Univariate Time Series

Fall 2008
A time series $Y_t$ is a process (or data) observed in sequence over time, $t = 1, \ldots, T$. 

- Macroeconomic series: e.g., inflation rate, GDP, unemployment, etc.
- Temperature over time in global warming
- Monthly or annual returns of financial assets

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Example: Disposal income and consumption in USA
Example: Average temperature in Central London
Example: Return on three-month T-bills in USA
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We can separate time series into two categories: univariate ($Y_t$ is scalar) and multivariate ($Y_t$ is vector-valued).
Stationarity: future will be like past

A time series \( \{Y_t\} \) is **covariance stationary** if its mean and (co-)variances are constant across time periods:

\[
E(Y_t) = \mu, \quad Var(Y_t) = \sigma^2 \quad \text{for all } t
\]

\[
Cov(Y_t, Y_{t+k}) = \gamma(k) \quad \text{for all } t \text{ and } k
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\( \gamma(k) \) is called the **autocovariance** function and \( \rho(k) = \gamma(k) / \gamma(0) \) is the **autocorrelation** function.
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\( \{ Y_t \} \) is said to be *strictly stationary* if the joint distribution of \( (Y_t,\ldots,Y_{t+k}) \) is independent of all \( t \) and \( k \).
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- \( \{ Y_t \} \) is said to be strictly stationary if the joint distribution of \( (Y_t, \ldots, Y_{t+k}) \) is independent of all \( t \) and \( k \).

- Thus, a stationary time series is one whose probability distributions are stable over time.
A time series process that is not stationary is called a *nonstationary* process.
Nonstationarity

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  - changing mean:
    \[ Y_t = \beta_0 + \beta_1 t + u_t \]
  - changing variance:
    \[ Y_t = Y_{t-1} + u_t \]
    This process is called the random walk.
  - changing mean and variance:
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Example 1: Changing mean

\[ Y_t = \beta_0 + \beta_1 t + u_t, \text{ where } \beta_0 = 0, \beta_1 = 1 \text{ and } u_t \sim iid N(0,1) \]
Example 2: Random walk

- $Y_t = Y_{t-1} + u_t$, where $Y_0 = 0$ and $u_t \sim iid N(0,1)$
Example 3: Random walk with drift

\[ Y_t = \beta_0 + Y_{t-1} + u_t, \text{ where } \beta_0 = 1, Y_0 = 0 \text{ and } u_t \sim iid \ N(0, 1) \]
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(Ergodic Theorem) If $Y_t$ is strictly stationary and ergodic and $E|Y_t| < \infty$, then as $T \to \infty$,

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\frac{1}{T} \sum_{t=1}^{T} Y_t \to_p E(Y_t)
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Using the Ergodic theorem, we can establish the consistency and asymptotic normality of OLS estimators in the regression with stationary and ergodic time series.
We first focus on the case with univariate time series, \{Y_t\}^T_{t=1}. Let \(I_{t-1} = \{Y_{t-1}, Y_{t-2}, \ldots\}\) denote the past history of the series.
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The primary example in the univariate time series is a model of autoregression specifying that only a finite number of past lags matter:

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E(Y_t|I_{t-1}) = E(Y_t|Y_{t-1}, \ldots, Y_{t-k}).
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An autoregressive process of order \( k \), called \( AR(k) \), is

\[
Y_t = \mu + \rho_1 Y_{t-1} + \ldots + \rho_k Y_{t-k} + u_t,
\]

where

\[
E ( u_t | I_{t-1} ) = 0.
\]
AR(1) Model

- AR(1) Model:

\[ Y_t = \mu + \rho Y_{t-1} + u_t \]

- if \(|\rho| < 1\), then

\[ E(Y_t) = \frac{\mu}{1 - \rho} \]
\[ \text{Var}(Y_t) = \frac{\sigma^2}{1 - \rho^2} \]
\[ \text{Cov}(Y_t, Y_{t-k}) = \sigma^2 \frac{\rho^k}{1 - \rho^2} \]

- If \(|\rho| < 1\), the process \(\{Y_t\}\) is stationary and ergodic.

- (Estimation) In order to estimate \(\mu\) and \(\rho\), we just need to run the OLS regression of \(Y_{t-1}\) on \(Y_t\).
We model the consumption growth series in USA as AR(1):

$$gc_t = \mu + \rho gc_{t-1} + u_t$$

<table>
<thead>
<tr>
<th></th>
<th>Coeff.</th>
<th>Std. Err.</th>
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</thead>
<tbody>
<tr>
<td>gc_1</td>
<td>0.45</td>
<td>0.156</td>
</tr>
<tr>
<td>constant</td>
<td>0.01</td>
<td>0.003</td>
</tr>
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Another example is a model of moving average process of order $q$, $\{Y_t\}$, that is a weighted sum of lagged i.i.d. shocks:

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Thus, an $MA(1)$ process is stationary and ergodic.

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