Limited Dependent Variable Models III

Fall 2008

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  - the number of times a person is arrested in a year.

Poisson Distribution

- A random variable Y, which only takes on nonnegative integer values, follows the Poisson distribution if, for $k = 0, 1, 2, ...$

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- The mean and the variance of Poisson random variable is λ:

$$\mathbb{E}(Y) = Var(Y) = \lambda.$$

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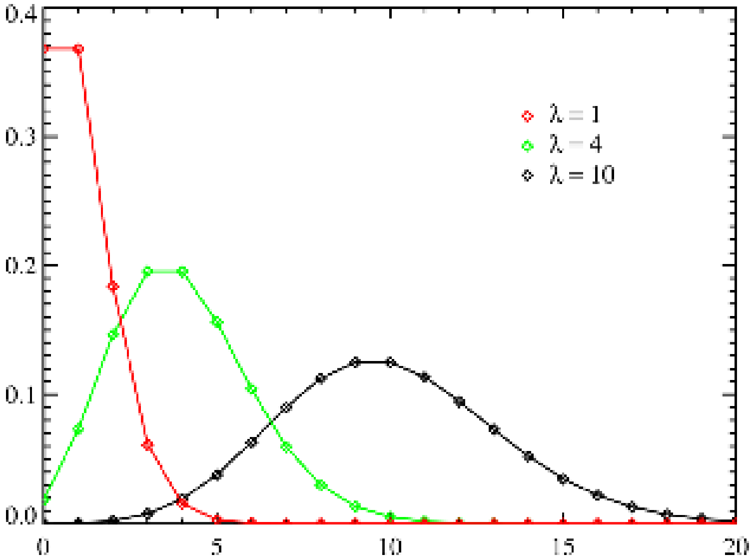

Poisson Regression Model

- The poisson regression model specifies that

$$\Pr\left(Y_i = k | X_i\right) = \frac{\exp\left(-\lambda_i\right)\lambda_i^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$\begin{aligned} \lambda_i &= \mathbb{E}\left(Y_i | X_i\right) = \exp\left(\beta_0 + \beta_1 X_i\right) \text{ or} \\ \ln\left(\lambda_i\right) &= \beta_0 + \beta_1 X_i \end{aligned}$$

- Interpretation of β_1: When there is a one-unit increase in X, the percentage change of $\mathbb{E}\left(Y | X\right)$ is $100 \times \beta_1$.

Estimation

- In principle, the Poisson model is simply a nonlinear regression. It is much easier to estimate the parameter with maximum likelihood method.
- The log-likelihood function is

$$
\begin{aligned}
& \ln L\left(\beta_0, \beta_1; \{Y_i, X_i\}_{i=1}^{N}\right) \\
= & \sum_{i=1}^{N} \ln \Pr\left(Y_i = y_i | X_i\right) \\
= & \sum_{i=1}^{N} \left[- \exp\left(\beta_0 + \beta_1 X_i\right) + Y_i \left(\beta_0 + \beta_1 X_i\right) - \ln(Y_i!)\right]
\end{aligned}
$$

- Note that the Poisson model could be too restrictive since the variance is equal to the mean.

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- The dependent variable is the number of times a man is arrested. The frequency of this variable is

\# of times arrested

	0	1	2	3	4	5	≥ 6	Total
Freq	1970	559	121	42	12	13	8	2,725
(%)	(72.3)	(20.5)	(4.4)	(1.5)	(0.4)	(0.5)	(0.3)	(100)

Example: Number of Arrests

- The independent variables are various individual characteristics such as proportion of prior convicted arrests (pcnv) and the number of quarters a man employed in 1986 (qemp86).

	pcnv	qemp86	black	constant	log_lik
Coeff.	-0.403	-0.203	0.532	-0.480	-2325.77
Std. Err.	0.08	0.02	0.07	0.06	

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    - (corner solution) consumption of a good (smoking/alcohol);
    - (data collection censoring) top-coding of income.

Structure of the Model

- An underlying latent variable has the following relationship:

$$Y_i^* = \beta_0 + \beta_1 X_i + u_i, \;\; u_i|X \sim N\left(0, \sigma^2\right)$$

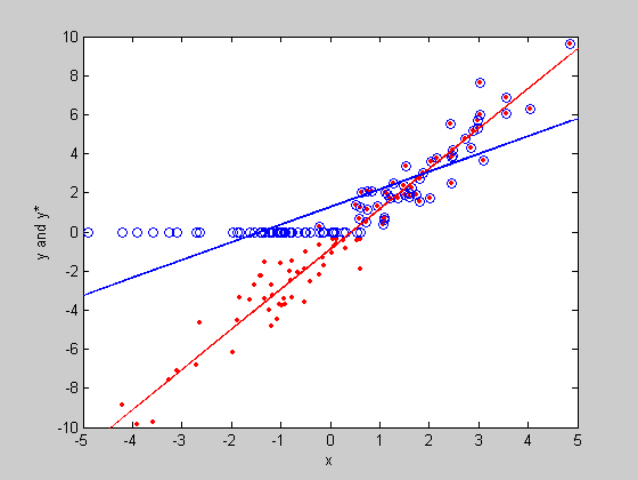
$$Y_i = \begin{cases} Y_i^* & \text{if} \quad Y_i^* > 0 \\ 0 & \text{otherwise} \end{cases}$$

- The model is called **Tobit**. Then, the probability of censoring is given

$$\begin{aligned} \Pr\left(Y = 0|X\right) &= \Pr\left(Y^* \leq 0|X\right) \\ &= 1 - \Phi\left(\frac{\beta_0 + \beta_1 X}{\sigma}\right) \end{aligned}$$

Example:

$\beta_0 = -1$, $\beta_1 = 2$, and $\sigma = 1$



Marginal Effects

- The marginal effect of X_i and $\mathbb{E}\left(Y_i^*|X\right)$ is

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$$\begin{aligned} \frac{\partial \mathbb{E}\left(Y_i|X\right)}{\partial X_i} &= \frac{\partial \Pr\left(Y_i > 0|X\right)}{\partial X_i}\mathbb{E}\left(Y_i|Y_i > 0, X\right) \\ &\quad + \Pr\left(Y_i > 0|X\right)\frac{\partial \mathbb{E}\left(Y_i|Y_i > 0, X\right)}{\partial X_i} \\ &= \beta_1 \Phi\left(\frac{\beta_0 + \beta_1 X_i}{\sigma}\right) = \beta_1 \Pr\left(Y_i > 0|X\right) \end{aligned}$$

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- Therefore, due to censoring, $\partial \mathbb{E}\left(Y_i|X\right)/\partial X_i < \partial \mathbb{E}\left(Y_i^*|X\right)/\partial X_i$.

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- The log-likelihood function is a mixture of the probit and the normal density:

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$$= \sum_{Y_i=0} \ln\left[1 - \Phi\left(\frac{\beta_0 + \beta_1 X_i}{\sigma}\right)\right] + \sum_{Y_i>0} \ln\left[\frac{1}{\sigma}\phi\left(\frac{Y_i - (\beta_0 + \beta_1 X_i)}{\sigma}\right)\right]$$

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- The ML estimators maximize the log-likelihood function with respect to the parameters.

Example: Willingness to Pay

- The willingness to pay is censored at zero. We can compare the two regressions:

$$OLS : WTP_i = \beta_0 + \beta_1 \ln y_i + \beta_2 age_i + \beta_3 sex_i + \beta_4 smell_i + u_i,$$

$$\begin{aligned} Tobit \quad &: \quad WTP_i^* = \beta_0 + \beta_1 \ln y_i + \beta_2 age_i + \beta_3 sex_i + \beta_4 smell_i + u_i, \\ &: \quad WTP_i = \max\left(WTP_i^*, 0\right). \end{aligned}$$

	OLS		Tobit		
	Coeff.	t-stat.	Coeff.	t-stat.	Mar. Eff.
ln(y)	2.52	2.74	2.70	2.5	1.93
age	-0.12	-2.0	-0.21	-3.0	-0.15
sex	0.41	0.28	0.14	0.0	0.10
smell	-1.43	-0.90	-1.80	-0.9	-1.29
const	-4.01	-0.50	-3.68	-0.4	