Online Supplementary Material for "The Identification Region of the Potential Outcome Distributions under Instrument Independence"

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Abstract

This online material supplements *The Identification Region of the Potential Outcome Distributions under Instrument Independence*. In Appendix C, the identification region of the potential outcome distributions and of the Average Treatment Effect (ATE) are shown to coincide with the Balke-Pearl bounds (Balke and Pearl, 1997) when the outome is binary. In Appendix D, a geometric illustration is provided that shows how to construct the ATE bounds under Random Assignment (RA) when the outcome is continuous, which provides further intuition as to the source of the identification gain in strengthening Marginal Statistical Independence to RA.

Appendix C: Comparison with the Balke-Pearl Bounds in the Binary Outcome Case

In this appendix, we show that $IR_{ATE}(P, Q|RA)$, which is presented in Proposition 4.1, coincides with the expression for the ATE that is derived from the Balke-Pearl bounds (Balke and Pearl, 1997) when the outcome is binary. Since the dominating measure μ puts point mass on $\{1, 0\}$, each $p_{Y_j}(y_j)$ or $q_{Y_j}(y_j)$ for $y_j \in \{1, 0\}$ and $j = 1, 0$ represents $\Pr(Y = y_j, D = j|Z = 1)$ or $\Pr(Y = y_j, D = j|Z = 0)$.

By solving a certain linear optimization problem, Balke and Pearl (1997) derives the following

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bound formulas for $E(Y_1)$ and $E(Y_0)$.

$$\max\left\{\begin{array}{c} p_{Y_1}(1) \\ q_{Y_1}(1) \\ p_{Y_0}(0)+p_{Y_1}(1)-q_{Y_0}(0)-q_{Y_1}(0) \\ p_{Y_0}(1)+p_{Y_1}(1)-q_{Y_1}(0)-q_{Y_0}(1) \end{array}\right\} \leq E(Y_1), \tag{C.1}$$

$$\min\left\{\begin{array}{c} 1-p_{Y_1}(0) \\ 1-q_{Y_1}(0) \\ p_{Y_0}(1)+p_{Y_1}(1)+q_{Y_0}(0)+q_{Y_1}(1) \\ p_{Y_0}(0)+p_{Y_1}(1)+q_{Y_0}(1)+q_{Y_1}(1) \end{array}\right\} \geq E(Y_1), \tag{C.2}$$

and

$$\max\left\{\begin{array}{c} q_{Y_0}(1) \\ p_{Y_0}(1) \\ q_{Y_0}(1)+q_{Y_1}(1)-p_{Y_0}(0)-p_{Y_1}(1) \\ q_{Y_1}(0)+q_{Y_0}(1)-p_{Y_0}(0)-p_{Y_1}(0) \end{array}\right\} \leq E(Y_0), \tag{C.3}$$

$$\min\left\{\begin{array}{c} 1-q_{Y_0}(0) \\ 1-p_{Y_0}(0) \\ q_{Y_1}(0)+q_{Y_0}(1)+p_{Y_0}(1)+p_{Y_1}(1) \\ q_{Y_0}(1)+q_{Y_1}(1)+p_{Y_1}(0)+p_{Y_0}(1) \end{array}\right\} \geq E(Y_0). \tag{C.4}$$

In addition, Balke and Pearl (1997) shows that the ATE bounds are equal to the difference between each upper and lower bound. That is, the lower bound of $E(Y_1) - E(Y_0)$ is equal to the lower bound of $E(Y_1)$ less the upper bound of $E(Y_0)$, and the upper bound of $E(Y_1) - E(Y_0)$ is equal to the upper bound of $E(Y_1)$ less the lower bound of $E(Y_0)$.

To facilitate comparison, we note that $E(Y_1) = f_{Y_1}(1)$ and $E(Y_0) = f_{Y_0}(1)$, and use this notation interchangeably. Moreover, we rewrite the Balke-Pearl bounds, exploiting the following results. For each $j = 1, 0$,

$$p_{Y_j}(\cdot) + q_{Y_j}(\cdot) = \max\{p_{Y_j}(\cdot), q_{Y_j}(\cdot)\} + \min\{p_{Y_j}(\cdot), q_{Y_j}(\cdot)\}, \tag{C.5}$$

and

$$p_{Y_0}(0) + p_{Y_1}(0) + p_{Y_0}(1) + p_{Y_1}(1) = 1, \tag{C.6}$$

$$q_{Y_0}(0) + p_{Y_1}(0) + q_{Y_0}(1) + q_{Y_1}(1) = 1. \tag{C.7}$$

We also combine the first and second element of each lower bound and each upper bound into a single element.

$$\max\left\{\begin{array}{c}\max\{p_{Y_1}(1),q_{Y_1}(1)\}\\ \max\{p_{Y_1}(1),q_{Y_1}(1)\}+\min\{p_{Y_1}(1),q_{Y_1}(1)\}+p_{Y_0}(0)+q_{Y_0}(1)-1\\ \max\{p_{Y_1}(1),q_{Y_1}(1)\}+\min\{p_{Y_1}(1),q_{Y_1}(1)\}+q_{Y_0}(0)+p_{Y_0}(1)-1\end{array}\right\}\leq E(Y_1), \tag{C.8}$$

$$\min\left\{\begin{array}{c}\max\{p_{Y_1}(1),q_{Y_1}(1)\}+(1-\delta_{Y_1})\\ \max\{p_{Y_1}(1),q_{Y_1}(1)\}+\min\{p_{Y_1}(1),q_{Y_1}(1)\}+p_{Y_0}(0)+q_{Y_0}(1)\\ \max\{p_{Y_1}(1),q_{Y_1}(1)\}+\min\{p_{Y_1}(1),q_{Y_1}(1)\}+q_{Y_0}(0)+p_{Y_0}(1)\end{array}\right\}\geq E(Y_1), \tag{C.9}$$

and

$$\max\left\{\begin{array}{c}\max\{p_{Y_0}(1),q_{Y_0}(1)\}\\ \max\{p_{Y_0}(1),q_{Y_0}(1)\}+\min\{p_{Y_0}(1),q_{Y_0}(1)\}+p_{Y_1}(0)+q_{Y_1}(1)-1\\ \max\{p_{Y_0}(1),q_{Y_0}(1)\}+\min\{p_{Y_0}(1),q_{Y_0}(1)\}+q_{Y_1}(0)+p_{Y_1}(1)-1\end{array}\right\}\leq E(Y_0), \tag{C.10}$$

$$\min\left\{\begin{array}{c}\max\{p_{Y_0}(1),q_{Y_0}(1)\}+(1-\delta_{Y_0})\\ \max\{p_{Y_0}(1),q_{Y_0}(1)\}+\min\{p_{Y_0}(1),q_{Y_0}(1)\}+p_{Y_1}(0)+q_{Y_1}(1)\\ \max\{p_{Y_0}(1),q_{Y_0}(1)\}+\min\{p_{Y_0}(1),q_{Y_0}(1)\}+q_{Y_1}(0)+p_{Y_1}(1)\end{array}\right\}\geq E(Y_0). \tag{C.11}$$

We refer to the first element of each bound as the Manski bounds (Manski, 1990). The remaining two elements of each bound are due to the additional identifying information that is provided by RA (versus MSI). If the first element of every bound is binding then we say that the Manski bounds are all binding and, if so, then we conclude that RA provides no additional identification gain beyond MSI for the given data generating process. There are several cases to consider.

The first case is where the data generating process reveals $\delta_{Y_1} > 1$ or $\delta_{Y_0} > 1$. In this case, the Balke-Pearl bounds and $IR_{(f_{Y_1},f_{Y_0})}(P,Q|RA)$ both yield empty sets, and so trivially coincide. Equally, $IR_{ATE}(P,Q|RA)$ is empty. In all other cases, we assume that this condition is satisfied so that the identification region is non-empty.

The second case is where the data generating process reveals $\lambda_{Y_1} = 1-\delta_{Y_0}$. Consider (C.8). Observe that

$$\max\{p_{Y_0}(0)+q_{Y_0}(1),q_{Y_0}(0)+p_{Y_0}(1)\}\leq\delta_{Y_0}=1-\lambda_{Y_1} \tag{C.12}$$

and

$$\min\{p_{Y_0}(1),q_{Y_0}(1)\}-1\leq\lambda_{Y_1}-1. \tag{C.13}$$

Adding (C.12) and (C.13), it follows that the second and third elements of (C.8) are less than or equal to the first element. Next, consider (C.9). Observe that

$$\min\{p_{Y_0}(0) + q_{Y_0}(1), q_{Y_0}(0) + p_{Y_0}(1)\} \geq \lambda_{Y_0} = 1 - \delta_{Y_1} \tag{C.14}$$

by Lemma A.2, and so the first element of (C.9) must be less than or equal to its second and third elements. Similarly, it is straightforward to show that the first element of (C.10) is greater than or equal to its second and third elements, and that the first element of (C.11) is less than or equal to its second and third elements. The Manski bounds are all binding.

To show that $IR_{ATE}(P, Q|RA)$ coincides with the expression that is derived from the Balke-Pearl bounds, it is sufficient to show that $IR_{(f_{Y_1}, f_{Y_0})}(P, Q|MSI)$ is equivalent to the Balke-Pearl bounds above (since Proposition 3.2 establishes that $IR_{(f_{Y_1}, f_{Y_0})}(P, Q|RA)$ coincides with $IR_{(f_{Y_1}, f_{Y_0})}(P, Q|MSI)$ in this case). For convenience, we write

$$\mathcal{F}^{env}_{f_{Y_1}}(P, Q) = \{E(Y_1) : E(Y_1) \geq \max\{p_{Y_1}(1), q_{Y_1}(1)\}, 1 - E(Y_1) \geq \max\{p_{Y_1}(0), q_{Y_1}(0)\}\}\,, \tag{C.15}$$

$$\mathcal{F}^{env}_{f_{Y_0}}(P, Q) = \{E(Y_0) : E(Y_0) \geq \max\{p_{Y_0}(1), q_{Y_0}(1)\}, 1 - E(Y_0) \geq \max\{p_{Y_0}(0), q_{Y_0}(0)\}\}\,. \tag{C.16}$$

Notice though that these sets coincide with the set of values of $E(Y_1)$ and $E(Y_0)$ that are defined by the Manski bounds. Therefore, we have shown coincidence with the Balke-Pearl bounds in this case.

The third case is where the data generating process reveals $\lambda_{Y_1} > 1 - \delta_{Y_0}$. This is the final case that we consider since the opposite case, where the data generating process reveals $\lambda_{Y_1} < 1 - \delta_{Y_0}$, is comparable. If $\lambda_{Y_1} > 1 - \delta_{Y_0}$, the data generating process reveals that p_{Y_0} and q_{Y_0} cross, and we rewrite (C.10) and (C.11) as

$$\max\left\{\begin{array}{c} \max\{p_{Y_1}(1), q_{Y_1}(1)\} \\ \max\{p_{Y_1}(1), q_{Y_1}(1)\} + \min\{p_{Y_1}(1), q_{Y_1}(1)\} - (1 - \delta_{Y_0}) \end{array}\right\} \leq E(Y_1), \tag{C.17}$$

$$\min\left\{\begin{array}{c} \max\{p_{Y_1}(1), q_{Y_1}(1)\} + (1 - \delta_{Y_1}) \\ \max\{p_{Y_1}(1), q_{Y_1}(1)\} + \min\{p_{Y_1}(1), q_{Y_1}(1)\} + \lambda_{Y_0} \end{array}\right\} \geq E(Y_1). \tag{C.18}$$

That is, either $p_{Y_0}(0) > q_{Y_0}(0)$ and $q_{Y_0}(1) > p_{Y_0}(1)$ or $p_{Y_0}(0) < q_{Y_0}(0)$ and $q_{Y_0}(1) < p_{Y_0}(1)$. If, instead, p_{Y_0} and q_{Y_0} were to nest then either $\lambda_{Y_1} = 1 - \delta_{Y_0}$ (if p_{Y_1} and q_{Y_1} nest) or $\lambda_{Y_1} < 1 - \delta_{Y_0}$ (if p_{Y_1} and q_{Y_1} cross), in violation of the maintained assumption. We now show that the Balke-Pearl bounds define a set of values of $E(Y_1)$ that coincides with $\mathcal{F}^{*}_{f_{Y_1}}(P, Q)$. There are three possibilities.

Note that, since $\mathcal{F}^{*}_{f_{Y_1}}(P,Q) \subseteq \mathcal{F}^{env}_{f_{Y_1}}(P,Q)$, common to all three possibilities is the requirement that $f_{Y_1} \geq \underline{f_{Y_1}}$. Any feasible f_{Y_1} must then satisfy this property, allocating the remaining mass $1-\delta_{Y_1}$ between the two points of support.

The first possibility is where $1-\delta_{Y_0} < \min\{p_{Y_1}(0), q_{Y_1}(0)\}$ and $1-\delta_{Y_0} < \min\{p_{Y_1}(1), q_{Y_1}(1)\}$. Note that if these two conditions hold then the second elements of (C.17) and (C.18) are binding (i.e., the Manski bounds are not all binding). Note that the Manski bounds allocate the remaining mass $1-\delta_{Y_1}$ to one of the two points of support.[1] Such an allocation is not feasible here since

$$\begin{aligned}\min\{1-\delta_{Y_1}, \min\{p_{Y_1}(0), q_{Y_1}(0)\}\} + \min\{0, \min\{p_{Y_1}(1), q_{Y_1}(1)\}\} &= \min\{p_{Y_1}(0), q_{Y_1}(0)\} \\ &< \lambda_{Y_1} - (1-\delta_{Y_0}),\end{aligned} \tag{C.19}$$

$$\begin{aligned}\min\{0, \min\{p_{Y_1}(0), q_{Y_1}(0)\}\} + \min\{1-\delta_{Y_1}, \min\{p_{Y_1}(1), q_{Y_1}(1)\}\} &= \min\{p_{Y_1}(1), q_{Y_1}(1)\} \\ &< \lambda_{Y_1} - (1-\delta_{Y_0}),\end{aligned} \tag{C.20}$$

which imply that the remaining mass $1-\delta_{Y_1}$ must be split between both points of support. In this setting,

$$[\min\{p_{Y_1}(1), q_{Y_1}(1)\}]^{rtrim}_{1-\delta_{Y_0}} = \min\{p_{Y_1}(1), q_{Y_1}(1)\} - (1-\delta_{Y_0}), \tag{C.21}$$

$$[\min\{p_{Y_1}(0), q_{Y_1}(0)\}]^{ltrim}_{1-\delta_{Y_0}} = \min\{p_{Y_1}(0), q_{Y_1}(0)\} - (1-\delta_{Y_0}), \tag{C.22}$$

but these are precisely the amounts that (C.19) and (C.20) are, respectively, less than $\lambda_{Y_1} - (1-\delta_{Y_0})$ by. As such, we obtain

$$f_{Y_1}(1) \geq \underline{f_{Y_1}}(1) + [\min\{p_{Y_1}(1), q_{Y_1}(1)\}]^{rtrim}_{1-\delta_{Y_0}}, \tag{C.23}$$

$$f_{Y_1}(1) \leq \underline{f_{Y_1}}(1) + (1-\delta_{Y_1}) - [\min\{p_{Y_1}(1), q_{Y_1}(1)\}]^{ltrim}_{1-\delta_{Y_0}}, \tag{C.24}$$

which simplify to yield the second elements of (C.17) and (C.18) respectively. In particular,

$$[\min\{p_{Y_1}(0), q_{Y_1}(0)\}]^{ltrim}_{1-\delta_{Y_0}} = (1-\delta_{Y_1}) - \lambda_{Y_0} - \min\{p_{Y_1}(1), q_{Y_1}(1)\}, \tag{C.25}$$

by Lemma A.2.

The second possibility is where $1-\delta_{Y_0} < \min\{p_{Y_1}(0), q_{Y_1}(0)\}$ and $1-\delta_{Y_0} \geq \min\{p_{Y_1}(1), q_{Y_1}(1)\}$. That is, the lower Manski bound for $E(Y_1)$ is binding but the upper Manski bound is not. In this

[1] If $\delta_{Y_1} \leq 1$ and $\delta_{Y_0} \leq 1$ then $1-\delta_{Y_1} \geq \lambda_{Y_1} - (1-\delta_{Y_0})$ due to Lemma A.2. Furthermore, if $1-\delta_{Y_0} < \min\{p_{Y_1}(0), q_{Y_1}(0)\}$ and $1-\delta_{Y_0} < \min\{p_{Y_1}(1), q_{Y_1}(1)\}$, then $1-\delta_{Y_1} \geq \min\{p_{Y_1}(0), q_{Y_1}(0)\}$ and $1-\delta_{Y_1} \geq \min\{p_{Y_1}(0), q_{Y_1}(0)\}$. The lower Manski bound on $E(Y_1)$ allocates all of the remaining mass $1-\delta_{Y_1}$ to $y=0$ while the upper Manski bound on $E(Y_1)$ allocates all of the remaining mass $1-\delta_{Y_1}$ to $y=1$.

setting,

$$[\min\{p_{Y_1}(1), q_{Y_1}(1)\}]^{rtrim}_{1-\delta_{Y_0}} = 0, \tag{C.26}$$

$$[\min\{p_{Y_1}(0), q_{Y_1}(0)\}]^{ltrim}_{1-\delta_{Y_0}} = \min\{p_{Y_1}(0), q_{Y_1}(0)\} - (1-\delta_{Y_0}), \tag{C.27}$$

which is compatible with the lower Manski bound for $E(Y_1)$ being binding.

The third possibility is where $1-\delta_{Y_0} \geq \min\{p_{Y_1}(0), q_{Y_1}(0)\}$ and $1-\delta_{Y_0} < \min\{p_{Y_1}(1), q_{Y_1}(1)\}$. That is, the upper Manski bound for $E(Y_1)$ is binding but the lower Manski bound is not. In this setting,

$$[\min\{p_{Y_1}(1), q_{Y_1}(1)\}]^{rtrim}_{1-\delta_{Y_0}} = \min\{p_{Y_1}(1), q_{Y_1}(1)\} - (1-\delta_{Y_0}), \tag{C.28}$$

$$[\min\{p_{Y_1}(0), q_{Y_1}(0)\}]^{ltrim}_{1-\delta_{Y_0}} = 0, \tag{C.29}$$

which is compatible with the upper Manski bound for $E(Y_1)$ being binding.

With regard to f_{Y_0} and its identified set: there are two possibilities, depending upon whether p_{Y_1} and q_{Y_1} nest or cross. In either case, the second and third elements of (C.10) are bounded from above by

$$\underline{f_{Y_0}}(1) + \lambda_{Y_0} - \min\{p_{Y_0}(0), q_{Y_0}(0)\} - (1-\delta_{Y_1}), \tag{C.30}$$

which is less than $\underline{f_{Y_0}}(1)$ due to the maintained assumption that $\lambda_{Y_1} > 1-\delta_{Y_0}$ and by Lemma A.2. Similarly, the second and third elements of (C.11) are bounded from below by

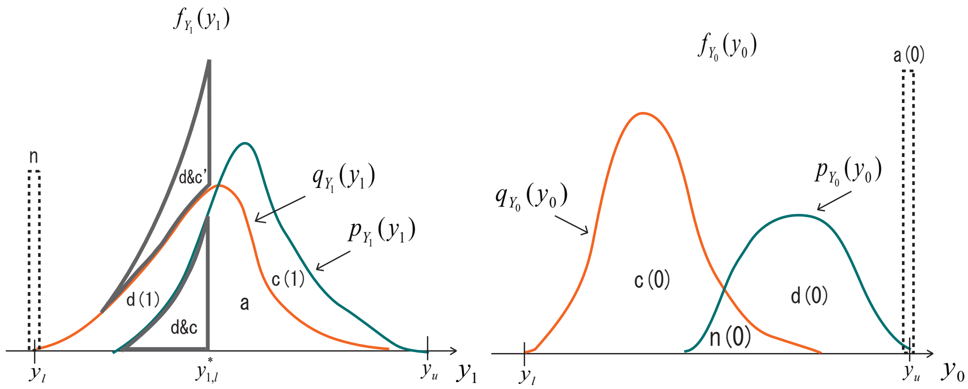
$$\underline{f_{Y_0}}(1) + \lambda_{Y_0} - \min\{p_{Y_0}(0), q_{Y_0}(0)\} + \lambda_{Y_1}, \tag{C.31}$$

which is greater than $\underline{f_{Y_1}}(1) + (1-\delta_{Y_1})$ due to the maintained assumption that $\lambda_{Y_1} > 1-\delta_{Y_0}$. Hence, the Manski bounds are binding for $E(Y_0)$. Since we have already shown that the Manski bounds define a set of values of $E(Y_0)$ that coincides with $\mathcal{F}^{env}_{f_{Y_0}}(P,Q)$, we conclude that the Balke-Pearl bounds also coincide.

We have shown that the Balke-Pearl bounds define a set of values of $E(Y_1)$ and of $E(Y_0)$ that coincide with $\mathcal{F}^{*}_{f_{Y_1}}(P,Q)$ and $\mathcal{F}^{env}_{f_{Y_0}}(P,Q)$ respectively. Therefore, we have shown coincidence with the Balke-Pearl bounds in this case. For completeness, note that, in this third case, $IR_{ATE}(P,Q|RA)$ can otherwise be written as

$$\begin{aligned} IR_{ATE}(P,Q|RA) = \Big[&\underline{f_{Y_1}}(1) - \underline{f_{Y_0}}(1) + [\min\{p_{Y_0}(1), q_{Y_0}(1)\}]^{rtrim}_{1-\delta_{Y_0}} - (1-\delta_{Y_0}) \\ &\underline{f_{Y_1}}(1) - \underline{f_{Y_0}}(1) + [\min\{p_{Y_0}(1), q_{Y_0}(1)\}]^{ltrim}_{1-\delta_{Y_0}} + \lambda_{Y_0}\Big], \end{aligned} \tag{C.32}$$

Figure D.1: A geometric illustration of the ATE bounds under RA.



The figure depicts a data generating process with $\lambda_{Y_1} > 1 - \delta_{Y_0}$ (the area of $a(0)$ is less than the area of $a \cup d\&c$). Here, f_{Y_1} and f_{Y_0} are purposefully selected to attain the lower bound of the ATE under RA.

which firmly establishes the coincidence of the Balke-Pearl bounds and $IR_{ATE}(P, Q|RA)$ given the specific values that the trimmed sub-densities take.

What is clear from the preceding analysis is that, unlike the case where the data generating process reveals continuous sub-densities, it is not guaranteed that RA yields an identification gain beyond MSI. In particular, in the binary outcome case if $1 - \delta_{Y_0} \geq \min\{p_{Y_1}(0), q_{Y_1}(0)\}$ and $1 - \delta_{Y_0} \geq \min\{p_{Y_1}(1), q_{Y_1}(1)\}$ then the Manski bounds are all binding and there is no identification gain, even if the data generating process reveals $\lambda_{Y_1} > 1 - \delta_{Y_0}$. While $\lambda_{Y_1} > 1 - \delta_{Y_0}$ (or its reverse) is a necessary condition for the $IR_{(f_{Y_1}, f_{Y_0})}(P, Q|RA) \subset IR_{(f_{Y_1}, f_{Y_0})}(P, Q|MSI)$, it is not a sufficient condition when the marginal distributions of Y_1 and Y_0 have point mass.

Appendix D: A Geometric Illustration of the ATE Bounds

In this appendix, we provide a geometric illustration of how the definition of $IR_{ATE}(P, Q|RA)$ emerges in constructing the identification region under RA. We consider the case where the data generating process reveals continuous sub-densities with support on $[y_l, y_u]$ and $\lambda_{Y_1} > 1 - \delta_{Y_0}$. We restrict attention to the lower bound of the ATE since the upper bound is constructed in a similar way, and begin by discussing how $IR_{ATE}(P, Q|MSI)$ is constructed.

Under MSI, when $IR_{(f_{Y_1}, f_{Y_0})}(P, Q|RA)$ is non-empty, $f_{Y_1} \geq \underline{f_{Y_1}}$ and $f_{Y_0} \geq \underline{f_{Y_0}}$. As per Proposition

4.1, the ATE attains its lower bound when

$$f_{Y_1}(y_l) = \underline{f_{Y_1}}(y_l) + (1 - \delta_{Y_1}) \text{ and, for all } y_1 \in (y_l, y_u],\ f_{Y_1}(y_1) = \underline{f_{Y_1}}(y_1), \tag{D.1}$$

$$f_{Y_0}(y_u) = \underline{f_{Y_0}}(y_u) + (1 - \delta_{Y_0}) \text{ and, for all } y_0 \in [y_l, y_u),\ f_{Y_0}(y_0) = \underline{f_{Y_0}}(y_0), \tag{D.2}$$

or, in words, when all the remaining mass is allocated to the end-points of the support. This allocation is depicted for the marginal distribution of Y_0 in the right-hand panel of Figure D.1, given the specific data generating process that is imagined. However, under RA, such an allocation would violate the area constraints due to the fact that $\lambda_{Y_0} < 1-\delta_{Y_1}$, as implied by $\lambda_{Y_1} > 1-\delta_{Y_0}$ and Lemma A.2.

Under RA, $f_{Y_0} \in \mathcal{F}^{env}_{f_{Y_0}}(P, Q)$ and so the marginal distribution of Y_0 is not constrained beyond MSI. As such, the marginal distribution of Y_0 that is described in (D.2) remains valid and is the distribution that attains the lower bound of the ATE. We focus on construction of the marginal distribution of Y_1 that attains the lower bound of the ATE.

Firstly, we know that the area under $\min\{p_{Y_1}, q_{Y_1}\}$ cannot solely comprise always-takers, and must comprise a mixture of always-takers and defiers or compliers. If, instead, the area under $\min\{p_{Y_1}, q_{Y_1}\}$ were to comprise always-takers then this would violate the area constraint, as the area labelled $a(0)$ is smaller than the areas labelled a and $d\&c$ combined. The question then is how to partition the area under $\min\{p_{Y_1}, q_{Y_1}\}$.

Secondly, we know that wherever $\min\{p_{Y_1}, q_{Y_1}\} > f_{Y_1,T}(y_1, a)$ (the support of the region labelled $d\&c$ in the left-hand panel of Figure D.1),

$$f_{Y_1}(y_1) \geq \underline{f_{Y_1}}(y_1) + \min\{p_{Y_1}, q_{Y_1}\} - f_{Y_1,T}(y_1, a) > \underline{f_{Y_1}}(y_1) \tag{D.3}$$

so as to preserve the compatibility constraints. For instance, in the left-hand panel of Figure D.1, we partition the area under $\min\{p_{Y_1}, q_{Y_1}\}$ into the two regions labelled a and $d\&c$, and to preserve the compatibility constraints we add the region labelled $d\&c'$. Note that the region labelled $d\&c'$ is the translation of the region labelled $d\&c$ above $\underline{f_{Y_1}}$, and so is otherwise identical to that region. Integrating over the support, we ascertain that the remaining mass that is left over is $(1 - \delta_{Y_1}) - \lambda_{Y_1} + (1 - \delta_{Y_0})$, which is the area of the region labelled $n(0)$.

Thirdly, we know that the ATE attains its lower bound for the marginal distribution of Y_1 that allocates as much mass as is possible to the left of the support. The fact that f_{Y_1} must envelope the region labelled $d\&c'$ together with the fact that the area of this region is $\lambda_{Y_1} - (1 - \delta_{Y_0})$ means that the constraint

$$\int_{\mathcal{Y}} \min\{f_{Y_1} - \underline{f_{Y_1}}, \min\{p_{Y_1}, q_{Y_1}\}\} d\mu \geq \lambda_{Y_1} - (1 - \delta_{Y_0}) \tag{D.4}$$

is satisfied. Moreover, the location of the never-takers above this envelope is then unrestricted, with the area constraint automatically satisfied for this remaining type. As such, the never-takers concentrate at y_l, which yields the region labelled n. Similarly, although the number of compliers and defiers inside the area under $\min\{p_{Y_1}, q_{Y_1}\}$ is fixed, the location of these two types is not. Specifying the regions labelled a and $d\&c$ as shown in Figure D.1 minimises $E(Y_1)$ since the region labelled $d\&c'$ is then as far to the left of the support as is possible.

Note that the data generating process reveals continuous sub-densities. As such, there exists a value, labelled y_{1l}^* in Figure D.1, satisfying

$$y_{1l}^* = \sup \left\{ t : \int_t^{\infty} \min\{p_{Y_1}, q_{Y_1}\} d\mu \geq 1 - \delta_{Y_0} \right\}, \tag{D.5}$$

above which the area under $\min\{p_{Y_1}, q_{Y_1}\}$ is precisely $1 - \delta_{Y_0}$. Furthermore, y_{1l}^* is clearly distinct from y_l, which is a general property when the data generating process reveals continuous sub-densities. The immediate implication is that $IR_{(f_{Y_1}, f_{Y_0})}(P, Q|RA) \subset IR_{(f_{Y_1}, f_{Y_0})}(P, Q|MSI)$. If, instead, the data generating process reveals point mass at y_l or y_u then, while it is possible to find y_{1l}^* satisfying (D.5), it is not necessarily the case that y_{1l}^* partitions the area under $\min\{p_{Y_1}, q_{Y_1}\}$ into two regions with areas $\lambda_{Y_1} - (1 - \delta_{Y_0})$ and $1 - \delta_{Y_0}$ respectively. This possibility explains the added complexity of the expressions that appear in Proposition 4.1.

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