Robust inference about partially identified SVARs*

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Abstract

Most empirical applications using partially identified Structural Vector Autoregressions (SVARs) adopt Bayesian inference. Known drawbacks of the approach in partially identified models are its sensitivity to the choice of priors even in large samples and the fact that even apparently uninformative priors lead to informative inference, in the sense of resulting in credible regions that asymptotically lie strictly within the true identified set. We consider the general case of SVARs that are partially identified due to an insufficient number of equality restrictions and/or to the use of sign restrictions and propose a method for conducting posterior inference on impulse responses that is robust to the choice of priors. The method considers multiple priors for the non-identified parts of the model and delivers a class of multiple posteriors, which we propose to summarize by reporting the posterior mean upper and lower bounds, together with the associated robustified credible region. In practice, the posterior bounds can be obtained by a simple modification of the numerical algorithms already used in the literature. For general equality and/or sign restrictions, the identified set is not necessarily convex, and our approach can be viewed as conducting inference about the convex hull of the identified set. When the identified set is convex, we show that the posterior bounds converge to the true identified set, thereby overcoming the main drawback of single-prior Bayesian inference. We provide easily verifiable conditions on the type of equality and sign restrictions that guarantee convexity of the identified set. Useful diagnostic tools delivered by our procedure are the ability to report the posterior belief about the plausibility of the imposed restrictions and to disentangle the information contained in the identifying restriction from that introduced through the choice of a prior.

Keywords: Ambiguous beliefs, Partial causal ordering, Posterior bounds, Credible region

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1 Introduction

There is a growing demand in the empirical literature on structural vector autoregressions (SVARs) for econometric methods that relax the requirement of point identification and allow a researcher to impose only the identifying restrictions that she considers credible. For example, one may wish to use only a subset of the commonly used causal ordering restrictions (Sims (1980) and Bernanke (1986)) or long-run neutrality restrictions (e.g., Blanchard and Quah (1993)), and/or to impose sign restrictions (Canova and Nicolo (2002), Faust (1998), and Uhlig (2005)). Even though there is a large literature in microeconometrics that proposes frequentist methods for conducting inference in partially identified models, most existing macroeconomic applications adopt a Bayesian approach. This might be partly explained by the high computational cost of frequentist methods and the fact that their validity in SVAR models has been shown for only a limited class of identifying restrictions (Moon, Schorfheide, and Granziera (2013), Gafarov and Montiel-Olea (2014)), which means that, at present, the frequentist approach allows little flexibility in the number and type of restrictions to impose. Moreover, the use of informative priors is a common device used in reduced-form VAR modelling for overcoming the curse of dimensionality that affects these models, and thus Bayesian methods have the advantage of allowing a researcher to consider larger and more realistic SVAR models than would be viable in a frequentist setting.

A feature of Bayesian inference in partially identified models that could be considered problematic is the sensitivity of the conclusions to the choice of priors for the non-identified aspects of the model, an effect that, unlike in the point identified case, is present even in large samples (Poirier (1998)). For SVARs this means that, whereas the prior for the reduced form parameters is updated through the likelihood, the prior for the rotation matrix that transforms reduced form shocks into structural shocks is never updated. Choosing a prior for the rotation matrix is a difficult task and an uninformative prior does not necessarily provide a solution as it can lead to unintentionally informative inference about the impulse responses. This issue has been actively debated in the literature, and several important contributions have discussed the challenges that Bayesian inference poses for the interpretation of empirical findings. For example, Baumeister and Hamilton (2013) show that Bayesian inference in sign-identified models can be exclusively driven by priors that may be difficult to defend and the results in Moon and Schorfheide (2012) imply that any prior, no matter how uninformative it may appear, will lead to "overly informative" inference, in the sense of resulting in credible regions that asymptotically lie strictly within the true identified set.

The goal of this paper is to propose a new approach to posterior inference that overcomes these drawbacks. We consider the general case of SVARs that are partially identified due to an insufficient number of equality restrictions and/or to the use of sign restrictions and propose a method for conducting posterior inference on individual impulse responses that is robust to the choice of priors. The method adopts a single prior for the reduced form parameters but allows for multiple priors for the rotation matrices. It then applies Bayes rule to deliver a class of multiple posteriors, which can be interpreted as characterising the posterior distribution of the impulse response identified set. The multiple prior approach can be motivated in our context by assuming that the only prior knowledge about the rotation matrices is that they must be compatible with the identifying restrictions. This restricts the set of possible rotation matrices, and our analysis assumes that the researcher has "ambiguous beliefs" over the elements of this set, in the sense of not having any additional prior information that allow her to judge whether any rotation matrix is more credible than the others. In practice, we suggest summarizing the class of posteriors by reporting two intervals for a given impulse response: a posterior mean bounds interval, which can be interpreted as an estimator of the identified set, and an associated robustified credible region, which is a measure of the posterior uncertainty about the identified set. The fact that we limit attention to upper and lower bounds means that, in practice, the bounds can be obtained by adding a simple optimization step to the numerical algorithms typically used in the literature on sign-identified SVARs.

In order to analyse the behaviour of our bound analysis in large samples and aid the interpretation of empirical results, it is important to understand which types of equality and/or sign restrictions give rise to a convex identified set. We do so by providing sufficient conditions that can be used to verify if a collection of equality and/or sign restrictions imply a convex identified set for the impulse response of interest. These results are new to the literature and may be of separate interest regardless of whether one favours a Bayesian or a frequentist approach. Our main theorem shows that, when the identified set is convex, the posterior bounds we propose converge asymptotically to the identified set, thereby overcoming the limitations of single-prior Bayesian analysis in partially identified models. When the identified set is not convex, our method can still be applied and can be viewed as providing posterior inference about the convex hull of the identified set.

We envision two possible uses of our method in empirical work. First, the method can be used to perform robust Bayesian inference in partially identified SVARs without specifying a prior for the rotation matrix. Second, even if a user has a prior for the rotation matrix (for example the uniform prior typically used in the literature on sign-restricted SVARs) our method allows the researcher to disentangle the information contained in the identifying restrictions from that introduced by the choice of the prior. To this purpose, we consider two useful measures that can be reported in empirical applications: 1) the informativeness of the identifying restrictions, measured by how much the restrictions tighten the identified set estimator, relative to the case without restrictions; and 2) the informativeness of the prior, measured by how much the choice of a single prior for the rotation matrix tightens the credible region relative to the multiple-prior case. Finally, a useful diagnostic tool that is a by-product of our analysis is the ability to separately report the posterior belief for the plausibility of the imposed identifying restrictions and the posterior belief for the impulse responses, conditional on the imposed assumptions being plausible (in the sense of not contradicting the observed data). Note that if one were to adopt a frequentist approach it would generally be difficult to separate these two types of sample information, as discussed by Sims and Zha (1998).

We apply our method to a standard monetary SVAR and consider various subsets and combinations of the equality and sign restrictions that are typically imposed in the literature. Our findings illustrate that the commonly used sign restrictions have little identifying power, and that standard Bayesian inference in this case is largely driven by the choice of the prior for the rotation matrix. The addition of even a single equality restriction tightens the credible sets considerably, it makes standard Bayesian inference less sensitive to the choice of priors and it leads to informative inference about the sign of the output response to monetary policy shocks.

The remainder of the paper is organized as follows. Section 2 introduces the notation and the general analytical framework of SVARs with equality and/or sign restrictions. Section 3 characterizes the impulse response identified set. Section 4 introduces the robust Bayes approach and shows how to compute the posterior bounds and the robustified credible region. Section 5 shows conditions on which identifying restrictions guarantee convexity of the identified set. Section 6 shows that our bounds converge asymptotically to the true identified set. An empirical example is contained in Section 7. The proofs are collected in the Appendix. A reader who is mostly interested in the practical implementation of the procedure can focus on Sections 2, 3 and 4.2.

2 The Econometric Framework

Consider a SVAR(p) model

$$A_0 y_t = a + \sum_{j=1}^{p} A_j y_{t-j} + \epsilon_t \quad \text{for } t = 1, \ldots, T,$$

where y_t is an $n \times 1$ vector, ϵ_t an $n \times 1$ vector white noise process, normally distributed with mean zero and variance-covariance matrix I_n, the $n \times n$ identity matrix. Note that we assume the structural shocks to be uncorrelated, as is common in the SVAR literature. The initial conditions $y_1, \ldots, y_p$ are given.

The reduced form VAR representation of the model is

$$y_t = b + \sum_{j=1}^{p} B_j y_{t-j} + u_t, \tag{2.1}$$

where $b = A_0^{-1} a$, $B_j = A_0^{-1} A_j$, $u_t = A_0^{-1} \epsilon_t$, and $E(u_t u_t') \equiv \Sigma = A_0^{-1} \left(A_0^{-1}\right)'$. We denote the reduced form parameters by $\phi = (B, \Sigma) \in \Phi \subset \mathcal{R}^{n+n^2 p} \times \Omega$, where $B = [b, B_1, \ldots, B_p]$ and Ω is the space of symmetric positive-semidefinite matrices. We restrict the domain Φ to the set of $\phi's$ such that the reduced form VAR(p) model can be inverted into a VMA(∞) model.

We denote the h-th horizon impulse response matrix by the $n \times n$ matrix IR^h, $h = 0, 1, 2, \ldots$, where the (i, j)-element of IR^h represents the effect on the i-th variable in y_{t+h} of a unit shock to the j-th element of ϵ_t. Assuming the reduced form lag polynomial $\left(I_n - \sum_{j=1}^{p} B_j L^p\right)$ is invertible, the VMA(∞) representation of the reduced form model (2.1) is

$$\begin{aligned} y_t &= c + \sum_{j=0}^{\infty} C_j(B) u_{t-j} \qquad (2.2) \\ &= c + \sum_{j=0}^{\infty} C_j(B) A_0^{-1} \epsilon_{t-j}, \end{aligned}$$

where $C_j(B)$ is the j-th coefficient matrix of the inverted lag polynomial $\left(I_n - \sum_{j=1}^{p} B_j L^j\right)^{-1}$, which depends only on B. The impulse response IR^h is then given by

$$IR^h = C_h(B) A_0^{-1}. \qquad (2.3)$$

The long-run impulse response matrix is defined as

$$IR^{\infty} = \lim_{h\to\infty} IR^h = \left(I_n - \sum_{j=1}^{p} B_j\right)^{-1} A_0^{-1} \qquad (2.4)$$

and the long-run cumulative impulse response matrix is defined as

$$CIR^{\infty} = \sum_{h=0}^{\infty} IR^h = \left(\sum_{h=0}^{\infty} C_h(B)\right) A_0^{-1}. \qquad (2.5)$$

In the absence of any identifying restrictions, knowledge of the reduced form parameters ϕ does not pin down a unique A_0. We can express the set of observationally equivalent A_0's given Σ using an orthonormal matrix $Q \in \mathcal{O}(n)$, where $\mathcal{O}(n)$ is the set of $n \times n$ orthonormal matrices. The individual column vectors in Q are denoted by $[q_1, q_2, \ldots, q_n]$. Denote the Cholesky decomposition of Σ by $\Sigma = \Sigma_{tr}\Sigma_{tr}'$, where Σ_{tr} is the unique lower-triangular Cholesky factor with nonnegative diagonal elements. Since any A_0 of the form $A_0 = Q'\Sigma_{tr}^{-1}$ satisfies $\Sigma = (A_0' A_0)^{-1}$, in the absence of any identifying restrictions $\left\{A_0 = Q'\Sigma_{tr}^{-1} : Q \in \mathcal{O}(n)\right\}$ forms the set of A_0's that are consistent with the reduced-form variance-covariance matrix Σ (Uhlig (2005) Proposition A.1). Since the likelihood function only depends on the reduced form parameters ϕ, the data do not contain any information about Q, which leads to ambiguity in decomposing Σ. If the imposed identifying restrictions fail to identify A_0, it means that for a given Σ there are multiple Q's yielding the structural parameter matrix A_0 which satisfies the imposed restrictions.

In the absence of any identifying restrictions on A_0, the only restrictions to be imposed on Q are the sign normalization restrictions for the structural shocks. Following the identification analysis in Christiano, Eichenbaum, and Evans (1999), we impose the sign normalization restrictions on A_0, such that the diagonal elements of A_0 are all nonnegative. This means that a unit positive change in a structural shock is interpreted as a one standard-deviation positive shock to the corresponding endogenous variable.

Once the sign normalization restrictions on A_0 are imposed, the set of observationally equivalent A_0's corresponding to Σ can be expressed as

$$\left\{A_0 = Q'\Sigma_{tr}^{-1} : Q \in \mathcal{O}(n), \quad diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0\right\}, \tag{2.6}$$

where the inequality restriction $diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0$ means that all diagonal elements of $A_0 = Q'\Sigma_{tr}^{-1}$ are nonnegative. By denoting the column vectors of Σ_{tr}^{-1} as $\left[\sigma^1, \sigma^2, \ldots, \sigma^n\right]$, the sign normalization restriction can be written as a collection of linear inequalities

$$\left(\sigma^i\right)' q_i \geq 0 \quad \text{for all } i = 1, \ldots, n.$$

Suppose one is interested in a specific impulse response, say the (i, j^*)-th element of IR^h,

$$r_{ij^*}^h \equiv e_i' C_h(B)\,\Sigma_{tr} Q e_{j^*} \equiv c_{ih}'(\phi)\, q_{j^*},$$

where e_i is the i-th column vector of I_n and $c_{ih}'(\phi)$ is the i-th row vector of $C_h(B)\,\Sigma_{tr}$. For simplicity, we sometimes make i, j^*, and h implicit in our notation unless any confusion arises, and use $r \in \mathcal{R}$ to denote the impulse response of interest, i.e., $r \equiv r_{ij^*}^h$. When we want to emphasize the dependence of r on the reduced form parameters ϕ and the rotation matrix Q, we express r as $r(\phi, Q)$. Note that the analysis developed below for the impulse responses can be easily extended to the structural parameters A_0 and $[A_1, \ldots, A_p]$, since each structural parameter can be expressed by the inner product of a vector depending on ϕ and a column vector of Q, e.g., the (i, j)-th element of A_l can be obtained as $e_j'\left(\Sigma_{tr}^{-1} B_l\right)' q_i$.

3 The Impulse Response Identified Set

In this section we characterize the impulse response identified set obtained by imposing a collection of under-identifying equality restrictions and/or sign restrictions.

3.1 Under-identifying Equality Restrictions

Examples of under-identifying equality restrictions are zero restrictions on off-diagonal elements of A_0^{-1}, e.g., a subset of the restrictions imposed by the common Sims-Bernanke recursive identification strategy that sets the upper-triangular components of A_0^{-1} to zero. This amounts to assuming only a partial causal ordering for the variables in the model while allowing contemporaneous relationships among the remaining variables (see Example 3.1 below). More in general, if some equation in the system represents the behavioral response of a sector or an economic agent, zero restrictions can be placed on the elements of A_0 by invoking economic theory or available institutional knowledge. Note that, since the contemporaneous impulse response matrix is $IR^0 = A_0^{-1}$, zero restrictions on the contemporaneous impulse responses can be seen as zero restrictions on the corresponding elements of A_0^{-1} and hence can be treated as part of the causal ordering restrictions. Our framework also accommodates zero restrictions on the lagged coefficients $\{A_l : l = 1, \dots p\}$ as well as restrictions on the long-run impulse responses, which are zero restrictions on some elements of the long-run impulse response $IR^\infty = \left(I - \sum_{j=1}^{p} B_j\right)^{-1} \Sigma_{tr} Q$ or the long-run cumulative impulse response, $CIR^\infty = \sum_{h=0}^{\infty} C_h(B) \Sigma_{tr} Q$.

Since A_0^{-1}, A_0, $\{A_l : l = 1, \dots, p\}$, and $\{IR^h : h = 1, 2 \dots, \infty\}$ are products of Q and a matrix that depends only on the reduced-form parameters, all the zero restrictions above can be viewed as imposing linear constraints on the columns of Q, with coefficients depending on the reduced-form parameters $\phi = (\Sigma, B)$. For example:

$$\begin{aligned}
\left((i,j)\text{-th element of } A_0^{-1}\right) &= 0 \iff \left(e_i' \Sigma_{tr}\right) q_j = 0, \\
\left((i,j)\text{-th element of } A_0\right) &= 0 \iff \left(\Sigma_{tr}^{-1} e_j\right)' q_i = 0, \\
\left((i,j)\text{-th element of } A_l\right) &= 0 \iff \left(\Sigma_{tr}^{-1} B_l e_j\right)' q_i = 0, \\
\left((i,j)\text{-th element of } CIR^\infty\right) &= 0 \iff \left[e_i' \sum_{h=0}^{\infty} C_h(B) \Sigma_{tr}\right] q_j = 0.
\end{aligned} \tag{3.1}$$

We can thus represent a collection of zero restrictions in the following general form:

$$F(\phi, Q) \equiv \begin{pmatrix} F_1(\phi) q_1 \\ F_2(\phi) q_2 \\ \vdots \\ F_n(\phi) q_n \end{pmatrix} = \mathbf{0}, \tag{3.2}$$

where $F_i(\phi)$ is an $f_i \times n$ matrix that depends only on the reduced form parameters $\phi = (B, \Sigma)$. Each row vector in $F_i(\phi)$ corresponds to the coefficient vector of a zero restriction that constrains q_i as in (3.1), and $F_i(\phi)$ stacks all the coefficient vectors that multiply q_i into a matrix. Hence, f_i is the number of zero restrictions constraining q_i. If the set of zero restrictions does not constrain q_i, $F_i(\phi)$ does not exist and thus $f_i = 0$.

In order to implement the method, one must first order the variables in the model.

Notation 3.1 *(Ordering of variables) The variables in the SVAR are ordered so that the number of equality restrictions f_i imposed on the $i-th$ column of Q (i.e., the rows of $F_i(\phi)$ in (3.2)) satisfy $f_1 \geq f_2 \geq \dots \geq f_n \geq 0$. In case of ties, if the impulse response of interest is that to the j^*-th structural shock, let the j^*-th variable be ordered first. That is, set $j^* = 1$ when no other column vector has a larger number of restrictions than q_{j^*}. If $j^* \geq 2$, then order the variables so that $f_{j^*-1} > f_{j^*}$.*

Note that our assumption for the ordering of the variables pins down a unique j^*, while it does not necessarily yield a unique ordering for the other variables if some of them admit the same number of constraints. However, the condition for the convexity of the identified set for the impulse responses to the j^*-th structural shock that we provide in Lemma 5.1 is not affected by the ordering chosen for the other variables as long as the f_i's are in decreasing order.

Rubio-Ramirez et. al. (2010) focus on point identification in SVARs subject to equality restrictions of the form (3.2) and their conditions for point identification provide a starting point for our analysis. Rubio-Ramirez et. al. (2010) define the parameters to be exactly identified if for almost every $\phi \in \Phi$, there exist unique $(A_0, A_1, \ldots, A_p)$ satisfying the identifying restrictions, which can be equivalently stated as saying that there is a unique Q satisfying $F(\phi, Q) = \mathbf{0}$ and the sign normalizations**.** They then show that under regularity assumptions, a necessary and sufficient condition for point identification is that $f_i = n - i$ for all $i = 1, \ldots, n$. Here we consider restrictions that make the SVAR partially identified because

$$f_i \leq n - i \quad \text{for all } i = 1, \ldots, n, \tag{3.3}$$

with strict inequality for at least one $i \in \{1, \ldots, n\}$. This means that there are multiple Q's satisfying $F(\phi, Q) = \mathbf{0}$ and the sign normalizations at almost every value of ϕ. Denote by $\mathcal{Q}(\phi|F)$ the set of Q's that satisfy the restrictions (3.2) and the sign normalization given ϕ,

$$\mathcal{Q}(\phi|F) = \left\{ Q \in \mathcal{O}(n) : F(\phi, Q) = \mathbf{0},\, diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0 \right\}.$$

The identified set for the impulse response is a set-valued map from ϕ to a subset in $\mathcal{R}$ that gives the range of $r(\phi, Q)$ when Q varies over its domain $\mathcal{Q}(\phi|F)$,

$$IS_r(\phi|F) = \{ r(\phi, Q) : Q \in \mathcal{Q}(\phi|F) \}.$$

The class of under-identified models that we consider here does not exhaust the universe of all possible non-identified SVARs, since there exist models that do not satisfy (3.3), but for which the structural parameters are not globally identified for some values of the reduced form parameters with a positive measure. For instance, the example given in Section 4.4 of Rubio-Ramirez, Waggoner, and Zha (2010) provides an example with $n = 3$ and $f_1 = f_2 = f_3 = 1$, where the structural parameters are locally identified but their global identification fails. Such locally-, but not globally-identified models are ruled out from the class of partially-identified SVARs considered in this paper. For another example, the zero restrictions given in page 77 of Christiano, Eichenbaum, and Evans (1999) correspond to a case with $n = 3$ and $f_1 = f_2 = f_3 = 1$, where even local identification fails. This case is also ruled out by the condition (3.3).

We now provide an example to illustrate how to order the variables in order to satisfy Notation 3.1.

Example 3.1 *(Partial causal ordering) Consider a SVAR with quarterly observations of* $(\pi_t, \Delta gdp_t, m_t, i_t)'$, *where* π_t *is inflation,* Δgdp_t *real GDP growth,* m_t *the monetary aggregate and* i_t *the nominal interest rate. Consider the under-identifying restrictions imposed on* A_0^{-1},

$$\begin{pmatrix} u_t^{\pi} \\ u_t^{\Delta gdp} \\ u_t^{m} \\ u_t^{i} \end{pmatrix} = \begin{pmatrix} a^{11} & a^{12} & 0 & 0 \\ a^{21} & a^{22} & 0 & 0 \\ a^{31} & a^{32} & a^{33} & a^{34} \\ a^{41} & a^{42} & a^{43} & a^{44} \end{pmatrix} \begin{pmatrix} \epsilon_t^{\pi} \\ \epsilon_t^{\Delta gdp} \\ \epsilon_t^{m} \\ \epsilon_t^{i} \end{pmatrix}. \tag{3.4}$$

As in Example 1 in Kilian (2013), we interpret the first two equations as an aggregate supply and an aggregate demand equation, the third equation as a money demand equation, and the last equation as a monetary policy reaction function. The zero restrictions imply that the private sectors in the economy do not react to the contemporaneous money demand and interest rate, which may be a credible assumption if the private sectors cannot react within the quarter. If in addition we set $a^{12} = a^{34} = 0$, *we would have the classical recursive identification restrictions which guarantee point identification. These additional restrictions are however often difficult to justify, as* $a^{12} = 0$ *implies a horizontal supply curve and* $a^{34} = 0$ *implies a money demand that is inelastic to the nominal interest rate.*

If the object of interest are the impulse responses to the monetary policy shock ϵ_t^i. *Let* $\left[q^{\pi}, q^{\Delta gdp}, q^{m}, q^{i}\right]$ *be a* 4×4 *orthogonal matrix with the order of columns same as in (3.4). By (3.1), the imposed restrictions imply two restrictions on* q^m *and two restrictions on* q^i. *Hence, an ordering of the variables that is consistent with Notation 3.1 is* $y_t = (i_t, m_t, \pi_t, \Delta gdp_t)'$, *and the corresponding numbers of restrictions are* $(f_1, f_2, f_3, f_4) = (2, 2, 0, 0)$ *with* $j^* = 1$. *Note that the current zero restrictions satisfy (3.3). If the objects of interest are the impulse responses to a demand shock* $\epsilon^{\Delta gdp}$, *we order the variables as* $y_t = (i_t, m_t, \Delta gdp_t, \pi_t)$, *and* $j^* = 3$.

3.2 Sign Restrictions

Sign restrictions on the impulse responses could be considered alone or could be added to the zero restrictions as a way to tighten the impulse response identified set. It is straightforward to incorporate sign restrictions on the impulse responses into the current framework. Given the zero restrictions $F(\phi, Q) = \mathbf{0}$, we maintain the order of the variables as specified in Notation 3.1. When only imposing sign restrictions, the order of the variables can be arbitrary, while we let the variable whose structural shock is of interest appear first, $j^* = 1$. For a vector $x = (x_1, \ldots, x_m)'$, $x \geq \mathbf{0}$ means $x_i \geq 0$ for all $i = 1, \ldots, m$, and $x > \mathbf{0}$ means $x_i \geq 0$ for all $i = 1, \ldots, m$ and $x_i > 0$ for some $i \in \{1, \ldots, m\}$.

Suppose that sign restrictions are placed on the responses to the j-th structural shock and let $s_{h,j}$ be the number of sign restrictions placed on the h-th horizon impulse responses. Since the impulse response vector to the j-th structural shock is given by the j-th column vector of $IR^h = C_h(B)\Sigma_{tr}Q$, we can write the sign restrictions on the h-th horizon response vector as

$$S_{h,j}(\phi)\, q_j \geq \mathbf{0},$$

where $S_{h,j}(\phi) \equiv D_{h,j}C_h(B)\Sigma_{tr}$ is a $s_{h,j} \times n$ matrix, and $D_{h,j}$ is the $s_{h,j} \times n$ selection matrix that selects the sign restricted responses from the $n \times 1$ response vector $C_h(B)\Sigma_{tr}q_j$. The nonzero elements of $D_{h,j}$ equal 1 or -1 depending on whether the corresponding impulse responses are restricted to be positive or negative. By stacking the coefficient matrices $S_{h,j}(\phi)$ over multiple horizons, we express the whole set of sign restrictions on the responses to the j-th shock as

$$S_j(\phi)\, q_j \geq \mathbf{0}, \tag{3.5}$$

where $S_j(\phi)$ is a $\left(\sum_{h=0}^{\bar{h}_j} s_{h,j}\right) \times n$ matrix defined by $S_j(\phi) = \left[S_{0,j}(\phi)', \ldots, S_{\bar{h}_j,j}(\phi)\right]'$. If no sign restrictions are placed on the $\tilde{h}$-th horizon responses, $0 \leq \tilde{h} \leq \bar{h}$, we set $s_{\tilde{h},j} = 0$ and interpret $S_{\tilde{h},j}(\phi)$ as not present in the construction of $S_j(\phi)$.

Note that the sign restrictions considered here do not have to be limited to the impulse responses. Since $A_0' = \Sigma_{tr}^{-1\prime}Q$ and $A_l' = B_l'\left(\Sigma_{tr}^{-1}\right)'Q$, $l = 1, \ldots, p$, any sign restrictions on structural parameters appearing in the j-th row of A_0 or A_l take the form of linear inequalities for q_j, so these sign restrictions could be appended to $S_j(\phi)$ in (3.5).

Let $\mathcal{I}_S \subset \{1, 2, \ldots, n\}$ be the set of indices such that $j \in \mathcal{I}_S$ if some of the impulse responses to the j-th structural shock are sign-constrained. The set of all the sign constraints can be accordingly expressed by

$$S_j(\phi)\, q_j \geq \mathbf{0} \quad \text{for } j \in \mathcal{I}_S. \tag{3.6}$$

As a shorthand notation, we represent the entire set of sign restrictions by $S(\phi, Q) \geq \mathbf{0}$.

Given $\phi \in \Phi$, let $\mathcal{Q}(\phi|F,S)$ be the set of Q's that jointly satisfy the sign restrictions (3.6), zero restrictions (3.2), and the sign normalizations,

$$\mathcal{Q}(\phi|F,S) = \left\{Q \in \mathcal{O}(n) : S(\phi,Q) \geq \mathbf{0},\ F(\phi,Q) = \mathbf{0},\ diag\left(Q'\Sigma_{tr}^{-1}\right) \geq \mathbf{0}\right\}. \tag{3.7}$$

Unlike in the case with only under-identifying zero restrictions, $\mathcal{Q}(\phi|F,S)$ can be an empty set depending on ϕ and the imposed sign restrictions. If $\mathcal{Q}(\phi|F,S)$ is nonempty, the identified set for r denoted by $IS_r(\phi|F,S)$ is given by the range of r with the domain of Q given by $\mathcal{Q}(\phi|F,S)$. If $\mathcal{Q}(\phi|F,S)$ is empty, the identified set of r is defined as an empty set.

4 Inference on the Identified Set: a Robust Bayes Approach

In this section we consider a robust Bayes approach to conducting inference on the impulse response identified set. The procedure delivers two intervals: a posterior mean bounds interval, interpreted as an estimator of the identified set, and a robustified credible region, interpreted as a measure of the posterior uncertainty about the identified set. It is important to remark here that the robust Bayes interpretation of these intervals is valid regardless of whether the identified set is convex or not.

Robust Bayes inference has been considered in the statistics literature (Berger and Berliner (1986), DeRobertis and Hartigan (1981), and Wasserman (1990)) and in econometrics (Chamberlain and Leamer (1976) and Leamer (1982)) for the linear regression model, but in both cases only for point identified models. We should note that this paper is an outgrowth of the retired working paper Kitagawa (2012), in the sense that this paper contains and extends the main theoretical results that were in the working paper to the SVAR setting. At a later date, all the material that overlaps with this paper will be deleted from Kitagawa (2012) and the working paper will have a different focus.

4.1 Multiple Priors and Posterior Bounds

Let $\tilde{\pi}_\phi$ be a probability measure on the reduced form parameter space Φ. To construct a prior distribution for ϕ consistent with the zero restrictions $F(\phi, Q) = \mathbf{0}$ and the sign restrictions $S(\phi, Q) \geq \mathbf{0}$, we trim the support of $\tilde{\pi}_\phi$ as follows:

$$\pi_\phi \equiv \tilde{\pi}_{\phi|\Phi_{F,S}} \equiv \frac{\tilde{\pi}_\phi 1\left\{\mathcal{Q}\left(\phi|F,S\right) \neq \emptyset\right\}}{\tilde{\pi}_\phi\left(\left\{\mathcal{Q}\left(\phi|F,S\right) \neq \emptyset\right\}\right)}, \tag{4.1}$$

where the conditioning event $\Phi_{F,S}$ in the notation of $\tilde{\pi}_{\phi|\Phi_{F,S}}$ is the set of reduced form parameter values that are consistent with the imposed restrictions, $\Phi_{F,S} = \{\phi \in \Phi : \mathcal{Q}\left(\phi|F,S\right) \neq \emptyset\}$. By construction, the prior π_ϕ assigns probability one to the distribution of data that is consistent with the identifying restrictions, i.e., $\pi_\phi\left(\left\{\mathcal{Q}\left(\phi|F,S\right) \neq \emptyset\right\}\right) = 1$. A joint prior for $(\phi, Q) \in \Phi \times \mathcal{O}(n)$ that has ϕ-marginal π_ϕ can be expressed as $\pi_{\phi Q} = \pi_{Q|\phi}\pi_\phi$, where $\pi_{Q|\phi}$ is supported only on $\mathcal{Q}\left(\phi|F,S\right) \subset \mathcal{O}(n)$. Since the structural parameters $(A_0, A_1, \ldots, A_p)$ and the impulse responses are functions of (ϕ, Q), $\pi_{\phi Q}$ induces a unique prior distribution for the structural parameters and the impulse responses. Conversely, a prior distribution for $(A_0, A_1, \ldots, A_p)$ that incorporates the sign normalizations induces a prior for $\pi_{\phi Q}$. If one conducts SVAR analysis with a prior distribution for $(A_0, A_1, \ldots, A_p)$, the prior for ϕ induced by the prior for $(A_0, A_1, \ldots, A_p)$ is updated by the data, while the conditional prior $\pi_{Q|\phi}$, which is implicitly induced by the prior for $(A_0, A_1, \ldots, A_p)$, remains unchanged.

In the exact identification case where the imposed restrictions and the sign normalizations can pin down a unique Q (i.e., $\mathcal{Q}\left(\phi|F,S\right)$ is a singleton), $\pi_{Q|\phi}$ is degenerate and gives a point mass at such Q. As a result, specifying π_ϕ suffices to induce a single posterior distribution for the structural coefficients and the impulse responses. In contrast, in the partially identified case where $\mathcal{Q}\left(\phi|F,S\right)$ is non-singleton for ϕ's with a positive measure, specifying only π_ϕ cannot yield a unique posterior distribution for the impulse responses. To obtain a posterior distribution for the impulse responses, as desired in the standard Bayesian approach, we need to specify $\pi_{Q|\phi}$, which is supported only on $\mathcal{Q}(\phi|F,S) \subset \mathcal{O}(n)$ at each $\phi \in \Phi$. In empirical practice, however, it is a challenging task for a researcher to come up with a "reasonable" specification for $\pi_{Q|\phi}$ when the prior knowledge that she considers credible is exhausted by the zero restrictions and the sign restrictions. Even when it is feasible to specify $\pi_{Q|\phi}$, the fact that $\pi_{Q|\phi}$ is never updated by the data makes the posterior distribution for the impulse response sensitive to the choice of $\pi_{Q|\phi}$ even asymptotically, so that a limited confidence in the choice of $\pi_{Q|\phi}$ leads to an equally limited credibility in the posterior inference. Since (ϕ, Q) and the structural parameters $(A_0, A_1, \ldots, A_p)$ are one-to-one (under the sign normalizations), the difficulty of specifying a prior for $\pi_{Q|\phi}$ can be equivalently stated as the difficulty of specifying a joint prior for all structural parameters with fixing the prior for ϕ at π_ϕ.

The robust Bayes procedure considered in this paper aims to make the posterior inference free from the choice of $\pi_{Q|\phi}$. More specifically, we specify a single prior for the reduced form parameters ϕ, which the data are always informative about, whereas we introduce a set of priors (ambiguous belief) for $\pi_{Q|\phi}$. Let $\Pi_{Q|\phi}$ denote a collection of conditional priors $\pi_{Q|\phi}$. Given a single prior for ϕ, π_ϕ, let $\pi_{\phi|Y}$ be the posterior distribution for ϕ obtained by the Bayesian reduced-form VAR, where Y stands for a sample. The class of conditional priors that impose no restrictions other than the zero restrictions and/or the sign restrictions is defined as

$$\Pi_{Q|\phi} = \left\{\pi_{Q|\phi} : \pi_{Q|\phi}\left(\mathcal{Q}\left(\phi|F,S\right)\right) = 1,\ \pi_\phi\text{-almost surely}\right\}. \tag{4.2}$$

In words, it consists of arbitrary $\pi_{Q|\phi}$'s as far as they assign probability one to the set of Q's that satisfy the imposed restrictions.

The posterior for ϕ combined with the prior class $\Pi_{Q|\phi}$ generates the class of joint posteriors for (ϕ, Q),

$$\Pi_{\phi Q|Y} = \left\{\pi_{\phi Q|Y} = \pi_{Q|\phi}\pi_{\phi|Y} : \pi_{Q|\phi} \in \Pi_{Q|\phi}\right\},$$

which coincides with the class of posteriors obtained by applying Bayes rule to each prior in the class $\left\{\pi_{\phi,Q} = \pi_\phi \pi_{Q|\phi} : \pi_{Q|\phi} \in \Pi_{Q|\phi}\right\}$. This class of posteriors for (ϕ, Q) induces the class of posteriors for impulse response, $r = r(\phi, Q)$,

$$\Pi_{r|Y} \equiv \left\{\pi_{r|Y}(\cdot) = \pi_{\phi,Q|Y}\left(r(\phi,Q) \in \cdot\right) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}\right\}. \tag{4.3}$$

We summarize the posterior class for r by constructing the bounds of the posterior means of r and the posterior probabilities.

Proposition 4.1 *Let a prior for ϕ, π_ϕ, be given, and assume $\pi_\phi\left(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}\right) = 1$. Let a prior class for $\pi_{Q|\phi}$ be given by (4.2).*

(i) The bounds of the posterior probabilities for an event $\{r \in G\}$, where G is a measurable subset in $\mathcal{R}$, are given by $\left[\pi_{r|Y*}(G), \pi^*_{r|Y}(G)\right]$, where

$$\begin{aligned}
\pi_{r|Y*}(G) &\equiv \inf\left\{\pi_{r|Y}(G) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}\right\} \\
&= \pi_{\phi|Y}\left(IS_r(\phi|S,F) \subset G\right), \\
\pi^*_{r|Y}(G) &\equiv \sup\left\{\pi_{r|Y}(G) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}\right\} \\
&= \pi_{\phi|Y}\left(IS_r(\phi|S,F) \cap G \neq \emptyset\right), \\
&= 1 - \pi_{r|Y*}(G^c).
\end{aligned}$$

(ii) The range of the posterior means $E(r|Y)$ with the posterior class $\Pi_{r|Y}$ given in (4.3) is

$$\left[\int_\Phi \ell(\phi) d\pi_{\phi|Y}, \int_\Phi u(\phi)\, d\pi_{\phi|Y}\right], \tag{4.4}$$

where $\ell(\phi)$ is the lower bound of $IS_r(\phi|F,S)$, $\ell(\phi) = \inf\{r(\phi,Q) : Q \in \mathcal{Q}(\phi|F,S)\}$, and $u(\phi)$ is the upper bound of $IS_r(\phi|F,S)$, $u(\phi) = \sup\{r(\phi,Q) : Q \in \mathcal{Q}(\phi|F,S)\}$.

Proof. The first claim is a corollary of Theorem 3.1 in Kitagawa (2012). For a proof of the second claim, see Appendix A. ∎

Note that the construction of these bounds is valid irrespective of whether $IS_r(\phi|S,F)$ is a convex interval or not, so the formulas of the posterior probability bounds and the mean bounds apply to any set-identified SVARs. The presented posterior probability bounds are convex in the sense that every value in $\left[\pi_{r|Y*}(G), \pi^*_{r|Y}(G)\right]$ is attained by some posterior in $\Pi_{\phi,Q|Y}$ (see Lemma B.1 of Kitagawa (2012) for a proof of this statement). As the expressions for $\pi_{r|Y*}(G)$ and $\pi^*_{r|Y}(G)$ suggest, the bounds of the posterior probabilities can be computed by the posterior probability that G contains and intersects with the identified set of r, respectively. If the impulse response is point-identified in the sense of $IS_r(\phi|S,F)$ being $\pi_{\phi|Y}$-almost surely a singleton, the posterior probability bounds collapse to a point for every G, leading to a single posterior.

As the analytical expressions of the posterior bounds show, we can approximate these posterior probability bounds if we can compute $IS_r(\phi|S,F)$ at values of ϕ randomly drawn from its posterior $\pi_{\phi|Y}$. Computation of $IS_r(\phi|S,F)$ can be greatly simplified if $IS_r(\phi|S,F)$ is guaranteed to be convex, e.g., the cases where Lemma 5.1 and Lemma 5.2 apply, since obtaining convex $IS_r(\phi|S,F)$ is reduced to computing $\ell(\phi)$ and $u(\phi)$.

The posterior mean bounds (4.4) are given by the mean of the lower and upper bounds of $IS_r(\phi|S,F)$ taken with respect to the posterior of ϕ. The range of posterior means is convex irrespective of whether the identified set of r is convex or not.

Building on Proposition 4.1, our robust Bayes inference proposes to report the posterior mean bounds of (4.4). As a robustified credible region, we consider reporting an interval satisfying

$$\pi_{r|Y*}(C_\alpha) \geq \alpha. \tag{4.5}$$

C_α is interpreted as an interval estimate for the impulse response r, such that the posterior probability put on C_α is greater than or equal to α uniformly over the posteriors in the class (4.3). There are multiple ways to construct C_α satisfying (4.5). One proposal is to consider the interval that has shortest width (Kitagawa (2012)) and satisfies (4.5) with equality. We hereafter refer to it as the *robustified credible region with lower credibility* α. We can also define C_α by mapping the highest posterior density region of ϕ to the real line via the set-valued map $IS_r(\cdot|S,F)$ (Moon and Schorfheide (2011)), which can be conservative in the sense that (4.5) can hold with inequality See also Kline and Tamer (2013) and Liao and Simoni (2013) for alternative proposals to constructing C_α.

4.2 Computing Posterior Bounds

This subsection presents an algorithm to numerically approximate the posterior mean bounds and the robustified credible region discussed in Proposition 4.1 using random draws of ϕ from its posterior. The algorithm assumes that the variables are ordered according to Notation 3.1 and the imposed zero restrictions satisfy (3.3). Therefore, they should be checked prior to implementation.

Algorithm 4.1 *Let* $F(\phi, Q) = \mathbf{0}$ *and* $S(\phi, Q) \geq \mathbf{0}$ *be the set of identifying restrictions, and let* $r = c_{ih}'(\phi)\, q_{j^*}$ *be the impulse response of interest.*

(Step 1) Specify $\tilde{\pi}_\phi$*, a prior for the reduced form parameters* ϕ*. The proposed* $\tilde{\pi}_\phi$ *need not satisfy* $\tilde{\pi}_\phi(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}) = 1$*. Run a Bayesian reduced form VAR to obtain the posterior* $\tilde{\pi}_{\phi|Y}$*.*

(Step 2) Draw a reduced form parameter vector ϕ *from* $\tilde{\pi}_{\phi|Y}$*. Given the draw of* ϕ*, examine if* $\mathcal{Q}(\phi|F,S)$ *is empty or not by following the subroutine (Step 2.1) - (Step 2.3) below.*[1]

(Step 2.1) Let $z_1 \sim \mathcal{N}(0, I_n)$ *be a draw of an n-variate standard normal random variable. Let* $\mathcal{M}_1 z_1$ *be the* $n \times 1$ *residual vector in the linear projection of* z_1 *onto a* $n \times f_1$ *regressor matrix* $F_1(\phi)'$*. Set* $\tilde{q}_1 = \mathcal{M}_1 z_1$*. For* $i = 2, 3, \ldots, n$*, run the following procedure sequentially: draw* $z_i \sim \mathcal{N}(0, I_n)$*, and compute* $\tilde{q}_i = \mathcal{M}_i z_i$*, where* $\mathcal{M}_i z_i$ *is the residual vector in the linear projection of* z_i *onto the* $n \times (f_i + i - 1)$ *regressor matrix,* $\left[F_i(\phi)', \tilde{q}_1, \ldots, \tilde{q}_{i-1}\right]$*.*

(Step 2.2) Given $\tilde{q}_1, \ldots, \tilde{q}_n$ *obtained in the previous step, define*

$$Q = \left[sign\left((\sigma^1)' \tilde{q}_1 \right) \frac{\tilde{q}_1}{\|\tilde{q}_1\|}, \ldots, sign\left((\sigma^n)' \tilde{q}_n \right) \frac{\tilde{q}_n}{\|\tilde{q}_n\|} \right],$$

[1] Instead of our algorithm for drawing Q from $\mathcal{Q}(\phi|F,S)$ (Step 2.1 - 2.3), one could alternatively use an algorithm that Arias, Rubio-Ramirez, and Waggoner (2013) developed in their Theorem 4, which draws Q from the Haar measure condtional on the zero restrictions.

where $\|\cdot\|$ is the Euclidian metric in $\mathcal{R}^n$. Q can be seen as a draw of an orthogonal matrix from $\mathcal{Q}(\phi|F)$.[2]

(Step 2.3) [3]*If Q obtained in (Step 2.2) satisfies the sign restrictions $S(\phi, Q) \geq \mathbf{0}$, retain this Q and proceed to (Step 3). Otherwise, repeat (Step 2.1) and (Step 2.2) at most L times (e.g., $L = 10000$), until obtaining Q satisfying $S(\phi, Q) \geq \mathbf{0}$. If none of L number of draws of Q satisfies $S(\phi, Q) \geq \mathbf{0}$, approximate $\mathcal{Q}(\phi|F, S)$ to be empty, and go back to Step 2 to obtain a new draw of ϕ.*

(Step 3) Given ϕ and Q obtained in (Step 2) and (Step 2.3), compute the lower and upper bounds of $IS_r(\phi|S, F)$ by solving the following nonlinear optimization with equality and inequality constraints,[4]

$$
\begin{aligned}
& \ell(\phi) = \arg\min_{Q} c'_{ih}(\phi)\, q_{j^*}, \\
s.t. \quad & Q'Q = I_n, \quad F(\phi, Q) = \mathbf{0}, \\
& diag(Q'\Sigma_{tr}^{-1}) \geq 0, \text{ and } S(\phi, Q) \geq \mathbf{0},
\end{aligned}
$$

and $u(\phi) = \arg\max_Q c'_{ih}(\phi)\, q_{j^*}$ under the same set of constraints.

(Step 4) Repeat (Step 2) - (Step 3) M times, and obtain M draws of the intervals, $[\ell(\phi_m), u(\phi_m)]$, $m = 1, \ldots, M$. Approximate the posterior mean bounds of Proposition 4.1 by the sample averages of $(\ell(\phi_m) : m = 1, \ldots, M)$ and $(u(\phi_m) : m = 1, \ldots, M)$.

(Step 5) To obtain an approximation of the robustified credible region with credibility $\alpha \in (0, 1)$, define $d(r, \phi) = \max\{|r - \ell(\phi)|, |r - u(\phi)|\}$, and let $\hat{z}_\alpha(r)$ be the sample α-th quantile of $(d(r, \phi_m) : m = 1, \ldots, M)$. An approximated robustified credible region for r is obtained as an interval centered at $\arg\min_r \hat{z}_\alpha(r)$ with radius $\min_r \hat{z}_\alpha(r)$ (Proposition 5.1 of Kitagawa (2012)).

In the above algorithm, the non-linear optimization part of (Step 3) can be computationally unstable and time-consuming, especially when the number of variables and constraints are large and convergence to the optimum is slow. If one encounters such computational challenges in a given application, a more computationally stable algorithm can be used, in which (Step 3) above is

[2]If $(\sigma^i)' \tilde{q}_i$ is zero for some i, we can set $sign\left((\sigma^i)' \tilde{q}_i\right)$ at 1 or -1 randomly.

[3]Skip this step if there are no sign restrictions imposed.

[4]In the empirical application in section 7, we used the "auglag" function available in an R package "alabama", which implements the augumented Lagrangean multiplier method for a nonlinear optimization with equality and inequality constraints. At each ϕ, we used Q obtained in (Step 2.3) as an initial value for the nonlinear optimization. For all the models considered, the optimization algorithm converged under the default convergence criterion at every draw of ϕ.

replaced with (Step 3') below. A downside of this alternative algorithm is that the approximated identified set is smaller than $IS_r(\phi|F,S)$ at every draw of ϕ, resulting in approximated posterior bounds that are shorter than the actual ones. Nonetheless, these alternative bounds still provide a consistent estimator of the identified set, as the number of draws of Q's goes to infinity.

(Step 3') Iterate (Step 2.1) - (Step 2.3) K times and let $\left(Q_l : l = 1, \ldots, \tilde{K}\right)$ be the draws that satisfy the sign restrictions. (If none of the draws satisfy the sign restrictions, we draw a new ϕ and iterate (Step 2.1) - (Step 2.3) again). Let $q_{j^*,k}$, $k = 1, \ldots, \tilde{K}$, be the j^*-th column vector of Q_k. We then approximate $[\ell(\phi), u(\phi)]$ by $[\min_k c'_{ih}(\phi)\, q_{j^*,k}, \max_k c'_{ih}(\phi)\, q_{j^*,k}]$.

In a situation where the zero and sign restrictions satisfies their parsimony condition in Gafarov and Montiel-Olea (2014), closed form expressions for the optimum in (Step 3) obtained in Gafarov and Montiel-Olea (2014) can be used and they can lead to a faster implementation of Algorithm 4.1.

4.3 Diagnostic tools

4.3.1 Informativeness of identifying restrictions and of priors

We propose two measures that can be usefully reported in empirical applications and can help disentangle the information contained in the identifying restriction from that introduced through the choice of priors. Let model 0 denote the SVAR model without imposing any identifying restriction, and let model k be the SVAR model that imposes a set of under-identifying restrictions. The identifying power of the imposed restrictions for one impulse response at a fixed horizon r_{ij}^h, can be measured by

$$\text{Informativeness of restrictions in model } k = 1 - \frac{\text{width of posterior mean bounds of } r_{ij}^h \text{ in model } k}{\text{width of posterior mean bounds of } r_{ij}^h \text{ in model } 0}, \tag{4.6}$$

which measures by how much the restrictions imposed in model k reduce the posterior mean bounds of r_{ij}^h compared to the case with no restrictions. When the posterior probability for the nonemptiness of the identified set is close to one, this index measures the informational gain in posterior inference about r_{ij}^h by the reduction in the amount of ambiguity (the size of the set of priors for Q).

In order to quantify the amount of posterior information supplied by the choice of a single prior for Q, we define the following measure of prior informativeness for the posterior of r_{ij}^h:

$$\begin{aligned} & \text{Informativeness of prior in model } k \qquad (4.7) \\ = \;& 1 - \frac{\text{width of the 90\% Bayesian highest posterior density region of } r_{ij}^h \text{ in model } k}{\text{width of the 90\% robustified credible region of } r_{ij}^h \text{ in model } k}. \end{aligned}$$

This prior informativeness measure captures by what fraction the highest posterior density region for r_{ij}^h is tightened by choosing a particular prior for Q, relative to our multiple-prior robustified credible region. With this measure of prior informativeness, one can learn and report how much of the posterior information for r_{ij}^h comes from the non-updated part of the prior.

4.3.2 Plausibility of the Identifying Restrictions

By calculating the proportion of drawn ϕ's that pass (Step 2.3) of Algorithm 5.1, we can obtain an approximation of the posterior probability (corresponding to the non-trimmed prior $\tilde{\pi}_\phi$) of having a nonempty identified set, $\tilde{\pi}_{\phi|Y}\left(\{\phi : \mathcal{Q}\left(\phi|F,S\right) \neq \emptyset\}\right)$. With only zero restrictions the set of admissible Q's, $\mathcal{Q}\left(\phi|F\right)$, is never empty as will be shown in Lemma 5.1 below, so the data cannot detect violation of the imposed assumptions irrespective of the choice of $\tilde{\pi}_\phi$. In contrast, with sign restrictions $\mathcal{Q}\left(\phi|F,S\right)$ can become empty for some ϕ, so that if we specify $\tilde{\pi}_\phi$ that supports the entire Φ (e.g., the normal -Wishart prior for $\phi = (B, \Sigma)$), the data allow us to update the belief about the *plausibility* of the imposed assumptions (i.e., the posterior probability of having a non-empty identified set). As is also discussed in Kline and Tamer (2013), the posterior *plausibility* of the imposed assumptions is an important quantity to report in empirical applications, since it can convey the *upper bound* of the credibility (most optimistic belief) of the imposed assumptions after observing data.[5] In fact, the posterior *plausibility* of the imposed assumptions is not unique to our setting, but in principle it can be computed in the standard Bayesian approach to SVAR analysis with sign restrictions, although it has been rarely reported in the literature. Note that in the frequentist approach Moon, Schorfheide, and Granziera (2013), it is instead not straightforward to separate the inferential statement about the plausibility of the assumptions from the confidence statement about the identified set.

5 Convexity of the Identified Set

In this section we provide sufficient conditions that guarantee that a set of equality and/or sign restrictions result in an identified set $IS_r\left(\phi|F,S\right)$ for the impulse response that is ϕ-a.s. convex. Having an easy-to-check condition for the convexity is useful in our posterior bound analysis, as it enables us to interpret the constructed posterior mean bounds as an estimator for the identified set rather than an estimator for the convex hull of a potentially nonconvex identified set. The analytical results shown in this section clarify a general topological property of the impulse response identified

[5] An alternative quantity that is informative for assessing the plausibility of the imposed restrictions is the prior-posterior odds of the nonemptiness of the identified set,

$$O_{F,S} = \frac{\tilde{\pi}_{\phi|Y}\left(\{\phi : \mathcal{Q}\left(\phi|F,S\right) \neq \emptyset\}\right)}{\tilde{\pi}_{\phi}\left(\{\phi : \mathcal{Q}\left(\phi|F,S\right) \neq \emptyset\}\right)}.$$

$O_{F,S}$ exceeding one indicates that the data are in favor of "plausibility of the imposed assumptions."

set and they may be of separate interest regardless of whether one favours a Bayesian or frequentist inference.

To gain some intuition behind our convexity results, consider the case of equality restrictions that restrict a single column q_j of the rotation matrix by linear constraints in the form of (3.2). In this case, convexity of the impulse response identified set for r_{ij}^h follows if the subspace constrained by the zero restrictions has dimension greater than one. The reason is that in this case the set of feasible q_j's becomes a subset on the unit sphere in $\mathcal{R}^n$ where any two elements q_j and $q_{j'}$ in the subset are path-connected, which in turn implies a convex identified set for the impulse response because the impulse response is a continuous function of q_j. When the subspace has dimension one, non-convexity can occur because, for example, the impulse response identified set consists of two disconnected points - meaning that the impulse response is locally, but not globally identified. This argument implies that a simple sufficient condition for ϕ-a.s. convexity of $IS_r(\phi|F)$ can be obtained by finding a condition on the number of zero restrictions that guarantees the linear subspace where feasible q_j lies to have dimension greater than one.

Lemma 5.1 *(Convexity of the impulse response identified set under equality restrictions) Let* $\{r = c_{ih}'(\phi)\, q_{j^*} : i = 1, \ldots, n,\ h = 0, 1, 2, \ldots\}$ *be the impulse responses to the* j^**-th structural shock. Consider a collection of zero restrictions of the form given by (3.2), where the order of the variables is such that* $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$ *and* $f_{j^*-1} > f_{j^*}$ *if* $j^* \geq 2$*. Assume* $f_i \leq n - i$ *holds for all* $i = 1, 2, \ldots, n$*. Then, the identified set for* $r = c_{ih}'(\phi)\, q_{j^*}$ *is non-empty and bounded for every* $i \in \{1, \ldots, n\}$ *and* $h = 0, 1, 2, \ldots$*,* ϕ*-a.s. In addition, the identified set is convex for every* $i \in \{1, \ldots, n\}$ *and* $h = 0, 1, 2, \ldots$*,* ϕ*-a.s., if any of the following mutually exclusive conditions holds:*

(i) $j^* = 1$ and $f_1 < n - 1$.

(ii) $j^* \geq 2$, and $f_i < n - i$ for all $i = 1, \ldots, (j^* - 1)$.

(iii) $j^* \geq 2$ and there exists $1 \leq i^* \leq (j^* - 1)$ such that $[q_1, \ldots, q_{i^*}]$ is exactly identified (as in Definition 5.1) and $f_i < n - i$ for all $i = i^* + 1, \ldots, j^*$.

Proof. See Appendix A. ■

Below we define exact identification for a subset of the column vectors of Q.

Definition 5.1 *(Exact identification of column vectors of Q) Consider a collection of zero restrictions of the form given by (3.2), where the order of the variables is consistent with* $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$*. We say that the first* j*-th column vectors of* Q*,* $[q_1, \ldots, q_j]$ *are exactly identified if, for almost every* $\phi \in \Phi$*,* $\left(\sum_{i=1}^{j} f_j\right)$*-number of constraints*

$$\begin{pmatrix} F_1(\phi)\, q_1 \\ F_2(\phi)\, q_2 \\ \vdots \\ F_j(\phi)\, q_j \end{pmatrix} = \mathbf{0}$$

and the sign-normalizations $\left(\sigma^i\right)' q_i \geq 0$, $i = 1, \ldots, j$, *pin down a unique* $[q_1, \ldots, q_j]$.

If $rank(F_i(\phi)) = f_i$ for all $i = 1, \ldots, j$, ϕ-a.s., a *necessary* condition for exact identification of $[q_1, \ldots, q_j]$ is that $f_i = n - i$ for all $i = 1, 2, \ldots, j$. One can check if the condition is also sufficient by assessing if the following algorithm developed in Rubio-Ramirez, Waggoner, and Zha (2010) yields a unique set of orthonormal vectors $[q_1, \ldots, q_j]$ for every ϕ randomly drawn from a prior supporting the whole Φ (e.g., a normal-Wishart prior for (B, Σ)).

Algorithm 5.1 *(Successive construction of orthonormal vectors, Algorithm 1 in Rubio-Ramirez, Waggoner, and Zha (2010)) Consider a collection of zero restrictions of the form given by (3.2), where the order of the variables is consistent with* $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$. *Assume* $f_i = n - i$ *for all* $i = 1, \ldots, j$, *and* $rank(F_i(\phi)) = f_i$ *for all* $i = 1, \ldots, j$, ϕ*-a.s. Let* q_1 *be a unit length vector satisfying* $F_1(\phi) q_1 = 0$, *which is unique up to sign since* $rank(F_1(\phi)) = n - 1$ *by assumption. Given* q_1, *find orthonormal vectors* $q_2, \ldots, q_j$, *by solving*

$$\begin{pmatrix} F_i(\phi) \\ q_1' \\ \vdots \\ q_{i-1}' \end{pmatrix} q_i = 0,$$

successively for $i = 2, 3, \ldots, j$. *If*

$$rank \begin{pmatrix} F_i(\phi) \\ q_1' \\ \vdots \\ q_{i-1}' \end{pmatrix} = n - 1 \text{ for } i = 2, \ldots, j, \tag{5.1}$$

and q_i, $i = 1, \ldots, j$, *obtained by this algorithm satisfies* $\left(\sigma^i\right)' q_i \neq 0$ *for almost all* $\phi \in \Phi$, *i.e., the sign normalization restrictions determine a unique sign for the* q_i'*s, then* $[q_1, \ldots, q_j]$ *is exactly identified.*[6]

Lemma 5.1 shows that when a set of zero restrictions satisfies $f_i \leq n - i$ for all $i = 1, 2, \ldots, n$, the identified set for the impulse response is never empty for all variables and horizons, so any of the zero restrictions cannot be refuted by data. Furthermore, convexity of the identified set is guaranteed under additional restrictions as summarized by conditions (i) - (iii) of the lemma.

The following examples illustrate how to verify the conditions for convexity of the impulse response identified set using Lemma 5.1.

[6] A special situation where the rank conditions of (5.1) are guaranteed at almost every ϕ is when σ^i is linearly independent of the row vectors in $F_i(\phi)$ for all $i = 1, \ldots, n$, and the row vectors of $F_i(\phi)$ are spanned by the row vectors of $F_{i-1}(\phi)$ for all $i = 2, \ldots, j$. This condition holds in the recursive identification scheme, where we impose a triangularity restriction on A_0^{-1}. See Example 5.2.

Example 5.1 *Recall the example of partial causal ordering given in Example 3.1. If the object of interest are the impulse responses to the monetary policy shock ϵ_t^i, we order the variables as $y_t = (i_t, m_t, \pi_t, \Delta gdp_t)'$ and have $(f_1, f_2, f_3, f_4) = (2, 2, 0, 0)$ with $j^* = 1$. Since $f_1 = 2$, condition (i) of Lemma 5.1 guarantees that the impulse response identified sets are ϕ-a.s. convex. If the objects of interest are the impulse responses to a demand shock $\epsilon^{\Delta gdp}$, we order the variables as $y_t = (i_t, m_t, \Delta gdp_t, \pi_t)$, and $j^* = 3$. None of the conditions Lemma 5.1 apply in this case, so Lemma 5.1 does not guarantee convexity of the impulse response identified sets.*

Example 5.2 *Consider adding to the case in Example 3.1 a long-run money neutrality restriction, which sets the long-run impulse response of output gdp to monetary policy shock ϵ^i to zero. This results in one more restriction on q^i, as we are adding a zero restriction on the $(2, 4)$-th element of the long-run cumulative impulse response matrix CIR^∞. Accordingly, we can order the variables as $y_t = (i_t, m_t, \pi_t, \Delta gdp_t)'$ and we have $(f_1, f_2, f_3, f_4) = (3, 2, 0, 0)$. It can be shown that in this case the first two columns $[q_1, q_2]$ are exactly identified,[7] implying that the impulse responses to ϵ^i and ϵ^m are point-identified. The impulse responses to $\epsilon^{\Delta gdp}$ are instead partially identified and their identified sets are convex, as condition (iii) of Lemma 5.1 applies to $y_t = (i_t, m_t, \Delta gdp_t, \pi_t)'$ with $j^* = 3$.*

As an alternative to the long-run money neutrality restriction, assume $a^{12} = 0$. Then, an ordering of the variables when the objects of interest are the impulse responses to ϵ^i is given by $y_t = (i_t, m_t, \Delta gdp_t, \pi_t)'$ with $j^* = 1$ and $(f_1, f_2, f_3, f_4) = (2, 2, 1, 0)$. Compared to Example 3.1, imposing $a^{12} = 0$ does not change j^*. An inspection of the proof of Lemma 5.1 shows that if adding restrictions does not change the order of the variables and the number of zero restrictions up to the j^*-th variable, the identified set for the impulse responses to the j^*-th shock does not change for every $\phi \in \Phi$. Hence, adding $a^{12} = 0$ does not bring any additional identifying information for the impulse responses to the monetary policy shock. We can generalize this observation as stated in the next corollary (see Appendix A for a proof).

Corollary 5.1 *Let a set of zero restrictions, an ordering of variables $(1, \ldots, j^*, \ldots, n)$, and the corresponding number of zero restrictions $(f_1, \ldots, f_n)$ satisfy $f_i \leq n - i$ for all i, $f_1 \geq \cdots \geq f_n \geq 0$, and $f_{j^*-1} > f_{j^*}$, as in Notation 3.1. Consider imposing additional zero restrictions. Let $\pi(\cdot) : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation that reorders the variables to be consistent with Notation 3.1 after adding the new restrictions, and let $\left(\tilde{f}_{\pi(1)}, \ldots, \tilde{f}_{\pi(n)}\right)$ be the new number of restrictions. If $\tilde{f}_{\pi(i)} \leq n - \pi(i)$ for all $i = 1, \ldots, n$, $(\pi(1), \ldots, \pi(j^*)) = (1, \ldots, j^*)$, and $(f_1, \ldots, f_{j^*}) = \left(\tilde{f}_1, \ldots, \tilde{f}_{j^*}\right)$, i.e., adding the zero restrictions does not change the order of the variables and the number of restrictions for the first j^* variables, then the additional restrictions do not tighten the identified sets for the impulse response to the j^*-th shock for every $\phi \in \Phi$.*

[7] In the current case $F_2(\phi)$ is a submatrix of $F_1(\phi)$, implying that the vector space spanned by the rows of $F_1(\phi)$ contains the vector space spanned by the rows of $F_2(\phi)$ for every $\phi \in \Phi$. Hence, the rank condition for exact identification (5.1) holds.

Example 5.3 *Consider relaxing one of the zero restrictions in (3.4),*

$$\begin{pmatrix} u_t^{\pi} \\ u_t^{\Delta gdp} \\ u_t^m \\ u_t^i \end{pmatrix} = \begin{pmatrix} a^{11} & a^{12} & 0 & 0 \\ a^{21} & a^{22} & 0 & a^{24} \\ a^{31} & a^{32} & a^{33} & a^{34} \\ a^{41} & a^{42} & a^{43} & a^{44} \end{pmatrix} \begin{pmatrix} \epsilon_t^{\pi} \\ \epsilon_t^{\Delta gdp} \\ \epsilon_t^m \\ \epsilon_t^i \end{pmatrix},$$

where the $(2,4)$*-th element of* A_0^{-1} *is now unconstrained, i.e., the aggregate demand equation is allowed to respond contemporaneously to the monetary policy shock. If our interest is in the impulse responses to monetary policy shock* ϵ_t^i*, the variables can be ordered as* $y_t = (m_t, i_t, \pi_t, \Delta gdp_t)'$ *with* $j^* = 2$*. Condition (ii) of Lemma 5.1 is satisfied and the impulse response identified sets are convex. In fact, Lemma A.1 in the appendix implies that in situations where condition (ii) of Lemma 5.1 applies, the zero restrictions imposed on the preceding shocks to the* j^**-th structural shocks do not tighten the identified sets for the* j^**-th shock impulse responses compared to the case with no zero restrictions. In the current context, this means that dropping the two zero restrictions on* q_m *does not change the identified sets for the impulse responses to* ϵ_t^i*. The next corollary shows invariance of the identified sets when relaxing the zero restrictions, which partially overlaps with the implications of Corollary 5.1.*

Corollary 5.2 *Let a set of zero restrictions, an ordering of variables* $(1, \ldots, j^*, \ldots, n)$*, and the corresponding number of zero restrictions* $(f_1, \ldots, f_n)$ *satisfy* $f_i \leq n - i$ *for all* i*,* $f_1 \geq \cdots \geq f_n \geq 0$*, and* $f_{j^*-1} > f_{j^*}$*, as in Notation 3.1. Under any of the conditions (i) - (iii) of Lemma 5.1, the identified set for the impulse responses to the* j^**-th structural shock does not change when relaxing any or all of the zero restrictions on* $q_{j^*+1}, \ldots, q_{n-1}$*. Furthermore, if condition (ii) of Lemma 5.1 is satisfied, the identified set for the impulse responses to the* j^**-th structural shock does not change when relaxing any or all of the zero restrictions on* $q_1, \ldots, q_{j^*-1}$*. When condition (iii) of Lemma 5.1 is satisfied, the identified set for the impulse responses to the* j^**-th shock does not change when relaxing any or all of the zero restrictions on* $q_{i^*+1}, \ldots, q_{j^*-1}$*.*

The next lemma extends Lemma 5.1 to the case with sign restrictions.

Lemma 5.2 *(Convexity of the impulse response identified set under equality and sign restrictions) Let* $\{r = c'_{ih}(\phi) q_{j^*} : i = 1, \ldots, n, \, h = 0, 1, 2, \ldots\}$ *be the impulse responses of interest. Assume* $\mathcal{I}_S = \{j^*\}$*, i.e., the sign restrictions are placed only on the impulse responses to the* j^**-th structural shock.*

(i) Suppose that the zero restrictions $F(\phi, Q) = \mathbf{0}$ *satisfy one of the conditions (i) and (ii) of Lemma 5.1. If there exists a unit length vector* $q \in \mathcal{R}^n$ *such that*

$$F_{j^*}(\phi) q = 0 \text{ and } \begin{pmatrix} S_{j^*}(\phi) \\ (\sigma^{j^*})' \end{pmatrix} q > \mathbf{0}, \tag{5.2}$$

then $IS_r(\phi|F,S)$ *is nonempty and convex for every* $i \in \{1,\ldots,n\}$ *and* $h = 0,1,2,\ldots$.

(ii) Suppose that the zero restrictions $F(\phi,Q) = \mathbf{0}$ *satisfy condition (iii) of Lemma 5.1. Accordingly, let* $[q_1(\phi),\ldots,q_{i^*}(\phi)]$ *be the first* i^**-th orthonormal vectors that are exactly identified. If there exists a unit length vector* $q \in \mathcal{R}^n$ *such that*

$$\begin{pmatrix} F_{j^*}(\phi) \\ q_1'(\phi) \\ \vdots \\ q_{i^*}'(\phi) \end{pmatrix} q = 0 \text{ and } \begin{pmatrix} S_{j^*}(\phi) \\ \left(\sigma^{j^*}\right)' \end{pmatrix} q > \mathbf{0}, \tag{5.3}$$

then $IS_r(\phi|F,S)$ *is nonempty and convex for every* $i \in \{1,\ldots,n\}$ *and* $h = 0,1,2,\ldots$.

Proof. See Appendix A. ■

Lemma B.1 of Moon, Schorfheide, and Granziera (2013) shows convexity of the impulse response identified set for the special case where $\mathcal{I}_S = \{j^*\}$ and zero restrictions are imposed only on q_{j^*}, i.e., $j^* = 1$ and $f_i = 0$ for all $i = 2,\ldots,n$ in our notation. Lemma 5.2 extends their result to the case where zero restrictions are placed on the column vectors of Q other than q_{j^*}. Assumptions (5.2) or (5.3) of Lemma 5.2 imply that the set of feasible q's subject to the zero and sign restrictions is not degenerate in the sense that it does not collapse to a one-dimensional subspace in $\mathcal{R}^n$. If the set of feasible q's becomes degenerate, a non-convex identified set arises since the intersection of a one-dimensional subspace in $\mathcal{R}^n$ with the unit sphere consists of two disconnected points only. If the set of ϕ's that leads to such degeneracy has measure zero in Φ, then, as a corollary of Lemma 5.2, we can claim that the impulse response identified set is convex, ϕ-a.s., conditional on it being nonempty.

If sign restrictions are imposed on impulse responses to some structural shock other than the j^*-th shock, i.e., $\mathcal{I}_S$ contains an index other than j^*, the identified set for an impulse response can become non-convex, as we show in the next example.[8]

Example 5.4 *Consider a SVAR(0) model,*

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = A_0^{-1} \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

Let $\Sigma_{tr} = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$, *where* $\sigma_{11} \geq 0$ *and* $\sigma_{22} \geq 0$. *Positive semidefiniteness of* $\Sigma = \Sigma_{tr}\Sigma_{tr}'$ *requires* $\sigma_{22} \geq 1$, *while* σ_{21} *is left unconstrained. Denoting an orthonormal matrix by* $Q =$

[8] Consider the example given in Section 4.4 of Rubio-Ramirez (2010), where $n = 3$ and zero restrictions satisfying $f_1 = f_2 = f_3 = 1$. Their paper shows that the identified set for an impulse response consists of two distinct points. If we interpret the zero restrictions on the second and third variables as pairs of linear inequality restrictions for q_2 and q_3 with opposite signs, convexity of $IS_r(\phi|F,S)$ fails. In this counterexample, the assumption of $\mathcal{I}_S = \{j\}$ fails.

$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, *we can express the contemporaneous impulse response matrix as*

$$IR^0 = \begin{pmatrix} \sigma_{11}q_{11} & \sigma_{11}q_{12} \\ \sigma_{21}q_{11} + \sigma_{22}q_{21} & \sigma_{21}q_{12} + \sigma_{22}q_{22} \end{pmatrix}.$$

Consider restricting the sign of the $(1,2)$*-th element of* IR^0 *to being positive,* $\sigma_{11}q_{12} \geq 0$. *Since* $\Sigma_{tr}^{-1} = (\sigma_{11}\sigma_{22})^{-1} \begin{pmatrix} \sigma_{22} & 0 \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$, *the sign normalization restrictions give* $\sigma_{22}q_{11} - \sigma_{21}q_{21} \geq 0$ *and* $\sigma_{11}q_{22} \geq 0$. *We now show that the identified set for the* $(1,1)$*-th element of* IR^0 *is non-convex for a set of* Σ *with a positive measure. Note first that the second column vector of* Q *is constrained to* $\{q_{12} \geq 0, q_{22} \geq 0\}$, *so that the set of* $(q_{11}, q_{21})'$ *orthogonal to* $(q_{12}, q_{22})'$ *is constrained to*

$$\{q_{11} \geq 0, q_{21} \leq 0\} \cup \{q_{11} \leq 0, q_{21} \geq 0\}.$$

When $\sigma_{21} < 0$, *intersecting this union set with the half-space defined by the first sign normalization restriction* $\{\sigma_{22}q_{11} - \sigma_{21}q_{21} \geq 0\}$ *yields two disconnected arcs,*

$$\left\{ \begin{pmatrix} q_{11} \\ q_{21} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : \theta \in \left(\left[\frac{1}{2}\pi, \frac{1}{2}\pi + \psi\right] \cup \left[\frac{3}{2}\pi + \psi, 2\pi\right] \right) \right\},$$

where $\psi = \arccos\left(\frac{\sigma_{22}}{\sqrt{\sigma_{22}^2 + \sigma_{21}^2}}\right) \in \left[0, \frac{1}{2}\pi\right]$. *Accordingly, the identified set for* $r = \sigma_{11}q_{11}$ *is given by the union of two disconnected intervals*

$$\left[\sigma_{11}\cos\left(\frac{1}{2}\pi + \psi\right), 0\right] \cup \left[\sigma_{11}\cos\left(\frac{3}{2}\pi + \psi\right), \sigma_{11}\right].$$

Since $\{\sigma_{21} < 0\}$ *has a positive measure in the space of* Σ, *the identified set is non-convex with a positive measure.*

6 Asymptotic Properties

This section analyses the asymptotic properties of our method in large samples and shows that the posterior bounds converge asymptotically to the true identified set, when the set is convex. Let $\phi_0 \in \Phi$ be the true value of the reduced form parameters, and let $Y^T = (y_1, \ldots, y_T)$ denote a sample of size T generated from the probability distribution of the data, $p(Y^T|\phi_0)$. We assume posterior consistency for the reduced form parameters, meaning $\lim_{T\to\infty} \pi_{\phi|Y^T}(G) = 1$ for every G open neighborhood of ϕ_0, $p(Y^T|\phi_0)$-a.s.

Proposition 6.1 *Suppose that* $IS_r(\phi|F,S)$ *is a non-empty and continuous correspondence at* $\phi = \phi_0$, *and let* $[\ell(\phi_0), u(\phi_0)]$ *be the convex hull of* $IS_r(\phi_0|F,S)$.

(i) $\lim_{T\to\infty} \pi_{\phi|Y^T} \left(\{\phi : d_H \left(IS_r(\phi|F,S), IS_r(\phi_0|F,S) \right) > \epsilon \} \right) = 0$, *$p(Y^T|\phi_0)$-a.s., where $d_H(\cdot,\cdot)$ is the Hausdorff distance.*

(ii) If $\ell(\phi)$ and $u(\phi)$, $\phi \sim \pi_{\phi|Y^T}$, are uniformly integrable, $p(Y^T|\phi_0)$-a.s.,[9]*, the range of the posterior means converges to $[\ell(\phi_0), u(\phi_0)]$ as $T \to \infty$, $p(Y^T|\phi_0)$-a.s., i.e.,*

$$\int_\Phi \ell(\phi)\, d\pi_{\phi|Y^T} \to \ell(\phi_0) \quad \text{and}$$
$$\int_\Phi u(\phi)\, d\pi_{\phi|Y^T} \to u(\phi_0), \quad \text{as } T \to \infty,\ p(Y^T|\phi_0)\text{-a.s.},$$

and the shortest-width robustified credible region with credibility $\alpha \in (0,1)$ converges to $[\ell(\phi_0), u(\phi_0)]$, $p(Y^T|\phi_0)$-a.s.

Proof. See Appendix A. ■

The first claim of this proposition shows that the identified set $IS_r(\phi|F,S)$, viewed as a random set induced by the posterior of ϕ, converges to the true identified set in the Hausdorff metric. This claim only relies on continuity of the identified set correspondence and does not rely on convexity of $IS_r(\phi_0|F,S)$. If $IS_r(\phi_0|F,S)$ is convex, as is implied under the conditions of Lemma 5.1 or 5.2, Proposition 6.1 (ii) shows that the posterior mean bounds and the robustified credible region constructed in (Step 5) of Algorithm 5.1 converge to the true convex identified set. On the other hand, if the true identified set is non-convex, then, the posterior mean bounds and the robustified credible regions converge to the convex hull of the true identified set.

The continuity of $IS_r(\phi|F,S)$ at $\phi = \phi_0$ assumed in this proposition is crucial for guaranteeing consistency of the posterior bounds. The continuity of $IS_r(\phi|F,S)$ can be ensured by imposing a set of more primitive conditions involving a rank condition for the coefficient matrices of the zero and sign restrictions. We clarify them in the next proposition. In the statement of the proposition, for $x = (x_1, \ldots, x_m) \in \mathcal{R}^m$, $x >> 0$ means $x_i > 0$ for all $i = 1, \ldots, m$.

Proposition 6.2 *Consider the set-up of Lemma 5.2, where $\mathcal{I}_S = \{j^*\}$, i.e., the sign restrictions are placed only on the j^*-th structural shock. Let $\{r = c'_{ih}(\phi) q_{j^*} : i = 1, \ldots, n, h = 0, 1, 2, \ldots\}$ be the impulse responses of interest.*

(i) Suppose that the zero restrictions $F(\phi, Q) = \mathbf{0}$ satisfy one of the conditions (i) and (ii) of Lemma 5.1. If there exists $G \subset \Phi$ an open neighborhood of ϕ_0 such that $\text{rank}(F_{j^*}(\phi)) = f_{j^*}$ for all $\phi \in G$, and if there exists a unit length vector $q \in \mathcal{R}^n$ such that

$$F_{j^*}(\phi_0)\, q = 0 \ \text{and} \ \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} q >> \mathbf{0},$$

[9] The uniform integrability of $\ell(\phi)$ and $u(\phi)$, $p(Y^T|\phi)$-a.s. means

$$\sup_T \int_{|\ell(\phi)|>c} |\ell(\phi)|\, d\pi_{\phi|Y^T} \quad \to \quad 0, \text{ and}$$
$$\sup_T \int_{|u(\phi)|>c} |u(\phi)|\, d\pi_{\phi|Y^T} \quad \to \quad 0,$$

as $c \to \infty$, $p(Y^T|\phi)$-a.s.

then the identified set correspondence $IS_r(\phi)$ is continuous at $\phi = \phi_0$ for every $i = 1, \ldots, n$ and $h = 0, 1, 2, \ldots$.

(ii) Suppose that the zero restrictions $F(\phi, Q) = \mathbf{0}$ satisfy condition (iii) of Lemma 5.1, and let $[q_1(\phi), \ldots, q_{i^*}(\phi)]$ be the first i^*-th column vectors of Q that are exactly identified. If there exists $G \subset \Phi$ an open neighborhood of ϕ_0 such that $\begin{pmatrix} F_{j^*}(\phi) \\ q_1'(\phi) \\ \vdots \\ q_{i^*}'(\phi) \end{pmatrix}$ is a full row-rank matrix for all $\phi \in G$, and if there exists a unit length vector $q \in \mathcal{R}^n$ such that

$$\begin{pmatrix} F_{j^*}(\phi_0) \\ q_1'(\phi_0) \\ \vdots \\ q_{i^*}'(\phi_0) \end{pmatrix} q = 0 \text{ and } \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} q >> \mathbf{0},$$

then the identified set correspondence $IS_r(\phi)$ is continuous at $\phi = \phi_0$ for every $i = 1, \ldots, n$ and $h = 0, 1, 2, \ldots$.

Proof. See Appendix A. ■

7 An Empirical Example

We illustrate the use of our method and show how it can be used to: 1) perform robust Bayesian inference that does not require specifying a prior for the rotation matrix Q; 2) if a prior for Q is available, disentangle the information contained in the identifying restrictions from that introduced by the choice of the prior for Q.

We consider a SVAR for the nominal interest rate i_t, real GDP growth Δy_t, inflation rate π_t, and real money balances m_t. The data set is from Aruoba and Schorfheide (2011), and it is the same as in Moon et al (2013). The data are quarterly observations for the period 1965:I to 2005:I from the FRED2 database of the Federal Reserve Bank of St. Louis. See Aruoba and Schorfheide (2011) for details.

We consider a SVAR with two lags:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} i_t \\ \Delta y_t \\ \pi_t \\ m_t \end{pmatrix} = c + \sum_{j=1}^{2} A_j \begin{pmatrix} i_{t-j} \\ \Delta y_{t-j} \\ \pi_{t-j} \\ m_{t-j} \end{pmatrix} + \begin{pmatrix} \epsilon_{i,t} \\ \epsilon_{y,t} \\ \epsilon_{\pi,t} \\ \epsilon_{m,t} \end{pmatrix},$$

We order the variables so that the set of zero restrictions introduced below are compatible with Notation 3.1. Suppose that the impulse response of interest is the output response to a monetary policy shock, $\frac{\partial y_{t+h}}{\partial \epsilon_{i,t}}$, i.e., $j^* = 1$. The sign normalizations restrict the diagonal elements of the matrix in the left-hand side to being nonnegative, so that the output response is estimated with respect to a unit standard deviation contractionary monetary policy shock.

We consider different models that use different combinations of the following zero and sign restrictions. All models are partially identified.

Restrictions:

(i) The monetary authority does not respond to contemporaneous GDP growth, $a_{12} = 0$.

(ii) The instantaneous impulse response of the GDP growth rate to a monetary policy shock is zero, $IR^0(\Delta y, i) = 0$.

(iii) The long-run impulse response of the GDP level to a monetary policy shock is zero, $CIR^\infty(\Delta y, i) \approx \sum_{h=1}^{H} \frac{\partial \Delta y_{t+h}}{\delta \epsilon_{i,t}} = 0$, with $H = 80$.

(iv) The inflation response to a contractionary monetary policy shock is nonpositive for one quarter, $\frac{\partial \pi_{t+h}}{\partial \epsilon_{i,t}} \leq 0$ for $h = 0, 1$, the interest rate response is nonnegative for one quarter, $\frac{\partial i_{t+h}}{\partial \epsilon_{i,t}} \geq 0$ for $h = 0, 1$, and the response of the real money balances is nonpositive for one quarter, $\frac{\partial m_{t+h}}{\partial \epsilon_{i,t}} \geq 0$, for $h = 0, 1$.

We start from a model (model 0) which does not impose any identifying restrictions. We then impose seven different combinations of the restrictions, summarized in Table 1. The restrictions (i) through (iii) are zero restrictions that constrain the first column vector of Q, so $f_1 = 1$ if only one restriction out of (i) - (iii) is imposed (models II to IV), and $f_1 = 2$ if two restrictions are imposed (models V to VII). No zero restrictions are placed on the remaining columns of Q, so that for all models $f_2 = f_3 = f_4 = 0$.[10] The sign restrictions on the impulse responses are given by (iv), and are the same as those considered in Moon, Schorfheide, and Granziera (2013). We impose the sign restrictions on all the specifications.

Table 1: Model definition and Posterior Plausibility

Restrictions \ Model	0	I	II	III	IV	V	VI	VII
(i) $a_{12} = 0$	-	-	x	-	-	x	x	-
(ii) $IR^0(\Delta y, i) = 0$	-	-	-	x	-	x	-	x
(iii) $CIR^\infty(\Delta y, i) = 0$	-	-	-	-	x	-	x	x
(iv) sign restrictions	-	x	x	x	x		x	x
$\Pr(IS_r(\phi \mid F, S) \neq \emptyset \mid data)$	1.00	1.00	1.00	1.00	1.00	0.99	0.93	0.98

Note: "x" indicates the restriction is imposed

[10] Corollary 5.1 shows that adding one zero restriction to one of the other columns Q do not tighten the identified set in any of our Models II to VII. In the models with one zero restriction (Models II to VI), adding two zero restrictions to one of the other columns of Q can tighten the identified set of the responses to the monetary policy shock, as the ordering of variables consistent to Notation 3.1 changes once they are imposed.

The prior for the reduced form parameters ($\tilde{\pi}_\phi$, as defined in Section 4) is common to all the models and it is specified to be improper $d\tilde{\pi}_\phi(B, \Sigma) \propto |\Sigma|^{-\frac{4+1}{2}}$. This prior for ϕ corresponds to the Jeffreys' prior for the reduced form Gaussian VAR, and the posterior for ϕ is nearly identical to the likelihood with the current sample size. The bottom row of Table 1 reports the posterior probabilities for the plausibility of the imposed restrictions (nonemptiness of the identified set). In all the specifications considered, these probabilities are approximately one or nearly one.

In addition to the posterior bound analysis, we further consider standard Bayesian inference based on a single prior, for the purpose of assessing how much extra information is added to the posterior inference by the choice of a prior for the non-updated part of the model. We introduce a prior for Q that builds on the agnostic prior of Uhlig (2005). Specifically, we obtain the approximated posterior for the impulse responses based on the MCMC draws of the impulse responses. The draws for the impulse responses are obtained by iterating Step (2.1) - (2.3) of Algorithm 5.1, and retaining the draws of Q that satisfy the sign restrictions.

Figures 1 and 2 show the posterior bounds for the impulse responses and their credible region for both multiple-prior and single-prior approaches.[11] In implementing Algorithm 5.1, we draw ϕ's until we obtain 1000 realizations of the nonempty identified set $IS(\phi|S, F)$. Note that for all the models with the zero restriction(s), condition (i) of Lemma 5.1 holds. We also check existence of q satisfying (5.2) in Lemma 5.2 condition (i) at every MCMC draw of ϕ, so the draws of $IS(\phi|S, F)$ that the posterior bounds build on are all convex. In all the models considered, we employ the non-linear optimization step of Algorithm 5.1 (Step 3). Since we use the same prior for ϕ in every model and the posterior probabilities of having nonempty identified sets are close to one for all the models, the posterior bounds differ across the models mainly due to the different identifying restrictions. Table 2 provides the posterior inference results for the output responses at $h = 1$ (3 months), $h = 10$ (2 year and 6 months), and $h = 20$ (4 years) in each model. The table also shows the model and prior informativeness measures defined in (4.6) and (4.7).

Model I in Figure 1 shows that the posterior bounds using only the sign restrictions do not lead to informative inference for output responses. In fact, the measure of the informativeness of restrictions for model I indicates that the sign restrictions have little identifying power . Drawing informative posterior inference based only on these sign restrictions is therefore not feasible unless one introduces a specific prior for Q. The commonly used uniform prior for Q introduces information in the analysis, as the measure of the informativeness of prior shows that it narrows the impulse response credible regions by 30% to 40% relative to the robustified credible regions. This gain comes from a shift from ambiguous belief to a "noninformative" single prior for Q and highlights how the two different formulations of the "lack of prior knowledge" lead to different posterior inferences. Note that the posterior mean bounds and the robustified credible regions are as wide

[11] These figures summarize the marginal distribution of the impulse response at each horizon, and do not capture the dependence of the responses across different horizons.

as the point estimates of the identified sets and the frequentist confidence intervals reported in Moon, Schorfheide, and Granziera (2013). This similarity with frequentist inference for impulse response identified sets is compatible with the consistency property of the posterior mean bounds (Proposition 6.1). When one zero restriction is additionally imposed (model II - IV), the posterior mean bounds and the robustified credible regions get substantially tighter. The identifying power of these zero restrictions varies across the horizons. The restriction on the contemporaneous response (restriction (ii)) combined with the sign restrictions in model III are very informative for impulse responses at short horizons, as illustrated by high numbers for the measure of informativeness of restrictions. Long-run restrictions (restriction (iii)), on the other hand, are informative for long-horizon impulse responses. The zero restriction on the (1,2)-element of the A_0 matrix (restriction (i)) helps tighten up the posterior mean bounds for both short- and long-horizon impulse responses.

With two additional zero restrictions (model V - VII), the posterior mean bounds become informative for the sign of the output impulse response at short to middle-range horizons. Specifically, when the imposed zero restrictions include $IR^0(\Delta y, i) = 0$ (models V and VII), the range of posterior means of output responses is negative for $h = 0$ up to $h = 10$. On the other hand, if restrictions (i) and (iii) are jointly imposed (model VI), the range of posterior means is positive for short horizons, and we obtain the opposite conclusion to models V and VII. These results on relatively more informative posterior bounds show that, despite the lack of point-identification, the posterior inference is less sensitive to the choice of prior for Q once any of the two zero restrictions is imposed.

A noteworthy observation is that in models V to VII the posterior mean bounds lie strictly inside the 90% single-prior Bayesian credible region. These observations might appear to be contradicting the asymptotic result of Moon and Schorfheide (2012) which states that any Bayesian posterior credible region asymptotically lies inside the true identified set that our posterior mean bounds consistently estimate (Proposition 6.1 (ii)). Our explanation for these seemingly contradictory observations is as follows. When the identified set $IS_r(\phi) = [l(\phi), u(\phi)]$ is tight with high posterior probabilities and its width is small relative to the posterior variances of $(l(\phi), u(\phi))$, the single-prior Bayes credible region for the impulse response can become as wide as the posterior credible regions for $l(\phi)$ or $u(\phi)$, because the posterior of the impulse response is not so sensitive to the choice of prior for Q and it does not differ much from the posteriors of $l(\phi)$ or $u(\phi)$. On the other hand, the posterior mean bounds $\left[E_{\phi|Y}(l(\phi)), E_{\phi|Y}(u(\phi))\right]$ can remain tight even for large variances of $(l(\phi), u(\phi))$ as they are determined only by the locations of the posterior distributions of $l(\phi)$ and $u(\phi)$. This implies that when $l(\phi)$ and $u(\phi)$ are much more volatile than the width of $[l(\phi), u(\phi)]$, the single prior Bayesian credible region becomes substantially wider than the posterior mean bounds. As the sample size increases, the posterior distribution of $(l(\phi), u(\phi))$ becomes concentrated around their true values and the posterior distribution for the impulse response is eventually supported only on the true identified set, no matter how narrow the identified set is. Our findings in Figures

1 and 2 indicate that with a fixed sample size, whether or not Moon and Schorfheide's asymptotic results can well approximate the actual finite sample behavior of the Bayesian posterior depends on how accurately $l(\phi)$ and $u(\phi)$ are estimated and how tightly the imposed restrictions set-identify the object of interest. In contrast, the relationship between the Bayesian credible region (under the agnostic prior for Q) and the robustified credible region stays stable across the models. As shown by the measure of prior informativeness in Table 2, the Bayesian credible regions are 20% to 40% shorter than the robustified credible regions in every model. In terms of the their absolute length, this implies that the difference between the width of the Bayesian credible region and the width of the robustified credible region is smaller as the identified sets becomes tighter.

Both the posterior mean bounds and the robustified credible region become tighter as more restrictions are added. As long as the posterior probability of a nonempty identified set is one, this monotonic gain in the informativeness of the posterior bounds holds *irrespective of the realized values of the observations*, since adding identifying restrictions monotonically reduces the size of the prior class without changing the posterior of ϕ.[12] This property of "more restrictions, more informative inference" does not necessarily hold if we report frequentist confidence intervals for the true identified set.

We conclude this section by summarizing the recommended uses and advantages of our posterior bound analysis.

1. By reporting posterior mean bounds and robustified credible regions, one can learn what inferential conclusions can be supported solely by the imposed identifying restrictions and the posterior for the reduced form parameters. Even if a user has a credible choice of prior for Q, reporting our posterior bounds will help communicate with other users who may have different priors for Q.

2. By comparing the posterior bounds across different sets of identifying restrictions, one can learn and report which identifying restrictions are crucial in drawing a given inferential conclusion.

3. Our procedure can be a useful tool for separating the information about the impulse responses contained in the data from any prior input that is not updated by the data. Given that the shape of the likelihood is an object of interest for both Bayesians and frequentists, both may find the proposed analysis useful in summarizing and visualizing the information about the impulse responses contained in the *observed* likelihood, as is also advocated in Sims and Zha (1998) for the point-identified case.

[12] Manski (2003) calls the general principle of the trade-off between the strength of assumptions and the informativeness of the conclusion "the law of decreasing credibility." Manski defines this concept in terms of the true identified set, while our posterior bounds analysis respects this principle in the posterior inferential statement at every possible realization of data.

4. Even if the posterior bounds for a given set of identifying assumptions are too wide to draw any informative policy recommendation, this should not be considered a disadvantage of the method. The wide posterior bounds may encourage the analyst to search for additional credible assumptions and/or to refine the set of priors for Q[13] by further inspecting how the data are collected, any empirical evidence in other studies, and/or available economic theories. If any additional assumptions are not available, the posterior bounds inform the analyst about the amount of ambiguity that the policy decision will be subject to. As Manski (2013) argues, knowing what we do not know is an important step for a policy decision without incredible certitude.

[13] We leave for future research an investigation of analytically and computationally tractable ways to refine the set of priors of Q based on partial prior knowledge available for the structural parameters or the impulse responses.

Table 2: Output responses at $h = 1, 10$, and 20: Single Prior Bayes vs Posterior Bounds

	Model I			Model II		
	$h=1$	$h=10$	$h=20$	$h=1$	$h=10$	$h=20$
Bayes: post. mean	.10	-.09	.13	-.01	-.20	.06
Bayes: 90% CR	[-.53,.71]	[-.72,.55]	[-.55,.79]	[-.28,.27]	[-.57,.17]	[-.45,.57]
Post. mean bounds	[-.75,.84]	[-.79,.72]	[-.72,.85]	[-.31,.29]	[-.43,.11]	[-.40,.45]
90% robustified CR	[-.86,.98]	[-1.06,.97]	[-.96,1.23]	[-.46,.43]	[-.77,.44]	[-.80,.92]
Informativeness of restrictions*	.11	.21	.27	.66	.72	.60
Informativeness of prior**	.33	.37	.39	.38	.38	.40
	Model III			Model IV		
	$h=1$	$h=10$	$h=20$	$h=1$	$h=10$	$h=20$
Bayes: Post. mean	-.09	-.25	.02	-.02	-.27	-.11
Bayes: 90% CR	[-.21,.02]	[-.63,.12]	[-.53,.57]	[-.62,.58]	[-.74,.24]	[-.55,.30]
Post. mean bounds	[-.16,.01]	[-.53,.08]	[-.53,.48]	[-.68,.61]	[-.66,.17]	[-.42,.20]
90% robustified CR	[-.24,.12]	[-.82,.38]	[-.91,.88]	[-.85,.86]	[-.97,.64]	[-.73,.66]
Informativeness of restrictions	.90	.68	.53	.28	.57	.71
Informativeness of prior	.37	.38	.49	.30	.38	.39
	Model V			Model VI		
	$h=1$	$h=10$	$h=20$	$h=1$	$h=10$	$h=20$
Bayes: Post. mean	-.12	-.19	.15	.07	-.23	-.08
Bayes: 90% CR	[-.21,-.02]	[-.54,.15]	[-.31,.64]	[-.17,.31]	[-.59,.12]	[-.45,.30]
Post. mean bounds	[-.14,-.10]	[-.34,-.01]	[-.03,.35]	[-.04,.16]	[-.36,-.08]	[-.24,.06]
90% robustified CR	[-.22,.00]	[-.64,.31]	[-.52,.77]	[-.23,.37]	[-.69,.25]	[-.62,.43]
Informativeness of restrictions	.98	.82	.82	.93	.86	.86
Informativeness of prior	.17	.27	.27	.20	.25	28
	Model VII					
	$h=1$	$h=10$	$h=20$			
Bayes: Post. mean	-.08	-.31	-.12			
Bayes: 90% CR	[-.19,.04]	[-.63,.03]	[-.48,.26]			
Post. mean bounds	[-.19,..09]	[-.75,.17]	[-.70,.40]			
90% robustified CR	[-.19,.09]	[-.75,.17]	[-.70,.40]			
Model Informativeness	.97	.84	.84			
Prior Informativeness	.17	.29	.32			

Notes: * see eq. (4.6) for the definition. ** see eq. (4.7) for the definition.

8 Conclusion

We develop a robust Bayes inference procedure for a general class of structural vector autoregressions subject to under-identifying zero and/or sign restrictions. The proposed procedure reports

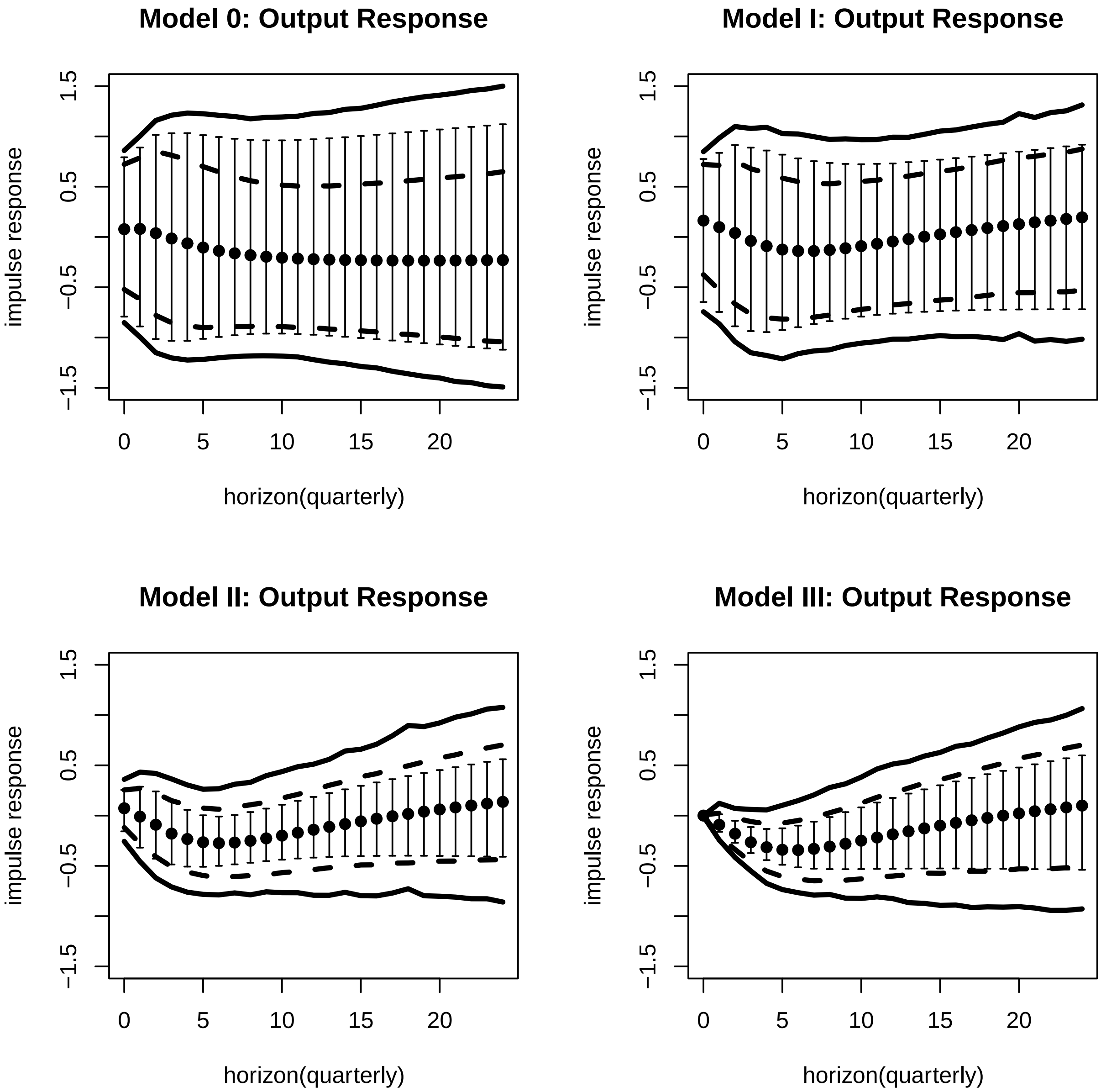

Figure 1: **Plots of Output Impulse Responses.** See Table 1 for the definition of each model. In each figure, the points plot the posterior means with the single prior for Q, the vertical bars show the posterior mean bounds with the multiple priors for Q, the dashed curves connect the upper/lower bounds of the highest posterior density regions with credibility 90% with the single prior for Q, and the solid curves connect the upper/lower bounds of our posterior robustified credible regions with credibility 90%.

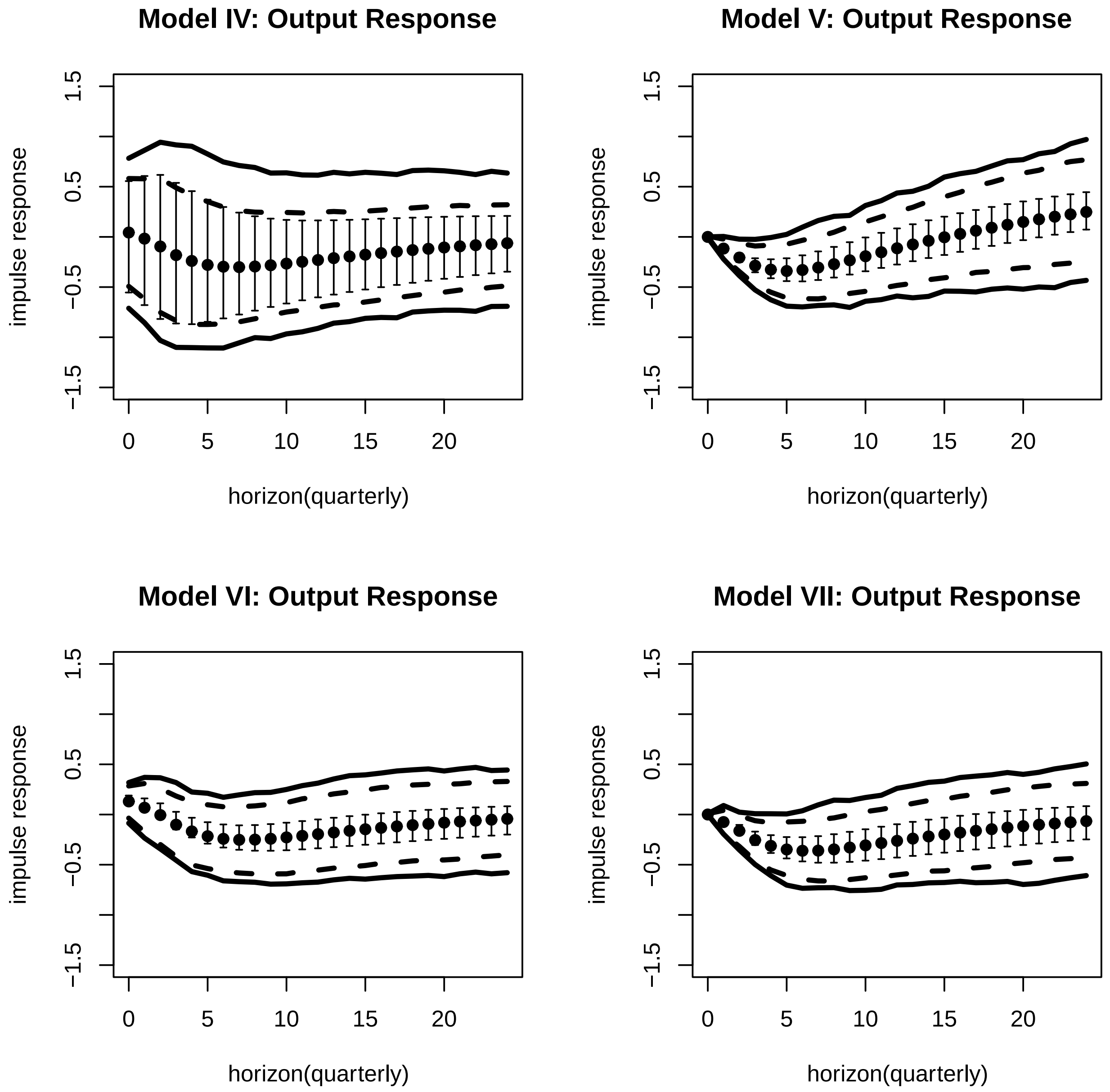

Figure 2: **Plots of Output Impulse Responses for Model IV - VII.** See Table 1 for the definition of each model. See the caption of Figure 1 for remarks.

the range of posterior means and posterior probabilities for a given impulse response, when the prior varies over the class that consists of any priors for the non-identified components of the model that satisfy the restrictions. The posterior bounds are easy to compute even for a large number of restrictions. The range of posterior quantities we derived can be interpreted as conducting Bayesian inference about the identified set, and the posterior mean bounds and the robustified credible region converge asymptotically to the true identified set when it is convex. We provide easy-to-check conditions for convexity that are verified for a large class of non-identified SVARs with zero and/or sign restrictions.

Note that the robustified credible region we provide are for a specific impulse response at a given horizon. If one wanted to provide inferential statements about multiple impulse responses, it would be in principle possible to define the range of posterior probabilities, but this presents challenges both in terms of visualization and computation. This is true in the point identified case (see the discussion in Inoue and Kilian (2013)), and it appears even more challenging in the set identified case. We thus leave this endeavour for future research.

Appendix

A Proofs

The proofs given below use the following notation. For given $\phi \in \Phi$ and $i = 1, \ldots, n$, let $\tilde{f}_i(\phi) \equiv rank(F_i(\phi))$. Since the rank of $F_i(\phi)$ is determined by its row rank, $\tilde{f}_i(\phi) \leq f_i(\phi)$ holds. Let $\mathcal{F}_i^{\perp}(\phi)$ be the linear subspace of $\mathcal{R}^n$ that is orthogonal to the row vectors of $F_i(\phi)$. If no zero restrictions are placed on q_i, we interpret $\mathcal{F}_i^{\perp}(\phi)$ to be $\mathcal{R}^n$. Note that the dimension of $\mathcal{F}_i^{\perp}(\phi)$ is equal to $n - \tilde{f}_i(\phi)$. We let $\mathcal{H}_i(\phi)$ be the half-space in $\mathcal{R}^n$ defined by the sign normalization restriction $\left\{ z \in \mathcal{R}^n : \left(\sigma^i\right)' z \geq 0 \right\}$, where σ^i is the i-th column vector of Σ_{tr}^{-1}. The unit sphere in $\mathcal{R}^n$ is denoted by $\mathcal{S}^{n-1}$. Given linearly independent vectors, $A = [a_1, \ldots, a_j] \in \mathcal{R}^{n \times j}$, denote the linear subspace in $\mathcal{R}^n$ that is orthogonal to the column vectors of A by $\mathcal{P}(A)$. Note that the dimension of $\mathcal{P}(A)$ is $n - j$.

Proof of Lemma 5.1. Fix $\phi \in \Phi$. Let $Q_{1:i} = [q_1, \ldots, q_i]$, $i = 2, \ldots, (n-1)$, be an $n \times i$ matrix of orthogonal vectors in $\mathcal{R}^n$. The set of feasible Q's satisfying the zero restrictions and the sign normalizations, $\mathcal{Q}(\phi|F)$, can be written in the following recursive manner,

$$
\begin{aligned}
Q &= [q_1, \ldots, q_n] \in \mathcal{Q}(\phi|F) \\
& \quad \text{if and only if } Q = [q_1, \ldots, q_n] \text{ satisfies} \\
q_1 &\in D_1(\phi) \equiv \mathcal{F}_1^{\perp}(\phi) \cap \mathcal{H}_1(\phi) \cap \mathcal{S}^{n-1}, \\
q_2 &\in D_2(\phi, q_1) \equiv \mathcal{F}_2^{\perp}(\phi) \cap \mathcal{H}_2(\phi) \cap \mathcal{P}(q_1) \cap \mathcal{S}^{n-1}, \\
q_3 &\in D_3(\phi, Q_{1:2}) \equiv \mathcal{F}_3^{\perp}(\phi) \cap \mathcal{H}_3(\phi) \cap \mathcal{P}(Q_{1:2}) \cap \mathcal{S}^{n-1}, \\
& \vdots \\
q_j &\in D_j\left(\phi, Q_{1:(j-1)}\right) \equiv \mathcal{F}_j^{\perp}(\phi) \cap \mathcal{H}_j(\phi) \cap \mathcal{P}(Q_{1:(j-1)}) \cap \mathcal{S}^{n-1}, \qquad \text{(A.1)} \\
& \vdots \\
q_n &\in D_n\left(\phi, Q_{1:(n-1)}\right) \equiv \mathcal{F}_n^{\perp}(\phi) \cap \mathcal{H}_n(\phi) \cap \mathcal{P}(Q_{1:(n-1)}) \cap \mathcal{S}^{n-1}.
\end{aligned}
$$

where $D_i\left(\phi, Q_{1:(i-1)}\right) \subset \mathcal{R}^n$ denotes the set of feasible q_i's given $Q_{1:(i-1)} = [q_1, \ldots, q_{i-1}]$, the set of $(i-1)$ orthonormal vectors in $\mathcal{R}^n$ preceding i. Nonemptiness of the identified set for $r_{ij}^h = c_{ih}(\phi)\, q_j$ follows if the feasible domain of the orthogonal vector $D_i\left(\phi, Q_{1:(i-1)}\right)$ is nonempty at every $i = 1, \ldots, n$.

Note that by the assumption $f_1 \leq n-1$, $\mathcal{F}_1^{\perp}(\phi) \cap \mathcal{H}_1(\phi)$ is the half-space of the linear subspace of $\mathcal{R}^n$ with dimension $n - \tilde{f}_1(\phi) \geq n - f_1 \geq 1$. Hence, $D_1(\phi)$ is nonempty for every $\phi \in \Phi$. For $i = 2, \ldots, n$, $\mathcal{F}_i^{\perp}(\phi) \cap \mathcal{H}_i(\phi) \cap \mathcal{P}(Q_{1:(i-1)})$ is the half-space of the linear subspace of $\mathcal{R}^n$ with dimension at least

$$
\begin{aligned}
n - \tilde{f}_i(\phi) - \dim(\mathcal{P}(Q_{1:(n-1)})) &\geq n - f_i - (i-1) \\
&\geq 1,
\end{aligned}
$$

where the last inequality follows by the assumption $f_i \leq n-i$. Hence, $D_i\left(\phi, Q_{1:(i-1)}\right)$ is non-empty for every $\phi \in \Phi$. We thus conclude that $\mathcal{Q}(\phi|F)$ is nonempty, and this implies nonemptiness of the impulse response identified sets for every $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, and $h = 0, 1, 2, \ldots$. The boundedness of the identified sets follows since $\left|r_{ij}^h\right| \leq \|c_{ih}(\phi)\| < \infty$ for any $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, and $h = 0, 1, 2, \ldots$, where the boundedness of $\|c_{ih}(\phi)\|$ is ensured by the restriction on ϕ such that the reduced form VAR is invertible to VMA(∞).

Next we show convexity of the identified set of the impulse response to the j^*-th shock under each one of conditions (i) - (iii). Suppose $j^* = 1$ and $f_1 < n-1$ (condition (i)). Since $\tilde{f}_1(\phi) < n-1$ for all $\phi \in \Phi$, $D_1(\phi)$ is a path-connected set because it is an intersection of the half-space with dimension at least 2 and the unit sphere. Since the impulse response is a continuous function of q_1, the identified set of $r_{i1}^h = c_{ih}(\phi)\, q_1$ is an interval, as the range of a continuous function with a path-connected domain is always an interval (see, e.g., Propositions 12.11 and 12.23 in Sutherland (2009)).

Suppose $j^* \geq 2$ and assume condition (ii) holds. Denote the set of feasible q_{j^*}'s by $\mathcal{E}_{j^*}(\phi) \equiv \left\{q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F)\right\}$. The next lemma provides a specific expression of $\mathcal{E}_{j^*}(\phi)$. We defer its proof to a later part of this appendix.

Lemma A.1 *Suppose $j^* \geq 2$ and assume condition (ii) of Lemma 5.1 holds. Then $\mathcal{E}_{j^*}(\phi) = \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{S}^{n-1}$.*

This lemma shows that $\mathcal{E}_{j^*}(\phi)$ is an intersection of a half-space of a linear subspace with dimension $n - f_{j^*} \geq j^* \geq 2$ with the unit sphere. Hence, $\mathcal{E}_{j^*}(\phi)$ is a path-connected set on $\mathcal{S}^{n-1}$ and convexity of $IS_r(\phi|F)$ follows.

Next, suppose condition (iii) holds. Let $Q_{1:i^*}(\phi) \equiv [q_1(\phi), \ldots, q_{i^*}(\phi)]$ be the first i^* columns of feasible $Q \in \mathcal{Q}(\phi|F)$ that are common for all $Q \in \mathcal{Q}(\phi|F)$, ϕ-a.s., by the assumption of exact identification of the first i^* columns. In this case, the set of feasible q_{j^*}'s can be expressed as in the next lemma (see a later part of this appendix section for its proof).

Lemma A.2 *Suppose $j^* \geq 2$ and assume condition (iii) of Lemma 5.1 holds. Then, whenever $Q_{1:i^*}(\phi) = (q_1(\phi), \ldots, q_{i^*}(\phi))$ is uniquely determined as a function of ϕ (this is the case ϕ-a.s. by the assumption of exact identification), $\mathcal{E}_{j^*}(\phi) = \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:i^*}(\phi)) \cap \mathcal{S}^{n-1}$.*

This lemma shows that $\mathcal{E}_{j^*}(\phi)$ is an intersection of a half-space of a linear subspace with dimension $n - f_{j^*} - i^* \geq j^* + 1 - i^* \geq 2$ with the unit sphere. Hence, $\mathcal{E}_{j^*}(\phi)$ is a path-connected set on $\mathcal{S}^{n-1}$ and convexity of $IS_r(\phi|F)$ follows.

For the cases under condition (i) or (ii), since $\phi \in \Phi$ is arbitrary, the convexity of the impulse response identified set holds for every $\phi \in \Phi$. As for the case of condition (iii), the exact identification of $[q_1(\phi), \ldots, q_{i^*}(\phi)]$ assumes its unique determination for only ϕ-a.s., so convexity of the identified set holds ϕ-a.s. ∎

Proof of Lemma 5.2. Suppose $j^* = 1$ and $f_1 < n-1$ (condition (i) of Lemma 5.1). Using the notation introduced in (A.1), the set of feasible q_1's can be denoted by $D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq 0\}$. Let $\tilde{q}_1 \in D_1(\phi)$ be a unit length vector that satisfies $\begin{pmatrix} S_1(\phi) \\ (\sigma^1)' \end{pmatrix} \tilde{q}_1 > \mathbf{0}$. Such $\tilde{q}_1$ is guaranteed to exist by the assumption. Let $q_1 \in D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq 0\}$ be arbitrary. Note that $q_1 \neq -\tilde{q}_1$ must hold, since otherwise some of the sign restrictions are violated. Consider

$$q_1(\lambda) = \frac{\lambda q_1 + (1-\lambda)\tilde{q}_1}{\|\lambda q_1 + (1-\lambda)\tilde{q}_1\|}, \quad \lambda \in [0,1],$$

which is a connected path in $D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ since the denominator is nonzero for all $\lambda \in [0,1]$ by the fact that $q_1 \neq -\tilde{q}_1$. Since q_1 is arbitrary, we can connect any points in $D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ by connected-paths via $\tilde{q}_1$. Hence, $D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ is path-connected, and convexity of the impulse response identified set follows.

Suppose $j^* \geq 2$ and assume that the imposed zero restrictions satisfy condition (ii) of Lemma 5.1. Let $\mathcal{E}_{j^*}(\phi) \equiv \left\{ q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F,S) \right\}$, and let $\tilde{q}_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be chosen so as to satisfy $\begin{pmatrix} S_{j^*}(\phi) \\ \left[\sigma^{j^*}(\phi)\right]' \end{pmatrix} \tilde{q}_{j^*} > \mathbf{0}$. Such $\tilde{q}_{j^*}$ exists by the assumption. For any $q_{j^*} \in \mathcal{E}_{j^*}(\phi)$, $q_{j^*} \neq -\tilde{q}_{j^*}$ must be true, since otherwise q_{j^*} would violate some of the imposed sign restrictions. Consider constructing a path between q_{j^*} and $\tilde{q}_{j^*}$ as follows. For $\lambda \in [0,1]$, let

$$q_{j^*}(\lambda) = \frac{\lambda \tilde{q}_{j^*} + (1-\lambda) q_{j^*}}{\left\| \lambda \tilde{q}_{j^*} + (1-\lambda) q_{j^*} \right\|}, \tag{A.2}$$

which is a continuous path on the unit sphere since the denominator is nonzero for all $\lambda \in [0,1]$ by the construction of $\tilde{q}_{j^*}$. Along this path, $F_{j^*}(\phi) q_{j^*}(\lambda) = \mathbf{0}$ and the sign restrictions hold. Hence, for every $\lambda \in [0,1]$, if there exists $Q(\lambda) \equiv [q_1(\lambda), \ldots, q_{j^*}(\lambda), \ldots, q_n(\lambda)] \in \mathcal{Q}(\phi|F,S)$, where the j^*-th column is set to $q_{j^*}(\lambda)$, then the path-connectedness of $\mathcal{E}_{j^*}(\phi)$ follows. The recursive construction of Algorithm 5.1 can be used to construct such $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$. For $i = 1, \ldots, (j^*-1)$, we recursively obtain $q_i(\lambda)$ that solves

$$\begin{pmatrix} F_i(\phi) \\ q_1'(\lambda) \\ \vdots \\ q_{i-1}'(\lambda) \\ q_{j^*}'(\lambda) \end{pmatrix} q_i(\lambda) = \mathbf{0}, \tag{A.3}$$

and satisfies $\left[\sigma^i(\phi)\right]' q_i(\lambda) \geq 0$. Such a $q_i(\lambda)$ always exists since the rank of the matrix multiplied to $q_i(\lambda)$ is at most $f_i + i$, which is less than n under condition (ii). For $i = (j^*+1), \ldots, n$, a direct application of Algorithm 5.1 yields a feasible $q_i(\lambda)$. Thus, existence of $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$, $\lambda \in [0,1]$, is established. We therefore conclude that $\mathcal{E}_{j^*}(\phi)$ is path-connected under condition (ii), and the convexity of impulse response identified sets holds for every variable and every horizon. This completes the proof of (i) in the lemma.

Now, suppose that the imposed zero restrictions satisfy condition (iii) of Lemma 5.1. Let $[q_1(\phi), \ldots, q_{i^*}(\phi)]$ be the first i^*-th columns of feasible $Q's$, that are common for all $Q \in \mathcal{Q}(\phi|F,S)$, ϕ-a.s., by exact identification of the first i^*-columns. Let $\tilde{q}_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be chosen so as to satisfy $\begin{pmatrix} S_{j^*}(\phi) \\ (\sigma^{j^*})' \end{pmatrix} \tilde{q}_{j^*} > \mathbf{0}$, and $q_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be arbitrary. Consider $q_{j^*}(\lambda)$ in (A.2) and construct $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$ as follows. The first i^*-th column of $Q(\lambda)$ must be $[q_1(\phi), \ldots, q_{i^*}(\phi)]$, ϕ-a.s., by the assumption of exact identification. For $i = (i^*+1), \ldots, (j^*-1)$, we can recursively obtain $q_i(\lambda)$

that solves

$$\begin{pmatrix} F_i(\phi) \\ q_1'(\phi) \\ \vdots \\ q_{i^*}'(\phi) \\ q_{i^*+1}'(\lambda) \\ \vdots \\ q_{i-1}'(\lambda) \\ q_{j^*}'(\lambda) \end{pmatrix} q_i(\lambda) = \mathbf{0} \tag{A.4}$$

and satisfies $\left[\sigma^i(\phi)\right]' q_i(\lambda) \geq 0$. There always exist such $q_i(\lambda)$ because $f_i < n - i$ for all $i = (i^* + 1), \ldots, (j^* - 1)$. The rest of column vectors $q_i(\lambda)$, $i = j^* + 1, \ldots, n$, of $Q(\lambda)$ are obtained successively by applying Algorithm 5.1. Having shown a feasible construction of $Q(\lambda) \in \mathcal{Q}(\phi|F, S)$ for $\lambda \in [0, 1]$, we conclude that $\mathcal{E}_{j^*}(\phi)$ is path-connected, and convexity of the impulse response identified sets follows for every variable and every horizon. ■

In what follows, we provide proofs for the lemmas used in the proof of Lemma 5.1.

Proof of Lemma A.1. Given zero restrictions $F(\phi, Q) = \mathbf{0}$ and the set of feasible orthogonal matrices $\mathcal{Q}(\phi|F)$, define the projection of $\mathcal{Q}(\phi|F)$ with respect to the first i-th column vectors,

$$\mathcal{Q}_{1:i}(\phi|F) \equiv \{[q_1, \ldots, q_i] : Q \in \mathcal{Q}(\phi|F)\}.$$

Following the recursive representation of (A.1), $\mathcal{E}_{j^*}(\phi) \equiv \left\{q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F)\right\}$ can be written as

$$\begin{aligned} \mathcal{E}_{j^*}(\phi) &= \bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \left[\mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:(j^*-1)}) \cap \mathcal{S}^{n-1}\right] \\ &= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \left[\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)})\right] \cap \mathcal{S}^{n-1}. \end{aligned}$$

Hence, the conclusion follows if we can show $\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)}) = \mathcal{S}^{n-1}$. To show this claim, let $q \in \mathcal{S}^{n-1}$ be arbitrary, and we construct $Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)$ such that $q \in \mathcal{P}(Q_{1:(j^*-1)})$ holds. Specifically, construct q_i, $i = 1, \ldots, (j^* - 1)$, successively, by solving

$$\begin{pmatrix} F_i(\phi) \\ q_1' \\ \vdots \\ q_{i-1}' \\ q' \end{pmatrix} q_i = \mathbf{0},$$

and choose the sign of q_i to satisfy its sign normalization. Under condition (ii) of Lemma 5.1, $q_i \in \mathcal{S}^{n-1}$ solving these equalities exist since the rank of the coefficient matrix is at most $f_i + i < n$. Thus-obtained $Q_{1:(j^*-1)} = [q_1, \ldots, q_{j^*-1}]$ belongs to $\mathcal{Q}_{1:(j^*-1)}(\phi|F)$ by construction, and it is orthogonal to q. Hence, $q \in \mathcal{P}(Q_{1:(j^*-1)})$. Since q is arbitrary, we obtain $\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)}) = \mathcal{S}^{n-1}$. ■

Proof of Lemma A.2. Let $Q_{1:i^*}(\phi) \equiv [q_1(\phi), \ldots, q_{i^*}(\phi)]$ be the first i^*-th columns of feasible $Q \in \mathcal{Q}(\phi|F)$, that are common for all $Q \in \mathcal{Q}(\phi|F)$, ϕ-a.s., by exact identification of the first i^*-columns. As in the proof of Lemma A.1, $\mathcal{E}_{j^*}(\phi)$ can be written as

$$\begin{aligned}
\mathcal{E}_{j^*}(\phi) &= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \left[\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)}) \right] \cap \mathcal{S}^{n-1} \\
&= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:i^*}(\phi)) \cap \bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)}) \cap \mathcal{S}^{n-1},
\end{aligned}$$

where $\mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F) = \left\{ Q_{(i^*+1):(j^*-1)} = [q_{i^*+1}, \ldots, q_{j^*-1}] : Q \in \mathcal{Q}(\phi|F) \right\}$ is the projection of $\mathcal{Q}(\phi|F)$ with respect to the (i^*+1)-th to (j^*-1)-th columns of Q. We now show that, under condition (iii) of Lemma 5.1, $\bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)}) = \mathcal{S}^{n-1}$ holds. Let $q \in \mathcal{S}^{n-1}$ be arbitrary, and we consider constructing $Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)$ such that $q \in \mathcal{P}(Q_{(i^*+1):(j^*-1)})$ holds. For $i = (i^*+1), \ldots, (j^*-1)$, we recursively obtain q_i by solving

$$\begin{pmatrix} F_i(\phi) \\ q_1'(\phi) \\ \vdots \\ q_{i^*}'(\phi) \\ q_{i^*+1}' \\ \vdots \\ q_{i-1}' \\ q' \end{pmatrix} q_i = \mathbf{0},$$

and choose the sign of q_i to be consistent with the sign normalization. Under condition (iii) of Lemma 5.1, $q_i \in \mathcal{S}^{n-1}$ solving these equalities exist since the rank of the coefficient matrix is at most $f_i + i < n$ for all $i = (i^*+1), \ldots, (j^*-1)$. Thus-obtained $Q_{(i^*+1):(j^*-1)} = [q_{i^*+1}, \ldots, q_{j^*-1}]$ belongs to $\mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)$ by construction, and it is orthogonal to q. Hence, $q \in \mathcal{P}(Q_{(i^*+1):(j^*-1)})$. Since q is arbitrary, $\bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)}) = \mathcal{S}^{n-1}$ is shown. ■

Proof of Corollaries 5.1 and 5.2. As for a proof of Corollary of 5.1, the successive construction of the feasible column vectors q_i, $i = 1, \ldots, n$, show that the additional zero restrictions that do not change the order of variables nor the zero restrictions for those preceding j^* does not constrain the set of feasible q_{j^*}'s.

As for Corollary 5.2, dropping the zero restrictions imposed for those following the j^*-th variable does not change the order of variables nor the construction of the set of feasible q_{j^*}'s. Under condition (ii) of Lemma 5.1, Lemma A.1 above shows that the set of feasible q_{j^*}'s does not depend on any of $F_i(\phi)$, $i = 1, \ldots, (j^* - 1)$. Hence, removing or altering them (as far as condition (ii) of Lemma 5.1 holds) does not affect the set of feasible q_{j^*}'s. Under condition (iii) of Lemma 5.1, Lemma A.2 shows that the set of feasible q_{j^*}'s does not depend on any of $F_i(\phi)$, $i = (i^* + 1), \ldots, (j^* - 1)$. Hence, relaxing the zero restrictions constraining $[q_{i^*+1}, \ldots, q_{j^*-1}]$ does not affect the set of feasible q_{j^*}'s. ■

Proof of Proposition 4.1 (ii). The proof proceeds by applying the proof of Proposition 4.1 of Kitagawa (2012). Let $r(\phi, Q) = c'_{ih}(\phi) q_j$ be the impulse response of interest. By Lemma A.4 of Kitagawa (2012) and Proposition 10.3 of Denneberg (1994), the upper bound of the posterior means of $r(\phi, Q)$ satisfies the following equality,

$$\sup_{\pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}} \int r(\phi, Q) d\pi_{\phi Q|Y} = \int r(\phi, Q) d\pi^*_{\phi Q|Y},$$

where the integral with respect to the upper probability $\int r(\phi, Q) d\pi^*_{\phi Q|Y}$ stands for the generalized Choquet integral (Denneberg (1994), pp62),

$$\int r(\phi, Q) d\pi^*_{\phi Q|Y} = \int_{-\infty}^{0} \left[\pi^*_{\phi Q|Y} \left(\{ r(\phi, Q) \geq \tilde{r} \} \right) - 1 \right] d\tilde{r} + \int_{0}^{\infty} \pi^*_{\phi Q|Y} \left(\{ r(\phi, Q) \geq \tilde{r} \} \right) d\tilde{r}.$$

By the current Proposition 4.1 (i), we have that

$$\begin{aligned} \pi^*_{\phi Q|Y} \left(\{ r(\phi, Q) \geq \tilde{r} \} \right) &= \pi^*_{r|Y} \left(\{ r \geq \tilde{r} \} \right) \\ &= \pi_{\phi|Y} \left(IS(\phi|F, S) \cap \{ r \geq \tilde{r} \} \neq \emptyset \right). \end{aligned}$$

Note that $IS(\phi|F, S) \cap \{ r \geq \tilde{r} \} \neq \emptyset$ is true if and only if $\{ u(\phi) \geq \tilde{r} \}$. Hence, we have

$$\begin{aligned} \int r(\phi, Q) d\pi^*_{\phi Q|Y} &= \int_{-\infty}^{0} \left[\pi_{\phi|Y} \left(u(\phi) \geq \tilde{r} \right) - 1 \right] d\tilde{r} + \int_{0}^{\infty} \pi_{\phi|Y} \left(u(\phi) \geq \tilde{r} \right) d\tilde{r} \\ &= -\int_{-\infty}^{0} \pi_{\phi|Y} \left(u(\phi) < \tilde{r} \right) d\tilde{r} + \int_{0}^{\infty} \pi_{\phi|Y} \left(u(\phi) \geq \tilde{r} \right) d\tilde{r} \\ &= E_{\phi|Y}(u(\phi)), \end{aligned}$$

where the last line follows by the identity $E(X) = -\int_{-\infty}^{0} \Pr(X < x) dx + \int_{0}^{\infty} \Pr(X \geq x) dx$ that holds for any integrable random variable X. The lower bound of the posterior means can be obtained similarly by replacing $r(\phi, Q)$ above with $-r(\phi, Q)$. Any posterior means between the lower and upper bounds can be obtained by a mixture of the priors putting probability masses at the lower and upper bounds, so the range of the posterior means is convex. ■

Proof of Proposition 6.1. (i) Let $\epsilon > 0$ be arbitrary, and denote the identified set of an impulse response by $IS(\phi)$ for short. Recall that $IS(\cdot)$ is a compact-valued correspondence as implied from Lemma 4.1. Accordingly, the assumption of continuity of the identified set correspondence at ϕ_0 is equivalent to Hausdorff continuity of $IS(\cdot)$ at ϕ_0 (see, e.g., Proposition 5 in Chapter E of Ok (2007)), implying that there exists an open neighborhood G of ϕ_0 such that $d_H(IS(\phi), IS(\phi_0)) < \epsilon$ holds for every $\phi \in G$. Consider

$$
\begin{aligned}
\pi_{\phi|Y^T}\left(\{\phi : d_H(IS(\phi), IS(\phi_0)) > \epsilon\}\right) &= \pi_{\phi|Y^T}\left(\{\phi : d_H(IS(\phi), IS(\phi_0)) > \epsilon\} \cap G\right) \\
&\quad + \pi_{\phi|Y^T}\left(\{\phi : d_H(IS(\phi), IS(\phi_0)) > \epsilon\} \cap G^c\right) \\
&\leq \pi_{\phi|Y^T}(G^c),
\end{aligned}
$$

where the last line follows because $\{\phi : d_H(IS(\phi), IS(\phi_0)) > \epsilon\} \cap G = \emptyset$ by the construction of G. The posterior consistency of ϕ yields $\lim_{T\to\infty} \pi_{\phi|Y^T}(G^c) = 0$, $p(Y^T|\phi_0)$-a.s.

(ii) The posterior consistency of ϕ implies that ϕ converges in probability (in terms of $\pi_{\phi|Y^T}$) to ϕ_0 as $T \to \infty$. Since continuity of the identified set correspondence implies that $\ell(\phi)$ and $u(\phi)$ are continuous at ϕ_0, $\ell(\phi)$ and $u(\phi)$ converge in probability to $\ell(\phi_0)$ and $u(\phi_0)$ as $T \to \infty$, respectively. Combined with the assumption of the uniform integrability of $\ell(\phi)$ and $u(\phi)$, the convergences in probability of $\ell(\phi)$ and $u(\phi)$ imply their convergences in mean (see, e.g., Proposition 4.12 in Kallenberg (2001)).

To show the convergence of robustified credible regions, recall the notation introduced in (Step 5) of Algorithm 5.1, $d(r,\phi) = \max\{|r - \ell(\phi)|, |r - u(\phi)|\}$, and let $z_\alpha(r)$ be the α-th quantile of the posterior distribution of $d(r,\phi)$. By Proposition 5.1 of Kitagawa (2012), the shortest width robustified credible region can be written as

$$
C_\alpha^{shortest} = \left[r^* - z_\alpha(r^*), r^* + z_\alpha(r^*)\right],
$$

where $r^* \in \arg\min_r z_\alpha(r)$. Note that the convex hull of $IS_r(\phi_0|F,S)$, $[\ell(\phi_0), u(\phi_0)]$, can be written as

$$
\left[\arg\min_r d(r,\phi_0) - \min_r d(r,\phi_0), \arg\min_r d(r,\phi_0) + \min_r d(r,\phi_0)\right].
$$

Note also that $d(r,\phi_0)$ is continuous in r and has a unique minimum. Hence, $r^* \pm z_\alpha(r^*)$ converges to $\arg\min_r d(r,\phi_0) \pm \min_r d(r,\phi_0)$, if $z_\alpha(r)$ converges to $d(r,\phi_0)$ uniformly over r, $p(Y^T|\phi_0)$-a.s.

To show this uniform convergence, consider

$$
\begin{aligned}
|z_\alpha(r) - d(r,\phi_0)| &\leq |z_\alpha(r) - d(r,\phi)| + |d(r,\phi) - d(r,\phi_0)| \\
&\leq \frac{1}{\alpha \wedge (1-\alpha)} \rho_\alpha(d(r,\phi) - z_\alpha(r)) + |d(r,\phi) - d(r,\phi_0)|, \qquad \text{(A.5)}
\end{aligned}
$$

where $\rho_\alpha(\cdot)$ is the check loss function, $\rho_\alpha(u) = \alpha u 1\{u \geq 0\} - (1-\alpha) u 1\{u < 0\}$, and the second line uses $|u| \leq \frac{1}{\alpha \wedge (1-\alpha)} \rho_\alpha(u)$. Let $\Delta(\phi) = \max\{|\ell(\phi) - \ell(\phi_0)|, |u(\phi) - u(\phi_0)|\}$. Since

$|d(r,\phi)-d(r,\phi_0)|\leq\Delta(\phi)$, taking the posterior expectation on (A.5) leads to

$$\begin{aligned}
|z_\alpha(r)-d\left(r,\phi_0\right)| &\leq \frac{1}{\alpha\wedge(1-\alpha)}E_{\phi|Y^T}\left[\rho_\alpha(d(r,\phi)-z_\alpha(r))\right]+E_{\phi|Y^T}\left(\Delta(\phi)\right)\\
&\leq \frac{1}{\alpha\wedge(1-\alpha)}E_{\phi|Y^T}\left[\rho_\alpha(d(r,\phi)-d(r,\phi_0))\right]+E_{\phi|Y^T}\left(\Delta(\phi)\right)\\
&\leq \frac{1}{\alpha\wedge(1-\alpha)}E_{\phi|Y^T}\left[\rho_\alpha(\Delta\left(\phi\right))+\rho_\alpha(-\Delta\left(\phi\right))\right]+E_{\phi|Y^T}\left(\Delta(\phi)\right)\\
&= \left[\left\{\left[1\vee\left(\frac{\alpha}{1-\alpha}\right)\right]+\left[1\vee\left(\frac{1-\alpha}{\alpha}\right)\right]\right\}+1\right]E_{\phi|Y^T}\left(\Delta\left(\phi\right)\right),
\end{aligned}$$

where the second line follows since posterior α-th quantile $z_\alpha(r)$ minimizes $E_{\phi|Y^T}\left[\rho_\alpha(d(r,\phi)-z)\right]$ in z. Since the left-hand side of this inequality does not depend on r, the uniform convergence of $|z_\alpha(r)-d\left(r,\phi_0\right)|$ follows if $E_{\phi|Y^T}\left(\Delta\left(\phi\right)\right)\to 0$ as $T\to\infty$. This holds true, because

$$\begin{aligned}
E_{\phi|Y^T}\left(\Delta\left(\phi\right)\right) &\leq E_{\phi|Y^T}\left(|\ell(\phi)-\ell(\phi_0)|\right)+E_{\phi|Y^T}\left(|u(\phi)-u(\phi_0)|\right)\\
&\to 0,\ \text{ as } T\to\infty,
\end{aligned}$$

where the last line follows from the convergences in probability of $\ell(\phi)$ and $u(\phi)$ and uniform integrability of $\ell(\phi)$ and $u(\phi)$. ■

Proof of Proposition 6.2. (i) Following the notation introduced in the proofs of Lemmas 5.1 and 5.2, the upper and lower bounds of the impulse response identified set for $r=c'_{ih}(\phi)q_{j^*}$ are written as

$$\begin{aligned}
u(\phi)/\ell(\phi) &= \max/\min_{q_{j^*}} c'_{ih}(\phi)q_{j^*}, \qquad \text{(A.6)}\\
\text{s.t., } q_{j^*} &\in \mathcal{E}_{j^*}(\phi) \text{ and } S_{j^*}(\phi)q_{j^*}\geq\mathbf{0}.
\end{aligned}$$

When $j^*=1$ (condition (i) of Lemma 5.1), $\mathcal{E}_1(\phi)$ is given by $D_1(\phi)$ defined in (A.1). On the other hand, when $j^*\geq 2$ and condition (ii) of Lemma 5.1 holds, Lemma A.1 given in the proof of Lemma 5.1 provides a specific expression for $\mathcal{E}_{j^*}(\phi)$. Accordingly, in either case, the constrained set of q_{j^*} in (A.6) can be expressed as

$$\tilde{\mathcal{E}}_{j^*}(\phi)\equiv\left\{q\in\mathcal{S}^{n-1}: F_{j^*}(\phi)q=\mathbf{0},\ \begin{pmatrix}S_{j^*}(\phi)\\ \left(\sigma^{j^*}(\phi)\right)'\end{pmatrix}q\geq\mathbf{0}\right\}.$$

The objective function of (A.6) is continuous in q_{j^*}, so, by the theorem of Maximum (see, e.g., Theorem 9.14 of Sundaram (1996)), the continuity of $u(\phi)$ and $\ell(\phi)$ is obtained if $\tilde{\mathcal{E}}_{j^*}(\phi)$ is shown to be a continuous correspondence at $\phi=\phi_0$.

To show continuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$, note first that $\tilde{\mathcal{E}}_{j^*}(\phi)$ is a closed and bounded correspondence, so upper-semicontinuity and lower-semicontinuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$ can be defined in terms of sequences (see, e.g., Propositions 21 of Border (2013)),

- $\tilde{\mathcal{E}}_{j^*}(\phi)$ is upper-semicontinuous (usc) at $\phi=\phi_0$ if and only if, for any sequence $\phi^v\to\phi_0$, $v=1,2,\ldots$, and any $q^v_{j^*}\in\tilde{\mathcal{E}}_{j^*}(\phi^v)$, there is a subsequence of $q^v_{j^*}$ with limit in $\tilde{\mathcal{E}}_{j^*}(\phi_0)$.

- $\tilde{\mathcal{E}}_{j^*}(\phi)$ is lower-semicontinuous (lsc) at $\phi = \phi_0$ if and only if, $\phi^v \to \phi_0$, $v = 1, 2, \ldots$, and $q_{j^*}^0 \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$ imply that there is a sequence $q_{j^*}^v \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ with $q_{j^*}^v \to q_{j^*}^0$.

In the proofs given below, we use the same index v to denote a subsequence, just to compress notation.

Usc: Since $q_{j^*}^v$ is a sequence on the unit-sphere, it has a convergent subsequence $q_{j^*}^v \to q_{j^*}$. Since $q_{j^*}^v \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$, $F_{j^*}(\phi^v) q_{j^*}^v = \mathbf{0}$ and $\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} q_{j^*}^v \geq \mathbf{0}$ hold for all v. Since $F_{j^*}(\cdot)$ and $\begin{pmatrix} S_{j^*}(\cdot) \\ \left(\sigma^{j^*}(\cdot)\right)' \end{pmatrix}$ are continuous in ϕ, these equality and sign restrictions hold at the limit as well. Hence, $q_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$.

Lsc: Our proof of lsc proceeds similarly to the proof of Lemma 3 in Moon et al (2013). Let $\phi^v \to \phi_0$ be arbitrary. Let $q_{j^*}^0 \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$, and define $\mathbf{P}^0 = F_{j^*}(\phi_0)' \left[F_{j^*}(\phi_0) F_{j^*}(\phi_0)'\right]^{-1} F_{j^*}(\phi_0)$ be the projection matrix onto the space spanned by the row vectors of $F_{j^*}(\phi_0)$. By the assumption, $F_{j^*}(\phi)$ has full row-rank in the open neighborhood of ϕ_0, so $\mathbf{P}^0$ and $\mathbf{P}^v = F_{j^*}(\phi^v)' \left[F_{j^*}(\phi^v) F_{j^*}(\phi^v)'\right]^{-1} F_{j^*}(\phi^v)$ are well-defined for all large v. Let $\boldsymbol{\xi}^* \in \mathcal{R}^n$ be a vector satisfying $\begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} \left[I - \mathbf{P}^0\right] \boldsymbol{\xi}^* >> \mathbf{0}$, which exists by the assumption. Let

$$\eta = \min \left\{ \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} \left[I - \mathbf{P}^0\right] \boldsymbol{\xi}^* \right\} > 0,$$

and define

$$\begin{aligned}
\boldsymbol{\xi} &= \frac{2}{\eta} \boldsymbol{\xi}^*, \\
\epsilon^v &= \left\| \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} \left[I - \mathbf{P}^v\right] - \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} \left[I - \mathbf{P}^0\right] \right\|, \\
q_{j^*}^v &= \frac{\left[I - \mathbf{P}^v\right] \left[q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}\right]}{\left\| \left[I - \mathbf{P}^v\right] \left[q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}\right] \right\|}.
\end{aligned}$$

Since $\mathbf{P}^v$ converges to $\mathbf{P}^0$, $\epsilon^v \to 0$. Furthermore, $\left[I - \mathbf{P}^0\right] q_{j^*}^0 = q_{j^*}^0$ implies that $q_{j^*}^v$ converges to

$q^0_{j^*}$ as $v \to \infty$. Note that $q^v_{j^*}$ is orthogonal to $F_{j^*}(\phi^v)$ by construction. Furthermore, note that

$$\begin{aligned}
&\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} q^v_{j^*} \\
= \; & \frac{1}{\left\| [I - \mathbf{P}^v] \left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi} \right] \right\|} \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} \left[[I - \mathbf{P}^v] \left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi} \right] \right] \\
\geq \; & \frac{1}{\left\| [I - \mathbf{P}^v] \left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi} \right] \right\|} \begin{pmatrix} \left(\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} [I - \mathbf{P}^v] - \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} \left[I - \mathbf{P}^0 \right] \right) q^0_{j^*} \\ + \epsilon^v \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} [I - \mathbf{P}^v] \boldsymbol{\xi} \end{pmatrix} \\
\geq \; & \frac{1}{\left\| [I - \mathbf{P}^v] \left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi} \right] \right\|} \left(-\epsilon^v \left\| q^0_{j^*} \right\| \mathbf{1} + \epsilon^v \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} [I - \mathbf{P}^v] \boldsymbol{\xi} \right) \\
= \; & \frac{\epsilon^v}{\left\| [I - \mathbf{P}^v] \left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi} \right] \right\|} \left(\frac{2}{\eta} \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} [I - \mathbf{P}^v] \boldsymbol{\xi}^* - \mathbf{1} \right),
\end{aligned}$$

where the third line follows by $\begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} \left[I - \mathbf{P}^0 \right] q^0_{j^*} = \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} q^0_{j^*} \geq \mathbf{0}$. By the construction of $\boldsymbol{\xi}^*$ and η, $\frac{2}{\eta} \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} [I - \mathbf{P}^v] \boldsymbol{\xi}^* > \mathbf{1}$ holds for all large v. This implies that $\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} q^v_{j^*} \geq 0$ holds for all large v, implying that $q^v_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ for all large v. Hence, $\tilde{\mathcal{E}}_{j^*}(\phi)$ is lsc at $\phi = \phi_0$.

(ii) *Usc*: Under condition (iii) of Lemma 5.1, Lemma A.2 implies that the constraint set of q_{j^*} in (A.6) can be expressed as

$$\tilde{\mathcal{E}}_{j^*}(\phi) \equiv \left\{ q \in \mathcal{S}^{n-1} : \begin{pmatrix} F_{j^*}(\phi) \\ q_1'(\phi) \\ \vdots \\ q_{i^*}'(\phi) \end{pmatrix} q = \mathbf{0}, \; \begin{pmatrix} S_{j^*}(\phi) \\ \left(\sigma^{j^*}(\phi)\right)' \end{pmatrix} q \geq \mathbf{0} \right\}.$$

Let $q^v_{j^*}$, $v = 1, 2, \ldots$, be a sequence on the unit-sphere, such that $q^v_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ holds for all v. This has a convergent subsequence $q^v_{j^*} \to q_{j^*}$. Since $F_i(\phi)$ are continuous in ϕ for all $i = 1, \ldots, i^*$, $q_i(\phi)$, $i = 1, \ldots, i^*$, are continuous in ϕ as well, implying that the equality restrictions and the sign restrictions, $\begin{pmatrix} F_{j^*}(\phi^v) \\ q_1'(\phi^v) \\ \vdots \\ q_{i^*}'(\phi^v) \end{pmatrix} q^v_{j^*} = \mathbf{0}$ and $\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} q^v_{j^*} \geq \mathbf{0}$ must hold at the limit $v \to \infty$. Hence, $q_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$.

Lsc: Define $\mathbf{P}^0$ and $\mathbf{P}^v$ as the projection matrices onto the row vectors of $\begin{pmatrix} F_{j^*}(\phi_0) \\ q_1'(\phi_0) \\ \vdots \\ q_{i^*}'(\phi_0) \end{pmatrix}$ and $\begin{pmatrix} F_{j^*}(\phi^v) \\ q_1'(\phi^v) \\ \vdots \\ q_{i^*}'(\phi^v) \end{pmatrix}$, respectively. The imposed assumptions imply that $\mathbf{P}^v$ and $\mathbf{P}^0$ are well-defined for all large v, and $\mathbf{P}^v \to \mathbf{P}^0$. With the current definition of $\mathbf{P}^v$ and $\mathbf{P}^0$, lower-semicontinuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$ can be shown by repeating the same argument as in the proof of part (i) of the current proposition. We omit details for brevity. ■

References

Arias, J. E., J. F. Rubio-Ramirez, and D. Waggoner (2013): "Inference Based on SVARs Identified with Sign and Zero Restrictions: Theory and Applications," *unpublished manuscript.*

Aruoba, B., and F. Schorfheide (2011): "Sticky Prices versus Monetary Frictions: An Estimation of Policy Trade-offs," *American Economic Journal: Macroeconomics*, 3, 60–90.

Baumeister, C., and J. Hamilton (2013): "Sign Restrictions, Structural Vector Autoregressions, and Useful Prior Information," *unpublished manuscript.*

Berger, J., and L. Berliner (1986): "Robust Bayes and Empirical Bayes Analysis with Îţ-contaminated Priors," *The Annals of Statistics*, 14, 461–486.

Bernanke, B. (1986): "Alternative Explorations of the Money-Income Correlation," *Carnegie-Rochester Conference Series on Public Policy*, 25, 49–99.

Blanchard, O., and D. Quah (1993): "The Dynamic Effects of Aggregate Demand and Supply Disturbances," *American Economic Review*, 83, 655–673.

Border, K. (2013): "Introduction to Correspondences," *unpublished manuscript.*

Canova, F., and G. D. Nicolo (2002): "Monetary Disturbances Matter for Business Fluctuations in the G-7," *Journal of Monetary Economics*, 49, 1121–1159.

Chamberlain, G., and E. Leamer (1976): "Matrix Weighted Averages and Posterior Bounds,," *Journal of Royal Statistical Society, Series B*, 38, 73–84.

Christiano, L. J., M. Eichenbaum, and C. L. Evans (1999): "Monetary Policy Shock: What Have We Learned and to What End?," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and M. Woodford, vol. 1, Part A, pp. 65–148. Elsevier.

Denneberg, D. (1994): *Non-additive Measures and Integral.* Kluwer Academic Publishers.

DeRobertis, L., and J. Hartigan (1981): "Bayesian Inference Using Intervals of Measures," *The Annals of Statistics*, 9, 235–244.

Faust, J. (1998): "The Robustness of Identified VAR Conclusions about Money," *Carnegie-Rochester Conference Series on Public Policy*, 48, 207–244.

Gafarov, B., and J.-L. Montiel-Olea (2014): "On the Maximum and Minimum Response to an Impulse in SVARs," *unpublished manuscript.*

Inoue, A., and L. Kilian (2013): "Inference on Impulse Response Functions in Structural VAR Models," *unpublished manuscript.*

Kallenberg, O. (2001): *Foundations of Modern Probability.* Springer-Verlag, second edn.

Kilian, L. (2013): "Structural Vector Autoregressions," in *Handbook of Research Methods and Applications in Empirical Macroeconomics*, ed. by M. Hashimzade, N. & Thornton, pp. 515–554. Edward Elgar, Cheltenham, UK.

Kitagawa, T. (2012): "Estimation and Inference for Set-identified Parameters Using Posterior Lower Probabilities," *unpublished manuscript.*

Kline, B., and E. Tamer (2013): "Default Bayesian Inference in a Class of Partially Identified Models," *unpublished manuscript.*

Leamer, E. (1982): "Sets of Posterior Means with Bounded Variance Priors," *Econometrica*, 50, 725–736.

Liao, Y., and A. Simoni (2013): "Semi-parametric Bayesian Partially Identified Models based on Support Function," *unpublished manuscript.*

Manski, C. F. (2003): *Partial Identification of Probability Distributions.* Springer-Verlag, New York.

——— (2013): *Public Policy in an Uncertain World.* Harvard University Press.

Moon, H., and F. Schorfheide (2011): "Bayesian and Frequentist Inference in Partially Identified Models," *NBER working paper.*

——— (2012): "Bayesian and Frequentist Inference in Partially Identified Models," *Econometrica*, 80, 755–782.

Moon, H., F. Schorfheide, and E. Granziera (2013): "Inference for VARs Identified with Sign Restrictions," *unpublished manuscript*.

Ok, E. A. (2007): *Real Analysis with Economic Applications*. Princeton University Press.

Poirier, D. (1998): "Revising Beliefs in Nonidentified Models," *Econometric Theory*, 14, 483–509.

Rubio-Ramirez, J., D. Waggoner, and T. Zha (2010): "Structural Vector Autoregressions: Theory of Identification and Algorithm for Inference," *Review of Economic Studies*, 77, 665–696.

Sims, C. (1980): "Macroeconomics and Reality," *Econometrica*, 48, 1–48.

Sims, C., and T. Zha (1998): "Error Bands for Impulse Responses," *Econometrica*, 67, 1113–1155.

Sundaram, R. K. (1996): *A First Course in Optimization Theory*. Cambridge University Press.

Sutherland, W. A. (2009): *Introduction to Metric and Topological Spaces*. Oxford University Press, second edn.

Uhlig, H. (2005): "What are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure," *Journal of Monetary Economics*, 52, 381–419.

Wasserman, L. (1990): "Prior Envelopes Based on Belief Functions," *The Annals of Statistics*, 18, 454–464.