input flows are constructed based on the CPA classification that defines 105 industries/products and which are different from the COICOP classification that we use in our model. To bridge this gap, we construct a mapping between the CPA classification and the COICOP one starting from the most disaggregated list of product classification (CPC10) of which there are more than 2000 products, although only 832 are for final consumption. The mapping consists in two steps. The first is to use the CPC10 to COICOP tables and assign weights to each product using the CPI weights available from ONS data. For example, if there are four CPC10 goods for a given COICOP category (we use the most disaggregated one for which we observe consumption weights) that has a weight of 1, each good will receive a weight of 0.25. Also note that the vast majority of CPC10 goods (more than 80%) map to a single COICOP category. Another 12% maps to two categories and only less than 5% maps to 3-5 COICOP categories.

Similarly in the other direction, we map the COICOP10 consumption goods to the CPA industry definitions using the concordance tables available from the UN's Statistics Division. [11] Unsurprisingly, the mapping of consumption goods to industries contains fewer one-to-one cases than with COICOP. Nonetheless, about 60% of goods only map to one or two CPA industries and another 30% map to 3 or 4.

Closed economy adjustment. The intermediate consumption tables provided by the ONS do not specify the share of inputs produced domestically vs what is imported. In our closed-economy world it must be

[10]Note that this formula is valid for any demand system and can be derived directly from the definition of $\bar{\epsilon}^s_k$ as the *elasticity* of the aggregate elasticity with respect to its own price. Taking the derivative wrt price gives $\left(\frac{p_k(j)}{\bar{\epsilon}_k} \right) \left(\int \left(\partial_{p_k} \epsilon_k(i) \frac{e_k(i)}{E_k} + \epsilon_k(i) \left(\frac{\partial_{p_k} e_k(i)}{E_k} - \frac{e_k(i) \int \partial_{p_k} e_k(i) di}{E^2_k} \right) \right) di \right)$. Use the fact that $\partial_{p_k} e_k(i) = c_k(i) (1 - \epsilon_k(i))$ and re-arrange to get the expression in the text.

[11]Note that this has to be done in a few steps that consists of the following chain of mapping CPC10 $\rightarrow$ ISIC3 $\rightarrow$ ISIC3.1 $\rightarrow$ ISIC4 $\rightarrow$ NACE2. That final classification contains 626 categories that can be aggregated to the 105 sectors used in the UK's IO tables.

the case that final demand (private consumption) plus intermediate consumption equals to total domestic output $[PY]$. To make this identity hold when we calibrate the model to the real-world data we adjust the vector of domestic total outputs with weights $\{\alpha_1, \alpha_2, \ldots, \alpha_K\}$ such that the following holds

$$[PC]_k + T\,[\alpha]_k = \mathcal{D}\,[\alpha]\,[PY]_k\,,$$

where the matrix T gives the flow of intermediate inputs and specifically $T_{i,j}$ is the amount of product i used in industry j. [12] This correction imposes that all production is done domestically (while not distorting the input mix used by different industries as given by T) and hence sectors in which the UK imports (exports) a lot will have a higher (lower) adjustment factor α.

The table below shows the IO matrix Ω for the eights sectors in our model. As is standard, we observe that sectors mostly tend to use goods produced by their own sector and so the diagonal entries dominate.

0.200	0.009	0.023	0.019	0.031	0.049	0.006	0.043
0.003	0.024	0.016	0.024	0.023	0.028	0.001	0.040
0.006	0.011	0.322	0.055	0.036	0.036	0.001	0.094
0.005	0.019	0.060	0.108	0.047	0.064	0.001	0.086
0.008	0.011	0.057	0.039	0.239	0.066	0.003	0.089
0.019	0.011	0.051	0.042	0.068	0.180	0.008	0.109
0.090	0.002	0.043	0.007	0.008	0.014	0.014	0.029
0.005	0.010	0.055	0.029	0.042	0.073	0.007	0.239

D.1 Model without heterogeneity in price stickiness and markups, and without I-O linkages

The baseline model includes various features other than non-homotheticities. In this appendix we study their quantitative importance. First, we shut down sectoral heterogeneity in prices stickiness and steady-state markups, as well as Input-Output linkages. Concretely, we achieve this by targeting in the calibration the (unweighted) average markup across sectors, setting all Calvo parameters equal to the average across sectors, and by setting intermediate input shares to zero.

Figure 3 shows impulse responses under a Taylor rule, with the aforementioned features shot down. As shown by the figure, we preserve the key result that the output gap declines in the two necessity sectors: *Food* and *Electricity & Gas*. In sector *Transport*, the output gap now increases. The increase observed in the baseline model is thus driven by the features that we shut down in this appendix. This is consistent with the fact that Transport is neither a luxury nor a necessity sector (the luxury index equals zero for this sector).

Figure 6 shows the Guidance experiment in the model version without I-O linkages and sectoral heterogeneity in markups price stickiness. The figure shows that the key result, that monetary policy is relatively loose in response to shocks in necessity sectors (Food and Electricity & Gas) is preserved.

Overall these results underscore the importance of non-homotheticities and show that our main results in the baseline model are not driven by sectoral heterogeneity in price setting, markups and I-O linkages.

Next, we study the wedges quantitatively. Figure 4 plots the baseline model. We observe that the $\mathcal{NH}$ wedge increases following negative shocks productivity shocks in the sectors *Food* and *Electricity & Gas*, and decreases following shocks to *Clothing*, *Furniture*, *Recreation*, and *Restaurants & Hotels*. We also observe that the movements tend to be relatively transitory. By contrast, the $\mathcal{M}$ wedge falls persistently following all negative productivity shocks. Finally, we observe that the relative price wedge $\mathcal{P}$ plays a quantitatively important role as well. Recall that this wedge enters the NKPC with a negative sign. The negative of the wedge increases for following negative shocks to *Food*, Clothing, *Electricity & Gas* and Transport as these are sectors with relatively flexible prices. For the remaining sectoral shocks, the negative of the wedge falls (on impact). From Table 4 in the main text it can be seen luxury sectors tend to have stickier prices. Thus, the $\mathcal{NH}$ and the $\mathcal{P}$ wedge tend to reinforce each other, although there are exceptions (e.g. *Clothing* shocks).

Figure 5 shows the wedges for the baseline model, but without heterogeneity in Calvo probabilities and without I-O linkages. In this model version, the weighted relative price wedge is zero. The movements in the $\mathcal{NH}$ and $\mathcal{M}$ wedges are qualitatively similar to those in the baseline model.

[12] Note that in terms of the Ω matrix one can write the flow matrix as $T = (\mathcal{D}\,[PY]\,\Omega)^T$.

D.2 Implementing optimal policy with a Taylor rule plus guidance

In this appendix, we show how we back out the "policy guidance" in the exercise of Section 5.3. Guidance is defined as a series of interest rate rule residuals, $\{u^R_{t+s}\}_{s=0}^{\infty}$, where $u^R_{t+s} = \hat{R_{t+s}} - \phi\pi_{t+s}$. These residuals are announced at the moment a certain shock hits (this could be e.g. a sectoral or aggregate productivity shock). The guidance may varies across shocks.

Our goal is to solve for the guidance which, for a certain shock, implements the optimal policy. Let IRF_{OP} be a column vector containing the Impulse Response Function (IRF) of some variable under optimal monetary policy, IRF_{TR} the IRF under a Taylor rule, and $IRF_{MP(s)}$ be the IRF to a purely transitory, unit news shock to the Taylor rule, hitting at date s and announced at date 0. We want to solve for $\{u^R_{t+s}\}_{s=0}^{S-1}$ such that

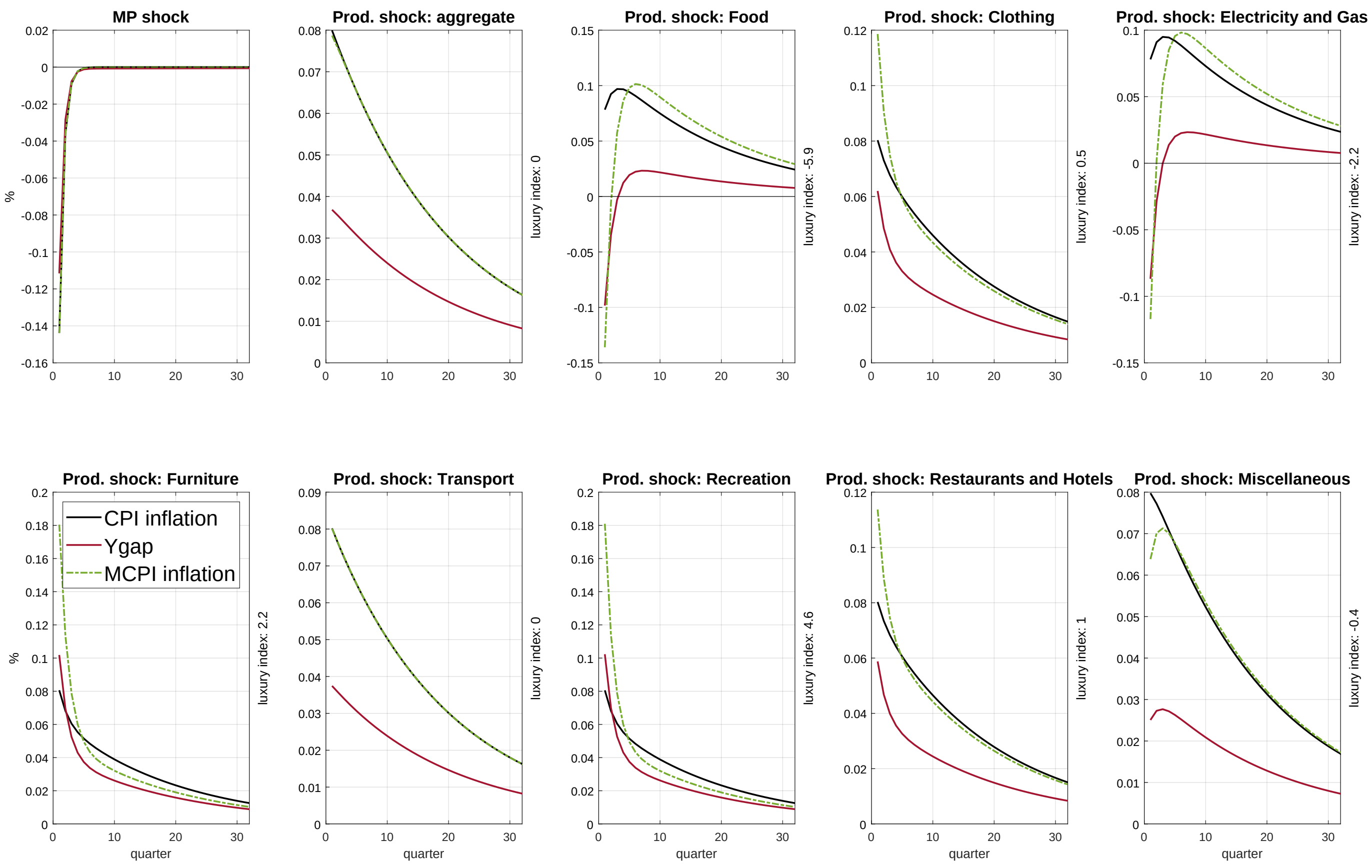
$$IRF_{OP} = IRF_{TR} + u^R_{t+s} \sum_{s=0}^{S} IRF_{MP(s)} = IRF_{TR} + \boldsymbol{IRF}_{MP}\boldsymbol{u}$$

where S is a truncation date, $\boldsymbol{IRF}_{MP}$ is an $S \times S$ matrix containing the IRFs to the monetary policy shocks on its columns, and $\boldsymbol{u^R}$ is a column vector containing the guidance. We solve for the guidance vector as:

$$\boldsymbol{u^R} = \boldsymbol{IRF}_{MP}^{-1} \left(IRF_{OP} - IRF_{TR} \right).$$

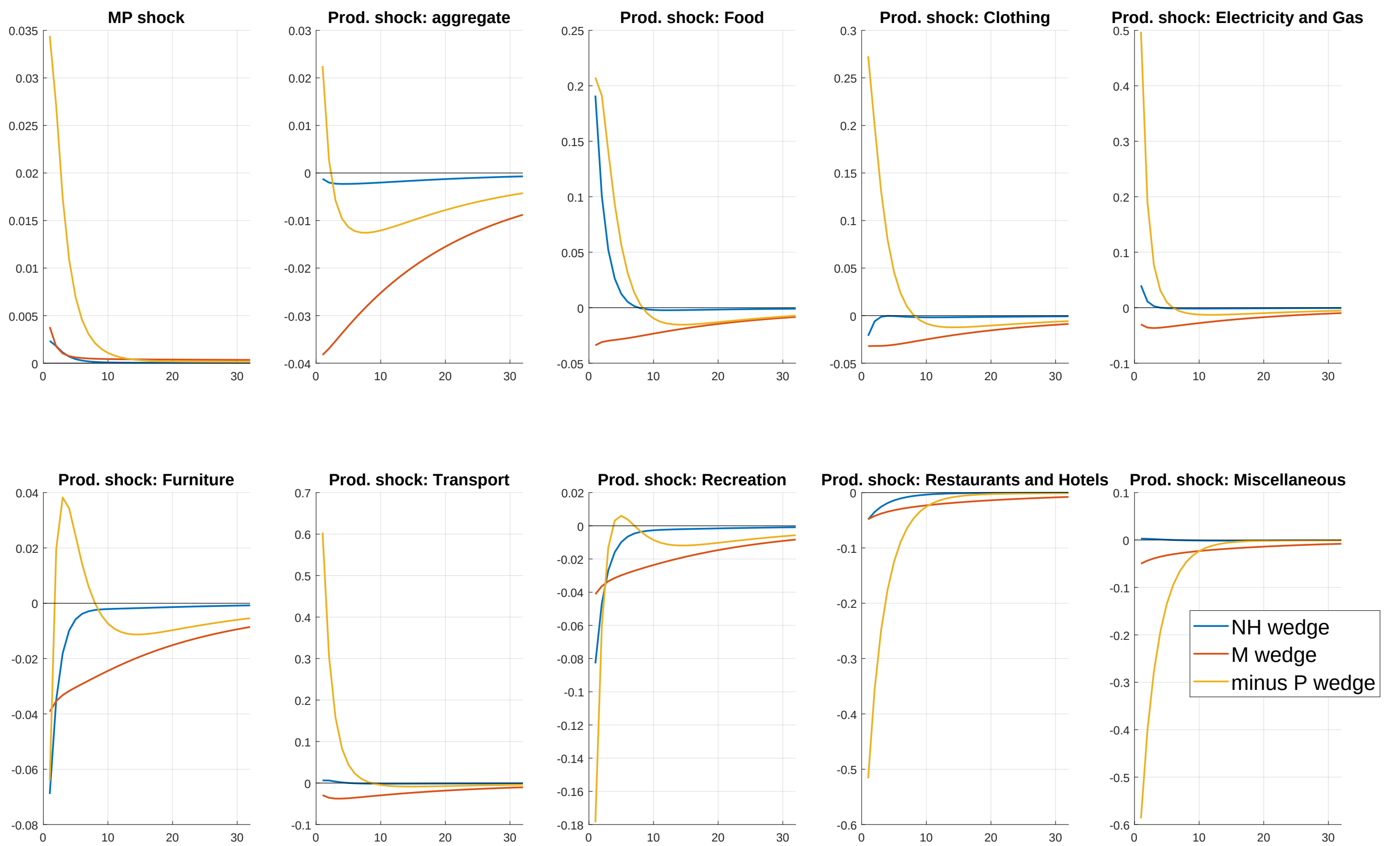
In our implementation, we use the IRF of CPI inflation to aggregate and sector-level shocks. We set the truncation horizon to 75 quarters. We verify ex post that the IRFs of variables are close to identical under optimal policy and the interest rate rule plus guidance.

Figure 3. Responses in the baseline model, but without heterogeneity in prices stickiness and steady-state markups across sectors, and without Input-Output linkages.



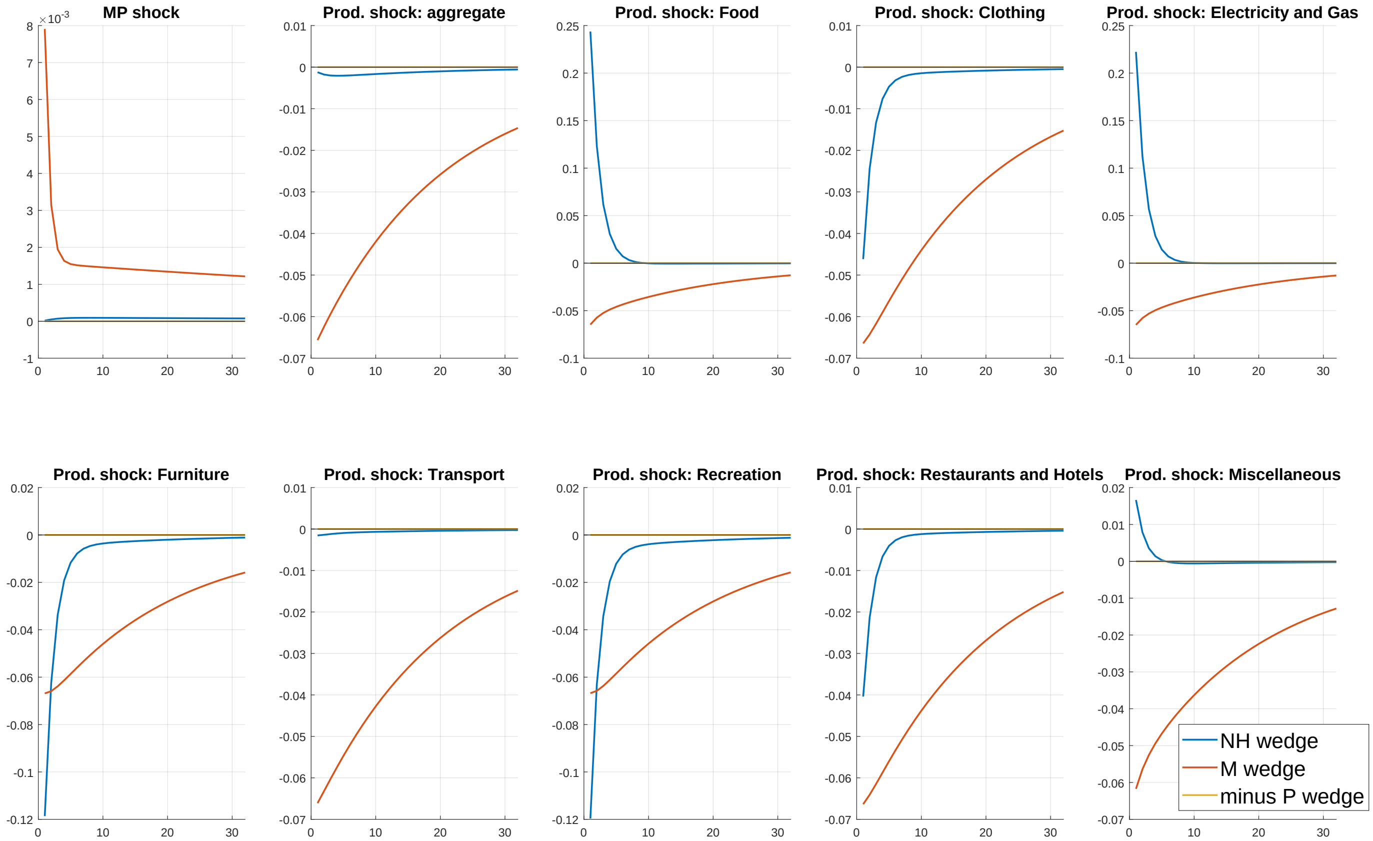
Notes: Responses for productivity shocks are for a 1 percent decline in productivity where scaled for comparability (see main text). On the right axis, the luxury index is defined as $100(\overline{\partial_e e_l} - \bar{s}_k)$.

Figure 4. Responses of the wedges in the baseline model.



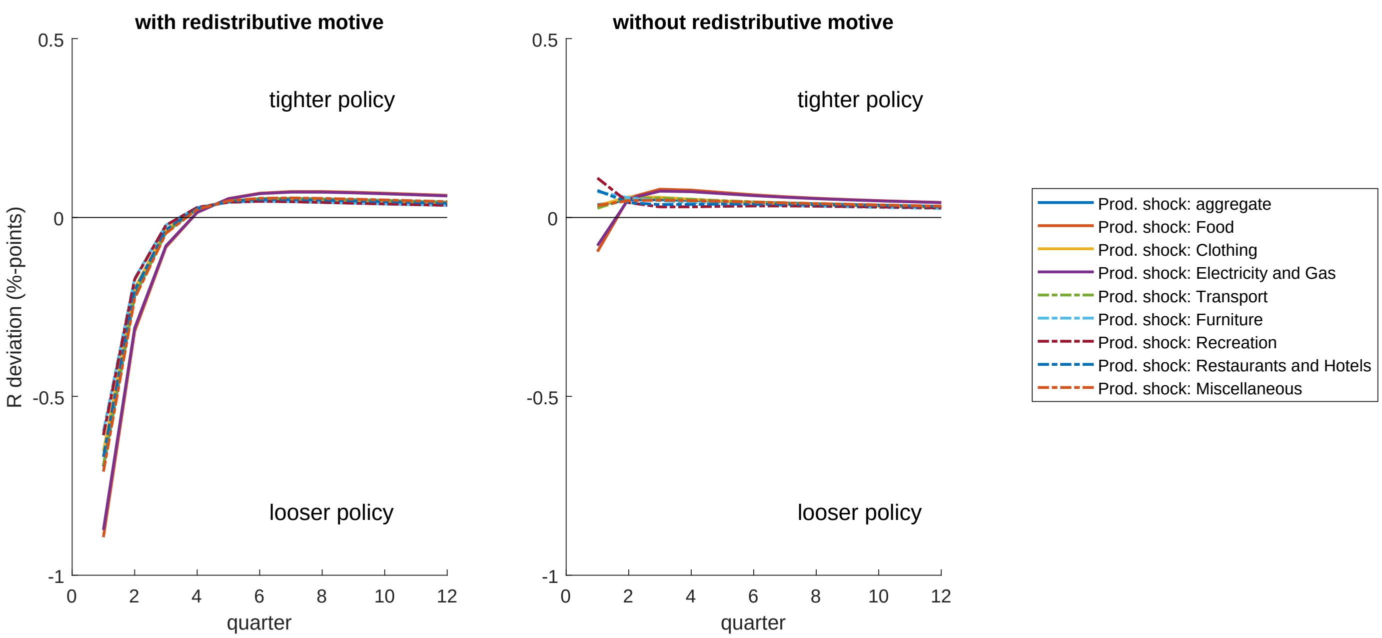
Notes: wedges are aggregated using NKPC weights given by $\frac{\bar{s}_k \lambda_k}{\sum_l \bar{s}_l \lambda_l}$.

Figure 5. Responses of the wedges in the baseline model, but without heterogeneity in prices stickiness and steady-state markups across sectors, and without Input-Output linkages.



Notes: wedges are aggregated using NKPC weights given by $\frac{\bar{s}_k \lambda_k}{\sum_l \bar{s}_l \lambda_l}$.

Figure 6. Optimal policy relative to Taylor rule in the model without heterogeneity in prices stickiness and steady-state markups across sectors, and without Input-Output linkages.



Notes: Deviations from the Taylor rule $\hat{R}_t = 1.5\pi_{cpi,t}$ which implement Optimal Policy ("optimal guidance"). Higher values mean that optimal monetary policy is tight relative to this rule. See the main text for details. All productivity shocks are negative.

E Optimal Policy

E.1 Optimal policy: derivations

As noted in the main text, the Central Bank (CB) values the utility of households according to the social welfare function $\mathcal{W}$ defined has:

$$\mathcal{W} = (1-\delta)\int G\left(V^{-}(i), i\right) di + \delta \mathbb{E}_0 \sum_{t_0=0}^{\infty} \beta^{t_0} \int G\left(V^{t_0}(i), i\right) di$$

Here, a superscript t_0 denotes the birth date of a cohort (within a household type i) and a superscript $_-$ denotes cohorts born before $t = 0$.[13] The value of a cohort t_0 in type i is given by:

$$V^{t_0}(i) = \mathbb{E}_{t_0} \sum_{s=0}^{\infty} \left((1-\delta)\beta\right)^s \left\{ (1-\varphi(i)) \left[U_i\left(\mathcal{U}_1\left(\boldsymbol{c}^{t_0,u}_{1,t_0+s}(i)\right), \ldots \mathcal{U}_K\left(\boldsymbol{c}^{t_0,u}_{K,t_0+s}(i)\right)\right) - \chi\left(\frac{n^{t_0,u}_{t_0+s}(i)}{\vartheta(i)}\right)\right] + \varphi(i)\left[U_i\left(\mathcal{U}_1\left(\boldsymbol{c}^{t_0,HtM}_{1,t_0+s}(i)\right), \ldots \mathcal{U}_K\left(\boldsymbol{c}^{t_0,HtM}_{K,t_0+s}(i)\right)\right) - \chi\left(\frac{n^{t_0,HtM}_{t_0+s}(i)}{\vartheta(i)}\right)\right]\right\}.$$

and note that within each cohort/type a fraction $\varphi(i)$ is HtM, and recall that non-HtM households are denoted by a superscript u. The value of pre-existing cohorts, $V_{-}(i)$, is defined analogously. The CB maximizes $\mathcal{W}$ under the following set of constraints (for any i, j, k, t, t_0):

- Optimality of intratemporal consumption decisions

$$c^{t_0,h}_{k,t}(i,j) = d_k\left(p_{k,t}(j), \boldsymbol{p}_{k,t}, e^*_k\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right)\right)$$

$$v_i\left(e^{t_0,h}_t(i), \boldsymbol{P}\right) = U_i\left(\mathcal{U}_1\left(\boldsymbol{d}_1\left(p_{1,t}(j), \boldsymbol{p}_{1,t}, e^*_1\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right)\right)\right), \ldots \mathcal{U}_K\left(\boldsymbol{d}_K\left(p_{K,t}(j), \boldsymbol{p}_{K,t}, e^*_K\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right)\right)\right)\right)$$

  for $h \in \{u, HtM\}$. Here, d_k and e^*_k are the solutions of the inner and outer consumption problem defined in the previous sections.

- Optimality of labor supply decisions, for $h \in \{u, HtM\}$:

$$\chi'\left(\frac{n^{t_0,h}_t(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)} = W_t \partial_e v_{i,t}\left(e^{t_0,h}_t(i), \boldsymbol{P}\right).$$

- Optimality of intertemporal expenditure decisions for non-HtM households (Euler equation and budget constraint):

$$\partial_e v_{i,t}\left(e^{t_0,u}_t(i), \boldsymbol{P}_t\right) = \beta(1-\delta)R_t \mathbb{E}_t\left[\partial_e v_{i,t+1}\left(e^{t_0,u}_{t+1}(i), \boldsymbol{P}_t\right)\right],$$

$$\frac{b^{t_0,u}_{t+1}(i)}{R_t} = b^{t_0,u}_t(i) + n^{t_0,u}_t(i) W_t + \sum_k \varsigma_k(i) Div_{k,t} - e^{t_0,u}_t(i),$$

  with $b^{t_0,u}_{t_0}(i) = b^{t_0,HtM}_{t_0}(i) = \left(1 + \sum_l \bar{s}_l\left(\frac{P_{l,t_0} - P_{l,-}}{P_{l,-}}\right)\right) b^{-}_0(i)$.

- HtM consumption:

$$\left(\frac{1}{R_t} - 1\right) b^{t_0,HtM}_{t_0}(i) = n^{t_0,HtM}_t(i) W_t + \sum_k \varsigma_k(i) Div_{k,t} - e^{t_0,HtM}_t(i).$$

- Optimal Price resetting:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \tilde{\beta}^s \theta^s_k \left(D_{k,t+s}\left(p^*_{k,t}(j^*), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s}\right) + \left(p^*_{k,t}(j^*) - (1-\tau_k) MC_{k,t+s}(j^*)\right) \partial_p D_{k,t+s}\left(p^*_{k,t}(j^*), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s}\right)\right) = 0$$

  where the aggregate demand for subvarieties is defined in the previous section.

[13] Note that it would be equivalent – to a first order approximation – to differentiate households born before t_0 according to their date of birth, that is consider the social welfare function $\mathcal{W} = \delta \mathbb{E}_0 \sum_{t_0=-\infty}^{\infty} \beta^{t_0} \int G\left(V^{t_0}(i), i\right) di$.

- Labor market clearing:

$$(1-\delta)^{t+1}\int(1-\varphi(i))\,n_t^{-,u}(i)+\varphi(i)\,n_t^{-,HtM}(i)di+\delta\sum_{t_0}(1-\delta)^{t-t_0}\int(1-\varphi(i))\,n_t^{t_0,u}(i)+\varphi(i)\,n_t^{t_0,HtM}(i)di=\sum_{k=1}^{K}\mathcal{N}_k(\boldsymbol{P}_t,W_t)\int\frac{D_{k,t}\left(p_{k,t}(j),\boldsymbol{p}_{k,t},\boldsymbol{e}_{k,t},\tilde{Y}_{k,t}\right)}{A_{k,t}}dj$$

The firm optimal choice of input, the market clearing conditions for intermediate goods and consumption goods and the government budget constraint will be used implicitly.

We denote by $\mathbb{E}_{\delta,t}\left(X_t^{t_0}\right)\equiv(1-\delta)^{t+1}X_t^{-}+\delta\sum_{t_0=0}^{t}(1-\delta)^{t-t_0}X_t^{t_0}$ the inter-generational average of variable $X_t^{t_0}$ at t. We further denote by $\breve{\Xi}_t$ and $\tilde{\mu}_{k,t}$ the Lagrange multipliers on the labor market clearing constraint and optimal price setting constraints, and by $\check{\lambda}_t^{t_0}(i)$, $\check{\zeta}_t^{t_0,u}(i)$,$\check{\zeta}_t^{t_0,HtM}(i)$, $\check{\alpha}_t^{t_0}(i)$ and $\check{\aleph}_t^{t_0}(i)$, the Lagrange multipliers on the Euler equation of unconstrained households ($\check{\lambda}_t^{t_0}(i)$), on the optimality of labor supply decisions ($\check{\zeta}_t^{t_0,u}(i)$,$\check{\zeta}_t^{t_0,HtM}(i)$) and on the budget constraints of households ($\check{\alpha}_t^{t_0}(i)$ for unconstrained households and $\check{\aleph}_t^{t_0}(i)$ for HtM households), The Lagrangian of the optimal policy problem is:

$$
\begin{aligned}
&(1-\delta)\int\frac{1}{E}G\left(V^{-}(i)(i),i\right)di+\delta\mathbb{E}_0\sum\beta^{t_0}\int G\left(V^{t_0}(i),i\right)di\\
&\quad+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\partial_e v_{t,i}\left(e_t^{t_0,u}(i),\boldsymbol{P}\right)\left(\check{\lambda}_t^{t_0}(i)-R_{t-1}\check{\lambda}_{t-1}^{t_0}(i)\right)di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i)\left(W_t\partial_e v_{t,i}\left(e_t^{t_0,u}(i),\boldsymbol{P}\right)-\chi'\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)}\right)di+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\left(W_t\partial_e v_{t,i}\left(e_t^{t_0,HtM}(i),\boldsymbol{P}\right)-\chi'\left(\frac{n_t^{t_0,HtM}(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)}\right)di\\
&\quad+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\beta^t\int(1-\varphi(i))\,\check{\alpha}_t^{t_0}(i)\left(\frac{b_{t+1}^{t_0,u}(i)}{R_t}-\left(b_t^{t_0,u}(i)+n_t^{t_0,u}(i)W_t+\sum_k\varsigma_k(i)Div_{k,t}-e_t^{t_0,u}(i)\right)\right)di\\
&\quad+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\check{\aleph}_t^{t_0}(i)\,\varphi(i)\left(\left(\frac{1}{R_t}-1\right)b_{t_0}^{t_0,HtM}(i)-\left(n_t^{t_0,HtM}(i)W_t+\sum_k\varsigma_k(i)Div_{k,t}-e_t^{t_0,HtM}(i)\right)\right)\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\breve{\Xi}_tW_t\left((1-\delta)^{t+1}\int(1-\varphi(i))\,n_t^{-,u}(i)+\varphi(i)\,n_t^{-,HtM}(i)di+\delta\sum_{t_0}(1-\delta)^{t-t_0}\int(1-\varphi(i))\,n_t^{t_0,u}(i)+\varphi(i)\,n_t^{t_0,HtM}(i)di-\sum_{k=1}^{K}\mathcal{N}_k(\boldsymbol{P}_t,W_t)\int\frac{D_{k,t}\left(p_{k,t}(j),\boldsymbol{p}_{k,t},\boldsymbol{e}_{k,t},\tilde{Y}_{k,t}\right)}{A_{k,t}}dj\right)\\
&\quad+\sum_{k=1}^{K}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\tilde{\mu}_{k,t}\left(\sum_{s=0}^{\infty}\tilde{\beta}^s\theta_k^s\left(D_{k,t+s}\left(p_{k,t}^*(j^*),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right)+\left(p_{k,t}^*(j^*)-(1-\tau_k)MC_{k,t+s}(j^*)\right)\partial_pD_{k,t+s}\left(p_{k,t}^*(j^*),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right)\right)\right)
\end{aligned}
$$

First-order conditions

Let us consider a steady state in which the CB targets zero inflation (and all goods prices and wages are constant), setting $R_t=1/\tilde{\beta}$. Recall also that we normalized $A_{k,t}=1$ and that we assumed that elasticities of substitution across varieties are equal for households and intermediate input producers, i.e. $\frac{P_k\partial_pD_k^C}{D_k^C}=\frac{P_k\partial_pD_k^{\mathcal{I}}}{D_k^{\mathcal{I}}}=-\bar{\epsilon}_k$. In such a steady state, wealth, expenditure and labor supply of households is constant across time and identical for unconstrained and HtM households of the same type i. We first show that, given the presence of a subsidy undoing markups, $(1-\tau_k)\frac{\bar{\epsilon}_k}{\bar{\epsilon}_k-1}=1$, and the first assumption on the social welfare function, $G'\left(V_{t_0}(i),i\right)\partial_ev(i)=1$, this steady state is efficient. We do so by first showing that the first-order conditions to the optimal policy problem hold at the steady state.[14] After doing so, we perturb the first-order conditions around the steady state, in order to solve for the optimal dynamics.

[14] When we derive the loss function, we also show that the second-order conditions are satisfied.

- First-order conditions for $b_t^{t_0,u}(i)$:

$$\tilde{\beta}\check{\alpha}_t^{t_0}(i) = \frac{1}{R_{t-1}}\check{\alpha}_{t-1}^{t_0}(i).$$
$$\Rightarrow \check{\alpha}_t^{t_0}(i) = \check{\alpha}^{t_0}(i)$$

where the second line gives the necessary optimality condition in a steady state with constant prices and $R_t = 1/\tilde{\beta}$.

- First-order conditions for the interest rate, R_t :

$$-\beta\mathbb{E}_{\delta,t+1}\left(\int(1-\varphi(i))\check{\lambda}_t^{t_0}(i)\partial_e v_{t+1,i}\left(e_{t+1}^{t_0,u}(i),\boldsymbol{P}\right)di\right) - \mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\check{\alpha}_t^{t_0}(i)\frac{1}{R_t^2}b_{t+1}^{t_0,u}(i)\,di\right) - \mathbb{E}_{\delta,t}\left(\int\varphi(i)\check{\aleph}_t^{t_0}(i)\frac{1}{R_t^2}b_{t_0}^{t_0,HtM}(i)\,di\right) = 0$$
$$\Rightarrow -\mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\check{\lambda}_t^{t_0}(i)\partial_e v_i(e(i),\boldsymbol{P})\,di\right) - \mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\check{\alpha}^{t_0}(i)\frac{b(i)}{R}di\right) - \mathbb{E}_{\delta,t}\left(\int\varphi(i)\check{\aleph}_t^{t_0}(i)\frac{1}{R}b(i)\,di\right) = 0$$

where the second line gives the necessary optimality condition in a steady state with constant prices and $R_t = 1/\tilde{\beta}$ (so wealth is constant across time and generations)

- First-order conditions for W_t. Denoting as before $\mathcal{Q}_{l,k} = \mathcal{Y}_{l,k}\frac{Y_k}{Y_l A_k}$ the matrix of intermediate shares, we have:

$$0 = \mathbb{E}_{\delta,t}\int(1-\varphi(i))\check{\zeta}_t^{t_0,u}(i)\partial_e v_{t,i}\left(e_t^{t_0,u}(i),\boldsymbol{P}_t\right)di + \mathbb{E}_{\delta,t}\int\varphi(i)\check{\zeta}_t^{t_0,HtM}(i)\partial_e v_{t,i}\left(e_t^{t_0,HtM}(i),\boldsymbol{P}_t\right)di$$
$$-\sum_{k=1}^{K}(1-\tau_k)\sum_{s=0}^{t}((1-\delta)\theta_k)^{t-s}\tilde{\mu}_{k,s}\mathcal{N}_k(\boldsymbol{P}_t,W_t)\partial_p D_{k,t}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t},\boldsymbol{e}_{k,t},\tilde{Y}_{k,t}\right)$$
$$+\sum_{k=1}^{K}\sum_{s=0}^{t}((1-\delta)\theta_k)^{t-s}\tilde{\mu}_{k,s}\left[d_k^{\mathcal{I}}Y_k + \left(p_{k,s}^*(j^*) - (1-\tau_k)MC_{k,t}(j^*)\right)\partial_p d_k^{\mathcal{I}}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t}\right)Y_k\right](Id-Q)^{-1}\left[\sum_k\frac{\partial\mathcal{Y}_{1,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_1},...,\sum_k\frac{\partial\mathcal{Y}_{K,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_K}\right]^T$$
$$-\mathbb{E}_{\delta,t}\beta^t\int(1-\varphi(i))\check{\alpha}_t^{t_0}(i)\left(n_t^{t_0,u}(i) - \varsigma(i)\sum_k\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)Y_{k,t}}{A_{k,t}}\right)di - \mathbb{E}_{\delta,t}\int\check{\aleph}_t^{t_0}(i)\varphi(i)\left(\left(n_t^{t_0,HtM}(i) - \sum_k\varsigma(i)\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)Y_{k,t}}{A_{k,t}}\right)\right)$$
$$-\mathbb{E}_{\delta,t}\int\left((1-\varphi(i'))\check{\alpha}_t^{t_0}(i') + \varphi(i')\check{\aleph}_t^{t_0}(i')\right)\varsigma(i')\sum_k\int d_k^I\partial_W\tilde{Y}_{k,t}(p_{k,t}(j) - MC_{k,t})\,dj\,di'$$
$$-\check{\Xi}_t\sum_{k=1}^{K}\partial_W\mathcal{N}_k(\boldsymbol{P}_t,W_t)\frac{Y_k}{A_{e_k,t}} - \check{\Xi}_t\left[\frac{\mathcal{N}_1(\boldsymbol{P}_t,W_t)Y_1}{A_{1,t}},...,\frac{\mathcal{N}_K(\boldsymbol{P}_t,W_t)Y_K}{A_{K,t}}\right](Id-Q)^{-1}\left[\sum_k\frac{\partial\mathcal{Y}_{1,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_1},...,\sum_k\frac{\partial\mathcal{Y}_{K,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_K}\right]^T,$$

where the change in demand for intermediary in response to a change in wage solves:

$$\partial_W\tilde{Y}_{l,t} = \sum_k\partial_W\mathcal{Y}_{l,k}(\boldsymbol{P_t},W_t)Y_{k,t} + \sum_k\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)\int d_{k,t}^I(j)\,dj\,\partial_W\tilde{Y}_{k,t}.$$

We use the market clearing condition for intermediary and the optimal input demand from firms to obtain the expression on the last line. Using the fact that subvariety prices are constant and equal, that $\frac{P_k\partial_p D_k^C}{D_k^C} = \frac{P_k\partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}} = -\bar{\epsilon}_k$ and $(1-\tau_k)\frac{\bar{\epsilon}_k}{\bar{\epsilon}_k-1} = 1$, we can use:

$$d_k^{\mathcal{I}}Y_k + \left(p_{k,s}^*(j^*) - (1-\tau_k)MC_{k,t+s}(j^*)\right)\partial_p d_k^{\mathcal{I}}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t}\right)Y_k = 0 \quad \forall k$$
$$[W\mathcal{N}_1(\boldsymbol{P},W)Y_1,...,W\mathcal{N}_K(\boldsymbol{P},W)Y_K](Id-Q)^{-1} = [P_1Y_1,...,P_KY_K]$$
$$W_t\partial_W\mathcal{N}_k(\boldsymbol{P},W) + \sum_l P_l\frac{\partial\mathcal{Y}_{l,k}}{\partial W} = 0 \quad \forall k.$$

In addition, we define $\check{\mu}_{k,T}$ which constrains the growth rate of sectoral inflation: :

$$\check{\mu}_{k,T} \equiv \frac{\theta_k}{1-\theta_k} \sum_{t=0}^{T} \left((1-\delta)\,\theta_k \right)^{T-t} \tilde{\mu}_{k,t} \sum_{s=0}^{\infty} \tilde{\beta}^s \theta_k^s \left(2p_{k,t}(j^*) \partial_p D_{k,t+s} + \left(p_{k,t}(j^*) - (1-\tau_k) MC_{k,t+s}(j^*) \right) p_{k,t}(j^*) \partial_{pp} D_{k,t+s} \right)$$

and note that around our steady steady state:

$$\begin{aligned} \check{\mu}_{k,T} &= \frac{\theta_k}{(1-\theta_k)\left(1-\tilde{\beta}\theta_k\right)} \left(2P_k \partial_p D_{k,} + \left(P_k - (1-\tau_k) MC_k \right) P_k \partial_{pp} D_k \right) \sum_{t=0}^{T} \left((1-\delta)\,\theta_k \right)^{T-t} \tilde{\mu}_{k,t} \\ &= P_k \partial_p D_k \frac{\bar{\epsilon}_k - 1}{\bar{\epsilon}_k} \frac{1}{\lambda_k} \sum_{t=0}^{T} \left((1-\delta)\,\theta_k \right)^{T-t} \tilde{\mu}_{k,t}. \end{aligned}$$

Using this, we rewrite the first order condition as:

$$\begin{aligned} 0 = \mathbb{E}_{\delta,t} \int (1-\varphi(i))\, \check{\zeta}_t^{t_0,u}(i)\, \partial_e v_{t,i}^{u,t_0} di + \mathbb{E}_{\delta,t} \int \varphi(i)\, \check{\zeta}_t^{t_0,HtM}(i)\, \partial_e v_{t,i}^{HtM,t_0} di - \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,s} \mathcal{N}_k(\boldsymbol{P}_t, W_t) \\ - \mathbb{E}_{\delta,t} \beta^t \int (1-\varphi(i))\, \check{\alpha}_t^{t_0}(i) \left(n(i) - \varsigma(i) \sum_k \mathcal{N}_k(\boldsymbol{P}_t, W_t) Y_{k,t} \right) di - \mathbb{E}_{\delta,t} \int \check{\aleph}_t^{t_0}(i)\, \varphi(i) \left(\left(n(i) - \sum_k \varsigma_k(i) \mathcal{N}_k(\boldsymbol{P}_t, W_t) Y_{k,t} \right) \right) di. \end{aligned}$$

Using the fact that $\varsigma(i) = n(i)/N$, the steady state equation is:

$$0 = \mathbb{E}_{\delta,t} \int (1-\varphi(i))\, \check{\zeta}_t^{t_0,u}(i)\, \partial_e v di + \mathbb{E}_{\delta,t} \int \varphi(i)\, \check{\zeta}_t^{t_0,HtM}(i)\, \partial_e v di - \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,s} \mathcal{N}_k(\boldsymbol{P}_t, W_t).$$

- First order conditions with respect to labor supply, $n_t^{t_0,u}(i)$ and , $n_t^{t_0,HtM}(i)$:

$$\begin{aligned} \check{\zeta}_t^{t_0,u}(i)\, \partial_e v_{t,i}^{u,t_0} = \psi n_t^{t_0,u}(i) \left\{ \check{\Xi}_t - \check{\alpha}_t^{t_0}(i) - G'\left(V_{t_0}(i)\right) \partial_e v_{t,i}^{u,t_0} \right\}. \\ \Rightarrow \check{\zeta}_t^{t_0,u}(i)\, \partial_e v = \psi n(i) \left\{ \check{\Xi}_t - \check{\alpha}^{t_0}(i) - 1 \right\} \\ \check{\zeta}_t^{t_0,HtM}(i)\, \partial_e v_{t,i}^{HtM,t_0} = \psi n_t^{t_0,HtM}(i) \left\{ \check{\Xi}_t - \check{\aleph}_t^{t_0}(i) - G'\left(V_{t_0}(i)\right) \partial_e v_{t,i}^{HtM,t_0} \right\} \\ \Rightarrow \check{\zeta}_t^{t_0,HtM}(i)\, \partial_e v = \psi n(i) \left\{ \check{\Xi}_t - \check{\aleph}_t^{t_0}(i) - 1 \right\} \end{aligned}$$

where we used the definition $\psi = \chi'\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right) / \left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)} \chi''\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right)\right)$ and the optimality of labor supply decisions, and the second and fourth line are the steady state equations.

- First order condition with respect to expenditure of the non-HtM, $e_t^u(i)$:

$$G'\left(V_{t_0}(i),i\right)\partial_e v_{t,i}^{u,t_0}+\left(\check{\lambda}_t^{t_0}(i)-R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_{ee}v_{t,i}^{u,t_0}+\check{\zeta}_t^{t_0}(i)\,W_t\partial_{ee}v_{t,i}^{u,t_0}$$
$$+\check{\alpha}_t^{t_0}(i)-\mathbb{E}_{\delta,t}\int\left(\left(1-\varphi\left(i'\right)\right)\check{\alpha}_t^{t_0}\left(i'\right)+\varphi\left(i'\right)\check{\aleph}_t^{t_0}\left(i'\right)\right)\varsigma(i')\sum_k\int\left(\partial_e e_k\partial_e d_k+d_k^I\partial_e\tilde{Y}_k\right)\left(p_{k,t}(j)-MC_{k,t}\right)djdi'$$
$$-\check{\Xi}_t\sum_{k=1}^K\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)}{A_k}\int\left(\partial_e e_k\partial_e d_k+d_k^I\partial_e\tilde{Y}_k\right)dj$$
$$+\sum_{k=1}^K\sum_{s=0}^t\left(\left(1-\delta\right)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left(\partial_e d_{k,t}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t},e_{k,t}\right)+\left(p_{k,s}^*(j^*)-(1-\tau_k)MC_{k,t}(j^*)\right)\partial_{pe}d_{k,t}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t},e_{k,t}\right)\right)$$
$$+\sum_{k=1}^K\sum_{s=0}^t\left(\left(1-\delta\right)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left[d_k^{\mathcal{I}}+\left(p_{k,s}^*(j^*)-(1-\tau_k)MC_{k,t+s}(j^*)\right)\partial_p d_k^{\mathcal{I}}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t}\right)\right]\partial_e\tilde{Y}_k=0$$

With some abuse of notation, $\partial_e\tilde{Y}_k$ is the Gateaux derivative (keeping prices fixed) of demand for intermediary output with respect to a change in $e_t^{t_0,u}(i)$. Note that we use $(1-\tau_k)\frac{\bar{\epsilon}_k}{\bar{\epsilon}_k-1}=1$ and the adjustment of the lump sum tax to express the total change in dividends. We have, denoting $\tilde{Q}_{l,k}=\mathcal{Y}_{l,k}/A_k$:

$$\left[\partial_e\tilde{Y}_1,...,\partial_e\tilde{Y}_K\right]^T=(Id-\tilde{Q})^{-1}\left[\sum_k\mathcal{Y}_{1,k}(\boldsymbol{P},W)\int\partial_e e_k\partial_e d_k(i,j)dj,...,\sum_k\mathcal{Y}_{K,k}(\boldsymbol{P},W)\int\partial_e e_k\partial_e d_k(i,j)dj\right]^T=(Id-\tilde{Q})^{-1}\tilde{Q}\left[\int\left(\partial_e e_1\partial_e d_1\right)dj,...,\int\left(\partial_e e_K\partial_e d_K\right)dj\right]^T,$$
$$\sum_{k=1}^K\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)}{A_k}\int\left(\partial_e e_k\partial_e d_k+d_k^I\partial_e\tilde{Y}_k\right)dj=\sum_{k=1}^K\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)+\sum_l\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)}{A_k}\int\left(\partial_e e_k\partial_e d_k\right)dj=1.$$

Simplifying we have, in steady state:

$$1+\left(\check{\lambda}_t^{t_0}(i)-R\lambda_{t-1}^{t_0}(i)\right)\partial_{ee}v+\check{\zeta}_t^{t_0,u}(i)\,W\partial_{ee}v+\check{\alpha}^{t_0}(i)-\check{\Xi}_t-\sum_{k=1}^K\frac{1}{P_kY_k}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)=0.$$

Following the same steps, the first order conditions for the expenditure of HtM households, $e_t^{HtM}(i)$, is

$$G'\left(V_{t_0}(i),i\right)\partial_e v_{t,i}^{HtM,t_0}+\check{\zeta}_t^{t_0,u}(i)\,W_t\partial_{ee}v_{t,i}^{HtM,t_0}+\check{\aleph}_t^{t_0}(i)-\mathbb{E}_{\delta,t}\int\left(\left(1-\varphi\left(i'\right)\right)\check{\alpha}_t^{t_0}\left(i'\right)+\varphi\left(i'\right)\check{\aleph}_t^{t_0}\left(i'\right)\right)\varsigma(i')\sum_k\int\left(\partial_e e_k\partial_e d_k+d_k^I\partial_e\tilde{Y}_k\right)\left(p_{k,t}(j)-MC_{k,t}\right)djdi'$$
$$-\check{\Xi}_t\sum_{k=1}^K\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)}{A_{e_k,t}}\int\left(\partial_e e_k\partial_e d_k+d_k^I\partial_e\tilde{Y}_k\right)dj+\sum_{k=1}^K\sum_{s=0}^t\left(\left(1-\delta\right)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left(\partial_e d_{k,t}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t},e_{k,t}\right)+\left(p_{k,s}^*(j^*)-(1-\tau_k)MC_{k,t}(j^*)\right)\partial_{pe}d_{k,t}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t},e_{k,t}\right)\right)$$
$$+\sum_{k=1}^K\sum_{s=0}^t\left(\left(1-\delta\right)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left[d_k^{\mathcal{I}}Y_k+\left(p_{k,s}^*(j^*)-(1-\tau_k)MC_{k,t+s}(j^*)\right)\partial_p d_k^{\mathcal{I}}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t}\right)Y_k\right]\partial_e\tilde{Y}_k=0,$$

and in steady state simplifies to:

$$1+\check{\zeta}_t^{t_0}(i)\,W\partial_{ee}v+\check{\aleph}_t^{t_0}(i)-\check{\Xi}_t-\sum_{k=1}^K\frac{1}{P_kY_k}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)=0.$$

- Finally consider the first order conditions for a compensated change in resetted prices $p_{k,t}^*(j^*)$ (That is, each household receives a transfer in period $t+s$, $s\geq 0$ which cancels the income effect of the price change. For a household consuming a bundle $\boldsymbol{d}_{k,t+s}(i,j)$of the varieties in sector k at $t+s$, the transfer would be $(1-\theta_k)\,\theta_k^s\int d_{k,t+s}(i,j)\,dj$. Note that we can alternatively consider an uncompensated change in prices, but the terms corresponding to the income effects can then be simplified using the first-order condition corresponding to the optimality of expenditure of unconstrained and HtM households.):

$$
\begin{aligned}
0 = & \tilde{\mu}_{k,t}\left(\sum_{s=0}^{\infty}\tilde{\beta}^{s}\theta_{k}^{s}\left(2\partial_{p}D_{k,t+s}\left(p_{k,t}^{*}(j^{*}),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right)+\left(p_{k,t}^{*}(j^{*})-(1-\tau_{k})MC_{k,t+s}(j^{*})\right)\partial_{pp}D_{k,t+s}\left(p_{k,t}^{*}(j^{*}),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right)\right)\right)\\
& +(1-\theta_{k})\,\mathbb{E}_{0}\sum_{T=t}(\beta\theta_{k})^{T-t}\sum_{s=0}^{T}((1-\delta)\,\theta_{l})^{T-s}\,\tilde{\mu}_{k,s}\left(\partial_{P}D_{k,T}+\int\partial_{e}d_{l,T}\left(p_{l,s}^{*}(j^{*}),\boldsymbol{p}_{l,T},e_{l,T}\right)\frac{e_{k,T}}{p_{k,T}(j)}di\right)\\
& +(1-\theta_{k})\,\mathbb{E}_{0}\sum_{T=t}(\beta\theta_{k})^{T-t}\sum_{s=0}^{T}((1-\delta)\,\theta_{l})^{T-s}\,\tilde{\mu}_{k,s}\left((p_{k,s}(j^{*})-(1-\tau_{k})MC_{k,T}(j^{*}))\left(\partial_{pP}D_{k,T}+\int\partial_{pe}d_{l,T}\left(p_{l,s}^{*}(j^{*}),\boldsymbol{p}_{l,T},e_{l,T}\right)\frac{e_{k,T}}{p_{k,T}(j)}di\right)\right)\\
& +(1-\theta_{k})\sum_{T=t}^{\infty}(\beta\theta_{k})^{T-t}\sum_{l=1}^{K}\sum_{s=0}^{T}((1-\delta)\,\theta_{l})^{T-s}\,\tilde{\mu}_{l,s}\\
& \cdot\int\left(\partial_{e}d_{l,T}\left(p_{l,s}^{*}(j^{*}),\boldsymbol{p}_{l,T},e_{l,T}\right)+\left(p_{l,s}^{*}(j^{*})-(1-\tau_{k})MC_{l,T}(j^{*})\right)\partial_{pe}d_{l,T}\left(p_{l,s}^{*}(j^{*}),\boldsymbol{p}_{l,T},e_{l,T}\right)\left(\partial_{p_{k}(j^{*})}e_{l,T}-\mathbb{1}_{l=k}\frac{e_{k,T}}{p_{k,T}(j)}+\partial_{e}e_{l,T}d_{k,T}(i,j^{*})\right)\right)di\\
& +(1-\theta_{k})\sum_{T=t}^{\infty}(\beta\theta_{k})^{T-t}\sum_{l=1}^{K}\sum_{s=0}^{T}((1-\delta)\,\theta_{l})^{T-s}\,\tilde{\mu}_{l,s}\left[d_{l}^{\mathcal{I}}\tilde{Y}_{l}+\left(p_{l,s}^{*}(j^{*})-(1-\tau_{l})MC_{l,T}(j^{*})\right)\partial_{p}d_{k}^{\mathcal{I}}\left(p_{l,T}^{*}(j^{*}),\boldsymbol{p}_{l,t}\right)\tilde{Y}_{l}\right](Id-Q)^{-1}\cdot\left[\sum_{l}\frac{\partial\mathcal{Y}_{1,l}}{\partial p(j^{*})}\frac{Y_{l}}{A_{l,t}Y_{1}},...,\sum_{l}\frac{\partial\mathcal{Y}_{K,l}}{\partial p(j^{*})}\frac{Y_{l}}{A_{l,t}Y_{K}}\right]^{T}\\
& +(1-\theta_{k})\sum_{T=t}^{\infty}(\beta\theta_{k})^{T-t}\sum_{l=1}^{K}\sum_{s=0}^{T}((1-\delta)\,\theta_{l})^{T-s}\,\tilde{\mu}_{l,s}\left[d_{l}^{\mathcal{I}}\left(p_{l,s}^{*}(j^{*})-(1-\tau_{l})MC_{l,T}(j^{*})\right)\partial_{p}d_{l}^{\mathcal{I}}\left(p_{l,s}^{*}(j^{*}),\boldsymbol{p}_{l,T}\right)\right]\partial_{p_{k}(j^{*})}\tilde{Y}_{l,T}\\
& -(1-\theta_{k})\sum_{T=t}^{\infty}(\beta\theta_{k})^{T-t}\sum_{l=1}^{K}\sum_{s=0}^{T}((1-\delta)\,\theta_{l})^{T-s}\,\tilde{\mu}_{l,s}(1-\tau_{k})\frac{\mathcal{Y}_{k,l}}{A_{l,T}}\partial_{p}D_{l,T}-(1-\theta_{k})\sum_{s}(\beta\theta_{k})^{s}\,\check{\Xi}_{t+s}\left(\sum_{l=1}^{K}\partial_{p_{k}(j^{*})}\mathcal{N}_{l}(\boldsymbol{P}_{t},W_{t})\frac{Y_{l}}{A_{l,t}}\right)\\
& -(1-\theta_{k})\sum_{s}(\beta\theta_{k})^{s}\,\check{\Xi}_{t+s}\left(\left[\frac{\mathcal{N}_{1}(\boldsymbol{P}_{t},W_{t})Y_{1}}{A_{1,t}},...,\frac{\mathcal{N}_{K}(\boldsymbol{P}_{t},W_{t})Y_{K}}{A_{K,t}}\right](Id-Q)^{-1}\left[\sum_{l}\frac{\partial\mathcal{Y}_{1,l}}{\partial p(j^{*})}\frac{Y_{l}}{A_{l,t}Y_{1}},...,\sum_{k}\frac{\partial\mathcal{Y}_{K,l}}{\partial p(j^{*})}\frac{Y_{l}}{A_{l,t}Y_{K}}\right]^{T}\right)\\
& -(1-\theta_{k})\sum_{s}(\beta\theta_{k})^{s}\,\check{\Xi}_{t+s}\sum_{l=1}^{K}\frac{\mathcal{N}_{l}(\boldsymbol{P}_{t+s},W_{t+s})}{A_{l,t}}\mathbb{E}_{\delta,t+s}\int\int\left(\left(\partial_{p_{k}(j^{*})}e_{l,t+s}+\partial_{e}e_{l,t+s}d_{k,t+s}(i,j^{*})\right)\partial_{e}d_{l,t+s}+d_{l,t+s}^{I}\partial_{p_{k}(j^{*})}\tilde{Y}_{l,t+s}\right)djdi\\
& -(1-\theta_{k})\sum_{s}(\beta\theta_{k})^{s}\,\check{\Xi}_{t+s}\frac{\mathcal{N}_{k}(\boldsymbol{P}_{t+s},W_{t+s})}{A_{e_{k},t}}\mathbb{E}_{\delta,t+s}\int\partial_{p}d_{k,t+s}+\int\partial_{P}d_{k,t+s}djdi+\left(\partial_{p_{k}(j^{*})}d_{k,t+s}^{I}+\int\partial_{P}d_{k,t+s}^{I}dj\right)\tilde{Y}_{k,t+s}\\
& -(1-\theta_{k})\sum_{s}(\beta\theta_{k})^{s}\,\mathbb{E}_{\delta,t+s}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}_{t+s}^{t_{0}}(i)-R\check{\lambda}_{t+s-1}^{t_{0}}(i)\right)+\check{\zeta}_{t+s}^{t_{0}}(i)\,W\right]\partial_{e}v_{t+s,i}^{t_{0},u}\partial_{e}d_{k_{t+s,i}}^{t_{0},u}\partial_{e}e_{k_{t+s,i}}^{t_{0},u}\right)\\
& -(1-\theta_{k})\sum_{s}(\beta\theta_{k})^{s}\,\mathbb{E}_{\delta,t+s}\left(\int\varphi(i)\,\check{\zeta}_{t+s}^{t_{0},HtM}(i)\,W\partial_{e}v_{t+s,i}^{t_{0},HtM}\partial_{e}d_{k_{t+s,i}}^{t_{0},HtM}\partial_{e}e_{k_{t+s,i}}^{t_{0},HtM}di\right)\\
& +(1-\theta_{k})\sum_{s}(\beta\theta_{k})^{s}\,\mathbb{E}_{\delta,t+s}\left(\int(1-\varphi(i))\,\check{\alpha}_{t}^{t_{0}}(i)\left[d_{k,t+s}(i,j^{*})-\varsigma(i)\,\partial_{p(j^{*})}Div_{t}\right]di\right)-\delta\int(1-\varphi(i))\,\check{\alpha}_{t+s}^{t+s}(i)\,b_{t+s}^{t+s,u}(i)\,di\bar{s}_{k}\frac{1}{P_{k}}\\
& +(1-\theta_{k})\sum_{s=0}(\beta\theta_{k})^{s}\left(-\mathbb{E}_{\delta t+s}\left(\int\varphi(i)\,\check{\aleph}_{t+s}^{t_{0}}\left[d_{k,t+s}(i,j^{*})-\varsigma(i)\,\partial_{p(j^{*})}Div_{t}\right]\right)di-\delta\sum_{u=0}((1-\delta)\,\beta)^{u}\int\varphi(i)\,\check{\aleph}_{t+u+s}^{t+s}(i)\,b_{t+s}^{t+s,HtM}(i)\,di\bar{s}_{k}\frac{R_{t+u}-1}{R_{t+u}}\frac{1}{P_{k}}\right).
\end{aligned}
$$

with

$$\left[\partial_{p_k(j^*)}\tilde{Y}_{1,t+s},...,\partial_{p_k(j^*)}\tilde{Y}_{K,t+s}\right]^T = (Id-\tilde{Q})^{-1}\tilde{Q}\left[\int\int\left(\partial_{p_k(j^*)}e_{1,t+s}+\partial_e e_{1,t+s}d_{k,t+s}(i,j^*)\right)\partial_e d_{1,t}djdi,...,\int\int\left(\partial_{p_k(j^*)}e_{K,t+s}+\partial_e e_{K,t}d_{k,t+s}(i,j^*)\right)\partial_e d_{K,t}djdi\right]^T$$

$$+(Id-\tilde{Q})^{-1}\tilde{Q}\left[0,...,\mathbb{E}_{\delta,t+s}\int\partial_p d_{k,t+s}+\int\partial_P d_{k,t+s}djdi+\left(\partial_{p_k(j^*)}d^I_{k,t+s}+\int\partial_P d^I_{k,t+s}dj\right)\tilde{Y}_{k,t+s},...,0\right]$$

$$\partial_{p(j^*)}Div_{t+s}=\sum_l\int\int\left(\left(\partial_{p_k(j^*)}e_{l,t+s}+\partial_e e_{l,s}d_{k,t+s}(i,j^*)\right)\partial_e d_{l,t+s}+d^I_{l,t+s}\partial_{p_k(j^*)}\tilde{Y}_{l,t+s}\right)(p_{l,t+s}(j)-MC_{l,t+s})\,didj$$

$$+\int\left(\partial_p d_{k,t+s}+\partial_p d^I_{k,t+s}\right)(p_{l,t+s}(j)-MC_{l,t+s})\,di+\int\int\left(\partial_P d_{k,t+s}+\partial_P d^I_{k,t+s}\right)(p_{l,t+s}(j)-MC_{l,t+s})\,didj+y_{k,t+s}(j)-\sum_l\mathcal{Y}_{k,l,t+s}Y_{l,t+s}$$

We define $\check{\mu}_{k,T}$, which constrains the growth rate of sectoral inflation:

$$\check{\mu}_{k,T}\equiv\frac{\theta_k}{1-\theta_k}\sum_{t=0}^{T}\left((1-\delta)\,\theta_k\right)^{T-t}\tilde{\mu}_{k,t}\sum_{s=0}^{\infty}\tilde{\beta}^s\theta_k^s\left(2p_{k,t}(j^*)\partial_p D_{k,t+s}+(p_{k,t}(j^*)-(1-\tau_k)MC_{k,t+s}(j^*))\,p_{k,t}(j^*)\partial_{pp}D_{k,t+s}\right)$$

Using the properties of the steady state, we get:

$$0=\frac{(1-\theta_k)}{\theta_k}\frac{1}{P_k}\left(\check{\mu}_{k,t}-((1-\delta)\,\theta_l)\,\check{\mu}_{k,t-1}\right)$$

$$-(1-\theta_k)\sum_{T=t}(\beta\theta_k)^{T-t}\left(\frac{(1-\theta_k)\left(1-\tilde{\beta}\theta_k\right)}{\theta_k}\check{\mu}_{k,T}\frac{1}{P_k}+\frac{1}{P_k}\lambda_k\check{\mu}_{k,T}\right)$$

$$-(1-\theta_k)\sum_{T=t}^{\infty}(\beta\theta_k)^{T-t}\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,T}}{P_lY_l}\int\left(\gamma_{e,l}(i)\frac{e_l(i)\,\rho_{l,k}(i)}{P_k}\right)di-(1-\theta_k)\sum_{T=t}^{\infty}(\beta\theta_k)^{T-t}\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,T}}{P_lY_l}\frac{\mathcal{Y}_{k,l}Y_l}{A_{l,T}}$$

$$-(1-\theta_k)\sum_s(\beta\theta_k)^s\,\mathbb{E}_{\delta,t+s}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}^{t_0}_{t+s}(i)-R\check{\lambda}^{t_0}_{t+s-1}(i)\right)+\check{\zeta}^{t_0,u}(i)\,W\right]\partial_e v\frac{1}{P_k}\partial_e e_k+\varphi(i)\,\check{\zeta}^{t_0,HtM}(i)\,W\partial_e v_i\frac{1}{P_k}\partial_e e_k di\right)$$

$$+(1-\theta_k)\sum_s(\beta\theta_k)^s\,\mathbb{E}_{\delta,t+s}\left(\int(1-\varphi(i))\,\check{\alpha}^{t_0}(i)\frac{1}{P_k}\left[e_k(i)-\varsigma(i)\,E_k\right]di\right)-\delta\int(1-\varphi(i))\,\check{\alpha}^{t+s}(i)\,b^{t+s}(i)\,di\bar{s}_k\frac{1}{P_k}$$

$$+(1-\theta_k)\sum_s(\beta\theta_k)^s\left(-\mathbb{E}_{\delta,t+s}\left(\int\varphi(i)\,\check{\aleph}^{t_0}_{t+s}(i)\frac{1}{P_k}\left[e_k(i)-\varsigma(i)\,E_k\right]\right)di-\delta\frac{R-1}{R}\sum_{u=0}\frac{1}{R^u}\int\varphi(i)\,\check{\aleph}^{t+s}_{t+u+s}(i)\,b(i)\,di\bar{s}_k\frac{1}{P_k}\right)$$

Taking the difference between the equation at $t+1$ times $\beta\theta_k$ and the equation at t we obtain

$$\left(\beta\check{\mu}_{k,t+1}-(1+((1-\delta)\,\beta))\,\check{\mu}_{k,t}+(1-\delta)\,\check{\mu}_{k,t-1}\right)=\lambda_k\check{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{Y_lP_l}\int\left(\gamma_{e,l}(i)\,e_l(i)\,\rho_{k,l}(i)\right)di-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_lY_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}$$

$$-\mathbb{E}_{\delta,t}\left(\int\left[(1-\varphi(i))\left((i)-R\lambda^{t_0}_{t-1}(i)\right)+\check{\zeta}^{t_0,u}_t(i)\right]\partial_e v\partial_e e_k+\varphi(i)\,\check{\zeta}^{t_0,HtM}_t(i)\,\partial_e v_i\partial_e e_k di\right)$$

$$+\mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\,\check{\alpha}^{t_0}(i)+\varphi(i)\,\check{\aleph}_t^{\,t_0}\right)\left[e_k(i)-\varsigma(i)\,E_k\right]di\right)-\delta\int\left((1-\varphi(i))\,\check{\alpha}^t(i)+\varphi(i)\,\frac{R-1}{R}\sum_{u=0}\frac{1}{R^u}\check{\aleph}^t_{t+u}(i)\right)b(i)\,di\bar{s}_k$$

We can now verify that a steady state with $R=1/\tilde{\beta}$, constant wages and prices (chosen such that the good markets and labor market clear, recall that this implies that wealth, expenditure and labor supply of households is constant across time and identical for unconstrained and HtM households) and $\check{\zeta}^{t_0,u}_t=\check{\zeta}^{t_0,HtM}_t=\check{\Xi}_t=\check{\mu}_{k,t}=\check{\lambda}^{t_0}_t=0$,

$\check{\alpha}^t(i) = \check{\aleph}_t^t(i) = \check{\aleph}^t(i) = -1$ solves the set of first-order conditions[15]

$$-\mathbb{E}_{\delta,t}\left((1-\varphi(i))\check{\lambda}_t^{t_0}(i)\partial_e v(e(i),\boldsymbol{P})\right) - \mathbb{E}_{\delta,t}\left((1-\varphi(i))\check{\alpha}^{t_0}(i)\frac{b(i)}{R}\right) - \mathbb{E}_{\delta,t}\left(\varphi(i)\check{\aleph}^{t_0}(i)\frac{1}{R}b(i)\right) = 0 \Rightarrow \mathbb{E}_{\delta,t}\left(\frac{b(i)}{R}\right) = 0$$

$$\check{\zeta}_t^{t_0,u}(i)\partial_e v = \psi n(i)\left\{\check{\Xi}_t - \check{\alpha}^{t_0}(i) - 1\right\} \quad \check{\zeta}_t^{t_0,HtM}(i)\partial_e v = \psi n(i)\left\{\check{\Xi}_t - \check{\alpha}^{t_0}(i) - 1\right\} \Rightarrow 0 = \psi n(i)\{1-1\} \quad 0 = \psi n(i)\{1-1\}$$

$$0 = \mathbb{E}_{\delta,t}\int(1-\varphi(i))\check{\zeta}_t^{t_0,u}(i)\partial_e v di + \mathbb{E}_{\delta,t}\int\varphi(i)\check{\zeta}_t^{t_0,HtM}(i)\partial_e di - \sum_{k=1}^{K}\lambda_k\check{\mu}_{k,s}\mathcal{N}_k(\boldsymbol{P}_t, W_t) \Rightarrow 0 = 0$$

$$1 + \left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_{ee}v + \check{\zeta}_t^{t_0,u}(i)W\partial_{ee}v + \check{\alpha}^{t_0}(i) - \check{\Xi}_t - \sum_{k=1}^{K}\frac{1}{P_kY_k}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\partial_e e_k(i) = 0 \Rightarrow 1-1 = 0$$

$$1 + \check{\zeta}_t^{t_0}(i)W\partial_{ee}v + \check{\aleph}^{t_0}(i) - \check{\Xi}_t - \frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\partial_e e_k(i) = 0 \Rightarrow 1-1 = 0$$

$$\left(\beta\check{\mu}_{k,t+1} - (1+((1-\delta)\beta))\check{\mu}_{k,t} + (1-\delta)\check{\mu}_{k,t-1}\right) = \lambda_k\check{\mu}_{k,t} - \sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{Y_lP_l}\int(\gamma_{e,l}(i)e_l(i)\rho_{k,l}(i))di$$

$$-\mathbb{E}_{\delta,t}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right) + \check{\zeta}_t^{t_0,u}(i)W\right]\partial_e v\partial_e e_k(i) + \varphi(i)\check{\zeta}_t^{t_0,HtM}(i)W\partial_e v_i\partial_e e_k(i)di\right) + \mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\check{\alpha}^{t_0}(i) + \varphi(i)\check{\aleph}^{t_0}\right)[e_k(i) - \varsigma(i)E_k]di\right)$$
$$-\delta\int\left((1-\varphi(i))\check{\alpha}^t(i) + \varphi(i)\check{\aleph}^t(i)\right)b(i)di\bar{s}_kP_k$$

$$\Rightarrow 0 = -\mathbb{E}_{\delta,t}\left(\int[e_k(i) - E_k]di\right) + \delta\int b(i)di\bar{s}_kP_k$$

Differentiating the first-order conditions

We now differentiate the first-order conditions around the steady state constructed in the previous section. Prices are in log-deviation while Lagrange multipliers are in absolute deviations.

- First order conditions with respect to $b_t^{t_0,u}(i)$:

$$\check{\alpha}_t^{t_0}(i) = \check{\alpha}_{t-1}^{t_0}(i) + \hat{R}_{t-1}$$
$$= \check{\alpha}^{t_0}(i) + \sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s}$$

- First Order conditions for the interest rate

$$-\mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\check{\lambda}_t^{t_0}(i)\partial_e v(e(i),\boldsymbol{P})di\right) = \mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\check{\alpha}^{t_0}(i)\frac{b(i)}{R}di\right) + \mathbb{E}_{\delta,t}\left(\int\varphi(i)\check{\aleph}_t^{t_0}(i)\frac{1}{R}b(i)di\right)$$
$$+\mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s}\frac{b(i)}{R}di\right)$$

[15] Note that to solve the steady state system we only need $\check{\Xi} - \check{\alpha}^{t_0} = -1$ and $\check{\Xi} - \check{\aleph}^{t_0} = -1$. It's direct to verify that choosing any values for $\check{\Xi}$, $\check{\alpha}^{t_0}$, and $\check{\aleph}^{t_0}$ that satisfy this would give the same system of differentiated first order conditions.

- First Order conditions for W_t:

$$0=\mathbb{E}_{\delta,t}\int\left(\left(1-\varphi\left(i\right)\right)\check{\zeta}_{t}^{t_{0},u}\left(i\right)+\varphi\left(i\right)\check{\zeta}_{t}^{t_{0},HtM}\left(i\right)\right)\partial_{e}vdi-\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,s}\mathcal{N}_{k}(\boldsymbol{P}_{t},W_{t})+\sum_{k,l}\frac{P_{l}\partial_{W}\mathcal{Y}_{l,k}}{A_{k}}Y_{k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{k}\frac{W\partial_{W}\mathcal{N}_{k}}{A_{k}}Y_{k}\hat{W}_{t}$$

- First-order conditions with respect to labor supply:

$$\begin{aligned}\check{\zeta}_{t}^{t_{0},u}\left(i\right)\partial_{e}v&=\psi n(i)\left\{\check{\Xi}_{t}-\sum_{s=0}^{t-t_{0}-1}\hat{R}_{t_{0}+s}-\check{\alpha}^{t_{0}}\left(i\right)-\frac{V\left(i\right)G''\left(V\left(i\right),i\right)}{G'\left(V\left(i\right),i\right)}\hat{V}_{t_{0}}\left(i\right)+\frac{1}{\sigma}\left(\hat{e}_{t}^{t_{0},u}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t}\right)+\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t}\right\}\\&=\psi n(i)\left\{\check{\Xi}_{t}-\check{\alpha}^{t_{0}}\left(i\right)-\frac{V\left(i\right)G''\left(V\left(i\right),i\right)}{G'\left(V\left(i\right),i\right)}\hat{V}_{t_{0}}\left(i\right)+\frac{1}{\sigma}\left(\hat{e}_{t_{0}}^{t_{0},u}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t_{0}}\right)+\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t_{0}}\right\}\\\check{\zeta}_{t}^{t_{0},HtM}\left(i\right)\partial_{e}v&=\psi n(i)\left\{\check{\Xi}_{t}-\check{\aleph}_{t}^{t_{0}}\left(i\right)-\frac{V\left(i\right)G''\left(V\left(i\right),i\right)}{G'\left(V\left(i\right),i\right)}\hat{V}_{t_{0}}\left(i\right)+\frac{1}{\sigma}\left(\hat{e}_{t}^{t_{0},HtM}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t}\right)+\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t}\right\}\end{aligned}$$

- First-order condition with respect to expenditure. We only need to rexpress the impact of individual consumption on profits.
- Log linearizing $\sum_k\int\left(\partial_e e_k\partial_e d_k+d_k^I\partial_e\tilde{Y}_k\right)\left(p_{k,t}(j)-MC_{k,t}\right)dj$ we have:

$$\begin{aligned}\frac{d\sum_k\int\left(\partial_e e_k\partial_e d_k+d_k^I\partial_e\tilde{Y}_k\right)\left(p_{k,t}(j)-MC_{k,t}\right)dj}{\sum_k\int\left(\partial_e e_k\partial_e d_k+d_k^I\partial_e\tilde{Y}_k\right)\left(p_{k,t}(j)-MC_{k,t}\right)dj}&=\left(\hat{P}_{k,t}+\hat{A}_{k,t}-\Omega_{N,k}\hat{W}_{t}+\sum_{l}\Omega_{k,l}\hat{P}_{l,t}\right)\mathcal{D}\left(P\right)\left(Id-\tilde{Q}\right)^{-1}\mathcal{D}^{-1}\left(P\right)\left[\partial_{e}e_{1},...,\partial_{e}e_{K}\right]^{T}\\&=\left[\partial_{e}e_{1},...,\partial_{e}e_{K}\right]^{T}\left(Id-\Omega\right)^{-1}\left[\left(\hat{P}_{k,t}+\hat{A}_{k,t}-\Omega_{N,k}\hat{W}_{t}-\sum_{l}\Omega_{k,l}\hat{P}_{l,t}\right)\right]\\&=-\hat{W}_{t}+\sum_{l}\partial_{e}e_{l}\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\end{aligned}$$

So we have:

$$\begin{aligned}&\frac{V\left(i\right)G''\left(V\left(i\right),i\right)}{G'\left(V\left(i\right),i\right)}\hat{V}_{t_{0}}\left(i\right)-\frac{1}{\sigma}\left(\hat{e}_{t}^{t_{0},u}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t}\right)-\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t}+\left(\lambda_{t}^{t_{0}}\left(i\right)-R\lambda_{t-1}^{t_{0}}\left(i\right)\right)\partial_{ee}v+\check{\zeta}_{t}^{t_{0},u}\left(i\right)W\partial_{ee}v\\&\quad+\check{\alpha}^{t_{0}}\left(i\right)+\sum_{s=0}^{t-t_{0}-1}\hat{R}_{t_{0}+s}-\check{\Xi}_{t}-\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)-\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)=0\\&\frac{V\left(i\right)G''\left(V\left(i\right),i\right)}{G'\left(V\left(i\right),i\right)}\hat{V}_{t_{0}}\left(i\right)-\frac{1}{\sigma}\left(\hat{e}_{t_{0}}^{t_{0},u}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t_{0}}\right)-\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t}+\left(\lambda_{t}^{t_{0}}\left(i\right)-R\lambda_{t-1}^{t_{0}}\left(i\right)\right)\partial_{ee}v+\check{\zeta}_{t}^{t_{0},u}\left(i\right)W\partial_{ee}v\\&\quad+\check{\alpha}^{t_{0}}\left(i\right)-\check{\Xi}_{t}-\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)-\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)=0\end{aligned}$$

For the expenditure of HtM households:

$$\begin{aligned}&\frac{V\left(i\right)G''\left(V\left(i\right),i\right)}{G'\left(V\left(i\right),i\right)}\hat{V}_{t_{0}}\left(i\right)-\frac{1}{\sigma}\left(\hat{e}_{t}^{t_{0},HtM}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t}\right)-\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t}+\check{\zeta}_{t}^{t_{0},HtM}\left(i\right)W\partial_{ee}v\\&\quad+\check{\aleph}_{t}^{t_{0}}\left(i\right)-\check{\Xi}_{t}-\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)-\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)=0\end{aligned}$$

- Finally for resetted prices, note that we have:

$$d\left(\sum_s (\beta\theta_k)^s \partial_{p(j^*)} Div_{t+s} - \sum_s (\beta\theta_k)^{s+1} \partial_{p(j^*)} Div_{t+1+s}\right) = \sum_l \left(\int e_l(i)\rho_{l,k}(i)\frac{1}{P_l P_k}di + \partial_{p_k(j^*)}\tilde{Y}_l\right) P_l\left(\hat{P}_{l,t} + \hat{A}_{l,t} - \hat{M}C_{l,t+s}\right)$$
$$+\frac{Y_k}{P_k}\bar{\epsilon}_k \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}\left(\beta\pi_{k,t+1} - \pi_{k,t}\right) + dE_{k,t}$$

Defining $\vartheta_k = \bar{\epsilon}_k \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}$, we obtain:

$$\left(\beta\check{\mu}_{k,t+1} - (1+((1-\delta)\beta))\check{\mu}_{k,t} + (1-\delta)\check{\mu}_{k,t-1}\right) = \lambda_k\check{\mu}_{k,t} - \sum_{l=1}^K \lambda_l \frac{\check{\mu}_{l,t}}{Y_l P_l}\int\left(\gamma_{e,l}(i)e_l(i)\rho_{l,k}(i)\right)di - \sum_{l=1}^K \lambda_l \frac{\check{\mu}_{l,t}}{P_l Y_l}\frac{P_k \mathcal{Y}_{k,l} Y_l}{A_l}$$
$$-\mathbb{E}_{\delta,t}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right) + \check{\zeta}_t^{t_0,u}(i)W\right]\partial_e v \partial_e e_k + \varphi(i)\check{\zeta}_t^{t_0,HtM}(i)W\partial_e v_i \partial_e e_k di\right)$$
$$+\mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\check{\alpha}^{t_0}(i) + \varphi(i)\check{\aleph}_t^{t_0}\right)\left[e_k(i) - \varsigma(i)E_k\right]di\right) + \mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\left[e_k(i) - \varsigma(i)E_k\right]di\sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s}\right) - \delta\int(1-\varphi(i))\check{\alpha}^t(i)b(i)di\bar{s}_k$$
$$-\delta\sum_{u=0}((1-\delta)\beta)^u\int\varphi(i)\check{\aleph}_{t+u}^t(i)b(i)di\bar{s}_k\frac{R-1}{R} + \delta\sum_{u=0}\frac{1}{R^u}\int\varphi(i)b(i)di\bar{s}_k\frac{\hat{R}_{t+u}}{R}$$
$$+\sum_{l,m}\frac{P_l P_k \partial_{P_k}\mathcal{Y}_{l,m}}{A_m}Y_m\left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_l \frac{W P_k \partial_{P_k}\mathcal{N}_l}{A_l}Y_l\hat{W}_t + \sum_l E_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) - P_k Y_k \vartheta_k\left(\pi_{k,t} - \beta\pi_{k,t+1}\right)$$

Solving the Labor market equation

The next step is to re-write the (infinite number of) linearized conditions, into a system of a limited number of equations and variables. Our first main equation is the optimality of the wage

$$0 = \mathbb{E}_{\delta,t}\int\left((1-\varphi(i))\check{\zeta}_t^{t_0,u}(i) + \varphi(i)\check{\zeta}_t^{t_0,HtM}(i)\right)W\partial_e v di - \sum_{k=1}^K \lambda_k\check{\mu}_{k,s}W\mathcal{N}_k + \sum_{k,l}\frac{P_l W \partial_W \mathcal{Y}_{l,k}}{A_k}Y_k\left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_k \frac{W^2\partial_W\mathcal{N}_k}{A_k}Y_k\hat{W}_t$$

Let us define the first component as $\tilde{Z}_t \equiv \mathbb{E}_{\delta,t}\int\left((1-\varphi(i))\check{\zeta}_t^{t_0,u}(i) + \varphi(i)\check{\zeta}_t^{t_0,HtM}(i)\right)W\partial_e v di$. Substituting out the Lagrange multipliers gives:

$$\tilde{Z}_t = \psi W N \check{\Xi}_t - \mathbb{E}_{\delta,t}\int\psi n(i)\frac{V(i)G''(V(i),i)}{G'(V(i),i)}\hat{V}_{t_0}(i)di + \mathbb{E}_{\delta,t}\int(1-\varphi(i))\psi W n(i)\left\{\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\hat{P}_{l,t_0} - \check{\alpha}^{t_0}(i)\right\}$$
$$+\varphi(i)\psi W n(i)\left\{\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i) - \sum_l s_l(i)\hat{P}_{l,t}\right) + \sum_l \partial_e e_l(i)\hat{P}_{l,t} - \check{\aleph}_t^{t_0}(i)\right\}di$$

Our first goal is to solve for $\check{\Xi}_t$, $\check{\alpha}^{t_0}(i)$ and $\check{\aleph}_t^{t_0}(i)$. Using the optimality of household's expenditure and substituting the $\check{\zeta}_t^{t_0,u}(i)$ term

$$\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_e v = \sigma e(i)\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - \sigma e(i)\left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\right) - \check{\zeta}_t^{t_0,u}(i)\,W\partial_e v + \sigma e(i)\,\check{\alpha}^{t_0}(i)$$

$$- \sigma e(i)\,\check{\Xi}_t - \sigma e(i)\left(\left(\hat{W}_t - \sum_l \partial_e e_l\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)$$

$$\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_e v = (\sigma e(i) + \psi n(i))\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - \left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\right)\right)$$

$$+ (\sigma e(i) + \psi n(i))\left(\check{\alpha}^{t_0}(i) - \check{\Xi}_t\right) - \sigma e(i)\left(\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)$$

So using, $\check{\lambda}_{t_0-1}^{t_0}(i) = 0\,, \frac{1}{R^t}\check{\lambda}_t^{t_0}(i) \to 0$, we have:

$$0 = (\sigma e(i) + \psi W n(i))\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - e(i)\left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\right)\right) + (\sigma e(i) + \psi W n(i))\,\check{\alpha}^{t_0}(i)$$

$$- \left(1 - \frac{1}{R}\right)(\sigma e(i) + \psi W n(i))\sum_{s=0}^{\infty}\frac{1}{R^s}\check{\Xi}_{t_0+s} - \sigma e(i)\left(1 - \frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s} - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t_0+s} + \hat{P}_{l,t_0+s}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t_0+s}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)$$

$$\check{\alpha}^{t_0}(i) = -\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) + \left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\,\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\right)$$

$$+ \sum_{s=0}^{\infty}\frac{1}{R^{s+1}}\Delta\Omega_{t_0+1+s} + \Omega_{t_0} + \frac{\sigma e(i)}{\sigma e(i) + \psi n(i)}\left(1 - \frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s} - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t_0+s} + \hat{P}_{l,t_0+s}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t_0+s}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)$$

Similarly for the budget constraint multiplier of HtM agents,

$$0 = (\sigma e(i) + \psi W n(i))\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - \left(\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,HtM}(i) - \sum_l s_l(i)\,\hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\right)\right)$$

$$+ (\sigma e(i) + \psi W n(i))\left(\check{\aleph}_t^{t_0}(i) - \check{\Xi}_t\right) - \sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)$$

$$\check{\aleph}_t^{t_0}(i) = -\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i) - \left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i) - \sum_l s_l(i)\,\hat{P}_{l,t}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t}\right)\right)$$

$$+ \check{\Xi}_t + \frac{\sigma e(i)}{\sigma e(i) + \psi W n(i)}\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)$$

Next, define

$$\tilde{\Lambda}_t \equiv \mathbb{E}_{\delta,t}\left((1 - \varphi(i))\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_e v\right)$$

Using the first-order condition for the optimality of expenditure of unconstrained households to substitute $\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_e v$ and the overlapping generation sturcture, the evolution of $\tilde{\Lambda}_t$ is given by:

$$\tilde{\Lambda}_{t+1}-(1-\delta)\tilde{\Lambda}_t=-(1-\delta)\left\{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di\Delta\check{\Xi}_{t+1}+\int(1-\varphi(i))\sigma e(i)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l(i)\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^K\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}(i)\partial_e e_k(i)\right)\right\}di$$
$$+\delta\int(1-\varphi(i))\check{\lambda}_{t+1}^{t+1}(i)\partial_e v di.$$

Next to derive the evolution of $\int(1-\varphi(i))\check{\lambda}_{t+1}^{t+1}(i)\partial_e v di$ we define

$$\tilde{\Lambda}_t^0\equiv\int(1-\varphi(i))\lambda_t^t(i)\partial_e v di-\tilde{\Lambda}_t.$$

The evolution of $\tilde{\Lambda}_t$ in terms of $\tilde{\Lambda}_t^0$ is simply:

$$\tilde{\Lambda}_{t+1}-\tilde{\Lambda}_t=-\left\{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di\Delta\check{\Xi}_{t+1}+\int(1-\varphi(i))\sigma e(i)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l(i)\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^K\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}(i)\partial_e e_k(i)\right)\right\}di+\frac{\delta}{1-\delta}\tilde{\Lambda}_{t+1}^0.$$

While using the definition of $\lambda_t^t(i)$from the FOC for expenditure, the evolution of $\tilde{\Lambda}_t^0$ is given by:

$$\begin{aligned}\int(1-\varphi(i))\lambda_t^t\partial_e v di&=(\sigma e(i)+\psi Wn(i))\left(\frac{V(i)G''(V(i))}{G'(V(i))}\hat{V}_t(i)-\left(\frac{1}{\sigma}\left(\hat{e}_t^{t,u}(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)+\sum_l\partial_e e_l(i)\hat{P}_{l,t}\right)\right)\\&\quad+(\sigma e(i)+\psi Wn(i))\left(\check{\alpha}^t(i)-\check{\Xi}_t\right)-\sigma e(i)\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^K\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\partial_e e_k(i)\right)\\&=\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))\,di\sum_{s=0}^\infty\frac{1}{R^{s+1}}\Delta\Omega_{t+1+s}\\&\quad+\int(1-\varphi(i))\sigma e(i)\sum_{s=0}^\infty\frac{1}{R^s}\left(\Delta\hat{W}_{t+1+s}-\sum_l\partial_e e_l(i)\Delta\left(\tilde{A}_{l,t+1+s}+\hat{P}_{l,t+1+s}\right)+\frac{1}{P_kY_k}\sum_{k=1}^K\lambda_k\Delta\check{\mu}_{k,t+1+s}\gamma_{e,k}(i)\partial_e e_k(i)\right)di\\\Rightarrow\tilde{\Lambda}_t^0-\frac{1}{(1-\delta)R}\tilde{\Lambda}_{t+1}^0&=-\left(1-\frac{1}{R}\right)\tilde{\Lambda}_t\end{aligned}$$

Coming back to $\tilde{Z}_t$, and defining a new variable $\tilde{\tilde{Z}}_t$ which captures the contribution of the unconstrained households to the variable $\tilde{Z}_t$, we can write

$$\begin{aligned}\tilde{Z}_t&=\psi WN\check{\Xi}_t-\mathbb{E}_{\delta,t}\int\psi Wn(i)\frac{V(i)G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)\,di\\&\quad+\mathbb{E}_{\delta,t}\int(1-\varphi(i))\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i)-\sum_l s_l(i)\hat{P}_{l,t_0}\right)+\sum_l\partial_e e_l(i)\hat{P}_{l,t_0}-\check{\alpha}^{t_0}(i)\right\}+\varphi(i)\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)+\sum_l\partial_e e_l(i)\hat{P}_{l,t}-\check{\aleph}_t^{t_0}(i)\right\}di\\\tilde{\tilde{Z}}_t&\equiv\tilde{Z}_t+\int\varphi(i)\psi Wn(i)\left(\frac{\sigma e(i)}{\sigma e(i)+\psi n(i)}\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^K\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\partial_e e_k(i)\right)\right)di\\&=\int(1-\varphi(i))\psi Wn(i)di\check{\Xi}_t+\mathbb{E}_{\delta,t}\int(1-\varphi(i))\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i)-\sum_l s_l(i)\hat{P}_{l,t_0}\right)+\sum_l\partial_e e_l(i)\hat{P}_{l,t_0}-\frac{V(i)G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)-\check{\alpha}^{t_0}(i)\right\}.\end{aligned}$$

The evolution of $\tilde{\tilde{Z}}_t$ is given by (using the fact that the second term in the definition of $\tilde{\tilde{Z}}_t$ is independent of t):

$$\tilde{\tilde{Z}}_{t+1}-(1-\delta)\,\tilde{\tilde{Z}}_{t}=(1-\delta)\int(1-\varphi(i))\,Wn(i)\,\psi di\Delta\check{\Xi}_{t+1}$$
$$+\delta\left(\int(1-\varphi(i))\,\psi Wn(i)di\check{\Xi}_{t+1}+\int(1-\varphi(i))\,\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t+1,u}(i)-\sum_{l}s_{l}(i)\,\hat{P}_{l,t+1}\right)+\sum_{l}\partial_{e}e_{l}(i)\,\hat{P}_{l,t+1}-\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t+1}(i)-\check{\alpha}^{t+1}(i)\right\}\right).$$

The second line correspond to the contribution of unconstrained households born at $t+1$ to $\tilde{\tilde{Z}}_{t+1}$. To characterize the dynamics of this term, we define:

$$\tilde{\tilde{Z}}_{t+1}^{0}\equiv\int(1-\varphi(i))\,\psi Wn(i)di\check{\Xi}_{t+1}+\int(1-\varphi(i))\,\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t+1,u}(i)-\sum_{l}s_{l}(i)\,\hat{P}_{l,t+1}\right)+\sum_{l}\partial_{e}e_{l}(i)\,\hat{P}_{l,t+1}-\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t+1}(i)-\check{\alpha}^{t+1}(i)\right\}-\tilde{\tilde{Z}}_{t+1}$$

Using the definition of $\tilde{\tilde{Z}}_{t+1}^{0}$, the joint evolution of $\tilde{\tilde{Z}}_{t}$ and $\tilde{\tilde{Z}}_{t}^{0}$ is given by:

$$\tilde{\tilde{Z}}_{t+1}-\tilde{\tilde{Z}}_{t}=\int(1-\varphi(i))\,Wn(i)\,\psi di\Delta\check{\Xi}_{t+1}+\frac{\delta}{1-\delta}\tilde{\tilde{Z}}_{t+1}^{0}$$
$$\tilde{\tilde{Z}}_{t}^{0}-\frac{1}{(1-\delta)\,R}\tilde{\tilde{Z}}_{t+1}^{0}=-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,\psi Wn(i)\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t+s}\gamma_{e,k}(i)\,\partial_{e}e_{k}(i)\right)di-\left(1-\frac{1}{R}\right)\tilde{\tilde{Z}}_{t}$$

Finally, define

$$Z_{t}\equiv\tilde{\tilde{Z}}_{t}+\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\tilde{\Lambda}_{t}$$
$$+\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\int(1-\varphi(i))\,\sigma e(i)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_{e}e_{k}(i)\right)di$$
$$Z_{t}^{0}\equiv\tilde{\tilde{Z}}_{t}^{0}+\tilde{\Lambda}_{t}^{0}$$

The dynamics of Z_t are characterized by the following two equations:

$$Z_{t+1}-Z_{t}=\frac{\delta}{1-\delta}Z_{t+1}^{0}$$

$$Z_{t}^{0}-\frac{1}{(1-\delta)\,R}Z_{t+1}^{0}+\left(1-\frac{1}{R}\right)Z_{t}$$
$$=\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,\sigma e(i)\left(\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}-\frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t+s}\gamma_{e,k}(i)\,\partial_{e}e_{k}(i)\right)di$$

And we can rewrite our original equation characterizing the optimality of the nominal wage as:

$$\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,s}W\mathcal{N}_{k}(\boldsymbol{P}_{t},W_{t})=Z_{t}-\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\tilde{\Lambda}_{t}$$
$$-\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\int(1-\varphi(i))\,\sigma e(i)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_{e}e_{k}(i)\right)di$$
$$-\int\varphi(i)\,\psi Wn(i)\left(\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_{e}e_{k}(i)\right)\right)di+\sum_{k,l}\frac{P_{l}W\partial_{W}\mathcal{Y}_{l,k}}{A_{k}}Y_{k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{k}\frac{W^{2}\partial_{W}\mathcal{N}_{k}}{A_{k}}Y_{k}\hat{W}_{t}$$

Next, we derive the evolution of $\tilde{\Lambda}_t$. The optimality of $\hat{R}_t$ allows us to express $\tilde{\Lambda}_t$ in terms of the Lagrange multipliers of the budget constraints of unconstrained and HtM households:

$$-\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\lambda}_{t}^{t_0}\left(i\right)\left(i\right)\partial_{e}v\left(e\left(i\right),\boldsymbol{P}\right)\right)=\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t}^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}_{t}^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)$$

$$\begin{aligned}-\tilde{\Lambda}_{t}=&\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t}^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}_{t}^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)\\&-\left(1-\delta\right)R\mathbb{E}_{\delta,t-1}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t-1}^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}_{t-1}^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)\end{aligned}$$

Define

$$\tilde{A}_{b,t}\equiv\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t}^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}_{t}^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)$$

Using our formulas for $\check{\alpha}_{t}^{t_0}$ and $\check{\aleph}_{t}^{t_0}$ derived above, the evolution of $\tilde{A}_{b,t}$ is given by:

$$\begin{aligned}\tilde{A}_{b,t+1}-\left(1-\delta\right)\tilde{A}_{b,t}=&\left(1-\delta\right)\hat{R}_{t}\int\left(1-\varphi\left(i\right)\right)\frac{b\left(i\right)}{R}di+\left(1-\delta\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di\Delta\check{\Xi}_{t+1}\\&+\left(1-\delta\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\Delta\hat{W}_{t+1}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\Delta\tilde{A}_{l,t+1}+\Delta\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_{k}\Delta\check{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)di\\&+\left(1-\delta\right)\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\frac{1}{R}b\left(i\right)\Delta\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t_0,HtM}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t+1}\right)+\Delta\sum_{l}\partial_{e}e_{l}\left(i\right)\hat{P}_{l,t+1}\right)\\&+\delta\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\check{\aleph}_{t+1}^{t+1}\left(i\right)\right)\frac{1}{R}b\left(i\right)di\end{aligned}$$

The last line gives the contribution of the newborn households, to characterize its evolution, we define:

$$\tilde{A}_{b,t+1}^{0}=\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\check{\aleph}_{t+1}^{t+1}\left(i\right)\right)\frac{1}{R}b\left(i\right)di-\tilde{A}_{b,t+1}$$

The decisions of households born at t in terms of expenditure at t and their change in welfare at t (usingRoy's identity) are given by:

$$\hat{e}_{t}^{t_0,HtM}\left(i\right)-\sum_{l}s_{l}\left(i\right)\hat{P}_{l,t}=\frac{\sigma}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\hat{R}_{t}\frac{b\left(i\right)}{R}+Wn(i)\left(\psi\hat{W}_{t}+\sum\bar{s}_{k}\tilde{A}_{k,t}\right)-\sum_{k}\left(e(i)\left(s_{k}(i)-\bar{s}_{k}\right)+\psi Wn(i)\partial_{e}e_{k}(i)\right)\hat{P}_{k,t}+\left(1-\frac{1}{R}\right)b(i)\sum_{l}\bar{s}_{l}\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right\}$$

$$\begin{aligned}dV_{t_0}\left(i\right)&=\partial_{e}v\sum_{s=0}\frac{1}{R_s}\left\{\frac{b\left(i\right)}{R}\hat{R}_{t+s}+\hat{W}_{t+s}Wn\left(i\right)-\sum_{k}e_{k}\left(i\right)\hat{P}_{k,t}+\varsigma\left(i\right)\sum_{k}E_{k}\left(\hat{P}_{k,t+s}+\tilde{A}_{k,t+s}-\hat{W}_{t+s}\right)+\frac{R-1}{R}b\left(i\right)\sum_{k}\bar{s}_{k}\hat{P}_{k,t_0}\right\}\\&=\partial_{e}v\sum_{s=0}\frac{1}{R_s}\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_{k}\left(s_{k}\left(i\right)-\bar{s}_{k}\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,t+s}\right\}\end{aligned}$$

$$dV_{-}\left(i\right)=\partial_{e}v\left(\sum_{s=0}\frac{1}{R_s}\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{s}-\pi_{cpi,1+s}\right)-e\left(i\right)\sum_{k}\left(s_{k}\left(i\right)-\bar{s}_{k}\right)\hat{P}_{k,s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,s}\right\}-b\left(i\right)\sum_{k}\bar{s}_{k}\hat{P}_{k,0}\right)$$

$$\begin{aligned}\hat{e}_{t}^{t,u}-\sum_{k}s_{k}(i)\hat{P}_{k,t}=&-\sigma\sum\frac{1}{R^{s+1}}\left(\hat{R}_{t+s}-\sum_{k}\partial_{e}e_{k}(i)\pi_{k,t+s+1}\right)\\&+\frac{\left(1-\frac{1}{R}\right)\sigma}{\left(\sigma e(i)+\psi Wn(i)\right)}\sum\frac{1}{R^{s}}\left(\frac{b(i)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+s+1}\right)+\psi Wn(i)\hat{W}_{t+s}-\sum_{k}\left(e(i)\left(s_{k}(i)-\bar{s}_{k}\right)+\psi Wn(i)\partial_{e}e_{k}(i)\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,t+s}\right)\end{aligned}$$

.

$$\hat{e}_0^{-,u} - \sum_k s_k(i)\hat{P}_{k,t} = -\sigma \sum \frac{1}{R^{s+1}} \left(\hat{R}_{t+s} - \sum_k \partial_e e_k(i) \pi_{k,t+s+1} \right)$$
$$+ \frac{\left(1-\frac{1}{R}\right)\sigma}{(\sigma e(i) + \psi Wn(i))} \left\{ \sum \frac{1}{R^s} \left(\frac{b(i)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+s+1}\right) + \psi Wn(i)\hat{W}_{t+s} - \sum_k \left(e(i)\left(s_k(i) - \bar{s}_k\right) + \psi Wn(i)\partial_e e_k(i)\right)\hat{P}_{k,t+s} + Wn\left(i\right)\sum_k \bar{s}_k \tilde{A}_{k,t+s} \right) - b\left(i\right)\sum_k \bar{s}_k \hat{P}_{k,0} \right\}$$

Using these expressions, $\check{\alpha}_t^t$ can be rewritten as:

$$\check{\alpha}_t^t\left(i\right) = \left(-\frac{\partial_e v G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)} + \frac{\left(1-\frac{1}{R}\right)}{(\sigma e(i) + \psi Wn(i))} \right) \sum_{s=0} \frac{1}{R_s} \left\{ \frac{b\left(i\right)}{R}\left(\hat{R}_{t+s} - \pi_{cpi,t+1+s}\right) - e\left(i\right)\sum_k \left(s_k\left(i\right) - \bar{s}_k\right)\hat{P}_{k,t+s} + Wn\left(i\right)\sum_k \bar{s}_k \tilde{A}_{k,t+s} \right\}$$
$$+ \left(1-\frac{1}{R}\right)\sum \frac{1}{R^s}\hat{W}_{t+s} - \sum \frac{1}{R^{s+1}}\hat{R}_{t+s} + \sum_{s=0}^{\infty} \frac{1}{R^{s+1}}\Delta\Omega_{t+1+s} + \check{\Xi}_t$$
$$+ \frac{\sigma e\left(i\right)}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \left(1-\frac{1}{R}\right) \sum_{s=0}^{\infty} \frac{1}{R^s} \left(-\sum_l \partial_e e_l\left(i\right)\tilde{A}_{l,t+s} + \frac{1}{P_k Y_k}\sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t+s}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right) \right)$$

The evolution of $\tilde{A}_{b,t}$ and $\tilde{A}_{b,t}^0$ is therefore characterized by:

$$\tilde{A}_{b,t+1} - \tilde{A}_{b,t} = \hat{R}_t \int \left(1-\varphi\left(i\right)\right)\frac{b\left(i\right)}{R}di + \int \varphi\left(i\right)\frac{1}{R}b\left(i\right)di\Delta\check{\Xi}_{t+1} + \int \varphi\left(i\right)\frac{b\left(i\right)}{R}\Delta\hat{W}_{t+1}di$$
$$+ \int \varphi\left(i\right)\frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \left(-\sum_l \partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \Delta\check{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right) \right) di$$
$$+ \mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\frac{b\left(i\right)}{R}\frac{1}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \left\{ \frac{b\left(i\right)}{R}\left(\Delta\hat{R}_{t+1} - \left(R-1\right)\pi_{cpi,t+1}\right) + Wn(i)\sum \bar{s}_k \Delta\tilde{A}_{k,t+1} - \sum_k e(i)\left(s_k(i) - \bar{s}_k\right)\pi_{k,t+1} \right\} \right) + \frac{\delta}{1-\delta}\tilde{A}_{b,t+1}^0$$

$$\tilde{A}_{b,t}^0 - \frac{1}{\left(1-\delta\right)R}\tilde{A}_{b,t+1}^0 + \left(1-\frac{1}{R}\right)\tilde{A}_{b,t} = \int \frac{b\left(i\right)}{R}\left(-\frac{\partial_e v G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)} + \frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \right) \left\{ \frac{b\left(i\right)}{R}\left(\hat{R}_{t+s} - \pi_{cpi,t+1+s}\right) - e\left(i\right)\sum_k\left(s_k\left(i\right) - \bar{s}_k\right)\hat{P}_{k,t+s} + Wn\left(i\right)\sum_k \bar{s}_k\tilde{A}_{k,t+s} \right\} di$$
$$+ \left(1-\frac{1}{R}\right)\int \frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \left(-\sum_l \partial_e e_l\left(i\right)\tilde{A}_{l,t} + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right) \right) di$$

Using the relationship between $\tilde{\Lambda}_t$ and $\check{\Xi}_t$, we obtain:

$$
\left(1-\frac{\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t+1}-\left(1-\frac{\left(1-\delta+R\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t}-\frac{\left(1-\delta\right)R\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}A_{b,t-1}-\frac{\delta}{1-\delta}A^{0}_{b,t+1}=
$$

$$
-\sum\left(\int\varphi\left(i\right)\frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)\partial_{e}e_{l}\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_{e}e_{l}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\Delta\tilde{A}_{l,t+1}
$$

$$
+\sum\left(\int\varphi\left(i\right)\frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\gamma_{e,k}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{\lambda_{k}}{P_{k}Y_{k}}\lambda_{k}\Delta\check{\mu}_{k,t+1}
$$

$$
+\left(\int\varphi\left(i\right)\frac{\left(\frac{b(i)}{R}\right)^{2}}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\left(\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\right)^{2}}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)
$$

$$
+\left(\int\varphi\left(i\right)\frac{\frac{b(i)}{R}Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\int\left(1-\varphi\left(i\right)\right)Wn(i)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\sum\bar{s}_{k}\Delta\tilde{A}_{k,t+1}
$$

$$
-\sum\left(\int\varphi\left(i\right)\frac{\frac{b(i)}{R}e(i)\left(s_{k}(i)-\bar{s}_{k}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_{k}(i)-\bar{s}_{k}\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\pi_{k,t+1}
$$

$$
A^{0}_{b,t}-\frac{1}{\left(1-\delta\right)R}A^{0}_{b,t+1}+\left(1-\frac{1}{R}\right)A_{b,t}=\int\frac{b\left(i\right)}{R}\left(-\frac{\partial_{e}vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_{k}\left(s_{k}\left(i\right)-\bar{s}_{k}\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,t+s}\right\}di
$$

$$
+\left(1-\frac{1}{R}\right)\int\frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_{l}\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,t}+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)di
$$

And the original wage equation in terms of Z_t and $A_{b,t}$ is

$$
\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,s}W\mathcal{N}_{k}(\boldsymbol{P}_{t},W_{t})=Z_{t}+\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi n\left(i\right)\right)di}\left(A_{b,t}-(1-\delta)RA_{b,t-1}\right)
$$

$$
-\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi n\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)di
$$

$$
-\int\varphi\left(i\right)\psi Wn(i)\left(\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)\right)di
$$

$$
+\sum_{k,l}\frac{P_{l}W\partial_{W}\mathcal{Y}_{l,k}}{A_{k}}Y_{k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{k}\frac{W^{2}\partial_{W}\mathcal{N}_{k}}{A_{k}}Y_{k}\hat{W}_{t}
$$

Solving the Price Setting equation

The second set of main equations are given by the optimality of price setting:

$$\begin{aligned}
\left(\beta\check{\mu}_{k,t+1}-\left(1+\left((1-\delta)\beta\right)\right)\check{\mu}_{k,t}+(1-\delta)\check{\mu}_{k,t-1}\right) &= \lambda_k\check{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{Y_lP_l}\int\left(\gamma_{e,l}(i)\,e_l(i)\,\rho_{l,k}(i)\right)di-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_lY_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}\\
&-\mathbb{E}_{\delta,t}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}_t^{t_0}(i)-R\check{\lambda}_{t-1}^{t_0}(i)\right)+\check{\zeta}_t^{t_0,u}(i)\,W\right]\partial_e v\partial_e e_k(i)+\varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\,W\partial_e v\partial_e e_k(i)\,di\right)\\
&+\mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\,\check{\alpha}_t^{t_0}(i)+\varphi(i)\,\check{\aleph}_t^{t_0}(i)\right)\left[e_k(i)-\varsigma(i)\,E_k\right]di\right)-\delta\int(1-\varphi(i))\,\check{\alpha}^t(i)\,b(i)\,di\bar{s}_k\\
&-\delta\sum_{u=0}\left((1-\delta)\beta\right)^u\int\varphi(i)\,\check{\aleph}_{t+u}^t(i)\,b(i)\,di\bar{s}_k\frac{R-1}{R}+\delta\sum_{u=0}\frac{1}{R^u}\int\varphi(i)\,b(i)\,di\bar{s}_k\frac{\hat{R}_{t+u}}{R}\\
&+\sum_{l,m}\frac{P_lP_k\partial_{P_k}\mathcal{Y}_{l,m}}{A_m}Y_m\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_l\frac{WP_k\partial_{P_k}\mathcal{N}_l}{A_l}Y_l\hat{W}_t+\sum_l E_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-Y_k\vartheta_k\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)
\end{aligned}$$

As in the previous subsection the goal here is to derive the dynamics of the components of these equation using only a finite number of variables.
First note that we have

$$\begin{aligned}
\mathbb{E}_{\delta,t}\left(\int\varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\,W\partial_e v\partial_e e_k(i)\,di\right) &= \mathbb{E}_{\delta,t}\left(\int\varphi(i)\,\partial_e e_k(i)\,\psi Wn(i)\left\{\check{\Xi}_t-\check{\aleph}_t^{t_0}(i)-\frac{G''(V(i))}{G'(V(i))}dV_{t_0}(i)+\frac{1}{\sigma}e_t^{t_0,HtM}(i)+\partial_e e_l(i)\cdot\hat{P}_{l,t}\right\}di\right)\\
&= -\mathbb{E}_{\delta,t}\left(\int\varphi(i)\,\partial_e e_k(i)\,\psi Wn(i)\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)di\right)\\
&= -\int\varphi(i)\,\partial_e e_k(i)\,\psi Wn(i)\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)di
\end{aligned}$$

Next, we solve for $\tilde{L}_{k,t}\equiv\mathbb{E}_{\delta,t}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}_t^{t_0}(i)-R\check{\lambda}_{t-1}^{t_0}(i)\right)+\check{\zeta}_t^{t_0,u}(i)\right]\partial_e v\partial_e e_k(i)\right)$, that we can re-express as:

$$\tilde{L}_{k,t}=\mathbb{E}_{\delta,t}\left(\int(1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\left\{\frac{G''(V(i))}{G'(V(i))}-\frac{1}{\sigma(i)}\hat{e}_{t_0}^{t_0,u}(i)-\partial_e e_l(i)\cdot\hat{P}_{l,t_0}+\check{\alpha}_{t_0}(i)-\left(\hat{W}_t-\partial_e e_l(i)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)-\check{\Xi}_t-\frac{1}{P_lY_l}\sum_{l=1}^{K}\lambda_l\check{\mu}_{l,t}\gamma_{e,l}(i)\,\partial_e e_l(i)\right\}di\right)$$

Define

$$\begin{aligned}
\tilde{\tilde{L}}_{k,t} &= \mathbb{E}_{\delta,t}\left((1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\left\{\frac{G''(V(i))}{G'(V(i))}-\frac{1}{\sigma(i)}\hat{e}_{t_0}^{t_0,u}(i)-\partial_e e_l(i)\cdot\hat{P}_{l,t_0}+\check{\alpha}_{t_0}(i)-\check{\Xi}_t\right\}\right)\\
&= \tilde{L}_{k,t}+\int(1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\left\{\left(\hat{W}_t-\partial_e e_l(i)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)+\frac{1}{P_lY_l}\sum_{l=1}^{K}\lambda_l\check{\mu}_{l,t}\gamma_{e,l}(i)\,\partial_e e_l(i)\right\}di
\end{aligned}$$

We have

$$\Delta\tilde{\tilde{L}}_{k,t+1}=-\int(1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di\Delta\check{\Xi}_{t+1}-\frac{\lambda_l}{P_lY_l}\sum_{l=1}^{K}\int(1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,\partial_e e_l(i)\,di\Delta\check{\mu}_{l,t+1}+\frac{\delta}{1-\delta}\tilde{\tilde{L}}_{k,t+1}^0$$

With

$$\tilde{\tilde{L}}_{k,t}^0-\frac{1}{R(1-\delta)}\tilde{\tilde{L}}_{k,t+1}^0+\left(1-\frac{1}{R}\right)\tilde{\tilde{L}}_{k,t}=\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\frac{\sigma e(i)}{\psi Wn(i)+\sigma e(i)}\sigma e(i)\,\partial_e e_k(i)\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_lY_l}\sum_{l=1}^{K}\lambda_l\check{\mu}_{l,t}\gamma_{e,l}(i)\,\partial_e e_l(i)\right)$$

Define

$$L_{k,t} \equiv \tilde{\tilde{L}}_{k,t} - \frac{\int (1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di}{\int (1-\varphi(i))\,(\sigma e(i)+\psi W n(i))\,di}\tilde{\Lambda}_t$$
$$- \frac{\int (1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di}{\int (1-\varphi(i))\,(\sigma e(i)+\psi W n(i))\,di}\int (1-\varphi(i))\,\sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)di$$
$$L^0_{k,t} \equiv \tilde{\tilde{L}}^0_{k,t} - \frac{\int (1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di}{\int (1-\varphi(i))\,(\sigma e(i)+\psi W n(i))\,di}\tilde{\Lambda}^0_t$$

We have

$$\Delta L_{k,t+1} = \frac{\delta}{1-\delta}L^0_{k,t+1}$$
$$L^0_{k,t} - \frac{1}{R(1-\delta)}L^0_{k,t+1} + \left(1-\frac{1}{R}\right)L_{k,t} = \left(1-\frac{1}{R}\right)\int (1-\varphi(i))\,\frac{\sigma e(i)}{\psi w n(i)+\sigma e(i)}\sigma e(i)\,\partial_e e_k(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right) + \frac{1}{P_l Y_l}\sum_{l=1}^{K}\lambda_l \check{\mu}_{l,t}\gamma_{e,l}(i)\,\partial_e e_l(i)\right)$$
$$- \frac{\int (1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di}{\int (1-\varphi(i))\,(\sigma e(i)+\psi W n(i))\,di}\int (1-\varphi(i))\,\sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)di$$

The resetting equation becomes

$$\left(\beta\check{\mu}_{k,t+1} - \left(1+\left((1-\delta)\beta\right)\right)\check{\mu}_{k,t} + (1-\delta)\,\check{\mu}_{k,t-1}\right) = \lambda_k\check{\mu}_{k,t} - \sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{Y_l P_l}\int\left(\gamma_{e,l}(i)\,e_l(i)\,\rho_{l,k}(i)\right)di - \sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_l Y_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}$$
$$- L_{k,t} + \int \varphi(i)\,\partial_e e_k \psi n(i)\,\frac{\sigma e(i)}{\sigma e(i)+\psi W n(i)}\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)di$$
$$+ \int (1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\left\{\left(\hat{W}_t - \partial_e e_l(i)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right) + \frac{1}{P_l Y_l}\sum_{l=1}^{K}\lambda_l\check{\mu}_{l,t}\gamma_{e,l}(i)\,\partial_e e_l(i)\right\}di$$
$$- \frac{\int (1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di}{\int (1-\varphi(i))\,(\sigma e(i)+\psi W n(i))\,di}\tilde{\Lambda}_t$$
$$- \frac{\int (1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di}{\int (1-\varphi(i))\,(\sigma e(i)+\psi W n(i))\,di}\int (1-\varphi(i))\,\sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)di$$
$$+ \mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\,\check{\alpha}^{t_0}_t(i) + \varphi(i)\,\check{\aleph}^{t_0}_t(i)\right)\left[e_k(i) - \varsigma(i)\,E_k\right]di\right) - \delta\int (1-\varphi(i))\,\check{\alpha}^t(i)\,b(i)\,di\bar{s}_k$$
$$- \delta\sum_{u=0}\left((1-\delta)\beta\right)^u\int \varphi(i)\,\check{\aleph}^t_{t+u}(i)\,b(i)\,di\bar{s}_k\frac{R-1}{R} + \delta\sum_{u=0}\frac{1}{R^u}\int \varphi(i)\,b(i)\,di\bar{s}_k\frac{\hat{R}_{t+u}}{R}$$
$$+ \sum_{l,m}\frac{P_l P_k \partial_{P_k}\mathcal{Y}_{l,m}}{A_m}Y_m\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right) + \sum_l \frac{W P_k \partial_{P_k}\mathcal{N}_l}{A_l}Y_l\hat{W}_t + \sum_l E_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right) - Y_k\vartheta_k\left(\pi_{k,t} - \beta\pi_{k,t+1}\right)$$

Next note that we have

$$\mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\,\check{\alpha}^{t_0}_t(i) + \varphi(i)\,\check{\aleph}^{t_0}_t(i)\right)\left[e_k(i) - \varsigma(i)\,E_k\right]di\right) = \mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\,\check{\alpha}^{t_0}_t(i) + \varphi(i)\,\check{\aleph}^{t_0}_t(i)\right)\left[e_k(i) - e(i)\bar{s}_k\right]di\right)$$
$$+ \left(1-\frac{1}{R}\right)\bar{s}_k\mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\,\check{\alpha}^{t_0}_t(i) + \varphi(i)\,\check{\aleph}^{t_0}_t(i)\right)b(i)di\right)$$

Define

$$\tilde{A}_{e_k,t} \equiv \mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_t^{t_0}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_t^{t_0}\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\right)$$

We have

$$\begin{aligned}\tilde{A}_{e_k,t+1}-\left(1-\delta\right)\tilde{A}_{e_k,t} &= \left(1-\delta\right)\hat{R}_t\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di+\left(1-\delta\right)\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\Delta\breve{\Xi}_{t+1}\\&+\left(1-\delta\right)\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}+\Delta\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^K\lambda_k\Delta\breve{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di\\&+\left(1-\delta\right)\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\Delta\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t_0,HtM}\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t+1}\right)+\Delta\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t+1}\right)\\&+\delta\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t+1}^{t+1}+\varphi\left(i\right)\breve{\aleph}_{t+1}^{t+1}\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\end{aligned}$$

Define

$$\tilde{A}^0_{e_k,t+1}=\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t+1}^{t+1}\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di-\tilde{A}_{e_k,t+1}$$

We have

$$\begin{aligned}\tilde{A}_{e_k,t+1}-\tilde{A}_{e_k,t} &= \hat{R}_t\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\Delta\breve{\Xi}_{t+1}+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\Delta\hat{W}_{t+1}di\\&+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^K\lambda_k\Delta\breve{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di\\&+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{1}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\frac{b\left(i\right)}{R}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)+Wn(i)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}-\sum_k e(i)\left(s_k(i)-\bar{s}_k\right)\pi_{k,t+1}\right\}\right)+\frac{\delta}{1-\delta}\tilde{A}^0_{e_k,t+1}\end{aligned}$$

$$\begin{aligned}\tilde{A}^0_{e_k,t}-\frac{1}{\left(1-\delta\right)R}\tilde{A}^0_{e_k,t+1}+\left(1-\frac{1}{R}\right)\tilde{A}_{e_k,t} &= +\left(1-\frac{1}{R}\right)\int\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}+\frac{1}{P_kY_k}\sum_{k=1}^K\lambda_k\breve{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di\\&+\int\left(e_k(i)-e(i)\bar{s}_k\right)\left(-\frac{\partial_e vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}di\end{aligned}$$

Define

$$A_{e_k,t}\equiv\tilde{A}_{e_k,t}+\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}di\tilde{\Lambda}_t$$

$$A^0_{e_k,t}\equiv\tilde{A}^0_{e_k,t}+\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}di\tilde{\Lambda}^0_t$$

We have

$$A_{e_k,t+1}-A_{e_k,t}=\hat{R}_t\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di$$

$$-\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l\left(i\right)\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)\right\}$$

$$+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\Delta\hat{W}_{t+1}di$$

$$+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di$$

$$+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{1}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\frac{b\left(i\right)}{R}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)+Wn(i)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}-\sum_k e(i)\left(s_k(i)-\bar{s}_k\right)\pi_{k,t+1}\right\}\right)+\frac{\delta}{1-\delta}A^0_{e_k,t+1}$$

$$A^0_{e_k,t}-\frac{1}{\left(1-\delta\right)R}A^0_{e_k,t+1}+\left(1-\frac{1}{R}\right)A_{e_k,t}=\left(1-\frac{1}{R}\right)\int\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di$$

$$+\int\left(e_k(i)-e(i)\bar{s}_k\right)\left(-\frac{\partial_e vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}di$$

Using the evolution of the wage (derived from the output gap Euler)

$$\left(1-\varphi^N\right)\psi\Delta\hat{W}_{t+1}=\left(\left(1-\varphi^N\right)\left(\sigma+\psi\right)-\sigma\left(1-\frac{1}{R}\right)\int\varphi\left(i\right)\frac{b(i)}{RE}\right)\hat{R}_t-\left(1-\varphi^N\right)\bar{s}_k\cdot\Delta\tilde{A}_{l,t+1}$$

$$+\int\varphi\left(i\right)\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)-\frac{e(i)}{E}\sum_k\left(\left(s_k(i)-\bar{s}_k\right)\right)\pi_{k,t+1}\right\}di$$

$$+\sigma\left(\sum_k-\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\partial_e e_k(i)\pi_{k,t+1}\right)$$

we have

$$A_{e_k,t+1}-A_{e_k,t}-\frac{\delta}{1-\delta}A^0_{e_k,t+1}=-\sum\left(\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)\partial_e e_l\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\Delta\tilde{A}_{l,t+1}$$

$$+\sum\left(\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\gamma_{e,l}\left(i\right)\partial_e e_l\left(i\right)di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\gamma_{e,l}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{\lambda_l}{P_lY_l}\lambda_l\Delta\check{\mu}_{l,t+1}$$

$$+\left(\int\varphi\left(i\right)\frac{\left(\frac{b(i)}{R}\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\left(\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\right)^2}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)$$

$$+\left(\int\varphi\left(i\right)\frac{\left(e_k(i)-e(i)\bar{s}_k\right)Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)Wn(i)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\sum\bar{s}_l\Delta\tilde{A}_{l,t+1}$$

$$-\sum\left(\int\varphi\left(i\right)\frac{\left(e_k(i)-e(i)\bar{s}_k\right)e(i)\left(s_l(i)-\bar{s}_l\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\pi_{k,t+1}$$

$$A^0_{e_k,t} - \frac{1}{(1-\delta)R} A^0_{e_k,t+1} + \left(1-\frac{1}{R}\right) A_{e_k,t} = \int (e_k(i) - e(i)\bar{s}_k) \left(- \frac{\partial_e v G''(V(i))}{G'(V(i))} + \frac{\left(1-\frac{1}{R}\right)}{\sigma e(i) + \psi W n(i)} \right) \left\{ \frac{b(i)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+1+s} \right) - e(i) \sum_k (s_k(i) - \bar{s}_k) \hat{P}_{k,t+s} + W n(i) \sum_k \bar{s}_k \tilde{A}_{k,t+s} \right\}$$
$$+ \left(1-\frac{1}{R}\right) \int (e_k(i) - e(i)\bar{s}_k) \frac{\sigma e(i)}{\sigma e(i) + \psi W n(i)} \left(- \sum_l \partial_e e_l(i) \tilde{A}_{l,t} + \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t} \gamma_{e,k}(i) \partial_e e_k(i) \right) di$$

The resetting equation becomes

$$\left(\beta \check{\mu}_{k,t+1} - (1 + ((1-\delta)\beta)) \check{\mu}_{k,t} + (1-\delta) \check{\mu}_{k,t-1}\right) = \lambda_k \check{\mu}_{k,t} - \sum_{l=1}^{K} \lambda_l \frac{\check{\mu}_{l,t}}{Y_l P_l} \int (\gamma_{e,l}(i) e_l(i) \rho_{l,k}(i)) di - \sum_{l=1}^{K} \lambda_l \frac{\check{\mu}_{l,t}}{P_l Y_l} \frac{P_k \mathcal{Y}_{k,l} Y_l}{A_l}$$
$$- L_{k,t} + \int \varphi(i) \partial_e e_k \psi n(i) \frac{\sigma e(i)}{\sigma e(i) + \psi W n(i)} \left(\hat{W}_t - \sum_l \partial_e e_l(i) \left(\tilde{A}_{l,t} + \hat{P}_{l,t} \right) + \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t} \gamma_{e,k}(i) \partial_e e_k(i) \right) di$$
$$+ \int (1 - \varphi(i)) \partial_e e_k(i) \sigma e(i) \left\{ \left(\hat{W}_t - \partial_e e_l(i) \cdot \left(\tilde{A}_{l,t} + \hat{P}_{l,t} \right) \right) + \frac{1}{P_l Y_l} \sum_{l=1}^{K} \lambda_l \check{\mu}_{l,t} \gamma_{e,l}(i) \partial_e e_l(i) \right\} di$$
$$- \frac{\int (1-\varphi(i)) \partial_e e_k(i) \sigma e(i) di}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) di} \tilde{\Lambda}_t$$
$$- \frac{\int (1-\varphi(i)) \partial_e e_k(i) \sigma e(i) di}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) di} \int (1-\varphi(i)) \sigma e(i) \left(\hat{W}_t - \sum_l \partial_e e_l(i) \left(\tilde{A}_{l,t} + \hat{P}_{l,t} \right) + \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t} \gamma_{e,k}(i) \partial_e e_k(i) \right) di$$
$$+ A_{e_k,t} - \frac{\int \varphi(i) (e_k(i) - e(i)\bar{s}_k)}{\int (1-\varphi(i)) (\sigma e(i) + \psi W n(i)) di} di \tilde{\Lambda}_t + \left(1-\frac{1}{R}\right) \bar{s}_k \mathbb{E}_{\delta,t} \left(\int \left((1-\varphi(i)) \check{\alpha}^{t_0}_t(i) + \varphi(i) \check{\aleph}^{t_0}_t(i) \right) b(i) di \right)$$
$$- \delta \int (1-\varphi(i)) \check{\alpha}^t(i) b(i) di \bar{s}_k - \delta \sum_{u=0} ((1-\delta)\beta)^u \int \varphi(i) \check{\aleph}^t_{t+u}(i) b(i) di \bar{s}_k \frac{R-1}{R} + \delta \sum_{u=0} \frac{1}{R^u} \int \varphi(i) b(i) di \bar{s}_k \frac{\hat{R}_{t+u}}{R}$$
$$+ \sum_{l,m} \frac{P_l P_k \partial_{P_k} \mathcal{Y}_{l,m}}{A_m} Y_m \left(\hat{P}_{l,t} + \tilde{A}_{l,t,} \right) + \sum_l \frac{W P_k \partial_{P_k} \mathcal{N}_l}{A_l} Y_l \hat{W}_t + \sum_l E_l \bar{\rho}_{l,k} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} \right) - Y_k \vartheta_k (\pi_{k,t} - \beta \pi_{k,t+1})$$

Finally, define

$$\mathcal{A}_{b,t} = \int (1-\varphi(i)) \check{\alpha}^t(i) b(i) di + \sum_{u=0} ((1-\delta)\beta)^u \int \varphi(i) \check{\aleph}^t_{t+u}(i) b(i) di \frac{R-1}{R} - \sum_{u=0} \frac{1}{R^u} \int \varphi(i) b(i) di \bar{s}_k \frac{\hat{R}_{t+u}}{R}$$

We have

$$\mathcal{A}_{b,t} - \frac{1}{R} \mathcal{A}_{b,t+1} = \left(1-\frac{1}{R}\right) \int b(i) \frac{\sigma e(i)}{\sigma e(i) + \psi W n(i)} \left(- \sum_l \partial_e e_l(i) \tilde{A}_{l,t} + \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t} \gamma_{e,k}(i) \partial_e e_k(i) \right) di$$
$$+ \int b(i) \left(- \frac{\partial_e v G''(V(i))}{G'(V(i))} + \frac{\left(1-\frac{1}{R}\right)}{\sigma e(i) + \psi W n(i)} \right) \left\{ \frac{b(i)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+1+s} \right) - e(i) \sum_k (s_k(i) - \bar{s}_k) \hat{P}_{k,t+s} + W n(i) \sum_k \bar{s}_k \tilde{A}_{k,t+s} \right\} di$$

Optimal Policy Equations: summary

We now collect and slightly simplify the optimal policy equations derived above.[16] We obtain a system of $6 + K * 6$ equations in the following variables: $\hat{W}_t, Z_t, Z^0_t$, $A_{b,t}, A^0_{b,t}$ and $\mathcal{A}_{b,t}$ and $M_{k,t}, \check{\mu}_{k,t}, L_{k,t}, L^0_{k,t}$ and $A_{e_k,t}, A^0_{e_k,t}$. These replace the interest rate rule. Note that the evolution of $\hat{W}_t$ is given by

$$\begin{aligned}\left(1-\varphi^N\right)\psi\Delta\hat{W}_{t+1} = &\left(\left(1-\varphi^E\right)\sigma + \left(1-\varphi^N\right)\psi\right)\hat{R}_t - \left(1-\varphi^N\right)\bar{s}_k \cdot \Delta\tilde{A}_{l,t+1} \\ &+ \int \varphi(i)\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1} - (R-1)\pi_{cpi,t+1}\right) - \frac{e(i)}{E}\sum_k\left((s_k(i)-\bar{s}_k)\right)\pi_{k,t+1}\right\}di \\ &+ \sigma\left(\sum_k - \int(1-\varphi(i))\frac{e(i)}{E}\partial_e e_k(i)\pi_{k,t+1}\right)\end{aligned}$$

We renormalize $\check{\mu}_{k,t} \equiv E\check{\mu}_{k,t}$, and define $g(i) = \left(-\frac{\partial_e v G''(V(i))\frac{R}{R-1}}{G'(V(i))} + \frac{1}{\sigma e(i)+\psi W n(i)}\right)E$. Our Labor Market equation equation becomes

$$\begin{aligned}\sum_{k=1}^{K}\Omega_{N,k}\lambda_k\check{\mu}_{k,t} = &Z_t + \frac{\int(1-\varphi(i))Wn(i)\psi di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di}\frac{1}{R}\left(A_{b,t}-(1-\delta)RA_{b,t-1}\right) \\ &- \left(\frac{\int(1-\varphi(i))\frac{Wn(i)}{WN}\psi di\int(1-\varphi(i))\sigma e(i)di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di} + \int\varphi(i)\psi\frac{Wn(i)}{WN}\frac{\sigma e(i)}{\sigma e(i)+\psi n(i)}di\right)\hat{W}_t \\ &+ \sum_k\left(\frac{\int(1-\varphi(i))\frac{Wn(i)}{WN}\psi di\int(1-\varphi(i))\sigma e(i)\partial_e e_k(i)di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di} + \int\varphi(i)\psi\frac{Wn(i)}{WN}\frac{\sigma e(i)\partial_e e_k(i)}{\sigma e(i)+\psi n(i)}di\right)\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right) \\ &- \sum_k\lambda_k\frac{E_k}{P_kY_k}\left(\frac{\int(1-\varphi(i))Wn(i)\psi di\int(1-\varphi(i))\sigma\frac{e(i)}{E_k}\gamma_{e,k}(i)\partial_e e_k(i)di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di} + \int\varphi(i)\frac{\sigma\psi Wn(i)}{\sigma e(i)+\psi n(i)}\frac{e(i)}{E_k}\gamma_{e,k}(i)\partial_e e_k(i)di\right)\check{\mu}_{k,t} \\ &+ \sum_{k,l}\frac{P_lW\partial_W\mathcal{Y}_{l,k}Y_k}{A_kE}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right) + \sum_k\frac{W\partial_W\mathcal{N}_k}{A_kN}Y_k\hat{W}_t\end{aligned}$$

With

$$Z_{t+1} - Z_t = \frac{\delta}{1-\delta}Z^0_{t+1}$$

$$\begin{aligned}Z^0_{t-1} - \frac{1}{(1-\delta)R}Z^0_t + \left(1-\frac{1}{R}\right)Z_{t-1} = &\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\sigma\frac{e(i)}{E}\left(\frac{\int(1-\varphi(i))Wn(i)\psi di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di} - \frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)di\hat{W}_{t-1} \\ &- \sum_k\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\sigma\frac{e(i)}{E}\partial_e e_k(i)\left(\frac{\int(1-\varphi(i))Wn(i)\psi di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di} - \frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)di\left(\tilde{A}_{k,t-1}+\hat{P}_{k,t-1}\right) \\ &+ \sum_k\frac{\lambda_kE_k}{P_kY_k}\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\sigma\frac{e(i)}{E_k}\partial_e e_k(i)\gamma_{e,k}(i)\left(\frac{\int(1-\varphi(i))Wn(i)\psi di}{\int(1-\varphi(i))(\sigma e(i)+\psi Wn(i))di} - \frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)di\check{\mu}_{k,t-1}\end{aligned}$$

[16] We also derived these equations in a different way, starting from the welfare loss function derived in the next appendix.

And

$$\left(1-\frac{\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t+1}-\left(1-\frac{\left(1-\delta+R\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t}-\frac{\left(1-\delta\right)R\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}A_{b,t-1}-\frac{\delta}{1-\delta}A^0_{b,t+1}=$$

$$-\sum_l\left(\int\varphi\left(i\right)\frac{b\left(i\right)}{E}\frac{\sigma e\left(i\right)\partial_e e_l\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{E}di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\Delta\tilde{A}_{l,t+1}$$

$$+\sum_k\left(\int\varphi\left(i\right)b\left(i\right)\frac{\sigma}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\frac{e\left(i\right)}{E_k}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di+\frac{\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e(i)}{E_k}\partial_e e_k\left(i\right)\gamma_{e,k}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{\lambda_k E_k}{P_k Y_k}\Delta\breve{\mu}_{k,t+1}$$

$$+\left(\int\varphi\left(i\right)\frac{\left(b\left(i\right)\right)^2}{ER}\frac{1}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{1}{ER}\frac{\left(\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\right)^2}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)$$

$$+\left(\int\varphi\left(i\right)\frac{\frac{b(i)}{E}Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{E}di\int\left(1-\varphi\left(i\right)\right)Wn(i)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}$$

$$-\sum\left(\int\varphi\left(i\right)\frac{\frac{b(i)}{E}e(i)\left(s_k(i)-\bar{s}_k\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{E}di\int\left(1-\varphi\left(i\right)\right)e(i)\left(s_k(i)-\bar{s}_k\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\pi_{k,t+1}$$

$$A^0_{b,t-1}-\frac{1}{\left(1-\delta\right)R}A^0_{b,t}+\left(1-\frac{1}{R}\right)A_{b,t-1}=\left(1-\frac{1}{R}\right)\int\frac{b\left(i\right)}{E}g(i)\frac{b\left(i\right)}{RE}di\left(\hat{R}_{t-1}-\pi_{cpi,t}\right)$$

$$-\left(1-\frac{1}{R}\right)\sum_k\int\frac{b\left(i\right)}{E}g(i)\frac{e\left(i\right)}{E}\left(s_k\left(i\right)-\bar{s}_k\right)di\hat{P}_{k,t-1}$$

$$+\left(1-\frac{1}{R}\right)\int\frac{b\left(i\right)}{E}g(i)\frac{Wn\left(i\right)}{WN}di\sum_k\bar{s}_k\tilde{A}_{k,t-1}$$

$$-\left(1-\frac{1}{R}\right)\sum_k\int\frac{b\left(i\right)}{E}\frac{\sigma e\left(i\right)\partial_e e_k\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di\tilde{A}_{k,t-1}$$

$$+\left(1-\frac{1}{R}\right)\sum_k\frac{\lambda_k E_k}{P_k Y_k}\int\frac{b\left(i\right)}{E_k}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di\breve{\mu}_{k,t-1}$$

The Price resetting equation becomes

$$M_{k,t}=\breve{\mu}_{k,t}-\left(1-\delta\right)\breve{\mu}_{k,t-1}$$

$$\beta M_{k,t+1} - M_{k,t} = \lambda_k \check{\mu}_{k,t} - \sum_{l=1}^{K} \frac{\lambda_l E_l}{P_l Y_l} \int \gamma_{e,l}(i) \frac{e_l(i)}{E_l} \rho_{l,k}(i)\, di \check{\mu}_{l,t} - \sum_{l=1}^{K} \lambda_l \frac{\check{\mu}_{l,t}}{P_l Y_l} \frac{P_k \mathcal{Y}_{k,l} Y_l}{A_l} + \sum_{l,m} \frac{P_l P_k \partial_{P_k} \mathcal{Y}_{l,m}}{A_m E} Y_m \left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_l \frac{P_k \partial_{P_k} \mathcal{N}_l}{A_l N} Y_l \hat{W}_t$$

$$+ \sum_l \bar{s}_l \bar{\rho}_{l,k} \left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) - \frac{P_k Y_k}{E} \vartheta_k \left(\pi_{k,t} - \beta \pi_{k,t+1}\right) - L_{k,t} + \left(\frac{\int (1-\varphi(i)) \partial_e e_k(i) \sigma e(i)\, di \int (1-\varphi(i)) \psi \frac{Wn(i)}{WN}}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} + \int \varphi(i) \partial_e e_k(i) \frac{\psi Wn(i)}{WN} \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} di \right) \hat{W}_t$$

$$- \sum_l \left(\int \varphi(i) \partial_e e_l(i) \partial_e e_k(i) \psi \frac{Wn(i)}{WN} \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} - \frac{\int (1-\varphi(i)) \partial_e e_k(i) \sigma e(i)\, di \int (1-\varphi(i)) \sigma \frac{e(i)}{E} \partial_e e_l(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} + \int (1-\varphi(i)) \sigma \frac{e(i)}{E} \partial_e e_k(i) \partial_e e_l(i)\, di \right) \left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)$$

$$+ \sum_{l=1}^{K} \frac{\lambda_l E_l}{P_l Y_l} \left(\int (1-\varphi(i)) \partial_e e_k(i) \frac{\sigma e(i)}{E_l} \gamma_{e,l}(i) \partial_e e_l(i)\, di + \int \varphi(i) \partial_e e_k \psi Wn(i) \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} \frac{1}{E_l} \gamma_{e,l}(i) \partial_e e_l(i)\, di \right) \check{\mu}_{l,t}$$

$$- \sum_{l=1}^{K} \frac{\lambda_l E_l}{P_l Y_l} \left(\frac{\int (1-\varphi(i)) \partial_e e_k(i) \sigma e(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \int (1-\varphi(i)) \partial_e e_l(i) \frac{\sigma e(i)}{E_l} \gamma_{e,l}(i)\, di \right) \check{\mu}_{l,t}$$

$$+ \left(\frac{\int (1-\varphi(i)) \partial_e e_k(i) \sigma e(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} + \frac{\int \varphi(i) (e_k(i) - e(i) \bar{s}_k)}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \frac{1}{R} \left(A_{b,t} - R(1-\delta) A_{b,t-1}\right) + \left(1 - \frac{1}{R}\right) \bar{s}_k A_{b,t} - \delta \bar{s}_k \mathcal{A}_{b,t} + A_{e_k,t}$$

With $\vartheta_k = \bar{\epsilon}_k \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}$ and:

$$\Delta L_{k,t+1} = \frac{\delta}{1-\delta} L^0_{k,t+1}$$

$$L^0_{k,t-1} - \frac{1}{R(1-\delta)} L^0_{k,t} + \left(1 - \frac{1}{R}\right) L_{k,t-1} = \left(1 - \frac{1}{R}\right) \int (1-\varphi(i)) \sigma \frac{e(i)}{E} \left(\frac{\sigma e(i) \partial_e e_k(i)}{\sigma e(i) + \psi Wn(i)} - \frac{\int (1-\varphi(i)) \sigma e(i) \partial_e e_k(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) di \hat{W}_{t-1}$$

$$- \sum_l \left(1 - \frac{1}{R}\right) \int (1-\varphi(i)) \sigma \frac{e(i)}{E} \partial_e e_l(i) \left(\frac{\sigma e(i) \partial_e e_k(i)}{\sigma e(i) + \psi Wn(i)} - \frac{\int (1-\varphi(i)) \sigma e(i) \partial_e e_k(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) di \left(\tilde{A}_{l,t-1} + \hat{P}_{l,t-1}\right)$$

$$+ \sum_l \frac{\lambda_l E_l}{P_l Y_l} \left(1 - \frac{1}{R}\right) \int (1-\varphi(i)) \sigma \frac{e(i)}{E_l} \partial_e e_l(i) \gamma_{e,l}(i) \left(\frac{\sigma e(i) \partial_e e_k(i)}{\sigma e(i) + \psi Wn(i)} - \frac{\int (1-\varphi(i)) \sigma e(i) \partial_e e_k(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) di \check{\mu}_{l,t-1}$$

$$A_{e_k,t+1} - A_{e_k,t} - \frac{\delta}{1-\delta} A^0_{e_k,t+1} = -\sum \left(\int \varphi(i) \frac{e(i)}{E} (s_k(i) - \bar{s}_k) \frac{\sigma e(i) \partial_e e_l(i)}{\sigma e(i) + \psi Wn(i)} di + \frac{\int (1-\varphi(i)) \frac{e(i)}{E} (s_k(i) - \bar{s}_k)\, di \int (1-\varphi(i)) \sigma e(i) \partial_e e_l(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \Delta \tilde{A}_{l,t+1}$$

$$+ \sum \left(\int \varphi(i) e(i) (s_k(i) - \bar{s}_k) \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} \frac{1}{E_l} \gamma_{e,l}(i) \partial_e e_l(i)\, di + \frac{\int (1-\varphi(i)) e(i) (s_k(i) - \bar{s}_k)\, di \int (1-\varphi(i)) \sigma e(i) \partial_e e_l(i) \frac{1}{E_l} \gamma_{e,l}(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \frac{\lambda_l E_l}{P_l Y_l} \lambda_l \Delta \check{\mu}_{l,t+1}$$

$$+ \left(\int \varphi(i) \frac{\left(\frac{b(i)}{R}\right) \frac{e(i)}{E} (s_k(i) - \bar{s}_k)}{\sigma e(i) + \psi Wn(i)} di + \frac{\int (1-\varphi(i)) b(i)\, di \int (1-\varphi(i)) \frac{e(i)}{E} (s_k(i) - \bar{s}_k)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \left(\Delta \hat{R}_{t+1} - (R-1) \pi_{cpi,t+1}\right)$$

$$+ \left(\int \varphi(i) \frac{\frac{e(i)}{E} (s_k(i) - \bar{s}_k) Wn(i)}{\sigma X_t \equiv e(i) + \psi Wn(i)} di + \frac{\int (1-\varphi(i)) \frac{e(i)}{E} (s_k(i) - \bar{s}_k)\, di \int (1-\varphi(i)) Wn(i) di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \sum \bar{s}_l \Delta \tilde{A}_{l,t+1}$$

$$- \sum \left(\int \varphi(i) \frac{\frac{e(i)}{E} (s_k(i) - \bar{s}_k) e(i) (s_l(i) - \bar{s}_l)}{\sigma e(i) + \psi Wn(i)} di + \frac{\int (1-\varphi(i)) \frac{e(i)}{E} (s_k(i) - \bar{s}_k)\, di \int (1-\varphi(i)) e(i) (s_l(i) - \bar{s}_l)}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \pi_{l,t+1}$$

$$
\begin{aligned}
A^0_{e_k,t-1} - \frac{1}{(1-\delta)R}A^0_{e_k,t} + \left(1-\frac{1}{R}\right)A_{e_k,t-1} = & \left(1-\frac{1}{R}\right)\int \frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)g(i)\frac{b(i)}{RE}di\left(\hat{R}_{t-1}-\pi_{cpi,t}\right) \\
& - \left(1-\frac{1}{R}\right)\sum_l \int \frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)g(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)di\hat{P}_{l,t-1} \\
& + \left(1-\frac{1}{R}\right)\int \frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)g(i)\frac{Wn(i)}{WN}di\sum_l \bar{s}_l\tilde{A}_{l,t-1} \\
& - \left(1-\frac{1}{R}\right)\sum_l \int \frac{e(i)\left(s_k(i)-\bar{s}_k\right)}{E}\frac{\sigma e(i)\,\partial_e e_l(i)}{\sigma e(i)+\psi Wn(i)}di\tilde{A}_{l,t-1} \\
& + \sum_k \lambda_l \frac{E_l}{P_l Y_l}\left(1-\frac{1}{R}\right)\int e(i)\left(s_k(i)-\bar{s}_k\right)\frac{\sigma e(i)/E_l}{\sigma e(i)+\psi Wn(i)}\gamma_{e,l}(i)\,\partial_e e_l(i)\,di\check{\mu}_{l,t-1}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_{b,t} - \frac{1}{R}\mathcal{A}_{b,t+1} = & \left(1-\frac{1}{R}\right)\int \frac{b(i)}{E}g(i)\frac{b(i)}{RE}di\left(\hat{R}_t-\pi_{cpi,t+1}\right) \\
& - \left(1-\frac{1}{R}\right)\sum_k \int \frac{b(i)}{E}g(i)\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)di\hat{P}_{k,t} \\
& + \left(1-\frac{1}{R}\right)\int \frac{b(i)}{E}g(i)\frac{e(i)}{E}\frac{Wn(i)}{WN}di\sum_k \bar{s}_k\tilde{A}_{k,t} \\
& - \left(1-\frac{1}{R}\right)\sum_k \int \frac{b(i)}{E}\frac{\sigma e(i)\,\partial_e e_k(i)}{\sigma e(i)+\psi Wn(i)}di\tilde{A}_{k,t} \\
& + \left(1-\frac{1}{R}\right)\sum_k \lambda_k \frac{E_k}{P_k Y_k}\int \frac{b(i)}{E_k}\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\gamma_{e,k}(i)\,\partial_e e_k(i)\,di\check{\mu}_{k,t}
\end{aligned}
$$

E.2 Optimal policy: proofs analytical results Section 5

Result 7

We first show that under (A.1) and (A.2), optimal policy attempts to jointly stabilize the output gap $\tilde{\mathcal{Y}}_t$ and an inflation index $\pi_t^\theta \equiv \sum_{k=1}^K \frac{\bar{s}_k \vartheta_k}{\vartheta} \pi_{k,t}$ (with $\vartheta = \sum_{k=1}^K \bar{s}_k \vartheta_k$), in the sense that optimal policy can equivalently be derived by solving:

$$\inf_{\{\tilde{\mathcal{Y}}_t, \pi_t^\theta\}_{t\geq 0}} \mathbb{E}_0 \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\frac{\sigma+\psi}{\sigma\psi} \tilde{\mathcal{Y}}_t^2 + \vartheta \left(\pi_t^\theta\right)^2 \right)$$

$$s.t. \quad \mathbb{E}_t \pi_{t+1}^\theta - R\pi_t^\theta = -R\kappa\tilde{\mathcal{Y}}_t - R\lambda^\theta \hat{\mathcal{W}}_t^\theta$$

We consider the general case in which $\bar{\epsilon}_k, \bar{\epsilon}_k^s$ and θ_k may vary across sectors. Note that under inner CES preferences the inflation index can be rewritten $\pi_t^\theta = \frac{1}{\sum_{k=1}^K \frac{\bar{s}_k \bar{\epsilon}_k}{\lambda_k}} \sum_{k=1}^K \frac{\bar{s}_k \bar{\epsilon}_k}{\lambda_k} \pi_{k,t}$, π_t^θ overweight larger sectors (higher $\bar{s}_k$) more rigid sectors (lower λ_k) and more elastic sector (higher $\bar{\epsilon}_k$). If we have that θ_k and $\bar{\epsilon}_k$ are equal across sector then π_t^θ is simply the CPI index. The NKPC associated with π_t^θ is given by

$$\mathbb{E}_t \pi_{t+1}^\theta - R\pi_t^\theta = -R\kappa\tilde{\mathcal{Y}}_t - R\lambda^\theta \hat{\mathcal{W}}_t^\theta$$

Where $\hat{\mathcal{W}}_t^\theta$ is a wedge that is independent from monetary policy (Result 1 of the positive section). Under the optimal policy, we have $\tilde{\mathcal{Y}}_0 = -\frac{\sigma\psi}{\sigma+\psi}\kappa\vartheta\pi_0^\theta$, and $\tilde{\mathcal{Y}}_t$ partially absorbs the wedge: if $\hat{\mathcal{W}}_t^\theta \geq 0$ at all t then $\tilde{\mathcal{Y}}_t \leq 0$ at all t. In addition, when ϑ goes to infinity keeping all other parameters fixed, we have $\tilde{\mathcal{Y}}_t = -\frac{\lambda^\theta}{\kappa}\hat{\mathcal{W}}_t^\theta$ and $\pi_t^\theta = 0$: the output gap fully absorbs the wedge. Inversely, when ϑ goes to 0, $\tilde{\mathcal{Y}}_t = 0$: the inflation index fully absorbs the wedge.

Note that under **(A.2)** we have $A_{b,t} = A_{b,t}^0 = 0$ for all t and since $e(i) = Wn(i)$, $Z_t = Z_t^0 = 0$ for all t. Defining $\check{\mu}_t \equiv \sum_{k=1}^K \check{\mu}_{k,t}$ We can rewrite the Labor Market equation as:

$$\frac{\sigma\psi}{\sigma+\psi}\kappa\check{\mu}_t = -\tilde{\mathcal{Y}}_t.$$

The system of price resetting equations becomes

$$M_{k,t} = \check{\mu}_{k,t} - (1-\delta)\check{\mu}_{k,t-1},$$

$$\begin{aligned}
\beta\mathbb{E}_t M_{k,t+1} - M_{k,t} = {} & -\bar{s}_k\vartheta_k\left(\pi_{k,t} - \beta\mathbb{E}_t\pi_{k,t+1}\right) \\
& + \overline{\partial_e e_k}\frac{\sigma\psi}{\sigma+\psi}\kappa\sum_{l=1}^K \check{\mu}_{l,t} + \overline{\partial_e e_k}\tilde{\mathcal{Y}}_t - L_{k,t} + A_{e_k,t} \\
& - \sum_{l=1}^K \lambda_l \int \gamma_{e,l}(i)\frac{e_l(i)}{E_l}\rho_{l,k}(i)\,di\check{\mu}_{l,t} + \sum_l \bar{s}_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) - \sigma\sum_l\left(\int\frac{e}{E}\partial_e e_k(i)\,\partial_e e_l(i)\,di - \overline{\partial_e e_k}\,\overline{\partial_e e_l}\right)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) \\
& \lambda_k\check{\mu}_{k,t} - \overline{\partial_e e_k}\sum_{l=1}^K\lambda_l\check{\mu}_{l,t} + \sum_{l=1}^K\lambda_l\left(\int\partial_e e_k(i)\frac{\sigma e(i)}{E_l}\gamma_{e,l}(i)\,\partial_e e_l(i)\,di - \overline{\partial_e e_k}\int\partial_e e_l(i)\frac{\sigma e(i)}{E_l}\gamma_{e,l}(i)\,di\right)\check{\mu}_{l,t}.
\end{aligned}$$

Note that we have:

$$\sum_{k=1}^K L_{k,t} = Z_t,$$

$$\sum_{k=1}^K e_l(i)\,\rho_{l,k}(i) = \sum_{k=1}^K \bar{s}_l\bar{\rho}_{l,k} = 0,$$

$$\sum_{k=1}^K s_k(i) = \sum_{k=1}^K \bar{s}_k = \sum_{k=1}^K \overline{\partial_e e_k} = \sum_{k=1}^K \partial_e e_k(i) = 1.$$

We therefore have $\sum_{k=1}^{K} L_{k,t} = \sum_{k=1}^{K} A_{e_k,t} = \sum_{k=1}^{K} A^0_{e_k,t} = 0$. Defining $M_t \equiv \sum_{k=1}^{K} M_{k,t}$,we have

$$M_t = \check{\mu}_t - (1-\delta)\,\check{\mu}_{t-1},$$

$$\beta \mathbb{E}_t M_{k,t+1} - M_{k,t} = -\sum_{k=1}^{K} \bar{s}_k \vartheta_k \left(\pi_{k,t} - \beta \mathbb{E}_t \pi_{k,t+1}\right),$$

Defining;

$$\vartheta \equiv \sum_{k=1}^{K} \bar{s}_k \vartheta_k,$$

$$\pi_t^\theta \equiv \sum_{k=1}^{K} \frac{\bar{s}_k \vartheta_k}{\vartheta} \pi_{k,t},$$

$$\lambda^\theta \equiv \sum_{k=1}^{K} \frac{\bar{s}_k \vartheta_k}{\vartheta} \lambda_k,$$

$$\hat{\mathcal{W}}_t^\theta \equiv \sum_{k=1}^{K} \frac{\bar{s}_k \vartheta_k \lambda_k}{\vartheta \lambda^\theta} \mathcal{M}_{k,t} + \sum_{k=1}^{K} \left(\overline{\partial_e e_k} - \frac{\bar{s}_k \vartheta_k \lambda_k}{\vartheta \lambda^\theta}\right) \tilde{P}_{k,t},$$

the evolution of the output gap under optimal policy is determined by:

$$\tilde{\mathcal{Y}}_t = -\frac{\sigma\psi}{\sigma+\psi} \kappa \check{\mu}_t,$$

$$\check{\mu}_t - (1-\delta)\,\check{\mu}_{t-1} = \vartheta \pi_t^\theta,$$

$$\mathbb{E}_t \pi_{t+1}^\theta - R\pi_t^\theta = -R\kappa \tilde{\mathcal{Y}}_t - R\lambda^\theta \hat{\mathcal{W}}_t^\theta.$$

Note that we would obtain the same system of equation if the central bank were instead to solve:

$$\inf_{\{\tilde{\mathcal{Y}}_t, \pi_t^\theta\}_{t\geq 0}} \mathbb{E}_0 \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\frac{\sigma+\psi}{\sigma\psi} \tilde{\mathcal{Y}}_t^2 + \vartheta \left(\pi_t^\theta\right)^2\right)$$

$$s.t. \quad \mathbb{E}_t \pi_{t+1}^\theta - R\pi_t^\theta = -R\kappa \tilde{\mathcal{Y}}_t - R\lambda^\theta \hat{\mathcal{W}}_t^\theta.$$

In the special case in which$\bar{\epsilon}_k, \bar{\epsilon}_k^s$ and θ_k are common across sectors we obtain the problem stated in result 6. Denoting by $\beta^t \check{\check{\mu}}_t$ the Lagrange multiplier on the NKPC, the first-order conditions are:

$$\tilde{\mathcal{Y}}_t = -\frac{\sigma\psi}{\sigma+\psi} R\kappa \check{\check{\mu}}_t$$

$$\vartheta \pi_t^\theta = R\check{\check{\mu}}_t - \beta^{-1} R\check{\check{\mu}}_{t-1}$$

Redefining $\check{\mu}_t = R\check{\check{\mu}}_t$we obtain:

$$\tilde{\mathcal{Y}}_t = -\frac{\sigma\psi}{\sigma+\psi} \kappa \check{\mu}_t,$$

$$\check{\mu}_t - (1-\delta)\,\check{\mu}_{t-1} = \vartheta \pi_t^\theta,$$

$$\mathbb{E}_t \pi_{t+1}^\theta - R\pi_t^\theta = -R\kappa \tilde{\mathcal{Y}}_t - R\lambda^\theta \hat{\mathcal{W}}_t^\theta,$$

which is the same system.

Note that under (**A.1**) and (**A.2**), the wedge $\hat{\mathcal{W}}_t^\theta$ evolves independently of monetary policy. The OP system can be rewritten as

$$\mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \left((1-\delta) + R\left(1 + \frac{\sigma\psi}{\sigma+\psi} \vartheta \kappa^2\right)\right) \tilde{\mathcal{Y}}_t - (1-\delta)\, R\tilde{\mathcal{Y}}_{t-1} = R\lambda^\theta \frac{\sigma\psi}{\sigma+\psi} \vartheta \kappa \hat{\mathcal{W}}_t^\theta.$$

Defining

$$\mu_{\pm} \equiv \frac{(1-\delta)+R\left(1+\frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa^2\right) \pm \sqrt{\left((1-\delta)+R\left(1+\frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa^2\right)\right)^2 - 4(1-\delta)R}}{2}$$

and noting that we have $0<\mu_-<1-\delta<R<\mu_+$, we have

$$\tilde{\mathcal{Y}}_t = -\mathbb{E}_t \frac{\lambda^\theta \frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa}{(1-\delta)} \sum_{s=0}^{t} \mu_-^{t+1-s} \sum_{u=0}^{+\infty} \mu_+^{-u} \hat{\mathcal{W}}_{s+u}^\theta$$

We directly obtain that if $\hat{\mathcal{W}}_t^\theta \gtreqless 0$ for all t then $\tilde{\mathcal{Y}}_t \lesseqgtr 0$ for all t. In addition we have $\lim_{\vartheta\to\infty}\mu_+^{-1} = \lim_{\vartheta\to\infty}\mu_- = 0$ and $\mu_- = (1-\delta)/\left(\frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa\right) + o(1/\vartheta)$, so as ϑ goes to infinity keeping all other parameters fixed, we have

$$\tilde{\mathcal{Y}}_t = -\frac{\lambda^\theta}{\kappa}\hat{\mathcal{W}}_t^\theta$$

Inversely when ϑ goes to 0, the output gap goes to 0 and π_t^θ fully absorbs the wedge $\hat{\mathcal{W}}_t^\theta$.

Result 8

In addition to (**A.1**) and (**A.2**), we now assume that there are no endogenous markups ($\gamma_{e,k}(i) = 0$ for all i, k) and that sectoral shocks in k follow vanish geometrically $\hat{A}_{k,t} = \rho_a^t \hat{A}_{k,0}$. We derive analytical formulas for the evolution of $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}$ and π_t^θ and characterize their sign. First note that for aggregate shocks, we have $\tilde{\mathcal{Y}}_t = \pi_{mcpi,t} = \pi_t^\theta = 0$. If $\overline{\partial_e e_k} < \frac{\bar{s}_k\vartheta_k}{\vartheta}$ (note that if ϑ_k are equal across sector the condition simply characterize necessity), following a negative shock in sector k, $\tilde{\mathcal{Y}}_t$ is negative on impact and there t^* such that for $t\geq t^*$, $\tilde{\mathcal{Y}}_t$ is positive. $\pi_{mcpi,t}$ is negative on impact and there t^* such that for $t\geq t^*$, $\pi_{mcpi,t}$ is positive. π_t^θ is positive on impact and if δ is small enough there t^* such that for $t\geq t^*$, π_t^θ is positive. In net present value term, we have $\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t, \sum_{t\geq 0}\frac{1}{R^t}\pi_{mcpi,t} > 0$ and$\sum_{t\geq 0}\frac{1}{R^t}\pi_t^\theta < 0$ following a negative shock in k with $\overline{\partial_e e_k} < \frac{\bar{s}_k\vartheta_k}{\vartheta}$.

Under $(A.2)$ and $\gamma_{e,k}(i) = 0$, we have $\lambda_k = \frac{\sigma\psi}{\sigma+\psi}\kappa \equiv \lambda$ and $\mathcal{M}_{k,t} = 0$ for all k, so we have that the exogenous wedge is given by:

$$\hat{\mathcal{W}}_t^\theta = \tilde{P}_t^\Delta,$$

with $P_t^\Delta = \sum_{k=1}^K \left(\overline{\partial_e e_k} - \frac{\bar{s}_k\vartheta_k}{\vartheta}\right)\hat{P}_{k,t}$, $A_t^\Delta = \sum_{k=1}^K \left(\overline{\partial_e e_k} - \frac{\bar{s}_k\vartheta_k}{\vartheta}\right)\hat{A}_{k,t}$, $\tilde{P}_t^\Delta = P_t^\Delta + A_t^\Delta$. The relative price $\tilde{P}_t^\Delta$ satisfies

$$\mathbb{E}_t\hat{P}_{t+1}^\Delta - (1+R(1+\lambda))\hat{P}_t^\Delta + R\hat{P}_t^\Delta = R\lambda\rho_a^t\hat{A}_0^\Delta$$

Denoting the roots of the equation polynomial as $\nu_\pm$, we have

$$\nu_\pm = \frac{1+R(1+\lambda)\pm\sqrt{(1+R(1+\lambda))^2-4R}}{2},$$

with $0<\nu_-<1<R<\nu_+$. And $\hat{P}_t^\Delta$ is given by:

$$P_t^\Delta = -\frac{R\lambda}{(\nu_--\rho_a)(\nu_+-\rho_a)}\left(\nu_-^{t+1}-\rho_a^{t+1}\right)\hat{A}_0^\Delta$$

P_t^Δ is independent of policy and always has the same sign as $-\hat{A}_{k,0}$. The wedge is then given by:

$$\tilde{P}_t^\Delta = -\frac{1}{(\nu_--\rho_a)(\nu_+-\rho_a)}\left((R-\nu_-)(1-\nu_-)\nu_-^t - (R-\rho_a)(1-\rho_a)\rho_a^t\right)\hat{A}_0^\Delta.$$

Noting that $(R-x)(1-x)$ is positive and decreasing on $[0,1]$, we conclude that the wedge (independently of policy) initially has the same sign as $\hat{A}_0^\Delta$ for $t<t^*$ (with t^* the smallest t such that $(R-\rho_a)(1-\rho_a)\rho_a^t > (R-\nu_-)(1-\nu_-)\nu_-^t$ if $\rho_a>\nu_-$,such that $(R-\rho_a)(1-\rho_a)\rho_a^t < (R-\nu_-)(1-\nu_-)\nu_-^t$ if $\rho_a<\nu_-$) and thas the same sign as $-\hat{A}_{k,0}$ for $t\geq t^*$. Note that $t^*=1$ for transitory shocks, $t^*=\infty$ for permanent shocks.

Plugging this formula in our general expression for the output gap and using the NKPC for the indices π_t^θ and $\pi_{mcpi,t}$, and the definition of the nominal interest rate, we obtain:

$$\tilde{\mathcal{Y}}_t = \frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left\{\frac{(R-\rho_a)(1-\rho_a)}{(\rho_a-\mu_+)(\rho_a-\mu_-)}\left\{\rho_a^{t+1}-\mu_-^{t+1}\right\}-\frac{(R-\nu_-)(1-\nu_-)}{(\nu_--\mu_+)(\nu_--\mu_-)}\left\{\nu_-^{t+1}-\mu_-^{t+1}\right\}\right\}\hat{A}_0^\Delta$$

$$\pi_{mcpi,t} = \frac{(R\lambda)^2\vartheta\kappa}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left\{\frac{(1-\rho_a)(R-\rho_a)}{(\rho_a-\mu_+)(\rho_a-\mu_-)}\left\{\frac{\rho_a}{R-\rho_a}\rho_a^t-\frac{\mu_-}{R-\mu_-}\mu_-^t\right\}-\frac{(R-\nu_-)(1-\nu_-)}{(\nu_--\mu_+)(\nu_--\mu_-)}\left\{\frac{\nu_-}{R-\nu_-}\nu_-^t-\frac{\mu_-}{R-\mu_-}\mu_-^t\right\}\right\}\hat{A}_0^\Delta$$

$$\pi_t^\theta = -\frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left\{\frac{(R-\rho_a)(1-\rho_a)}{(\rho_a-\mu_+)(\rho_a-\mu_-)}\left\{(\rho_a-(1-\delta))\rho_a^t-(\mu_--(1-\delta))\mu_-^t\right\}-\frac{(R-\nu_-)(1-\nu_-)}{(\nu_--\mu_+)(\nu_--\mu_-)}\left\{(\nu_--(1-\delta))\nu_-^t-(\mu_--(1-\delta))\mu_-^t\right\}\right\}\hat{A}_0^\Delta$$

$$\begin{aligned}\hat{R}_t = & -\frac{1+\psi}{\sigma+\psi}(1-\rho_a)\rho_a^t\sum_k \bar{s}_k A_{k,0} - \frac{\psi}{\sigma+\psi}(1-\rho_a)\rho_a^t\hat{A}_0^\Delta \\ & -\frac{1}{\sigma}\frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left\{\frac{(R-\rho_a)(1-\rho_a)}{(\rho_a-\mu_+)(\rho_a-\mu_-)}\left\{(1-\rho_a)\rho_a^{t+1}-(1-\mu_-)\mu_-^{t+1}\right\}-\frac{(R-\nu_-)(1-\nu_-)}{(\nu_--\mu_+)(\nu_--\mu_-)}\left\{(1-\nu_-)\nu_-^{t+1}-(1-\mu_-)\mu_-^{t+1}\right\}\right\}\hat{A}_0^\Delta \\ & +\frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left\{\frac{(1-\rho_a)(R-\rho_a)}{(\rho_a-\mu_+)(\rho_a-\mu_-)}\left\{\frac{R\kappa\rho_a}{R-\rho_a}\rho_a^{t+1}-\frac{R\kappa\mu_-}{R-\mu_-}\mu_-^{t+1}\right\}-\frac{(R-\nu_-)(1-\nu_-)}{(\nu_--\mu_+)(\nu_--\mu_-)}\left\{\frac{R\kappa\nu_-}{R-\nu_-}\nu_-^{t+1}-\frac{R\kappa\mu_-}{R-\mu_-}\mu_-^{t+1}\right\}\right\}\hat{A}_0^\Delta\end{aligned}$$

For aggregate shocks we have $\hat{A}_0^\Delta = 0$ so $\tilde{\mathcal{Y}}_t = \pi_{mcpi,t} = \pi_t^\theta = 0$.
On impact, after some algebra, we obtain

$$\tilde{\mathcal{Y}}_0 = -\frac{R\lambda^2\vartheta}{(\nu_+-\rho_a)(\mu_+-\rho_a)(\mu_+-\nu_-)}\left(-R+\rho_a\nu_- - \mu_+(\rho_a+\nu_-)+\mu_+(R+1)\right)\hat{A}_0^\Delta$$

$$\pi_{mcpi,0} = -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_+-\rho_a)(\mu_+-\rho_a)(\mu_+-\nu_-)}\frac{R}{R-\mu_-}(\mu_+-1)\hat{A}_0^\Delta$$

$$\pi_0^\theta = -\frac{R\lambda}{(\nu_+-\rho_a)(\mu_+-\rho_a)(\mu_+-\nu_-)}\left(-R+\rho_a\nu_- - \mu_+(\rho_a+\nu_-)+\mu_+(R+1)\right)\hat{A}_0^\Delta$$

Note that since $\rho_a, \nu_- < 1$, $\mu_+ > R$, we have:

$$\begin{aligned}-R+\rho_a\nu_- - \mu_+(\rho_a+\nu_-)+\mu_+(R+1) &= \mu_+(R+1-(\rho_a+\nu_-)) - R + \rho_a\nu_- \\ &\geq R(R+1-(\rho_a+\nu_-)) - R + \rho_a\nu_- \\ &= (R-\rho_a)(R-\nu_-) \geq 0\end{aligned}$$

$\tilde{\mathcal{Y}}_0, \pi_{mcpi,0} \gtrless 0$ and $\pi_0^\theta \lessgtr 0$ if $\hat{A}_0^\Delta \lessgtr 0$. In addition, $\tilde{\mathcal{Y}}_0 = -\lambda\vartheta\pi_0^\theta$. Note in particular that if $\bar{\epsilon}_k^s = 0$ (CES inner utility) and $\bar{\epsilon}_k = \bar{\epsilon}$ across sector, we have $\pi_t^\theta = \pi_{cpi,t}$ and $\hat{A}_t^\Delta = \sum_{k=1}^K\left(\overline{\partial_e e}_k - \bar{s}_k\right)\hat{A}_{k,t}$: $\hat{A}_t^\Delta$ is negative (positive) for negative shocks in luxury (necessity) sectors.
In the medium run the behavior of $\tilde{\mathcal{Y}}_t, \pi_t^\theta, \pi_{mcpi,t}$ a priori depends on which of the parameters ρ_a, μ_- or ν_- dominates. If $\rho_a > \mu_-, \nu_-$, we have:

$$\tilde{\mathcal{Y}}_t = \frac{R\lambda^2\vartheta}{(\nu_--\rho_a)(\nu_+-\rho_a)}\frac{(R-\rho_a)(1-\rho_a)}{(\rho_a-\mu_+)(\rho_a-\mu_-)}\rho_a^{t+1}\hat{A}_0^\Delta + o\left(\rho_a^t\right)$$

$$\pi_{mcpi,t} = \frac{(R\lambda)^2\vartheta\kappa}{(\nu_--\rho_a)(\nu_+-\rho_a)}\frac{(1-\rho_a)}{(\rho_a-\mu_+)(\rho_a-\mu_-)}\rho_a^{t+1}\hat{A}_0^\Delta + o\left(\rho_a^t\right)$$

$$\pi_t^\theta = -\frac{R\lambda}{(\nu_--\rho_a)(\nu_+-\rho_a)}\frac{(R-\rho_a)(1-\rho_a)}{(\rho_a-\mu_+)(\rho_a-\mu_-)}(\rho_a-(1-\delta))\rho_a^t\hat{A}_0^\Delta + o\left(\rho_a^t\right)$$

for t large enough we have $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t} \gtrless 0$ if $\hat{A}_0^\Delta \gtrless 0$. $\pi_t^\theta \gtrless 0$ ($\pi_t^\theta \lessgtr 0$) if $\hat{A}_0^\Delta \gtrless 0$ and $\rho_a < (1-\delta)$ ($\rho_a > (1-\delta)$). Similarly, if $\nu_- > \mu_-, \rho_a$, we have:

$$\tilde{\mathcal{Y}}_t = -\frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \nu_-^{t+1} \hat{A}_0^\Delta + o(\nu_-^t)$$

$$\pi_{mcpi,t} = -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \nu_-^{t+1} \hat{A}_0^\Delta + o(\nu_-^t)$$

$$\pi_t^\theta = \frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} (\nu_- - (1-\delta)) \nu_-^t \hat{A}_0^\Delta + o(\nu_-^t)$$

for t large enough we have $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t} \gtrless 0$ if $\hat{A}_0^\Delta \gtrless 0$. $\pi_t^\theta \gtrless 0$ ($\pi_t^\theta \lesseqgtr 0$) if $\hat{A}_0^\Delta \gtrless 0$ and $\nu_- < (1-\delta)$ ($\nu_- > (1-\delta)$). Finally, if $\mu_- > \nu_-, \rho_a$:

$$\tilde{\mathcal{Y}}_t = \frac{R\lambda^2\vartheta}{(\nu_+ - \rho_a)} \frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)} \mu_-^{t+1} \hat{A}_0^\Delta + o(\mu_-^t)$$

$$\pi_{mcpi,t} = \frac{(R\lambda)^2\vartheta\kappa}{(\nu_+ - \rho_a)} \frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)} \frac{1}{R-\mu_-} \mu_-^{t+1} \hat{A}_0^\Delta + o(\mu_-^t)$$

$$\pi_t^\theta = -\frac{R\lambda}{(\nu_+ - \rho_a)} \frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)} (\mu_- - (1-\delta)) \mu_-^t \hat{A}_0^\Delta + o(\mu_-^t)$$

Recall that $\mu_- < 1-\delta$ so we have $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}, \pi_t^\theta \gtrless 0$ if $\hat{A}_0^\Delta \gtrless 0$.

Finally we derive the net present value of $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}, \pi_t^\theta$ under optimal policy. We have:

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t = -\frac{(R\lambda)^2\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{R(\mu_+ - 1)}{R-\mu_-} \hat{A}_0^\Delta$$

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_{mcpi,t} = -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{R}{(R-\mu_-)^2(R-\rho_a)(R-\nu_-)} \left\{R^2(\mu_+(R-\delta) - R) + \delta R^2(\rho_a + \nu_-) + \left(\mu_-(R-1) + \delta R^2\right)\rho_a\nu_-\right\}$$

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_t^\theta = \frac{R^2\lambda(R-(1-\delta))}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{\mu_+ - 1}{R-\mu_-} \hat{A}_0^\Delta$$

Note that $\mu_+ > R$, $0 \leq \rho_a, \nu_- \leq 1$ and as $\beta(1-\delta)R = 1$, $R-\delta > 1$ so $R^2(\mu_+(R-\delta) - R) + \delta R^2(\rho_a + \nu_-) + \left(\mu_-(R-1) + \delta R^2\right)\rho_a\nu_- > 0$. We therefore have $\sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t, \sum_{t\geq 0} \frac{1}{R^t} \pi_{mcpi,t} \gtrless 0$ and$\sum_{t\geq 0} \frac{1}{R^t} \pi_t^\theta \lesseqgtr 0$ if $\hat{A}_0^\Delta \lesseqgtr 0$

Result 9

Under the assumption θ_k and $\bar{e}_k$ are equal across sector then $\pi_t^\theta = \pi_{cpi,t}$, using the result of the previous subsection, we have:

$$\tilde{\mathcal{Y}}_0 = -\frac{R\lambda\lambda\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} (-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1)) \hat{A}_0^\Delta$$

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t = -\frac{R^2\lambda\lambda\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{R(\mu_+ - 1)}{R-\mu_-} \hat{A}_0^\Delta$$

$$\pi_{cpi,0} = -\frac{R\lambda}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} (-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1)) \hat{A}_0^\Delta$$

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_{cpi,t} = \frac{R^2\lambda(R-(1-\delta))}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{\mu_+ - 1}{R-\mu_-} \hat{A}_0^\Delta$$

Note that using $(\mu_+ - 1)(\mu_+ - R) - R\lambda\vartheta\kappa\mu_+ = 0$ and $-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) \geq 0$ we have:

$$
\begin{aligned}
\tilde{\mathcal{Y}}_0 &= -\frac{\lambda}{\kappa}\frac{1}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(\mu_+ - R)\left(-R + \rho_a\nu_- - \mu_+\left(\rho_a + \nu_-\right) + \mu_+\left(R+1\right)\right)}{(\mu_+ - \nu_-)\,\mu_+}\hat{A}_0^{\Delta} \\
\left|\tilde{\mathcal{Y}}_0\right| &= \frac{\lambda}{\kappa}\frac{1}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(\mu_+ - R)\left(-R + \rho_a\nu_- - \mu_+\left(\rho_a + \nu_-\right) + \mu_+\left(R+1\right)\right)}{(\mu_+ - \nu_-)\,\mu_+}\left|\hat{A}_0^{\Delta}\right| \\
&< \frac{\lambda}{\kappa}\frac{1}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(\mu_+ - R)\left(-\left(\rho_a + \nu_-\right) + (R+1)\right)\mu_+}{(\mu_+ - \nu_-)\,\mu_+}\left|\hat{A}_0^{\Delta}\right| \\
&= \frac{\lambda}{\kappa}\frac{R+1-(\rho_a+\nu_-)}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(\mu_+ - R)}{(\mu_+ - \nu_-)}\left|\hat{A}_0^{\Delta}\right| \\
&\leq \frac{\lambda}{\kappa}\frac{R+1-(\rho_a+\nu_-)}{(\nu_+ - \rho_a)}\left|\hat{A}_0^{\Delta}\right|
\end{aligned}
$$

where the last line uses the fact that $\rho_a, \nu_- \leq 1 < R$. Similarly, using $(\mu_- - 1)(\mu_- - R) - R\lambda\vartheta\kappa\mu_- = 0$

$$
\begin{aligned}
\mathbb{E}_0 \sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t &= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(1-\mu_-)}{(\mu_+ - \nu_-)\,\mu_-}\hat{A}_0^{\Delta} \\
&= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+ - \rho_a)}\frac{(\mu_+ - 1)}{(\mu_+ - \rho_a)}\frac{(1-\mu_-)}{\left(\frac{1}{\beta} - \nu_-\mu_-\right)}\hat{A}_0^{\Delta} \\
\mathbb{E}_0\left|\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t\right| &< \frac{\lambda}{\kappa}\frac{R}{(\nu_+ - \rho_a)}\left|\hat{A}_0^{\Delta}\right|
\end{aligned}
$$

where the second line uses $\mu_+\mu_- = R(1-\delta) = 1/\beta$ and the last line uses $\frac{1}{\beta} - \nu_-\mu_- \geq 1 - \nu_-\mu_- \geq 1 - \mu_-$ and $\rho_a \leq 1$.
Under strict CPI targeting we have $\pi_{cpi,t} = 0$ at all dates and

$$
\tilde{\mathcal{Y}}_t = -\frac{\lambda}{\kappa}\mathcal{NH}_t = -\frac{1}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left((R-\rho_a)(1-\rho_a)\rho_a^t - (R-\nu_-)(1-\nu_-)\nu_-^t\right)\hat{A}_0^{\Delta}.
$$

So under strict CPI targeting:

$$
\begin{aligned}
\tilde{\mathcal{Y}}_0 &= --\frac{\lambda}{\kappa}\frac{1}{(\nu_+ - \rho_a)}\left(R+1-(\rho_a+\nu_-)\right)\hat{A}_0^{\Delta} \\
\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t &= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+ - \rho_a)}\hat{A}_0^{\Delta}.
\end{aligned}
$$

Denoting with a superscript CPI the variables under CPI targeting, OP the variables the variables under optimal policy we therefore have after a negative shock in a necessity sector:

$$
\begin{aligned}
\tilde{\mathcal{Y}}_0^{CPI} < \tilde{\mathcal{Y}}_t^{OP} &< 0 \\
\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t^{CPI} < \mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t^{OP} &< 0 \\
\pi_{cpi,0}^{OP} > \pi_{cpi,0}^{CPI} &= 0 \\
\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\pi_{cpi,t}^{OP} > \mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\pi_{cpi,t}^{CPI} &= 0
\end{aligned}
$$

monetary policy is more accomodative after a negative shock in a necessity sector than strict targeting. After a shock in a luxury sector, we have:

$$\tilde{\mathcal{Y}}_0^{CPI} > \tilde{\mathcal{Y}}_t^{OP} > 0$$

$$\mathbb{E}_0 \sum_{t \geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t^{CPI} > \mathbb{E}_0 \sum_{t \geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t^{OP} > 0$$

$$\pi_{cpi,0}^{OP} < \pi_{cpi,0}^{CPI} = 0$$

$$\mathbb{E}_0 \sum_{t \geq 0} \frac{1}{R^t} \pi_{cpi,t}^{OP} < \mathbb{E}_0 \sum_{t \geq 0} \frac{1}{R^t} \pi_{cpi,t}^{CPI} = 0$$

i.e. monetary policy is more strict.

E.3 Welfare loss function

Welfare Loss Function

In this appendix, we derive the second order approximation of our social welfare function:

$$\mathcal{W}=(1-\delta)\int G\left(V_{-}\left(i\right),i\right)di+\delta\mathbb{E}_{0}\sum_{t_0=0}^{\infty}\beta^{t_0}\int G\left(V_{t_0}\left(i\right),i\right)di.$$

Recall that the value function of household i born at $t_0$is given by:

$$V_{t_0}\left(i\right)=\mathbb{E}_{t_0}\sum_{s=0}^{\infty}\left(\left(1-\delta\right)\beta\right)^{s}\left\{\left(1-\varphi\left(i\right)\right)\left[U_i\left(\mathcal{U}_1\left(\boldsymbol{c}^{u}_{1,t_0+s}\left(i\right)\right),\dots\mathcal{U}_K\left(\boldsymbol{c}^{u}_{K,t_0+s}\left(i\right)\right)\right)-\chi\left(\frac{n^{u}_{t_0+s}(i)}{\vartheta(i)}\right)\right]+\varphi\left(i\right)\left[U_i\left(\mathcal{U}_1\left(\boldsymbol{c}^{HtM}_{1,t_0+s}\left(i\right)\right),\dots\mathcal{U}_K\left(\boldsymbol{c}^{HtM}_{K,t_0+s}\left(i\right)\right)\right)-\chi\left(\frac{n^{HtM}_{t_0+s}(i)}{\vartheta(i)}\right)\right]\right\}.$$

Where the quantities $\left\{\boldsymbol{c}^{u}_{k,t_0+s}\left(i\right),\boldsymbol{c}^{HtM}_{1,t_0+s}\left(i\right),n^{u}_{t_0+s}(i),n^{Htm}_{t_0+s}(i)\right\}_{s\geq0}$ are chosen optimally, as described in the derivation appendix. Applying Roy's identity (and using the fact that in steady state consumption and labor supply is constant across constrained and unconstrained households of the same type i) , the derivative of the value function is given by:

$$\begin{aligned}dV_{t_0}\left(i\right)=&\mathbb{E}_{t_0}\sum_{s=0}^{\infty}\left(\left(1-\delta\right)\beta\right)^{s}\partial_e v_{t_0+s}\left\{\frac{b_{t_0+s+1}\left(i\right)}{R_{t_0+s}}\hat{R}_{t_0+s}+\hat{W}_{t_0+s}W_{t_0+s}n_{t_0+s}\left(i\right)-\sum_{l=1}^{K}\int d^{t_0}_{l,t_0+s}(i,j)dp_{l,t_0+s}\left(j\right)dj+\varsigma\left(i\right)dDiv_{t_0+s}\right\}\\&+\left(1-\varphi\left(i\right)\right)\partial_e v_{t_0}b_{t_0}\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\varphi\left(i\right)\mathbb{E}_{t_0}\sum_{s=0}^{\infty}\left(\left(1-\delta\right)\beta\right)^{s}\partial_e v_{t_0+s}\left(1-\frac{1}{R_{t_0+s}}\right)b_{t_0}\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0},\end{aligned}$$

with the first order change in dividends given by:

$$\begin{aligned}dDiv_t=&\sum_{k=1}^{K}\int y_k\left(j\right)\left(dp_{k,t}\left(j\right)-\frac{1}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)dW_t+\sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)dP_{l,t}-\left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)W_t+\sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)P_{l,t}\right)\hat{A}_{k,t}\right)\right)\\&+\sum_{k=1}^{K}\int dy_k\left(j\right)\left(p_{k,t}\left(j\right)-\frac{1}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)W_t+\sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)P_{l,t}\right)\right),\end{aligned}$$

where the change in demand for variety j is given by:

$$\begin{aligned}dy_{k,t}\left(j\right)=&\mathbb{E}_{\delta,t}\left(\int\partial_p d_{k,t}(i,j)dp_{k,t}\left(j\right)+\int\partial_P d_{k,t}(i,j)\left[j^*\right]dp_{k,t}\left(j^*\right)dj^*+\partial_e d_{k,t}(i,j)\left(\left(1-\varphi\left(i\right)\right)de^{t_0,u}_{k,t}\left(i\right)+\varphi\left(i\right)de^{t_0,HtM}_{k,t}\right)di\right)\\&+\left(\partial_p d^{I}_{k,t}\left(j\right)dp_{k,t}\left(j\right)+\int\partial_P d^{I}_{k,t}\left(j\right)\left[j^*\right]dp_{k,t}\left(j^*\right)dj^*\right)\tilde{Y}_{k,t}+d\tilde{Y}_{k,t},\end{aligned}$$

and demand for intermediary $d\tilde{Y}_{k,t}$ solves the system

$$\begin{aligned}d\tilde{Y}_{k,t}&=\sum_{l}\left(\partial_W\mathcal{Y}_{k,l}(\boldsymbol{P}_t,W_t)dW_t+\sum_{m=1}^{K}\partial_{P_m}\mathcal{Y}_{k,l}(\boldsymbol{P}_t,W_t)dP_{m,t}\right)Y_{k,t}+\sum_{l}\mathcal{Y}_{k,l}(\boldsymbol{P}_t,W_t)dY_{k,t},\\dY_{k,t}&=\int dy_{k,t}\left(j\right)dj.\end{aligned}$$

Around a steady state where prices, consumption, wealth and labor supply are constant (and equal across generation, constrained and unconstrained households of the same type i) and with $p_k(j) = \frac{1}{A}\left(\mathcal{N}_k(\boldsymbol{P}, W)W + \sum_{l=1}^K \mathcal{Y}_{l,k}(\boldsymbol{P}, W)P_l\right)$, this simplifies to

$$dV_{t_0}(i) = \partial_e v(i)\,\mathbb{E}_{t_0}\sum_{s=0}^{\infty}((1-\delta)\beta)^s\left\{\frac{b(i)}{R}\hat{R}_{t_0+s} + \hat{W}_{t_0+s}Wn(i) - \sum_{l=1}^K e_l(i)\hat{P}_{l,t_0+s} + \varsigma(i)\sum_{k=1}^K P_kY_k\left(\hat{P}_{k,t_0+s} - \frac{1}{A_kP_k}\left(W\mathcal{N}_k(\boldsymbol{P}_t, W_t)\hat{W}_{t_0+s} + \sum_{l=1}^K P_l\mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)\hat{P}_{l,t}\right) + \hat{A}_{k,t_0+s}\right)\right\}$$
$$+ \partial_e v(i)\,b(i)\sum_{l=1}^K \bar{s}_l\hat{P}_{l,t_0}$$
$$= \partial_e v(i)\,\mathbb{E}_{t_0}\sum_{s=0}^{\infty}((1-\delta)\beta)^s\left\{\frac{b(i)}{R}\left(\hat{R}_{t_0+s} - \sum_{l=1}^K \bar{s}_l\pi_{l,t_0+s+1}\right) - \sum_{l=1}^K e(i)(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s} + Wn(i)\sum_{l=1}^K \bar{s}_l\tilde{A}_{l,t_0+s}\right\}$$

Next we derive the second order approximation of the social welfare function. We use the fact that in steady state, prices, consumption, wealth and labor supply are constant (and equal across generation, constrained and unconstrained households of the same type i), $G'(V_{t_0}(i), i)\,\partial_e v_{t_0} = 1$ and $p_k(j) = \frac{1}{A}\left(\mathcal{N}_k(\boldsymbol{P}, W)W + \sum_{l=1}^K \mathcal{Y}_{l,k}(\boldsymbol{P}, W)P_l\right)$. Using clearing of the bond market, we have:

$$d^2\mathcal{W} = (1-\delta)\int G''(i)(dV_-(i))^2\,di + \delta\mathbb{E}_0\sum_{t_0=0}^{\infty}\beta^{t_0}\int G''(i)(dV_{t_0}(i))^2\,di$$
$$- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{c}_t^{t_0,u} - \sum_k s_k(i)\hat{P}_{k,t}\right) + \sum_k\partial_e e_k(i)\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t - \sum_{l=1}^K(e(i)s_l(i) - Wn(i)\bar{s}_l)\hat{P}_{l,t} + Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t} + \mathbb{1}_{t=t_0}b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}\right\}di$$
$$- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{c}_t^{t_0,HtM} - \sum_k s_k(i)\hat{P}_{k,t}\right) + \sum_k\partial_e e_k(i)\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t - \sum_{l=1}^K(e(i)s_l(i) - Wn(i)\bar{s}_l)\hat{P}_{l,t} + Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t} + \left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}\right\}di$$
$$+ \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{dW_tdN_t - \sum_{l=1}^K\int d\left\{d_{l,t}^{t_0}(i,j)\right\}dp_{l,t_0+s}(j)\,dj + d^2Div_t\right\} + \mathbb{E}_0\sum\beta^t\mathbb{E}_{\delta,t}\left(\frac{dR_t}{R^2}b^{HtM,t_0}(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}\right),$$

with

$$\int d\left\{d_{l,t_0+s}^{t_0}(i,j)\right\}dp_{l,t_0+s}(j)\,dj = \int\partial_p d_{l,t_0+s}^{t_0}(i,j)dp_{l,t_0+s}(j)\left(dp_{l,t_0+s}(j) - dP_{l,t_0+s}\right) + \partial_e d_{l,t_0+s}^{t_0}(i,j)\left(de_{l,t_0+s}(i) - e_l(i)\hat{P}_{l,t_0+s}\right)dp_{l,t_0+s}(j)\,dj,$$

and

$$d^2Div_t = -dW_tdN_t^d - \sum_{k=1}^K d\tilde{Y}_{k,t}^d dP_{k,t} - \sum_{k=1}^K\tilde{Y}_{k,t}^d d^2P_{k,t} + \sum_{k=1}^K\int dy_k(j)\left(dp_{k,t}(j) + P_k\hat{A}_{k,t}\right)dj$$
$$+ \sum_{k=1}^K\frac{Y_k}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t, W_t)dW_t + \sum_{l=1}^K\mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)dP_{l,t} - \left(\mathcal{N}_k(\boldsymbol{P}_t, W_t)W_t + \sum_{l=1}^K\mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)P_{l,t}\right)\hat{A}_{k,t}\right)\hat{A}_{k,t}$$
$$+ \sum_{k=1}^K\int dy_k(j)\left(dp_{k,t}(j) - \frac{1}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t, W_t)dW_t + \sum_{l=1}^K\mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)dP_{l,t}\right) + p_{k,t}(j)\hat{A}_{k,t}\right)dj,$$

Simplifying – using market clearing conditions for labor and markets and properties of the steady state – and removing the terms independent of monetary policy, we have:

$$dW_tdN_t - \sum_{l=1}^K\int d\left\{d_{l,t}^{t_0}(i,j)\right\}dp_{l,t_0+s}(j)\,dj + d^2Div_t = -\sum_k P_kdC_{k,t}\left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t}\right)$$
$$- \left(\sum_{k,l}\frac{1}{A_l}P_kY_l\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) + \sum_{k,l,m}\frac{1}{A_l}P_kY_l\hat{P}_{m,t}\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t}\right)\right)$$
$$+ \hat{W}_t\sum_k E_k\tilde{\mathbf{A}}_{k,t} - \sum_{k=1}^K P_kY_k\left(\left(\frac{E_k}{P_kY_k}\bar{\epsilon}_k + \left(1-\frac{E_k}{P_kY_k}\right)\bar{\epsilon}_k^I\right)\int\left(\hat{p}_{k,t}(j) - \hat{P}_{k,t}\right)\hat{p}_{k,t}(j)\,dj\right)$$

Next we have, noting $\hat{p}^*_{k,t}$ the reset price at t and $\int \hat{p}^2_{k,t}(j)dj = (1-\theta_k)\sum_{m=0}^{t}\theta_k^m\left(\hat{p}^*_{k,t-m}\right)^2$:

$$\begin{aligned}
\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\int\left(\hat{p}_{k,t}(j)-\hat{P}_{k,t}\right)\hat{p}_{k,t}(j)\,dj &= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(\frac{1-\theta_k}{1-\beta\theta_k}\left(\hat{p}^*_{k,t}\right)^2-\left(\hat{P}_{k,t}\right)^2\right)\\
&= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{(1-\beta\theta_k)(1-\theta_k)}\left(\pi_{k,t}\right)^2+2\frac{1}{(1-\beta\theta_k)}\pi_{k,t}\hat{P}_{k,t-1}+\frac{(1-\theta_k)}{1-\beta\theta_k}\left(\hat{P}_{k,t-1}\right)^2-\left(\hat{P}_{k,t}\right)^2\right\}\\
&= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}\left(\pi_{k,t}\right)^2-\frac{\theta_k}{1-\beta\theta_k}\left(\hat{P}_{k,t-1}\right)^2+\frac{\beta\theta_k}{(1-\beta\theta_k)}\left(\hat{P}_{k,t}\right)^2\right\}\\
&= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}\left(\pi_{k,t}\right)^2.
\end{aligned}$$

We therefore have

$$\begin{aligned}
\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{dW_t dN_t-\sum_{l=1}^{K}\int d\left\{d^{t_0}_{l,t}(i,j)\right\}dp_{l,t_0+s}(j)\,dj+d^2Div_t\right\} &= -\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k P_k dC_{k,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(\sum_{k,l}\frac{1}{A_l}P_kY_lW\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)+\sum_{k,l,m}\frac{1}{A_l}P_mP_kY_l\hat{P}_{m,t}\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)\right)\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k E_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}-\frac{\theta_k\bar{\epsilon}_k}{(1-\beta\theta_k)(1-\theta_k)}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(\pi_{k,t}\right)^2
\end{aligned}$$

Let us first slightly rewrite the terms coming from substitution in production. We have:

$$\begin{aligned}
&-\left(\sum_{k,l}\frac{1}{A_l}P_kY_lW\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)+\sum_{k,l,m}\frac{1}{A_l}P_kP_mY_l\hat{P}_{m,t}\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)\right)\\
&\quad-\sum\frac{1}{A_l}Y_lW^2\partial_W\mathcal{N}_l\hat{W}_t^2-\sum_{k,l}\frac{1}{A_l}P_kY_lW\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)-\sum_{k,l}\frac{1}{A_l}WY_l\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)P_k\partial_{P_k}\mathcal{N}_l\hat{W}_t-\sum_{k,l,m}\frac{1}{A_l}P_mP_kY_l\left(\hat{P}_{m,t}+\tilde{\mathbf{A}}_{m,t}\right)\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)
\end{aligned}$$

where we used

$$\begin{aligned}
\sum_k P_k\partial_W\mathcal{Y}_{k,l} &= -W\partial_W\mathcal{N}_l\\
\sum_k P_k\partial_{P_m}\mathcal{Y}_{k,l} &= -W\partial_{P_m}\mathcal{N}_l\\
\partial_W\mathcal{Y}_{k,l} &= \partial_{P_k}\mathcal{N}_l\\
\partial_{P_m}\mathcal{Y}_{k,l} &= \partial_{P_k}\mathcal{Y}_{m,l}
\end{aligned}$$

We define

$$\mathcal{N}(W,\boldsymbol{P})\equiv\sum_l\frac{1}{A_l}Y_l\mathcal{N}_l(W,\boldsymbol{P})$$

$$\mathcal{Y}_k(W,\boldsymbol{P})=\sum_l\frac{1}{A_l}Y_l\mathcal{Y}_{k,l}(W,\boldsymbol{P})$$

We have

$$-\sum\frac{1}{A_l}Y_lW\partial_W\mathcal{N}_l\hat{W}_t^2-\sum_{k,l}\frac{1}{A_l}P_kY_lW\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)-\sum_{k,l}\frac{1}{A_l}WY_l\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)P_k\partial_{P_k}\mathcal{N}_l\hat{W}_t-\sum_{k,l,m}\frac{1}{A_l}P_mP_kY_l\left(\hat{P}_{m,t}+\tilde{\mathbf{A}}_{m,t}\right)\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)$$
$$=-W^2\partial_W\mathcal{N}\hat{W}_t^2-\sum_kP_kW\partial_W\mathcal{Y}_k\hat{W}_t\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)-\sum_kWP_k\partial_{P_k}\mathcal{N}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)\hat{W}_t-\sum_{k,l}P_lP_k\partial_{P_l}\mathcal{Y}_k\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)\left(\hat{P}_{l,t}+\tilde{\mathbf{A}}_{l,t}\right)$$

Note that we have

$$P_kdC_{k,t}=\mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))P_kdc_{k,t}^{t_0,u}(i)+\varphi(i)P_kdc_{k,t}^{t_0,HtM}(i)\right)\right)$$

with

$$P_kdc_{k,t}^{t_0}(i)=e(i)\partial_ee_k(i)\left(\hat{e}_t^{t_0}(i)-\sum_ks_k(i)\hat{P}_{k,t}\right)+e_k(i)\sum_l\rho_{k,l}(i)\hat{P}_{l,t}$$

So rearranging, we obtain:

$$P_kdC_{k,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)=\mathbb{E}_{\delta,t}\int\frac{1}{\sigma}\left(\hat{e}_t^{t_0}(i)-\sum_ks_k(i)\hat{P}_{k,t}\right)\sigma e(i)\partial_ee_k(i)di\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)$$
$$-\sum_kE_k\sum_l\bar{\rho}_{k,l}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)$$

So we can rewrite the welfare loss as

$$d^2\mathcal{W}=(1-\delta)\int G''(i)\left(dV_-(i)\right)^2di+\delta\mathbb{E}_0\sum_{t_0=0}^{\infty}\beta^{t_0}\int G''(i)\left(dV_{t_0}(i)\right)^2di$$
$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}-\sum_ks_k(i)\hat{P}_{k,t}\right)+\sum_k\partial_ee_k(i)\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di$$
$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}-\sum_ks_k(i)\hat{P}_{k,t}\right)+\sum_k\partial_ee_k(i)\hat{P}_{k,t}\right)\left\{\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_ee_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$
$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_ks_k(i)\hat{P}_{k,t}\right)+\sum_k\partial_ee_k(i)\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di$$
$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_ks_k(i)\hat{P}_{k,t}\right)+\sum_k\partial_ee_k(i)\hat{P}_{k,t}\right)\left\{\left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_ee_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$
$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{k,l}^{K}\int\sigma e(i)\partial_ee_k(i)\partial_ee_l(i)di\hat{P}_{l,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)$$
$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(-W^2\partial_W\mathcal{N}\hat{W}_t^2-\sum_kP_kW\partial_W\mathcal{Y}_k\hat{W}_t\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)-\sum_kWP_k\partial_{P_k}\mathcal{N}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)\hat{W}_t-\sum_{k,l}P_lP_k\partial_{P_l}\mathcal{Y}_k\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)\left(\hat{P}_{l,t}+\tilde{\mathbf{A}}_{l,t}\right)\right)$$
$$+\sum_kE_k\sum_l\bar{\rho}_{k,l}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_kE_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}-\frac{\theta_k\bar{\epsilon}_k}{(1-\beta\theta_k)(1-\theta_k)}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(\pi_{k,t}\right)^2$$
$$+\mathbb{E}_0\sum\beta^t\mathbb{E}_{\delta,t}\left(\frac{dR_t}{R^2}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}\right)$$

Simplification of the terms corresponding to the expenditure of unconstrained households

Here we simplify the second line of the expression. To do so, define

$$X_t \equiv \frac{1}{\int (1-\varphi(i)) \left(\sigma \frac{e(i)}{E} + \frac{Wn(i)}{WN}\psi\right) di} \int (1-\varphi(i)) \left\{ \frac{Wn(i)}{WN} \sum_l \bar{s}_l \tilde{A}_{l,t} + \frac{Wn(i)}{WN} \bar{\psi} \hat{W}_t + \sum_l \sigma \frac{e(i)}{E} \partial_e e_l(i) \, di \hat{P}_{l,t} \right\} di$$

$$- \frac{1}{\int (1-\varphi(i)) \left(\sigma \frac{e(i)}{E} + \frac{Wn(i)}{WN}\psi\right) di} \int \varphi(i) \left\{ \frac{b(i)}{ER} \left(\hat{R}_t - (R-1) \sum_l \bar{s}_l \hat{P}_{l,t} \right) - \sum_l \frac{e(i)}{E} (s_l(i) - \bar{s}_l) \hat{P}_{l,t} \right\}$$

X_t is an aggregate variable growing at rate $\hat{R}_t$. Indeed, recall that we have, from the labor market clearing condition:

$$\sum_l \bar{s}_l \tilde{A}_{l,t} + \psi \left(\hat{W}_t - \sum_l \int \frac{Wn(i)}{WN} \partial_e e_l(i) \hat{P}_{l,t} \right) = \mathbb{E}_{\delta,t} \left(\frac{e(i)}{E} \left(1 + \frac{Wn(i)\psi}{e(i)\sigma} \right) \left((1-\varphi(i)) \left(\hat{e}^{t_0,u}_{t_0+s} - \sum_k s_k(i) \hat{P}_{k,t_0+s} \right) + \varphi(i) \left(\hat{e}^{t_0,HtM}_{t_0+s} - \sum_k s_k(i) \hat{P}_{k,t_0} \right) \right) \right)$$

$$\equiv \mathcal{E}_t$$

And it is direct to show that:

$$\mathbb{E}_t \mathcal{E}_{t+1} - \mathcal{E}_t = \int (1-\varphi(i)) \left(\sigma \frac{e(i)}{E} + \frac{Wn(i)}{WN}\psi \right) \left(\hat{R}_t - \sum_l \partial_e e_l(i) \pi_{l,t+1} \right) di$$

$$+ \int \varphi(i) \left\{ \frac{b_0}{ER} \Delta \hat{R}_{t+1} + \frac{Wn(i)}{WN} \psi \Delta \hat{W}_{t+1} - \sum_l \frac{Wn(i)}{WN} \psi \partial_e e_l(i) \pi_{l,t+1} - \frac{e(i)}{E} s_k(i) \cdot \hat{\pi}_{l,t+1} + \frac{Wn(i)}{WN} \sum \left\{ \bar{s}_k \left(\hat{\pi}_{l,t+1} + \Delta \hat{A}_{l,t+1} \right) \right\} di \right\}$$

So we have:

$$\mathbb{E}_t X_{t+1} - X_t = \hat{R}_t.$$

Since:

$$\mathbb{E}_t \left(\frac{1}{\sigma} \left(\hat{e}^{t_0,u}_{t+1} - \sum_k s_k(i) \hat{P}_{k,t+1} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t+1} \right) - \frac{1}{\sigma} \left(\hat{e}^{t_0,u}_t - \sum_k s_k(i) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t} = \hat{R}_t,$$

we have

$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\delta,t} \int (1-\varphi(i)) \left(\frac{1}{\sigma} \left(\hat{e}^{t_0,u}_t - \sum_k s_k(i) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t} \right) \left\{ \frac{b(i)}{R} \hat{R}_t - \sum_{l=1}^K (e(i) s_l(i) - Wn(i) \bar{s}_l) \hat{P}_{l,t} + Wn(i) \sum_{l=1}^K \bar{s}_l \tilde{A}_{l,t} \right\} di$$

$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\delta,t} \int (1-\varphi(i)) \left(\frac{1}{\sigma} \left(\hat{e}^{t_0,u}_t - \sum_k s_k(i) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t} \right) \left\{ \mathbb{1}_{t=t_0} b(i) \sum_{l=1}^K \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^K \sigma e(i) \partial_e e_k(i) \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t} \right) \right\} di$$

$$= -\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\delta,t} \int (1-\varphi(i))$$

$$\left(\frac{1}{\sigma} \left(\hat{e}^{t_0,u}_t - \sum_k s_k(i) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t} - X_t \right) \left\{ \frac{b(i)}{R} \hat{R}_t - \sum_{l=1}^K (e(i) s_l(i) - Wn(i) \bar{s}_l) \hat{P}_{l,t} + Wn(i) \sum_{l=1}^K \bar{s}_l \tilde{A}_{l,t} + \mathbb{1}_{t=t_0} b(i) \sum_{l=1}^K \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^K \sigma e(i) \partial_e e_k(i) \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t} \right) \right\} di$$

$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t X_t \mathbb{E}_{\delta,t} \int (1-\varphi(i)) \left\{ \frac{b(i)}{R} \hat{R}_t - \sum_{l=1}^K (e(i) s_l(i) - Wn(i) \bar{s}_l) \hat{P}_{l,t} + Wn(i) \sum_{l=1}^K \bar{s}_l \tilde{A}_{l,t} + \mathbb{1}_{t=t_0} b(i) \sum_{l=1}^K \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^K \sigma e(i) \partial_e e_k(i) \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t} \right) \right\} di$$

Next we have:

$$\mathbb{E}_{t_0}\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k \partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$=\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{E}{\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}-X_{t_0+s}\right)$$

so

$$-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}$$

$$=-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left(\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+}\right.\right.$$

$$\left.\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\tilde{A}_{k,t_0+s}\right\}\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\sum_{l=1}^{K}\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}E\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

First note that we have, up to terms independent from monetary policy:

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\sum_{l=1}^{K}\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)\psi\sigma e\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)^2 di$$

$$+\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_l\partial_e e_l\left(i\right)\left(\hat{P}_{l,t_0+s}+\tilde{A}_{l,t_0+s}\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di$$

We therefore obtain:

$$-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}-\sum_k s_k(i)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k(i)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$\cdot\left\{\frac{b(i)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}(e(i)s_l(i)-Wn(i)\bar{s}_l)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\frac{E^2}{\sigma e(i)+Wn(i)\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b(i)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_{l=1}^{K}\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e(i)}{E}\sigma\sum_l\partial_e e_l(i)\tilde{A}_{k,t_0+s}\right\}\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\frac{Wn(i)\psi\sigma e(i)}{\sigma e(i)+Wn(i)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)^2 di$$

$$+\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\sigma e(i)\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_l\partial_e e_l(i)\left(\hat{P}_{l,t_0+s}+\tilde{A}_{l,t_0+s}\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\sigma e(i)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum\frac{1}{R^s}\sum_k\partial_e e_k(i)\hat{P}_{k,t_0+s}di$$

$$-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\sum_{s=0}^{\infty}\frac{1}{R^s}E\left\{\frac{b(i)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

Next we have:

$$\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_l\bar{s}_l\hat{P}_{l,t_0}$$

$$=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\bar{\psi}\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{k,t_0+s}\right\}di$$

$$-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi\right)^{-1}}{R^s}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\sum\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_l\bar{s}_l\hat{P}_{l,t_0}$$

where $\varphi^E \equiv \int \varphi(i)\frac{e(i)}{E}di, \varphi^N \equiv \int \varphi(i)\frac{Wn(i)}{WN}di$. Let us first rewrite the first term of this expression. We have:

$$\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\int(1-\varphi(i))\left\{\frac{Wn(i)}{WN}\sum_l \bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn(i)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e(i)}{E}\partial_e e_l(i)\,di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left\{\sum_{l=1}^{K}\sigma e(i)\,\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=\left(1-\frac{1}{R}\right)\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\left\{\sum\frac{1}{R^s}\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{1}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\sum_k\partial_e e_k(i)\,\hat{P}_{k,t_0+s}di+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\int(1-\varphi(i))\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\int(1-\varphi(i))\frac{Wn(i)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left\{\sum_{l=1}^{K}\sigma e(i)\,\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

Next, the second and third terms can be rewritten:

$$\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di$$

$$-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}di\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l di\hat{P}_{l,t_0+s}di\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}di$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di$$

$$+2\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}di$$

Using:

$$\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di=-\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di$$

Since in steady state:

$$\int b\left(i\right)di=\int\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)di=0$$

we have:

$$\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}$$

$$=\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_{k}\partial_e e_k\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum\frac{1}{R^s}\sum_{k}\partial_e e_k\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_{k}\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_{k}\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}\right\}di\right)^2$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_{l}\bar{s}_l\hat{P}_{l,t_0}$$

Putting everything together, we have:

$$-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}(i)-\sum_k s_k(i)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k(i)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$\cdot\left\{\frac{b(i)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}(e(i)s_l(i)-Wn(i)\bar{s}_l)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\frac{E^2}{\sigma e(i)+Wn(i)\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b(i)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e(i)}{E}\sigma\sum_l\partial_e e_l(i)\tilde{A}_{k,t_0+s}\right\}\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\frac{Wn(i)\psi\sigma e(i)}{\sigma e(i)+Wn(i)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)^2 di$$

$$+\left(1-\frac{1}{R}\right)\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\frac{1}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2$$

$$+\left(1-\frac{1}{R}\right)\frac{(1-\varphi^E)\sigma}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di$$

$$-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))b(i)di\sum_l\bar{s}_l\hat{P}_{l,t_0}di\sum\frac{1}{R^s}\hat{W}_{t_0+s}+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int(1-\varphi(i))b(i)di\sum_l\bar{s}_l\hat{P}_{l,t_0}$$

$$+\frac{1-\frac{1}{R}}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\left(R_{t_0+s}-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\tilde{A}_{k,t_0+s}\right\}di\right)^2$$

Finally, we rewrite some coefficients, we have:

$$\int(1-\varphi(i))\frac{Wn(i)\psi\sigma e(i)}{\sigma e(i)+Wn(i)\psi}di-\frac{\int(1-\varphi(i))\psi Wn(i)di\int(1-\varphi(i))\sigma e(i)di}{\int(1-\varphi(i))(\sigma e(i)+Wn(i)\psi)di}$$

$$=\frac{\sigma\psi}{\int(1-\varphi(i))(\sigma e(i)+Wn(i)\psi)di}\left\{\int(1-\varphi(i))e(i)\frac{Wn(i)\int(1-\varphi(i))(\sigma e(i)+Wn(i)\psi)di}{\sigma e(i)+Wn(i)\psi}di-\int(1-\varphi(i))Wn(i)di\int(1-\varphi(i))e(i)\frac{\sigma e(i)+Wn(i)\psi}{\sigma e(i)+Wn(i)\psi}di\right\}$$

$$=\frac{\sigma\psi}{\int(1-\varphi(i))(\sigma e(i)+Wn(i)\psi)di}\left\{\int(1-\varphi(i))\sigma e(i)\frac{Wn(i)\int(1-\varphi(i))e(i)di-e(i)\int(1-\varphi(i))Wn(i)di}{\sigma e(i)+Wn(i)\psi}di\right\}$$

$$=\left(1-\frac{1}{R}\right)\frac{\sigma\int(1-\varphi(i))\psi Wn(i)di}{\int(1-\varphi(i))(\sigma e(i)+Wn(i)\psi)di}\left\{\int(1-\varphi(i))\sigma e(i)\frac{\frac{Wn(i)}{\int(1-\varphi(i))\psi Wn(i)di}\int(1-\varphi(i))b(i)di-b(i)}{\sigma e(i)+Wn(i)\psi}di\right\}$$

$$=-\frac{\int(1-\varphi(i))\psi\frac{Wn(i)}{WN}di}{\left(1+\frac{\int(1-\varphi(i))\left(\frac{Wn(i)}{WN}\psi\right)di}{\int(1-\varphi(i))\frac{\sigma e(i)}{E}di}\right)}\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di}-\frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}$$

and we define the variance covariance matrix of marginal propensities to spend:

$$\mathcal{E}_{k,l}=\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\partial_e e_l(i)\,\partial_e e_k(i)\,di-\frac{\int(1-\varphi(i))\,e(i)\,\partial_e e_l(i)\,di\int(1-\varphi(i))\,e(i)\,\partial_e e_l(i)\,di}{\left(\int(1-\varphi(i))\,e(i)\,di\right)^2}$$

So we have:

$$-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}(i)-\sum_k s_k(i)\,\hat{P}_{k,t_0+s}\right)+\sum_k\partial e_k(i)\,\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$\cdot\left\{\frac{b(i)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e(i)\,s_l(i)-Wn(i)\,\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\,\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}$$

$$=-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\frac{E^2}{\sigma e(i)+Wn(i)\,\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b(i)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e(i)}{E}\left(s_l-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e(i)}{E}\sigma\sum_l\partial_e e_l(i)\,\tilde{A}_{k,t_0+s}\right\}\right\}^2 di$$

$$+\left(1-\frac{1}{R}\right)E\frac{\int(1-\varphi(i))\,\psi\frac{Wn(i)}{WN}di}{\left(1+\frac{\int(1-\varphi(i))\left(\frac{Wn(i)}{WN}\psi\right)di}{\int(1-\varphi(i))\frac{\sigma e(i)}{E}di}\right)}\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di}-\frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di$$

$$+\left(1-\frac{1}{R}\right)\frac{\left(\left(1-\varphi^E\right)\sigma\right)^2}{\int(1-\varphi(i))\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\sum_{k,l}\mathcal{E}_{k,l}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{P}_{l,t_0+s}+\tilde{A}_{l,t_0+s}\right)\right)$$

$$-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,b(i)\,di\sum_l\bar{s}_l\hat{P}_{l,t_0}di\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int(1-\varphi(i))\,b(i)\,di\sum_l\bar{s}_l\hat{P}_{l,t_0}$$

$$+\frac{1-\frac{1}{R}}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\left(R_{t_0+s}-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e(i)}{E}\left(s_l-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sum_{l=1}^{K}\sigma e(i)\,\partial_e e_k(i)\,\tilde{A}_{k,t_0+s}\right\}di\right)^2$$

Simplification of the terms corresponding to the expenditure of unconstrained households

Here we simplify the second line of the of the social welfare function. Recall that we have:

$$\frac{1}{\sigma}\left(\hat{e}^{t_0,HtM}_t-\sum_k s_k(i)\,\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\,\hat{P}_{k,t}$$

$$=\frac{E}{(\sigma e(i)+Wn(i)\,\psi)}\left\{\frac{b(i)}{RE}R_{t_0+s}+\frac{Wn(i)}{WN}\psi\sum_{l=1}^{K}\left(\hat{W}_t-\partial_e e_k(i)\,\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)b(i)\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right\}+\sum_k\partial_e e_k(i)\,\hat{P}_{k,t}$$

$$=\frac{E}{(\sigma e(i)+Wn(i)\,\psi)}\left\{\frac{b(i)}{RE}R_{t_0+s}+\frac{Wn(i)}{WN}\psi\hat{W}_t+\frac{\sigma e(i)}{E}\sum_{l=1}^{K}\partial_e e_k(i)\,\hat{P}_{l,t}-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)\right\}$$

We then get

$$-\sum_{s=0}^{\infty}\frac{1}{R^s}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,HtM}_{t_0+s}-\sum_k s_k(i)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k(i)\hat{P}_{k,t_0+s}\right)$$

$$\cdot\left\{\frac{b(i)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}(e(i)s_l(i)-Wn(i)\bar{s}_l)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\sum_{s=0}^{\infty}\frac{1}{R^s}\int\varphi(i)\frac{E^2}{(\sigma e(i)+Wn(i)\psi)}\left\{\frac{b(i)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)-\frac{\sigma e(i)}{E}\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\tilde{A}_{k,t_0+s}\right\}^2$$

$$-\sum_{s=0}^{\infty}\frac{1}{R^s}\int\varphi(i)\frac{E^2}{\sigma e(i)+Wn(i)\psi}\left\{\frac{Wn(i)}{WN}\psi\hat{W}_{t_0+s}+\frac{e(i)}{E}\sigma\sum_l\partial_e e_l(i)\hat{P}_{l,t_0+s}\right\}\left\{\sum_{l=1}^{K}\sigma\frac{e(i)}{E}\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$-\sum_{s=0}^{\infty}\hat{W}_{t_0+s}\frac{1}{R^s}\int\varphi(i)E\left\{\frac{b(i)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)\right\}$$

First note that we have, up to terms independent from monetary policy

$$-\mathbb{E}_0\sum_{s\geq 0}\frac{1}{R^s}\int\varphi(i)\frac{E^2}{\sigma e(i)+Wn(i)\psi}\left\{\frac{Wn(i)}{WN}\psi\hat{W}_{t_0+s}+\frac{e(i)}{E}\sigma\sum_l\partial_e e_l(i)\hat{P}_{l,t_0+s}\right\}\left\{\sum_{l=1}^{K}\sigma\frac{e(i)}{E}\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\mathbb{E}_0\sum_{s\geq 0}\frac{1}{R^s}\int\varphi(i)\frac{\sigma\psi e(i)Wn(i)}{\sigma e(i)+Wn(i)\psi}\left\{\hat{W}_{t_0+s}-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di$$

$$+\mathbb{E}_0\sum_{s\geq 0}\frac{1}{R^s}\int\varphi(i)\sigma e(i)\left\{\sum_{l,k}\partial_e e_l(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di-\mathbb{E}_0\sum_{s\geq 0}\frac{1}{R^s}\int\varphi(i)\sigma e(i)\hat{W}_{t_0+s}\sum_k\partial_e e_k(i)\hat{P}_{k,t_0+s}di$$

So we have:

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,HtM}_t-\sum_k s_k(i)\hat{P}_{k,t}\right)+\sum_k\partial e_k(i)\hat{P}_{k,t}\right)$$

$$\cdot\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}(e(i)s_l(i)-Wn(i)\bar{s}_l)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\frac{E^2}{(\sigma e(i)+Wn(i)\psi)}\left\{\frac{b(i)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)-\frac{\sigma e(i)}{E}\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\tilde{A}_{k,t_0+s}\right\}^2$$

$$-\mathbb{E}_0\sum\beta^t\int\varphi(i)\frac{\sigma\psi e(i)Wn(i)}{\sigma e(i)+Wn(i)\psi}\left\{\hat{W}_t-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2 di+\sum\beta^t\int\varphi(i)\sigma e(i)\left\{\sum_{l,k}\partial_e e_l(i)\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)\right\}^2 di-\sum\beta^t\int\varphi(i)\sigma e(i)\hat{W}_t\sum_k\partial_e e_k(i)\hat{P}_{k,t}di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\hat{W}_t\int\varphi(i)E\left\{\frac{b(i)}{RE}\hat{R}_t-\sum_{l=1}^{K}\frac{e(i)}{E}(s_l(i)-\bar{s}_l)\hat{P}_{l,t}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right\}$$

Next, we gather all the terms of the form $\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t x_t z_t$, that is:

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_k s_k(i)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\hat{P}_{k,t}\right)$$

$$\cdot\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}+\left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{k,l}^K\int\sigma e(i)\partial_e e_k(i)\partial_e e_l(i)\,di\hat{P}_{l,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k E_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

Following the same step as above, we rewrite the third line as:

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\left\{\int(1-\varphi(i))\frac{Wn(i)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn(i)}{WN}\bar{\psi}\hat{W}_t+\sum_l\sigma\frac{e(i)}{E}\partial_e e_l(i)\,di\hat{P}_{l,t}di\right\}\int(1-\varphi(i))\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\left\{\int(1-\varphi(i))\frac{Wn(i)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn(i)}{WN}\bar{\psi}\hat{W}_t+\sum_l\sigma\frac{e(i)}{E}\partial_e e_l(i)\,di\hat{P}_{l,t}di\right\}$$

$$\cdot\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}\right\}di$$

$$+\sum_{t=0}^{\infty}\beta^t\frac{1}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\int\varphi(i)\left\{\frac{b(i)}{ER}\left(\hat{R}_t-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t}di\right\}\int(1-\varphi(i))\left\{\sum_{l=1}^K\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{(1-\varphi^E)\sigma+(1-\varphi^N)\psi}\int\varphi(i)\left\{\frac{b(i)}{ER}\left(\hat{R}_t-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t}di\right\}$$

$$\cdot\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^K\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^K\bar{s}_l\tilde{A}_{l,t}\right\}di-\delta\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\int(1-\varphi(i))b(i)\sum_{l=1}^K\bar{s}_l\hat{P}_{l,t}di$$

First, we have

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}di\right\}\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di$$

$$=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\hat{W}_t-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2 di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2 di$$

$$+\sum_{t=0}^{\infty}\beta^t E\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)di\right)^2$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\sum_k\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)di\hat{W}_t\hat{P}_{k,t}$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}di\sum_l\bar{s}_l\tilde{A}_{l,t}di\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di$$

Next,

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}\right\}di$$

$$\cdot\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$=\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\hat{W}_t\int\varphi\left(i\right)E\left\{\frac{b\left(i\right)}{RE}\hat{R}_t-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}-\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\hat{P}_{l,t}\right\}di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\hat{W}_t\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int\left(1-\varphi\left(i\right)\right)\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}di\right\}\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}di\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di$$

$$-2\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t}di$$

Gathering the terms, we obtain:

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_k s_k(i)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\hat{P}_{k,t}\right)$$

$$\cdot\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\left(1-\frac{1}{R}\right)b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{k,l}^{K}\int\sigma e(i)\partial_e e_k(i)\partial e_l(i)di\hat{P}_{l,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k E_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e(i)s_l(i)-Wn(i)\bar{s}_l\right)\hat{P}_{l,t}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di$$

$$=-\sum_{s=0}^{\infty}\frac{1}{R^s}\int\varphi(i)\frac{E^2}{(\sigma e(i)+Wn(i)\psi)}\left\{\frac{b(i)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)-\frac{\sigma e(i)}{E}\sum_{l=1}^{K}\sigma e(i)\partial_e e_k(i)\tilde{A}_{k,t_0+s}\right\}^2$$

$$-\mathbb{E}_0\sum\beta^t\int\varphi(i)\frac{\sigma\psi e(i)Wn(i)}{\sigma e(i)+Wn(i)\psi}\left\{\hat{W}_t-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2 di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int(1-\varphi(i))\sigma\frac{e(i)}{E}\left\{\hat{W}_t-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2 di$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left(\left(1-\varphi^E\right)\sigma\right)^2}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\sum_{k,l}\mathcal{E}_{k,l}\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-\delta\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(1-\frac{1}{R}\right)\int(1-\varphi(i))b(i)di\sum_l\bar{s}_l\hat{P}_{l,t}di\sum_{s\geq 0}\frac{1}{R^s}\hat{W}_{t+s}$$

$$-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\int(1-\varphi(i))\left\{\frac{b(i)}{ER}\left(\hat{R}_t-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sigma\frac{e(i)}{E}\sum_{l=1}^{K}\partial_e e_k(i)\tilde{A}_{k,t}di\right\}\right)^2$$

$$-\delta\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\int(1-\varphi(i))b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t}di$$

Social Welfare Function

Note that we have, as $\mathbb{E}_t X_{t+1}-X_t=\hat{R}_t$

$$\mathbb{E}_0 X_t\int(1-\varphi(i))b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t}di+\mathbb{E}_0\left(1-\frac{1}{R}\right)\sum\frac{1}{R^s}X_{t+s}\int(1-\varphi(i))b(i)di\sum_l\bar{s}_l\hat{P}_{l,t}=\sum\frac{\hat{R}_s}{R^{s+1}}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}$$

In addition, for any variable x_t, z_t, we have

$$\sum_{t=0}^{\infty}\frac{1}{R^t}\left(x_{t_0+t}+\left(1-\frac{1}{R}\right)z_{t_0}\right)^2-\left(1-\frac{1}{R}\right)\left(\sum_{t=0}^{\infty}\frac{1}{R^t}x_{t_0+t}+z_{t_0}\right)^2=\mathbb{E}_0\sum_{t=0}^{\infty}\frac{1}{R^t}\left(x_{t_0+t}\right)^2-\mathbb{E}_0\left(1-\frac{1}{R}\right)\left(\sum_{t=0}^{\infty}\frac{1}{R^t}x_{t_0+t}\right)^2$$

Finally, we denote for any variable Z_{t_0},

$$\mathbb{E}_\beta\left(Z_{t_0}\right)\equiv\mathbb{E}_0(1-\beta)\left(\frac{1}{1-\frac{1}{R}}(1-\delta)Z_-+\delta\frac{1}{1-\frac{1}{R}}\sum_{t_0}\beta^{t_0}Z_{t_0}\right)$$

the social average over the population of the variable Z_{t_0}(Note that if $Z_{t_0}$is constant, we have $\mathbb{E}_\beta\left(Z_{t_0}\right)=Z$). Using this facts and the previous derivations, we can write – after normalization – the social welfare function as:

$$\mathcal{L} = (1-\beta)\sum \beta^s \frac{\left(1-\varphi^N\right)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)} \left\{ \underbrace{\int \frac{(1-\varphi(i))\, e(i)}{\int (1-\varphi(i))\, e(i)\, di} \left(\hat{W}_s - \sum_l \partial_e e_l(i) \left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)\right)^2 di}_{\text{Labor Distortions with average wealth effect}} \right\}$$

$$- (1-\beta)\sum \beta^s \frac{\left(1-\varphi^N\right)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)} \underbrace{\mathbb{E}_\beta \left(\left\{ \int \left(1-\frac{1}{R}\right) \frac{\frac{(1-\varphi(i))b(i)}{\int (1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int (1-\varphi(i))Wn(i)di}\frac{\int (1-\varphi(i))b(i)di}{\int (1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \left((1-\tilde{\beta}) \sum_{u\geq t_0} \tilde{\beta}^{u-t_0} \left(\hat{W}_u - \sum_l \partial_e e_l(i) \left(\hat{P}_{l,u} + \tilde{A}_{l,u}\right)\right)\right)^2 \right\}\right)}_{\text{Correctionfor the dispersion in wealth effects}}$$

$$+ (1-\beta)\sum \beta^s \underbrace{\int \varphi(i) \frac{Wn(i)}{WN} \frac{\psi}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \left(\hat{W}_s - \sum_l \partial_e e_l(i) \left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)\right)^2 di}_{\text{Labor distortion HtM}}$$

$$+ (1-\beta)\sum \beta^s \frac{\left(1-\varphi^E\right)\sigma}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)} \underbrace{\sum_{k,l} \mathcal{E}_{k,l} \left\{ \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right) - \mathbb{E}_\beta \left((1-\tilde{\beta}) \left(\sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right)\right) \left((1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)\right)\right)\right\}}_{\text{Intertemporal misallocation of expenditures}}$$

$$- (1-\beta)\sum \beta^s \underbrace{\sum_k \bar{s}_k \sum_l \bar{\mathsf{C}}_{k,l} \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)}_{\text{Intratemporal misallocation of expenditures}} + (1-\beta) \sum \beta^s \underbrace{\sum_k \bar{s}_k \vartheta_k \left(\pi_{k,s}\right)^2}_{\text{Within Sector misallocation}} + (1-\beta)\sum \beta^s \underbrace{\left(\hat{W}_s, \hat{P}_s + \tilde{\mathbf{A}}_s\right)^T \mathcal{O} \left(\hat{W}_s, \hat{P}_s + \tilde{\mathbf{A}}_s\right)}_{\text{Input misallocation}}$$

$$+ \underbrace{\mathbb{E}_\beta \left(\int g(i) \left(\nu_{t_0}(i) + (1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \sum_l \frac{Wn(i)}{WN} \bar{s}_l \tilde{A}_{l,s}\right)^2 di\right)}_{\text{Redistribution}} - \underbrace{2\mathbb{E}_\beta \left((1-\tilde{\beta}) \sum_{s\geq t_0} \int \left(\nu_{t_0}(i) \tilde{\beta}^{s-t_0} \sum_l \frac{(1-\varphi(i))\, \partial_e e_l(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \tilde{A}_{l,s}\right) di\right)}_{\text{"Bang for Buck non HtM"}}$$

$$+ \underbrace{\mathbb{E}_\beta \left((1-\tilde{\beta}) \sum \tilde{\beta}^{s-t_0} \int \varphi(i) \frac{E}{\sigma e(i) + Wn(i)\psi} \left\{ \aleph_s^{t_0}(i) + \sum_l \frac{Wn(i)}{WN} \bar{s}_l \tilde{A}_{l,s}\right\}^2 - \int \varphi(i) \frac{E}{\sigma e(i) + Wn(i)\psi} \left\{ \nu_{t_0}(i) + (1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \frac{Wn(i)}{WN} \bar{s}_l \tilde{A}_{l,s}\right\}^2\right)}_{\text{Non Smoothing compensation for HtM}}$$

$$- \underbrace{2\mathbb{E}_\beta \left((1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \int \aleph_s^{t_0}(i) \sum_l \frac{\varphi(i)\, \partial_e e_l(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \tilde{A}_{l,s} di\right)}_{\text{"Bang for Buck HtM"}}$$

$$+ \frac{E}{\int (1-\varphi(i))\left(\sigma e(i) + Wn(i)\psi\right) di} \mathbb{E}_\beta \underbrace{\left\{ (1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \left(\int (1-\varphi(i)) \left\{ \aleph_s^{t_0}(i) + \sum_l \frac{Wn(i)}{WN} \bar{s}_l \tilde{A}_{l,s} - \sum_l \frac{\sigma e(i)}{E} \partial_e e_l(i)\, \tilde{A}_{l,s}\right\}\right)^2 \right\}}_{\text{Smoothing Premium for non HtM}}$$

$$- \frac{E}{\int (1-\varphi(i))\left(\sigma e(i) + Wn(i)\psi\right) di} \mathbb{E}_\beta \underbrace{\left\{ \left(\int (1-\varphi(i)) \left\{ \nu_{t_0}(i) + (1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \sum_l \left(\frac{Wn(i)}{WN} \bar{s}_l \tilde{A}_{l,s} - \frac{\sigma e(i)}{E} \partial_e e_l(i)\, \tilde{A}_{l,s}\right)\right\} di\right)^2 \right\}}_{\text{Smoothing Premium for non HtM}}$$

where $\mathcal{O}$ denote the aggregate input substitution matrix and is given by $\mathcal{O} = \begin{bmatrix} \frac{W\partial_W \mathcal{N}}{N} & \left(\frac{P_k \partial_{P_k} \mathcal{N}}{N}\right)^T \\ \frac{P_k \partial_{P_k} \mathcal{N}}{N} & \frac{P_k \mathcal{Y}_k}{E} \frac{P_l \partial_{P_l} \mathcal{Y}_k}{\mathcal{Y}_k} \end{bmatrix}$.

The last five lines constitute the redistributive motive. the first term is standard it is 0 when households are compensated for their loss of income. the second term is an adjustment taking into account that it is relatively more efficient to compensate households who consume more in relatively more productive sectors. the last twho terms are adjustments taking into account that HtM households cannot smooth their consumption.
With

$$\mathcal{E}_{k,l} = \int \frac{(1-\varphi(i))\, e(i)}{\int (1-\varphi(i))\, e(i)\, di} \partial_e e_l(i)\, \partial_e e_k(i)\, di - \frac{\int (1-\varphi(i))\, e(i)\, \partial_e e_k(i)\, di \int (1-\varphi(i))\, e(i)\, \partial_e e_l(i)\, di}{\left(\int (1-\varphi(i))\, e(i)\, di\right)^2}$$

$$\nu_{t_0}(i) = (1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \left(\frac{b(i)}{ER} \left(\hat{R}_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_{l=1}^{K} \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,t_0+s} \right)$$

$$\nu_-(i) = (1-\tilde{\beta}) \sum_{s\geq 0} \tilde{\beta}^{s} \left(\frac{b(i)}{ER} \left(\hat{R}_{s} - (R-1) \sum_{l} \bar{s}_l \hat{P}_{l,s} \right) - \sum_{l} \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,s} \right)$$

$$\aleph_s^{t_0}(i) = \left(\frac{b(i)}{ER} \left(\hat{R}_{s} - (R-1) \sum_{l} \bar{s}_l \hat{P}_{l,s} \right) - \sum_{l} \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,s} \right) + \sum_l \frac{R-1}{R} \frac{b(i)}{E} \bar{s}_l \hat{P}_{l,t_0}$$

$$\aleph_-(i) = \left(\frac{b(i)}{ER} \left(\hat{R}_{s} - (R-1) \sum_{l} \bar{s}_l \hat{P}_{l,s} \right) - \sum_{l} \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,s} \right)$$

$$g(i) = -\frac{G''(V(i))\, v'(e(i))\, E}{(1-\tilde{\beta})\, G'(V(i))} + \frac{E}{\sigma e(i) + Wn(i)\psi}$$

Without HtM households and IO, this simplifies to:

$$\mathcal{L} = (1-\beta) \sum \beta^s \frac{\psi}{1+\frac{\psi}{\sigma}} \left\{ \underbrace{\int \frac{e(i)}{E} \left(\hat{W}_s - \sum_l \partial_e e_l(i) \left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right) \right)^2 di}_{\text{Labor Distortions with average wealth effect}} - \underbrace{\mathbb{E}_\beta \left(\left\{ \int \left(1 - \frac{1}{R}\right) \frac{\frac{b(i)}{E}}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \left((1-\tilde{\beta}) \sum_{u\geq t_0} \tilde{\beta}^{u-t_0} \hat{W}_u - \sum_l \partial_e e_l(i) \left(\hat{P}_{l,u} + \tilde{A}_{l,u}\right) \right)^2 \right\} \right)}_{\text{Correctionfor the dispersion in wealth effects}} \right\}$$

$$(1-\beta) \sum \beta^s \frac{\sigma}{1+\frac{\psi}{\sigma}} \underbrace{\sum_{k,l} \mathcal{E}_{k,l} \left\{ \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right) - \mathbb{E}_\beta \left((1-\tilde{\beta}) \left(\sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right) \right) \left((1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right) \right) \right) \right\}}_{\text{Intertemporal misallocation of expenditures}}$$

$$-(1-\beta) \sum \beta^s \underbrace{\sum_k \bar{s}_k \sum_l \bar{\complement}_{k,l} \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)}_{\text{Intratemporal misallocation of expenditures}} + (1-\beta) \sum \beta^s \underbrace{\sum_k \bar{s}_k \vartheta_k \left(\pi_{k,s}\right)^2}_{\text{Within Sector misallocation}}$$

$$+ \underbrace{\mathbb{E}_\beta \left(\int g(i) \left(\nu_{t_0}(i) + (1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \sum_l \frac{Wn(i)}{WN} \bar{s}_l \tilde{A}_{l,s} \right)^2 di \right)}_{\text{Redistribution}} - \underbrace{2\mathbb{E}_\beta \left((1-\tilde{\beta}) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \int \left(\nu_{t_0}(i) \sum_l \frac{\partial_e e_l(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \tilde{A}_{l,s} \right) di \right)}_{\text{"Bang for Buck"}}$$

Defining $\tilde{P}_{l,s} = \hat{P}_{l,s} + \tilde{A}_{l,s}$, the markup in k, and using $\tilde{\mathcal{Y}}_t = \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_l \overline{\partial_e e_l}\left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right)\right)$ we can re-express the loss function directly in terms of the output gap :

$$\mathcal{L} = (1-\beta)\sum \beta^s \frac{\psi}{1+\frac{\psi}{\sigma}}\left\{\left(\frac{1}{\sigma}+\frac{1}{\psi}\right)^2 \underbrace{\tilde{\mathcal{Y}}_s^2} + \mathcal{E}_{k,l}\tilde{P}_{k,s}\tilde{P}_{l,s} - \mathbb{E}_\beta\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{b(i)}{E}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left((1-\tilde{\beta})\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\frac{1}{\sigma}+\frac{1}{\psi}\right)\tilde{\mathcal{Y}}_u - \sum_l\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\tilde{P}_{l,s}\right)^2\right\}\right)\right\}$$

$$(1-\beta)\sum\beta^s\frac{\sigma}{1+\frac{\psi}{\sigma}}\sum_{k,l}\mathcal{E}_{k,l}\left\{\tilde{P}_{k,s}\tilde{P}_{l,s} - \mathbb{E}_\beta\left(\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\tilde{P}_{k,s}\right)\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\tilde{P}_{l,s}\right)\right)\right\}$$

$$-(1-\beta)\sum\beta^s\sum_k\bar{s}_k\sum_l\bar{\complement}_{k,l}\tilde{P}_{k,s}\tilde{P}_{l,s} + (1-\beta)\sum\beta^s\sum\bar{s}_k\vartheta_k\left(\pi_{k,s}\right)^2$$

$$+\mathbb{E}_\beta\left(\int g(i)\left(\nu_{t_0}(i) + (1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_l\frac{Wn(i)}{WN}\bar{s}_l\tilde{A}_{l,s}\right)^2 di\right) - 2\mathbb{E}_\beta\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\int\left(\nu_{t_0}(i)\sum_l\frac{\partial_e e_l(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\tilde{A}_{l,s}\right)di\right)$$

System of Equations without redistributive motive:

We now derive the system of optimal equation when the central bank ignores redistributive motives (last five lines of the loss function for the full model). Defining:

$$\mathcal{L}^{nd} \equiv (1-\beta)\sum\beta^s\frac{(1-\varphi^N)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)}\left\{\underbrace{\int\frac{(1-\varphi(i))e(i)}{\int(1-\varphi(i))e(i)di}\left(\hat{W}_s - \sum_l\partial_e e_l(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di}_{\text{Labor Distortions with average wealth effect}}\right\}$$

$$-(1-\beta)\sum\beta^s\frac{(1-\varphi^N)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)}\underbrace{\mathbb{E}_\beta\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left((1-\tilde{\beta})\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{W}_u - \sum_l\partial_e e_l(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)^2\right\}\right)}_{\text{Correctionfor the dispersion in wealth effects}}$$

$$+(1-\beta)\sum\beta^s\underbrace{\int\varphi(i)\frac{Wn(i)}{WN}\frac{\psi}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left(\hat{W}_s - \sum_l\partial_e e_l(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di}_{\text{Labor distortion HtM}}$$

$$(1-\beta)\sum\beta^s\frac{(1-\varphi^E)\sigma}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)}\underbrace{\sum_{k,l}\mathcal{E}_{k,l}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right) - \mathbb{E}_\beta\left((1-\tilde{\beta})\left(\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left((1-\tilde{\beta})\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}}_{\text{Intertemporal misallocation of expenditures}}$$

$$-(1-\beta)\sum\beta^s\underbrace{\sum_k\bar{s}_k\sum_l\bar{\complement}_{k,l}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)}_{\text{Intratemporal misallocation of expenditures}} + (1-\beta)\sum\beta^s\underbrace{\sum_k\bar{s}_k\vartheta_k\left(\pi_{k,s}\right)^2}_{\text{Within Sector misallocation}} + (1-\beta)\sum\beta^s\underbrace{\left(\hat{W}_s,\hat{P}_s+\tilde{\mathbf{A}}_s\right)^T\mathcal{O}\left(\hat{W}_s,\hat{P}_s+\tilde{\mathbf{A}}_s\right)}_{\text{Input misallocation}}$$

The central bank solves

$$\inf_{\left\{\hat{W}_t, \hat{\boldsymbol{P}}_t\right\}_{t\geq 0}} \mathcal{L}^{nd}$$

under the constraints

$$\begin{aligned}
\pi_{k,t} &= \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\Omega_{N,k} \sum_l \overline{\partial_e e_l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right)\right) + \sum_l \Omega_{k,l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right)\right) + s_k^C \mathcal{M}_{k,t} \right) + \beta \pi_{k,t+1}, \\
\hat{P}_{k,t} &= \hat{P}_{k,t-1} + \pi_{k,t}, \\
\mathcal{M}_{k,t} &= \sum_l \int \gamma_{e,k}(i) \frac{e_k(i)}{E_k} \rho_{k,l}(i) di \hat{P}_{l,t} + \frac{1 + \frac{\bar{\psi}}{\bar{\sigma}}}{1 - \frac{1}{R}} \int \gamma_{b,k}(i) \frac{Wn\left(i\right)}{WN} di \hat{\mathcal{Y}}_t^* + \mathcal{M}_{k,t}^D, \\
\mathbb{E}_t \mathcal{M}_{k,t+1}^D - \mathcal{M}_{k,t}^D &= \sum_l \sigma_{k,l}^{\mathcal{M},u} \left(\hat{R}_t - \pi_{l,t+1}\right) + \frac{\delta}{1-\delta} \mathbb{E}_t \mathcal{M}_{k,t+1}^0 \\
&+ \frac{R}{R-1} \int \left(\gamma_{b,k}^u(i) \left(\varphi(i) \frac{b(i)}{RE} - \frac{\left(1-\varphi(i)\right) Wn(i)}{\left(1-\varphi^N\right) WN} \int \left(\varphi(i) \frac{b(i)}{RE} \right) di \right) \right) di \mathbb{E}_t \Delta \hat{R}_{t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma_{b,k}^u(i) \left\{ (1 - \frac{1}{R}) \left(\varphi(i) \frac{b(i)}{E} - \frac{\left(1-\varphi(i)\right) Wn(i)}{\left(1-\varphi^N\right) WN} \int \left(\varphi(i) \frac{b(i)}{E} \right) di \right) \bar{s}_l \right\} di \mathbb{E}_t \pi_{l,t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma_{b,k}^u(i) \left\{ \left(\varphi(i) \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) - \frac{\left(1-\varphi(i)\right) Wn(i)}{\left(1-\varphi^N\right) WN} \int \varphi(i) \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) di \right) \right\} di \mathbb{E}_t \pi_{l,t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma_{b,k}^u(i) \left\{ \frac{Wn(i)}{WN} \psi \left(\varphi(i) \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) - \frac{1-\varphi(i)}{1-\varphi^E} \int \varphi(i) \frac{e(i)}{E} \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) di \right) \right\} di \mathbb{E}_t \pi_{l,t+1}, \\
\mathcal{M}_{k,t}^0 - \frac{1}{\left(1-\delta\right) R} \mathbb{E}_t \mathcal{M}_{k,t+1}^0 &= \int \gamma_{b,k}^u(i) \frac{b(i)}{RE} di \left(\hat{R}_t - \sum_l \bar{s}_l \mathbb{E}_t \pi_{l,t+1} \right) \\
&- \sum_l \int \gamma_{b,k}^u(i) \left(\frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) + \psi \frac{Wn(i)}{WN} \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) \right) di \hat{P}_{l,t} - \frac{R-1}{R} \mathcal{M}_{k,t}^D, \\
\left(1 - \varphi^N\right) \left(\mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t\right) &= \left(1-\varphi^N\right) \sigma \mathbb{E}_t \left(\hat{R}_t - \sum_k \overline{\partial_e e_k} \pi_{k,t+1} - r_t^* \right) + \mathbb{E}_t \int \frac{\sigma \varphi\left(i\right)}{\sigma + \psi} \left\{ \frac{b(i)}{RE} \left(\Delta \hat{R}_{t+1} - \sigma \left(R-1\right) \hat{R}_t\right) - \frac{e(i)}{E} \sum_k \left(\left(s_k(i) - \bar{s}_k\right) - \sigma(\partial_e e_k(i) - \overline{\partial_e e_k}) \right) \pi_{k,t+1} \right\} di \\
&- \mathbb{E}_t \int \frac{\sigma \varphi\left(i\right)}{\sigma + \psi} \left\{ \left(1 - \frac{1}{R}\right) \frac{b(i)}{E} \left(\pi_{cpi,t+1} - \sigma \pi_{mcpi,t+1}\right) \right\} di, \\
\tilde{\mathcal{Y}}_t &= \frac{\sigma \psi}{\sigma + \psi} \left(\hat{W}_t - \sum_k \overline{\partial_e e_k} \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right) \right).
\end{aligned}$$

Denoting $\check{\mu}_{k,t}$ the lagrange multiplier on the NKPC of sector k and $M_{k,t}$ the lagrange multiplier on the price evolution equation and taking derivatives with respect to the wage at t directly gives:

$$\sum_{k=1}^{K}\Omega_{N,k}\lambda_k\check{\mu}_{k,t}=Z_t+\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\frac{1}{R}\left(A_{b,t}-(1-\delta)RA_{b,t-1}\right)$$
$$-\left(\frac{\int(1-\varphi(i))\,\frac{Wn(i)}{WN}\psi di\int(1-\varphi(i))\,\sigma e(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}+\int\varphi(i)\,\psi\frac{Wn(i)}{WN}\frac{\sigma e(i)}{\sigma e(i)+\psi n(i)}di\right)\hat{W}_t$$
$$+\sum_k\left(\frac{\int(1-\varphi(i))\,\frac{Wn(i)}{WN}\psi di\int(1-\varphi(i))\,\sigma e(i)\,\partial_e e_k(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}+\int\varphi(i)\,\psi\frac{Wn(i)}{WN}\frac{\sigma e(i)\,\partial_e e_k(i)}{\sigma e(i)+\psi n(i)}di\right)\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right)$$
$$-\sum_k\lambda_k\frac{E_k}{P_kY_k}\left(\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di\int(1-\varphi(i))\,\sigma\frac{e(i)}{E_k}\gamma_{e,k}(i)\,\partial_e e_k(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}+\int\varphi(i)\,\frac{\sigma\psi Wn(i)}{\sigma e(i)+\psi n(i)}\frac{e(i)}{E_k}\gamma_{e,k}(i)\,\partial_e e_k(i)\,di\right)\check{\mu}_{k,t}$$
$$+\sum_{k,l}\frac{P_lW\partial_W\mathcal{Y}_{l,k}Y_k}{A_kE}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_k\frac{W\partial_W\mathcal{N}_k}{A_kN}Y_k\hat{W}_t$$

With

$$Z_{t+1}-Z_t=\frac{\delta}{1-\delta}Z^0_{t+1}$$
$$Z^0_{t-1}-\frac{1}{(1-\delta)\,R}Z^0_t+\left(1-\frac{1}{R}\right)Z_{t-1}=\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,\sigma\frac{e(i)}{E}\left(\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}-\frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)di\hat{W}_{t-1}$$
$$-\sum_k\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,\sigma\frac{e(i)}{E}\partial_e e_k(i)\left(\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}-\frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)di\left(\tilde{A}_{k,t-1}+\hat{P}_{k,t-1}\right)$$
$$+\sum_k\frac{\lambda_kE_k}{P_kY_k}\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,\sigma\frac{e(i)}{E_k}\partial_e e_k(i)\,\gamma_{e,k}(i)\left(\frac{\int(1-\varphi(i))\,Wn(i)\,\psi di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}-\frac{\psi Wn(i)}{\sigma e(i)+\psi Wn(i)}\right)di\check{\mu}_{k,t-1}$$

And

$$\left(1-\frac{\int\varphi(i)\,\frac{1}{R}b(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\right)A_{b,t+1}-\left(1-\frac{(1-\delta+R)\int\varphi(i)\,\frac{1}{R}b(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\right)A_{b,t}-\frac{(1-\delta)R\int\varphi(i)\,\frac{1}{R}b(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}A_{b,t-1}-\frac{\delta}{1-\delta}A^0_{b,t+1}=$$
$$+\sum_k\left(\int\varphi(i)\,b(i)\,\frac{\sigma}{\sigma e(i)+\psi Wn(i)}\frac{e(i)}{E_k}\gamma_{e,k}(i)\,\partial_e e_k(i)\,di+\frac{\int(1-\varphi(i))\,b(i)\,di\int(1-\varphi(i))\,\sigma\frac{e(i)}{E_k}\partial_e e_k(i)\,\gamma_{e,k}(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\right)\frac{\lambda_kE_k}{P_kY_k}\Delta\check{\mu}_{k,t+1}$$

$$A^0_{b,t-1}-\frac{1}{(1-\delta)\,R}A^0_{b,t}+\left(1-\frac{1}{R}\right)A_{b,t-1}=\left(1-\frac{1}{R}\right)\sum_k\frac{\lambda_kE_k}{P_kY_k}\int\frac{b(i)}{E_k}\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\gamma_{e,k}(i)\,\partial_e e_k(i)\,di\check{\mu}_{k,t-1}$$

Taking derivatives with respect to the price of k at t gives:

$$M_{k,t}=\check{\mu}_{k,t}-(1-\delta)\,\check{\mu}_{k,t-1}$$

$$\beta M_{k,t+1}-M_{k,t}=\lambda_k\check{\mu}_{k,t}-\sum_{l=1}^{K}\frac{\lambda_l E_l}{P_l Y_l}\int\gamma_{e,l}(i)\frac{e_l(i)}{E_l}\rho_{l,k}(i)\,di\check{\mu}_{l,t}-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_l Y_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}+\sum_{l,m}\frac{P_l P_k\partial_{P_k}\mathcal{Y}_{l,m}}{A_m E}Y_m\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_l\frac{P_k\partial_{P_k}\mathcal{N}_l}{A_l N}Y_l\hat{W}_t$$
$$+\sum_l\bar{s}_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-\frac{P_kY_k}{E}\vartheta_k\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)-L_{k,t}$$
$$+\left(\frac{\int(1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di\int(1-\varphi(i))\,\psi\frac{Wn(i)}{WN}}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}+\int\varphi(i)\,\partial_e e_k(i)\,\frac{\psi Wn(i)}{WN}\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}di\right)\hat{W}_t$$
$$-\sum_l\left(\int\varphi(i)\,\partial_e e_l(i)\,\partial_e e_k(i)\,\psi\frac{Wn(i)}{WN}\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}-\frac{\int(1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di\int(1-\varphi(i))\,\sigma\frac{e(i)}{E}\partial_e e_l di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}+\int(1-\varphi(i))\,\sigma\frac{e(i)}{E}\partial_e e_k(i)\,\partial_e e_l(i)\,di\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)$$
$$+\sum_{l=1}^{K}\frac{\lambda_l E_l}{P_l Y_l}\left(\int(1-\varphi(i))\,\partial_e e_k(i)\,\frac{\sigma e(i)}{E_l}\gamma_{e,l}(i)\,\partial_e e_l(i)\,di+\int\varphi(i)\,\partial_e e_k(i)\,\psi Wn(i)\,\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\frac{1}{E_l}\gamma_{e,l}(i)\,\partial_e e_l(i)\,di\right)\check{\mu}_{l,t}$$
$$-\sum_{l=1}^{K}\frac{\lambda_l E_l}{P_l Y_l}\frac{\int(1-\varphi(i))\,\partial_e e_k(i)\,\sigma e(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\int(1-\varphi(i))\,\partial_e e_l(i)\,\frac{\sigma e(i)}{E_l}\gamma_{e,l}(i)\,di\check{\mu}_{l,t}$$
$$+\left(\frac{\int(1-\varphi(i))\,\partial_e e_k\sigma e di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}+\frac{\int\varphi(i)\,(e_k(i)-e(i)\bar{s}_k)}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\right)\frac{1}{R}\left(A_{b,t}-R(1-\delta)\,A_{b,t-1}\right)+\left(1-\frac{1}{R}\right)\bar{s}_kA_{b,t}-\delta\bar{s}_k\mathcal{A}_{b,t}+A_{e_k,t}$$

With

$$\Delta L_{k,t+1}=\frac{\delta}{1-\delta}L^0_{k,t+1}$$
$$L^0_{k,t-1}-\frac{1}{R(1-\delta)}L^0_{k,t}+\left(1-\frac{1}{R}\right)L_{k,t-1}=\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,\sigma\frac{e(i)}{E}\left(\frac{\sigma e(i)\,\partial_e e_k(i)}{\sigma e(i)+\psi Wn(i)}-\frac{\int(1-\varphi(i))\,\sigma e(i)\,\partial_e e_k(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\right)di\hat{W}_{t-1}$$
$$-\sum_l\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,\sigma\frac{e(i)}{E}\partial_e e_l(i)\left(\frac{\sigma e(i)\,\partial_e e_k(i)}{\sigma e(i)+\psi Wn(i)}-\frac{\int(1-\varphi(i))\,\sigma e(i)\,\partial_e e_k(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\right)di\left(\tilde{A}_{l,t-1}+\hat{P}_{l,t-1}\right)$$
$$+\sum_l\frac{\lambda_l E_l}{P_l Y_l}\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,\sigma\frac{e(i)}{E_l}\partial_e e_l(i)\,\gamma_{e,l}(i)\left(\frac{\sigma e(i)\,\partial_e e_k(i)}{\sigma e(i)+\psi Wn(i)}-\frac{\int(1-\varphi(i))\,\sigma e(i)\,\partial_e e_k(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\right)di\check{\mu}_{l,t-1}$$
$$A_{e_k,t+1}-A_{e_k,t}-\frac{\delta}{1-\delta}A^0_{e_k,t+1}=\sum_l\left(\int\varphi(i)\,e(i)\,(s_k(i)-\bar{s}_k)\,\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\frac{1}{E_l}\gamma_{e,l}(i)\,\partial_e e_l(i)\,di\right)\frac{\lambda_l E_l}{P_l Y_l}\lambda_l\Delta\check{\mu}_{l,t+1}$$
$$+\sum_l\left(\frac{\int(1-\varphi(i))\,e(i)\,(s_k(i)-\bar{s}_k)\,di\int(1-\varphi(i))\,\sigma e(i)\,\partial_e e_l(i)\,\frac{1}{E_l}\gamma_{e,l}(i)\,di}{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di}\right)\frac{\lambda_l E_l}{P_l Y_l}\lambda_l\Delta\check{\mu}_{l,t+1}$$
$$A^0_{e_k,t-1}-\frac{1}{(1-\delta)R}A^0_{e_k,t}+\left(1-\frac{1}{R}\right)A_{e_k,t-1}=\sum_k\lambda_l\frac{E_l}{P_l Y_l}\left(1-\frac{1}{R}\right)\int e(i)\,(s_k(i)-\bar{s}_k)\,\frac{\sigma e(i)\,/E_l}{\sigma e(i)+\psi Wn(i)}\gamma_{e,l}(i)\,\partial_e e_l(i)\,di\check{\mu}_{l,t-1}$$
$$\mathcal{A}_{b,t}-\frac{1}{R}\mathcal{A}_{b,t+1}=\left(1-\frac{1}{R}\right)\sum_k\lambda_k\frac{E_k}{P_kY_k}\int\frac{b(i)}{E_k}\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\gamma_{e,k}(i)\,\partial_e e_k(i)\,di\check{\mu}_{k,t}$$

Efficiency of the Steady State

For first order change in prices, the change in welfare is

$$d\mathcal{W}=\mathbb{E}_\beta\int G'(V(i))\,\partial_e v(i)\,\mathbb{E}_{t_0}\sum_{s=0}^{\infty}((1-\delta)\beta)^s\left\{\left(\frac{b(i)}{ER}\left(\hat{R}_s-(R-1)\sum_l\bar{s}_l\hat{P}_{l,s}\right)-\sum_l\frac{e(i)}{E}(s_l-\bar{s}_l)\,\hat{P}_{l,s}\right)+\sum_l\frac{R-1}{R}\frac{b(i)}{E}\bar{s}_l\hat{P}_{l,t_0}\right\}di$$

Using the fact that $G'(V(i))\partial_e v(i) = 1$ and that the market for bonds clear, we have that

$$d\mathcal{W} = 0$$

For any change in prices (subvariety prices, wage and interest rate). Therefore there is no change in monetary policy that can improve upon the steady state. It is also direct to verify that there are no taxes at any date t, financed by a lump sum at t than can improve the steady state. To see this consider, for example a wage subsidy in sector k at t subsidized by an arbitrary lump sum. The total impact on welfare is given by

$$\beta^t \mathbb{E}_0 \int G'(V(i))\partial_e v(i)\left\{\varsigma(i)\frac{Y_k}{A_k}\mathcal{N}_k(\boldsymbol{P}_t, W_t)dW_{k,t} - d\tau_t(i)\right\} di$$

With

$$\int \frac{Y_k}{A_{k,t}}\mathcal{N}_k(\boldsymbol{P}_t, W_t)dW_{k,t}di = \int d\tau_t(i)\, di$$

Using again $G'(V(i))\partial_e v(i) = 1$ and $\int \varsigma(i)\, di = 1$, this subsidy has no impact on Welfare. Similarly, wage subsidy to workers, input taxes and commodity taxes have no impact on welfare. The economy is therefore constrained efficient: the only inefficiency is the uninsured death risk which could be corrected through an annuity market but neither through monetary policy or through taxes (if the government cannot have debt).

We now verify that the second order approximation of the Welfare Function is always negative. This will imply that a steady state with 0 inflation and output gap is a local maximum. First note that a direct application of Cauchy-Schwartz gives that for any variable x_t and any weight $0 < \alpha < 1$,

$$\mathbb{E}_0\left(\sum_{s\geq 0}\alpha^s x_{t+s}\right)^2 \leq \mathbb{E}_0\frac{1}{(1-\alpha)}\sum_{s\geq 0}\alpha^s (x_{t+s})^2$$

Given the definition of $\mathbb{E}_\beta$we have that $\sum_{s=0}^{\infty}\beta^s X_s = \mathbb{E}_\beta\left[\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}X_s\right]$ so we can rewrite the average labor distortion term (which we denote by L_1) as

$$(1-\beta)\sum\beta^s\left\{\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left(\hat{W}_s - \sum_l \partial_e e_l(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di\right\} = (1-\tilde{\beta})\,\mathbb{E}_\beta\left(\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left(\hat{W}_s - \sum_l \partial_e e_l(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di\right)$$

We can re-write the term that appears in the dispersion of wealth effect as

$$\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}} = \frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di} - (1-\varphi(i))\frac{e(i)\,Wn(i)\,/E}{\sigma e(i)+Wn(i)\,\psi}\left(\frac{\sigma}{(1-\varphi^N)}+\frac{\psi}{(1-\varphi^E)}\right),$$

where the second term is always weakly negative and hence

$$L_2 \equiv \mathbb{E}_\beta\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left((1-\tilde{\beta})\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{W}_u - \sum_l \partial_e e_l(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)^2 di\right\}\right) \leq$$

$$\mathbb{E}_\beta\left(\left\{\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left((1-\tilde{\beta})\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{W}_u - \sum_l \partial_e e_l(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)^2 di\right\}\right)$$

Putting these results together we have that

$$L_1 + L_2 \geq (1-\tilde{\beta})\,\mathbb{E}_\beta\left(\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{W}_s - \sum_l \partial_e e_l(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^2 di\right) - \mathbb{E}_\beta\left(\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left((1-\tilde{\beta})\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{W}_u - \sum_l \partial_e e_l(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)\right.$$

$$\geq 0$$

where the last line follows from our preliminary Cauchy-Schwartz result.

Next consider the term

$$(1-\beta)\sum\beta^{s}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\left(\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}$$

$$=\left(1-\tilde{\beta}\right)\mathbb{E}_{\beta}\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\left(\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)$$

$$=\left(1-\tilde{\beta}\right)\mathbb{E}_{\beta}\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}-\left(1-\tilde{\beta}\right)\left(\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{P}_{k,u}+\tilde{A}_{k,s}\right)\right)\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}-\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)$$

Since $\mathcal{E}$ is a variance-covariance matrix, it is positive semi-definite, therefore:

$$(1-\beta)\sum\beta^{s}\frac{\left(1-\varphi^{E}\right)\sigma}{\left(1+\frac{\left(1-\varphi^{N}\right)\psi}{\left(1-\varphi^{E}\right)\sigma}\right)}\sum_{k,l}\mathcal{E}_{k,l}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\left(\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}\geq 0$$

Next, we have:

$$-(1-\beta)\sum\beta^{s}\sum_{k}\bar{s}_{k}\sum_{l}\bar{\mathcal{C}}_{k,l}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\geq 0$$

$$-(1-\beta)\sum\beta^{s}\left\{\frac{W\partial_{W}\mathcal{N}}{N}\hat{W}_{s}^{\,2}+\sum_{k}\frac{P_{k}\mathcal{Y}_{k}}{E}\frac{W\partial_{W}\mathcal{Y}_{k}}{\mathcal{Y}_{k}}\hat{W}_{s}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)+\sum_{k}\frac{P_{k}\partial_{P_{k}}\mathcal{N}}{N}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)\hat{W}_{s}+\sum_{k,l}\frac{P_{k}\mathcal{Y}_{k}}{E}\frac{P_{l}\partial_{P_{l}}\mathcal{Y}_{k}}{\mathcal{Y}_{k}}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{\mathbf{A}}_{l,s}\right)\right\}\geq 0$$

Since both the substitution and the transformation matrix are negative semi-definite.
Finally, we can rewrite the redistribution terms as

$$+\mathbb{E}_{\beta}\left(\int-\frac{G''\left(V\left(i\right)\right)v'\left(e\left(i\right)\right)E}{\left(1-\tilde{\beta}\right)G'\left(V\left(i\right)\right)}\left(\nu_{t_0}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_{l}\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}\right)^{2}di\right)$$

$$+\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\int\left(1-\varphi\left(i\right)\right)\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left(\nu_{t_0}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\sum_{l}\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}-\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,s}\right)\right)di\right)$$

$$+\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum\tilde{\beta}^{s-t_0}\int\varphi\left(i\right)\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\aleph_{s}^{t_0}(i)+\sum_{l}\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}-\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,s}\right\}^{2}\right)$$

$$+\frac{E}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\mathbb{E}_{\beta}\left\{\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\int\left(1-\varphi\left(i\right)\right)\left\{\aleph_{s}^{t_0}(i)+\sum_{l}\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}-\sum_{l}\frac{\sigma e\left(i\right)}{E}\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,s}\right\}\right)^{2}\right\}$$

$$-\frac{E}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\mathbb{E}_{\beta}\left\{\left(\int\left(1-\varphi\left(i\right)\right)\left\{\nu_{t_0}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_{l}\left(\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}-\frac{\sigma e\left(i\right)}{E}\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,s}\right)\right\}di\right)^{2}\right\}$$

Note that

$$\left(1-\tilde{\beta}\right)\sum\tilde{\beta}^{s-t_0}\aleph_{s}^{t_0}(i)=\nu_{t_0}(i)$$

So

$$\mathbb{E}_\beta \left\{ \left(1-\tilde{\beta}\right) \sum_{s \geq t_0} \tilde{\beta}^{s-t_0} \left(\int \left(1-\varphi\left(i\right)\right) \left\{ \aleph_s^{t_0}\left(i\right) + \sum_l \frac{e\left(i\right)}{E} s_l\left(i\right) \tilde{A}_{l,s} - \sum_l \frac{\sigma e\left(i\right)}{E} \partial_e e_l\left(i\right) \tilde{A}_{l,s} \right\} \right)^2 \right\}$$

$$-\mathbb{E}_\beta \left\{ \left(\int \left(1-\varphi\left(i\right)\right) \left\{ \nu_{t_0}\left(i\right) + \left(1-\tilde{\beta}\right) \sum_{s \geq t_0} \tilde{\beta}^{s-t_0} \sum_l \left(\frac{e\left(i\right)}{E} s_l\left(i\right) \tilde{A}_{l,s} - \frac{\sigma e\left(i\right)}{E} \partial_e e_l\left(i\right) \tilde{A}_{l,s} \right) \right\} di \right)^2 \right\} \geq 0$$

Therefore, all terms in the social welfare function are positive: there are no second order deviations in prices, wage and interest rates improving on the steady state.

F Additional analytical results

Addition to Result 1: divine Coincidence indices without endogenous markups

We now derive an inflation index which can be fully stabilized alongside the output gap, for the case with the endogenous markup wedge. In this case, the sectoral NKPC can be written as:

$$\pi_{k,t} = \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\sum_l \left(\Omega_{N,k} \overline{\partial_e e_l} + \Omega_{k,l} \right) \left(\hat{P}_{l,t} - \hat{P}^*_{l,t} \right) - \left(\hat{P}_{k,t} - \hat{P}^*_{k,t} \right) \right) + \beta \mathbb{E}_t \pi_{k,t+1},$$

or in matrix form:

$$\boldsymbol{\pi_t} = \boldsymbol{\kappa} \tilde{\mathcal{Y}}_t + \mathcal{D}\left[\lambda\right] \left(\tilde{\Omega} - Id\right) \left(\boldsymbol{\hat{P}_t} - \boldsymbol{\hat{P}^*_t}\right) + \beta \mathbb{E}_t \boldsymbol{\pi_{t+1}}.$$

where $\boldsymbol{\kappa} = [\kappa_1, ..., \kappa_K]^T$, $\mathcal{D}[\lambda]$ is a $K \times K$ diagonal matrix with λ_k on the diagonal and $\tilde{\Omega}_{k,l} = \Omega_{N,k}\overline{\partial_e e_l} + \Omega_{k,l}$. Note that

$$\tilde{\Omega}_{k,l} \geq 0,$$
$$\sum_l \tilde{\Omega}_{k,l} = 1.$$

The Perron-Frobenius' theorem for row-stochastic matrices implies that we have an eigenvector $\tilde{\boldsymbol{\omega}} = [\tilde{\omega}_1, ..., \tilde{\omega}_K]$ with $\tilde{\omega}_k \geq 0$ and $\sum_k \tilde{\omega}_k = 1$ (normalization) such that

$$\tilde{\boldsymbol{\omega}} \tilde{\Omega} = \tilde{\boldsymbol{\omega}}.$$

Now, define

$$\boldsymbol{\omega} = \left[\frac{\lambda}{\lambda_1} \tilde{\omega}_1, ..., \frac{\lambda}{\lambda_K} \tilde{\omega}_K \right],$$
$$\frac{1}{\lambda} = \sum_k \frac{1}{\lambda_k} \tilde{\omega}_k.$$

Note that we have

$$\boldsymbol{\omega} \mathcal{D}[\lambda] \left(\tilde{\Omega} - Id\right) = \lambda \tilde{\boldsymbol{\omega}} \mathcal{D}^{-1}[\lambda] \mathcal{D}[\lambda] \left(\tilde{\Omega} - Id\right) = 0.$$

Now define

$$\pi_{d,t} = \sum \omega_k \pi_{k,t},$$

We have

$$\begin{aligned} \pi_{d,t} &= \kappa \tilde{\mathcal{Y}}_t + \boldsymbol{\omega} \mathcal{D}[\lambda] \left(\tilde{\Omega} - Id\right) \left(\boldsymbol{\hat{P}_t} - \boldsymbol{\hat{P}^*_t}\right) + \beta \mathbb{E}_t \pi_{d,t+1}, \\ &= \kappa \tilde{\mathcal{Y}}_t + \beta \mathbb{E}_t \pi_{d,t+1}. \end{aligned}$$

With $\kappa = \sum \omega_k \kappa_k$. Therefore $\pi_{d,t}$ can be stabilized jointly with the output gap.

Addition to Result 1: HtM and I-O

In this section we show how to extend Result 1 to the case with HtM households and Input-Output links. We slightly amend the assumption (**A.1**) and (**A.2**):

- **Assumption A1:** $\kappa_k = \kappa$ for all k (recall that with IO $\kappa_k = \lambda_k \left(\frac{1}{\sigma} + \frac{1}{\psi}\right) \left(\Omega_{N,k} + s^C_k \frac{\sigma\psi}{\sigma+\psi} \Gamma_k\right)$)
- **Assumption A2:** $\int \gamma^u_{b,k}(i) \left\{ \frac{(1-\varphi(i))b}{E} - \frac{(1-\varphi(i))Wn}{(1-\varphi^L)WN} \int (1-\varphi(i)) \frac{b(i)}{E} di \right\} di = \int \gamma^{HtM}_{b,k}(i) \left\{ \frac{\varphi(i)b}{E} - \frac{(1-\varphi(i))Wn}{(1-\varphi^L)WN} \int \varphi(i) \frac{b(i)}{E} di \right\} di = 0$

Note that (**A.2**) is slightly strengthened with HtM. Without HtM we only need to assume that $\gamma_{b,k}(i)$ is uncorrelated with wealth. With HtM we assume that $\gamma_{b,k}(i)$ is uncorrelated with both the wealth of the HtM and the unconstrained households. We rewrite once again the system of equation of relative prices $\tilde{P}_{k,t} = \hat{P}_{k,t} - \hat{P}_{d,t}$, with $\hat{P}_{d,t}$ defined in the previous section:

$$\tilde{\pi}_{k,t} = \left(\lambda_k s_k^C \mathcal{M}_{k,t} - \sum_l \lambda_l s_l^C \omega_l \mathcal{M}_{l,t}\right) + \lambda_k \left(+\sum \left(\Omega_{N,k}\overline{\partial_e e_l} + \Omega_{k,l}\right)\left(\tilde{P}_{l,t} - \tilde{P}^*_{l,t}\right) - \left(\tilde{P}_{k,t} - \tilde{P}^*_{k,t}\right)\right) + \beta \mathbb{E}_t \tilde{\pi}_{k,t+1}$$

$$\mathcal{M}_{k,t} = \Gamma_k \hat{\mathcal{Y}}^*_t + \mathcal{M}^P_{k,t} + \mathcal{M}^D_{k,t}$$

$$\mathcal{M}^P_{k,t} = \sum_l \mathcal{S}_{k,l} \tilde{P}_{l,t}$$

Therefore to prove that relative prices evolve independently of monetary policy, we only need show that under (**A.2**), $\mathcal{M}^D_{k,t}$ only depends on relative prices. Recall that we have:

$$\begin{aligned}
\mathbb{E}_t \mathcal{M}^D_{k,t+1} - \mathcal{M}^D_{k,t} = &\left(\sigma_k^{\mathcal{M},u} \hat{R}_t - \sum_l \sigma_{k,l}^{\mathcal{M},u} \pi_{l,t+1}\right) + \frac{\delta}{1-\delta} \mathbb{E}_t \mathcal{M}^0_{k,t+1} \\
&+ \frac{R}{R-1} \int \left(\gamma^u_{b,k}(i) \left(\varphi(i)\frac{b(i)}{RE} - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^L)\, WN} \int \left(\varphi(i)\frac{b(i)}{RE}\right) di\right)\right) di \mathbb{E}_t \Delta \hat{R}_{t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma^u_{b,k}(i) \left\{\left(1 - \frac{1}{R}\right)\left(\varphi(i)\frac{b(i)}{E} - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^L)\, WN} \int \left(\varphi(i)\frac{b(i)}{E}\right) di\right) \bar{s}_l\right\} di \mathbb{E}_t \pi_{l,t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma^u_{b,k}(i) \left\{\left(\varphi(i)\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^L)\, WN} \int \varphi(i)\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) di\right)\right\} di \mathbb{E}_t \pi_{l,t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma^u_{b,k}(i) \left\{\frac{Wn(i)}{WN}\psi\left(\varphi(i)\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) - \frac{1-\varphi(i)}{1-\varphi^E} \int \varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) di\right)\right\} di \mathbb{E}_t \pi_{l,t+1}
\end{aligned}$$

$$\mathcal{M}^0_{k,t} - \frac{1}{(1-\delta)R}\mathbb{E}_t \mathcal{M}^0_{k,t+1} = \int \gamma^u_{b,k}(i)\frac{b(i)}{RE} di \left(\hat{R}_t - \sum_l \bar{s}_l \mathbb{E}_t \pi_{l,t+1}\right) - \sum_l \int \gamma^u_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \psi \frac{Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right) di \hat{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}$$

Under (**A.2**) we can rewrite $\sigma_{k,l}^{\mathcal{M},u}$ as:

$$
\begin{aligned}
\sigma_{k,l}^{\mathcal{M},u} &= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \overline{\partial_e e_l}^u \frac{R}{R-1} \int \frac{(1-\varphi(i))\, Wn}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) di \left(\sigma \left(1-\varphi^E\right) + \psi \left(1-\varphi^L\right)\right) \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \frac{\left(\sigma \left(1-\varphi^E\right) + \psi \left(1-\varphi^L\right)\right)}{\sigma+\psi} \\
&+ \sigma \overline{\partial_e e_l}^u \int \frac{(1-\varphi(i))\, b}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) di \frac{\left(\sigma \left(1-\varphi^E\right) + \psi \left(1-\varphi^L\right)\right)}{\sigma+\psi} \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
&- \sigma \overline{\partial_e e_l}^u \int \gamma_{b,k}^u(i) \frac{(1-\varphi(i))\,(\sigma e + \psi Wn)}{(1-\varphi^L)\, WN} di \frac{\left(\left(1-\varphi^E\right) - \left(1-\varphi^L\right)\right)}{\sigma+\psi} \frac{R}{R-1} \\
&+ \sigma \overline{\partial_e e_l}^u \int \frac{(1-\varphi(i))\, b}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) di \frac{\left(\sigma \left(1-\varphi^E\right) + \psi \left(1-\varphi^L\right)\right)}{\sigma+\psi} \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
&+ \sigma \overline{\partial_e e_l}^u \int \frac{1}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) \left\{ (1-\varphi(i))\, b \int (1-\varphi(i)) \frac{\sigma e + \psi Wn}{(\sigma+\psi)\, E} di - (1-\varphi(i)) \frac{\sigma e + \psi Wn}{\sigma+\psi} \int (1-\varphi(i)) \frac{b(i)}{E} di \right\} di \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
&+ \sigma \overline{\partial_e e_l}^u \int \frac{1}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) \left\{ (1-\varphi(i))\, b \int (1-\varphi(i)) \left(\frac{Wn}{WN} + \frac{\sigma b}{\sigma+\psi} \frac{R-1}{RE} \right) - (1-\varphi(i)) \left(Wn + \frac{\sigma b}{\sigma+\psi} \frac{R-1}{R} \right) \int (1-\varphi(i)) \frac{b(i)}{E} di \right\} di \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
&+ \sigma \overline{\partial_e e_l}^u \frac{\sigma}{\sigma+\psi} \frac{R-1}{R} \int \gamma_{b,k}^u(i) \left\{ \frac{(1-\varphi(i))\, b}{E} - \frac{(1-\varphi(i))\, Wn}{(1-\varphi^L)\, WN} \int (1-\varphi(i)) \frac{b(i)}{E} di \right\} didi
\end{aligned}
$$

We therefore have under (**A.2**)

$$
\begin{aligned}
\sigma_{k,l}^{\mathcal{M},u} &= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
\sum_l \sigma_{k,l}^{\mathcal{M},u} &= \sigma_k^{\mathcal{M},u} = 0
\end{aligned}
$$

In addition we have

$$\begin{aligned}\int \gamma_{b,k}^{u}(i)\frac{b(i)}{RE}di &= \int \gamma_{b,k}^{u}(i)\frac{(1-\varphi(i))\,b}{E}di + \int \gamma_{b,k}^{u}(i)\frac{\varphi(i)b}{E}di\\ &= \int \gamma_{b,k}^{u}(i)\frac{(1-\varphi(i))\,b}{E}di + \int \gamma_{b,k}^{u}(i)\frac{\varphi(i)b}{E}di - \int \gamma_{b,k}^{u}(i)\frac{(1-\varphi(i))\,Wn}{(1-\varphi^{L})\,WN}\left(\int \frac{b(i)}{E}di\right)di\\ &= \int \gamma_{b,k}^{u}(i)\left\{\frac{(1-\varphi(i))\,b}{E} - \frac{(1-\varphi(i))\,Wn}{(1-\varphi^{L})\,WN}\int (1-\varphi(i))\,\frac{b(i)}{E}di\right\}di + \frac{R-1}{R}\int \gamma_{b,k}^{HtM}(i)\left\{\frac{\varphi(i)b}{E} - \frac{(1-\varphi(i))\,Wn}{(1-\varphi^{L})\,WN}\int \varphi(i)\frac{b(i)}{E}di\right\}di\\ &= 0\end{aligned}$$

Therefore the equations for $\mathcal{M}_{k,t}^{D}$ can be rewritten:

$$\begin{aligned}\mathbb{E}_t\mathcal{M}_{k,t+1}^{D} - \mathcal{M}_{k,t}^{D} &= -\sum_l \sigma_{k,l}^{\mathcal{M},u}\tilde{\pi}_{l,t+1} + \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}_{k,t+1}^{0}\\ &\quad - \frac{R}{R-1}\sum_l \int \gamma_{b,k}^{u}(i)\left\{\left(\varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) - \frac{(1-\varphi(i))\,Wn(i)}{(1-\varphi^{L})\,WN}\int \varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)di\right)\right\}di\mathbb{E}_t\tilde{\pi}_{l,t+1}\\ &\quad - \frac{R}{R-1}\sum_l \int \gamma_{b,k}^{u}(i)\left\{\frac{Wn(i)}{WN}\psi\left(\varphi(i)\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) - \frac{1-\varphi(i)}{1-\varphi^{E}}\int \varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)di\right)\right\}di\mathbb{E}_t\tilde{\pi}_{l,t+1}\\ \mathcal{M}_{k,t}^{0} - \frac{1}{(1-\delta)\,R}\mathbb{E}_t\mathcal{M}_{k,t+1}^{0} &= -\sum_l \int \gamma_{b,k}^{u}(i)\left(\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) + \psi\frac{Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}_{k,t}^{D}\end{aligned}$$

Where we use

$$\sum_l \int \gamma_{b,k}^{u}(i)\left(\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) + \psi\frac{Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di = 0$$

$$\begin{aligned}&\sum_l \int \gamma_{b,k}^{u}(i)\left\{\left(\varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) - \frac{(1-\varphi(i))\,Wn(i)}{(1-\varphi^{L})\,WN}\int \varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)di\right)\right\}\\ &\quad + \sum_l \int \gamma_{b,k}^{u}(i)\left\{\frac{Wn(i)}{WN}\psi\left(\varphi(i)\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) - \frac{1-\varphi(i)}{1-\varphi^{E}}\int \varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)di\right)\right\} = 0\end{aligned}$$

to replace $\hat{P}_{l,t}$, $\pi_{l,t}$ with $\tilde{P}_{l,t}$, $\tilde{\pi}_{l,t}$. Therefore, relative prices are determined by a system of $4(K-1)$ equations which independent of $\hat{R}_t$ and are therefore independent from monetary policy. We conclude, since $\mathcal{NH}_t, \mathcal{M}_{k,t}, \mathcal{P}_{k,t}, \mathcal{I}_{k,t}$ only depend on relative prices that the wedges are independent from monetary policy.

Additions to Result 3: Analytical Formulas with HtM

We first re-derive the evolution of any relative price $\tilde{P}_{k,t} = \hat{P}_{k,t} - \sum \overline{\partial_e e_l}\hat{P}_{l,t}$

$$R\tilde{\pi}_{k,t} = -\lambda R\left(\tilde{P}_{k,t} - \tilde{P}_{k,t}^{*}\right) + \tilde{\pi}_{k,t}$$

$$P_{\Delta,t+1} - (1+R+R\lambda)\,P_{\Delta,t} + RP_{\Delta,t-1} = \lambda R\hat{A}_{\Delta,t}$$

The eigenvalues of the system are:

$$\mu_{\pm} = \frac{R+R\lambda+1 \pm \sqrt{(R+R\lambda-1)^2 + 4R\lambda}}{2}$$

With $\mu_{+} > R+R\lambda$, $\mu_{-} < 1$. We obtain:

$$\tilde{P}_{k,t} = \lambda \sum_{0}^{t} \mu_-^{t-s+1} \sum \frac{1}{\mu_+^u} \tilde{P}_{k,t}^*$$

We now assume shock vanishes at a constant rate ρ_a, we have:

$$\tilde{P}_{k,t} = \frac{1}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \left(\mu_-^{t+1} - \rho_a^{t+1}\right) \tilde{P}_{k,0}^*$$

Next we slightly rewrite the output gap equation

$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma \left((1-\Phi_b)\hat{R}_t - (1-\Phi_b)\pi_{mcpi,t+1} - \hat{r}_t^*\right) + \Phi_b \left(\frac{1}{R-1}\left(\mathbb{E}_t \hat{R}_{t+1} - \hat{R}_t\right) - \bar{s}_l \pi_{l,t+1}\right) - \frac{(1-\varphi^E)}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l \left(\sigma\left(\overline{\partial_e e_l}^u - \overline{\partial_e e_l}\right) - (s_l^u - \bar{s}_l)\right)\tilde{\pi}_{l,t+1}$$

$$\Phi_b = \frac{\varphi^E - \varphi^L}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}$$

$$-1 < \Phi_b < 1$$

Response to aggregate shocks, inflation targeting. For aggregate shocks ($A_{k,t} = A_t$), all relative sectoral prices are constant so the response does not depend on which inflation index is targeted. Assume $\hat{R}_t = \phi \pi_t$ (with π_t an arbitrary index), we have

$$R\pi_t = R\kappa\tilde{\mathcal{Y}}_t + \pi_{t+1}$$

$$\pi_{t+2} - (1 + R + R\kappa\sigma(1-\Phi_b) + R\kappa\Phi_b)\pi_{t+1} + R\pi_t + R\kappa\sigma(1-\Phi_b)\hat{R}_t + R\kappa\Phi_b\frac{1}{R-1}\Delta\hat{R}_{t+1} - R\kappa\sigma\hat{r}_t^* = 0$$

$$\pi_{t+2} - \left(1 + R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{1}{R-1} - 1\right)\right)\right)\pi_{t+1} + R\left(1 + \kappa\phi\left(\sigma(1-\Phi_b) - \Phi_b\frac{1}{R-1}\right)\right)\pi_t - R\kappa\sigma\hat{r}_t^* = 0$$

The eigenvalues of the system are

$$\lambda_\pm^{HtM} = \frac{\left(1 + R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{1}{R-1} - 1\right)\right)\right) \pm \sqrt{\left(\left(R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{1}{R-1} - 1\right)\right)\right) - 1\right)^2 - 4R\kappa\left[(\phi-1)(\sigma(1-\Phi_b)) - \Phi_b\right]}}{2}$$

Note that when $\Phi_b < 0$, $\phi \geq 1$ implies that both eigenvalues of the system are larger than one[17]

$$\pi_t = \frac{R\kappa}{\rho_a^2 - \left(1 + R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{R}{R-1} - 1\right)\right)\right)\rho_a + R\left(1 + \kappa\phi\left(\sigma(1-\Phi_b) - \Phi_b\frac{R}{R-1}\right)\right)}\rho_a^t \hat{r}_0^*$$

$$= \frac{R\kappa\sigma}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t \hat{r}_0^*$$

$$\tilde{\mathcal{Y}}_t = \frac{\sigma(R - \rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t \hat{r}_0^*$$

with

$$\mathcal{C}(\rho_a) = -\Phi_b\left(\sigma(\phi - \rho_a) + \phi\frac{R}{R-1}(1-\rho_a) + \rho_a\right) > 0$$

[17]Note that we only need $\phi > \max\{1 + \frac{\Phi_b}{\sigma(1-\Phi_b)}, -(R-1)\left(\frac{1}{-\Phi_b}\left(\frac{1}{\kappa}\left(1 - \frac{1}{R}\right) + \sigma\right) + \sigma - 1\right)\}$in particular if $\kappa \leq 1$or $\sigma \geq 1$ this simplifies to $\phi > 1 + \frac{\Phi_b}{\sigma(1-\Phi_b)}$.

And $\mathcal{C}(\rho_a)$ is decreasing in ρ_A. For a given policy rule, the presence of HtM households decreases the impact of technology or monetary shocks on inflation and the output gap. Intuitively, as HtM have negative wealth on average they respond to an increase in inflation by cutting consumption, since they respond more strongly than non HtM this makes monetary policy more effective.

Response to sectoral shocks, inflation targeting. Now assume that CB targets CPI: $\hat{R}_t = \phi\pi_{cpi,t}$. The system becomes

$$\pi_{cpi,t+2} - \left(1 + R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{1}{R-1} - 1\right)\right)\right)\pi_{cpi,t+1} + R\left(1 + \kappa\phi\left(\sigma\left(1-\Phi_b\right) - \Phi_b\frac{1}{R-1}\right)\right)\pi_{cpi,t}$$
$$-R\kappa\sigma\hat{r}^*_t + R\lambda\left(\mathcal{NH}_{t+1} - \mathcal{NH}_t\right) - R\kappa\frac{(1-\varphi^E)}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l\left(\sigma\left(\overline{\partial_e e_l}^u - \overline{\partial_e e_l}\right) - \left(s^u_l - \bar{s}_l\right)\right)\tilde{\pi}_{l,t+1} - R\kappa\bar{\sigma}\left(\tilde{\pi}_{mcpi,t+1} - \tilde{\pi}_{cpi,t+1}\right) = 0$$

We have

$$\pi_t = \frac{R\lambda}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\left\{\left(1 - \frac{R\kappa\sigma\phi + R\kappa\mathcal{C}(\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\right)(1-\rho_a)\rho_a^t - \left(1 - \frac{R\kappa\sigma\phi + R\kappa\mathcal{C}(\mu_-)}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-) + R\kappa\mathcal{C}(\mu_-)}\right)(1-\mu_-)\mu_-^t\right\}\hat{A}_{\Delta,0}$$
$$+ \frac{R\kappa\sigma}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t\hat{r}^*_0$$
$$-\frac{R\kappa\sigma}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\frac{(1-\varphi^E)}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l\left(\sigma\left(\overline{\partial_e e_l}^u - \overline{\partial_e e_l}\right) - \left(s^u_l - \bar{s}_l\right)\right)\left(\frac{(1-\mu_-)}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-) + \mathcal{C}(\mu_-)}\mu_-^{t+1} - \frac{(1-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + \mathcal{C}(\rho_a)}\rho_a^{t+1}\right)\tilde{P}^*_{l,0}$$

$$\tilde{\mathcal{Y}}_t = -\underbrace{\frac{R\lambda}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\left\{\frac{(\sigma\phi + \mathcal{C}(\rho_a))(1-\rho_a)(R-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t - \frac{(\sigma\phi + \mathcal{C}(\mu_-))(1-\mu_-)(R-\mu_-)}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-) + R\kappa\mathcal{C}(\mu_-)}\mu_-^t\right\}\hat{A}_{\Delta,0}}_{\text{Impactof NH wedge}}$$
$$+ \frac{\sigma(R-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t\hat{r}^*_0$$
$$-\underbrace{\frac{1}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\frac{\varphi^E}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l\left(\left(s^{HtM}_l - \bar{s}_l\right) - \sigma\left(\overline{\partial_e e_l}^{HtM} - \overline{\partial_e e_l}\right)\right)\left(\frac{(1-\mu_-)(R-\mu_-)}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-) + R\kappa\mathcal{C}(\mu_-)}\mu_-^{t+1} - \frac{(1-\rho_a)(R-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\rho_a^{t+1}\right)\tilde{P}^*_{l,0}}_{\text{Impact of relative price on HtM consumption}}$$

The introduction of HtM has an ambiguous impact on both CPI inflation and the output gap. The response of the output gap is the sum of three terms. The first one is the contribution of the $\mathcal{NH}$ wedge:

$$-\frac{R\lambda}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\left\{\frac{(\sigma\phi + \mathcal{C}(\rho_a))(1-\rho_a)(R-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t - \frac{(\sigma\phi + \mathcal{C}(\mu_-))(1-\mu_-)(R-\mu_-)}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-) + R\kappa\mathcal{C}(\mu_-)}\mu_-^t\right\}\hat{A}_{\Delta,0}$$

As before, $\frac{(\sigma\phi+\mathcal{C}(\rho_a))(1-\rho_a)(R-\rho_a)}{(\lambda_+-\rho_a)(\lambda_--\rho_a)+R\kappa\mathcal{C}(\rho_a)}$ is decreasing in ρ_a, so the sign of this term is the same with or without HtM. The amplitude is however ambiguous. For transitory shocks in necessity sectors without HtM, as see in the previous subsection CPI inflation is positive and decreasing. This implies that the interest rate implemented is positive and decreasing which increases the growth rate of HtM demand which implies lower out output gap (output gap converges to 0in the long run): the response is amplified. By contrast, the response would be muted for a permanent shock

The second term summarizes the impact of the change in real rate. As explain previously, this response is always muted with HtM.

The third term corresponds to the difference in demand growth rate between HtM and unconstrained households in response to changes in sectoral prices:

$$-\frac{1}{(\mu_+ - \rho_a)(\mu_- - \rho_a)}\frac{\varphi^E}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l\left(\left(s^{HtM}_l - \bar{s}_l\right) - \sigma\left(\overline{\partial_e e_l}^{HtM} - \overline{\partial_e e_l}\right)\right)\left(\frac{(1-\mu_-)(R-\mu_-)}{(\lambda_+ - \mu_-)(\lambda_- - \mu_-) + R\kappa\mathcal{C}(\mu_-)}\mu_-^{t+1} - \frac{(1-\rho_a)(R-\rho_a)}{(\lambda_+ - \rho_a)(\lambda_- - \rho_a) + R\kappa\mathcal{C}(\rho_a)}\rho_a^{t+1}\right)\tilde{P}^*_{l,0}$$

For transitory shocks, after the first period, there is deflation of necessity goods. If the growth rate of HtM necessary good consumption is relatively higher in response to deflation in the necessity sector ($\left(s_l^{HtM} - \bar{s}_l\right) - \sigma \left(\overline{\partial_e e_l}^{HtM} - \overline{\partial_e e_l}\right) > 0$), the output gap is lower at all dates, which further amplifies the response of the output gap to a transitory necessity shock. This is reversed for close to permanent shocks: in that case there is inflation of necessity goods which reduces the growth rate of HtM demand and implies a relatively higher output gap.