Online Supplementary Material for: "Heterogeneous Peer Effects in the Classroom"

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1 A Theoretical Model of Social Interactions

The model is an application of the theory of conspicuous consumption in Hopkins and Kornienko (2004). In particular, the assumptions on the primitives of the model are such that the theorems in Hopkins and Kornienko (2004) hold true.

Students choose how much costly effort e to exert, and effort increases achievement y. Students are heterogeneous in terms of how costly it is for them to exert effort.[1] They belong to a reference group (e.g. classroom, school). The main model assumptions are the following:

A.1 Students' utility is increasing in own achievement.

A.2 Students' utility is increasing in achievement rank in their reference group.

A.3 (optional). There are technological spill-overs in the production of achievement working through peer mean effort cost.

Students differ in terms of a type c: those with a higher c incur a larger cost of effort. This is without loss of generality. For example, students could instead be heterogeneous in terms of how productive their effort is, and, under minor modifications to the assumptions on the utility function, the model would have conceptually the same implications. Type c captures all student characteristics, physical and/or psychological, that affect her ability to study, such as her cognitive or academic skills, her emotional well-being, access to a computer or books, availability of an appropriate space for studying, etc. Type c is distributed in the reference group according to c.d.f. $G(\cdot)$ on $[\underline{c}, \bar{c}]$. Each student's type c is private information, but the distribution of c in the reference group is common knowledge. There are no distributional assumptions on $G(\cdot)$.

The cost of effort is determined by an increasing and strictly quasi-convex function in effort: $q(e; c)$. Higher types c incur higher costs for every level of effort e, i.e. $\frac{\partial q(e;c)}{\partial c} > 0$ for all e. For this reason, type c is informally referred to as a student's cost of effort. Moreover, at higher types the marginal cost of effort is (weakly) higher: $\frac{\partial^2 q(e;c)}{\partial c \partial e} \geq 0$.

[1] This corresponds to income heterogeneity in Hopkins and Kornienko (2004), where individuals choose how much of their income to spend on a consumption good and how much on a positional good. Here, achievement is at the same time a consumption good and a positional good, and it can be produced at a cost.

Effort increases achievement according to the production function $y(e) = a(\mu)e + u(\mu)$, with $a(\mu) > 0$, where μ is the mean of c among peers. Parameters $a(\mu)$ and $u(\mu)$ capture technological spill-overs working through the mean of peer ability (assumption A.3). For example, teacher productivity may depend on student's average skills, or more academically skilled students (lower μ) may ask relevant questions in class that facilitate their peers' learning. These technological spill-overs are allowed to affect both the level of achievement ($u(\mu)$) and the productivity of effort ($a(\mu)$).

The utility function can be decomposed into two elements: a utility that depends only on own test score y (in absolute terms) and effort cost q, $V(y,q)$, embedding assumption A.1; and a utility that depends on rank in terms of achievement, embedding assumption A.2. The utility from achievement in absolute terms net of effort cost is non-negative, increasing and linear in achievement, decreasing and linear in q, and it admits an interaction between utility from achievement and cost of effort such that at higher costs, the marginal utility from achievement is (weakly) lower ($V_{12} \leq 0$).[2] No specific functional form assumptions are made on $q(\cdot)$ and on the interaction between y and q, therefore, results from the model are valid under a broad class of preferences. For example, students with lower effort cost c may (or may not) have higher marginal utilities from achievement.

A student's classroom rank in terms of achievement is given by the c.d.f. of achievement computed at her own achievement level, $F_Y(y)$. This is the fraction of students with achievement lower than one's own, and it is a standard way to model rank in theoretical models of status seeking (Frank 1985).[3] Because achievement is an increasing deterministic function of effort, rank in achievement is equal to rank in effort: $F_Y(y(e)) = F_E(e)$, where $F_E(\cdot)$ is the c.d.f. of effort. The utility from rank, $S(F_Y(y(e)))$, is given by $F_E(e)+\phi$, where ϕ is a positive constant. Overall utility $U(y,q;c)$ is the product of utility from achievement in absolute and in relative terms: $V(y,q;c)\,(F_E(e)+\phi)$.

Each student chooses effort to maximize overall utility. Focusing on symmetric Nash

[2] All results are valid under an alternative set of assumptions for the utility from achievement. These are: strictly quasi-concave utility of achievement, decreasing and linear utility from cost of effort ($V_2 < 0, V_{22} = 0$) with a linear cost function ($\frac{d^2q}{d^2e} = 0$) and additive separability between utility from achievement and cost of effort ($V_{12} = 0$).

[3] Related papers are Hoppe, Moldovanu, and Sela (2009) and Moldovanu, Sela, and Shi (2007).

equilibria in pure strategies, and initially assuming that the equilibrium strategy $e(c)$ is strictly decreasing and differentiable with inverse function $c(e)$, rank in equilibrium can be rewritten as $1-G(c(e_i))$, and i's utility as $V(y(e_i), q(e_i, c_i))(1-G(c(e_i)))$.[4] The first-order condition then is:

$$\underbrace{V_1 \qquad \overbrace{a(\mu)}^{\text{Mg. increase in achiev.}}}_{\text{mg. ut. from increased achiev.}} + \underbrace{\frac{V(y,q)}{1-G(c(e_i))+\phi}\overbrace{g(c(e_i))(-c^{'}(e_i))}^{\text{Mg. increase in rank}}}_{\text{mg. ut. from increased rank}} = \underbrace{-V_2\frac{\partial q}{\partial e}}_{\text{mg. cost}} \qquad (1)$$

and it implies the first-order differential equation reported in equation 2 in Appendix A. The solution to this differential equation is a function $e(c)$ that is a symmetric equilibrium of the game. The assumptions on the utility function, on the cost of effort function and on the achievement production function guarantee that the results in Hopkins and Kornienko (2004) apply under appropriate proof adaptations.[5] In particular, while the differential equation does not have an explicit closed-form solution, existence and uniqueness of its solution and comparative statics results concerning the equilibrium strategies can be proved for any distribution function $G(c)$ twice continuously differentiable and with a strictly positive density on some interval $[\underline{c}, \bar{c}]$, with $\underline{c} \geq 0$. This means that it is possible to trace how the equilibrium distribution of achievement in the reference group changes as the distribution of peer characteristics changes, without the need to explicitly solve for the equilibrium effort function $e(c)$. That is, it is possible to derive the shape of the peer effects.

The first theoretical result is summarized in the following Proposition:

Proposition 1.1 (Adapted from Proposition 1 in Hopkins and Kornienko (2004)). *The unique solution to the differential equation (2) with the boundary condition $e(\bar{c}) = e_{nr}(\bar{c})$, where e_{nr} solves the first order condition in the absence of rank concerns ($V_1 a(\mu)|_{e=e_{nr}} = -V_2\frac{\partial q}{\partial e}|_{e=e_{nr}}$), is a unique symmetric Nash equilibrium of the game of status. Equilibrium effort $e(c)$ and equilibrium achievement $y(c)$ are both continuous and strictly decreasing in type c.*

Proof: see Appendix A.
The empirical analysis tests the monotonicity of the achievement function.[6]

[4]The probability that a student i of type c_i with effort choice $e_i = e(c_i)$ chooses a higher effort than another arbitrarily chosen individual j is $F(e_i) = Pr(e_i > e(c_j)) = Pr(e^{-1}(e_i) < c_j) = Pr(c(e_i) < c_j) = 1-G(c(e_i))$, where $c(\cdot) = e^{-1}(\cdot)$. The function c maps e_i into the type c_i that chooses effort e_i under the equilibrium strategy. Strict monotonicity and differentiability of equilibrium $e(c)$ are initially assumed, and subsequently it is shown that equilibrium strategies must have these characteristics.

[5] One of the main differences with the model in Hopkins and Kornienko (2004) is that here equilibrium strategies $e(c)$ are decreasing in c, whereas there they are increasing. See the procurement auctions model in Hopkins and Kornienko (2007) for another example with decreasing strategies.

[6]Monotonicity rules out the case in which for large enough values of c students exert more effort, which would be akin to a backward-bending labor supply curve. For example, suppose that students have high disutility from very low values of achievement. Then, as the cost of effort increases, the "substitution"

Now consider two reference groups, A and B, with two distributions of c, $G_A(c)$ and $G_B(c)$, that are such that they have the same mean, but G_B has larger dispersion than G_A in the Unimodal Likelihood Ratio sense ($G_A \succ_{ULR} G_B$), defined in Appendix A. This happens when, for example, G_B is a mean-preserving spread of G_A. In informal terms, one can show that the effect on achievement of moving from group A to group B is heterogenous across individuals, depending on a student's type c. For a formal statement of this comparative statics result see Proposition (A.1) in Appendix A. This result provides the main testable implication of the theoretical model, which concerns the shape of peer effects generated by rank concerns. It can informally be stated as follows:
Testable Comparative Statics: When the dispersion of c in the reference group increases (keeping the mean constant), middle-c students perform more poorly in terms of achievement and high-c students perform better, while low-c students may perform better or worse, depending on the relative strength of the preference for achievement rank in the utility function. These patterns are represented graphically in Figure 1.

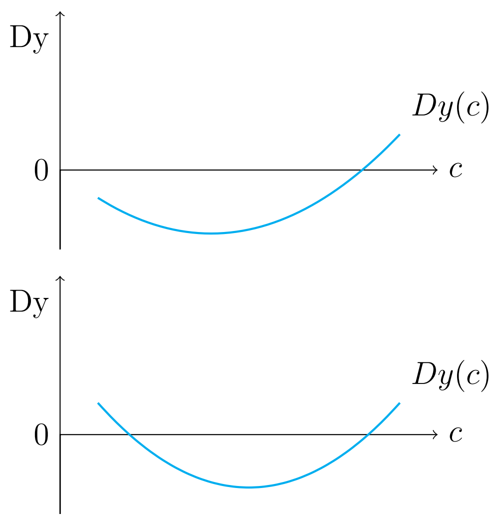

Figure 1: The function $Dy(c)$ traces the effect on achievement of increasing the dispersion (e.g.,the variance) of student type c, as a function of c. It can cross the x-axis once or twice. If it crosses it once (upper panel), the sequence of its signs, from low c to large c, is $-$, $+$. If it crosses it twice (lower panel), the sequence of its signs, from low c to large c, is $+$, $-$, $+$. As long as these crossing properties and signs are satisfied, the function $Dy(c)$ can admit any shape.

effect would induce individuals to exert less effort, but the "achievement effect" (like an income effect) would induce them to exert more effort to avoid very low values of achievement. The empirical test rejects such a scenario.

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A Appendix

A.1 Differential Equation

The first-order differential equation characterizing equilibrium strategies is obtained by rearranging the first order condition in 1, and substituting $c^{'}(e) = \frac{1}{e^{'}(c)}$:

$$\begin{aligned} e^{'}(c_i) &= \left(\frac{g(c_i)}{1-G(c_i)+\phi}\right)\left(\frac{V(y(e),q(e,c))}{a(\mu)V_1+V_2\frac{\partial q}{\partial e}}\right). \\ &= \frac{g(c_i)}{1-G(c_i)+\phi}\psi(e_i,c_i). \end{aligned} \tag{2}$$

A.2 Proof of Proposition 1.1

The proof is an adaptation of the proof in Hopkins and Kornienko (2004), where equilibrium strategies are strictly increasing and where the consumption and positional goods are two separate goods.

First, it is easy to show that the boundary conditions in the statement of the Proposition are optimal for the student with the highest cost, $\bar{c}$. The student with the highest type, $\bar{c}$, chooses the effort function that maxims utility V in the absence of rank concerns, as specified by the boundary condition in the statement of the Proposition. To see why, notice that in equilibrium her utility from rank is zero, therefore, she maximizes V because $V \times F + \phi \times V = V \times 0 + \phi \times V = \phi \times V$.

Next, I show that if the strategy $e^*(c)$ is a best response to other students' effort choices, then it is decreasing.If a student i of type c_i exerts effort $e_i = e^*(c_i)$ and this is a best response to the efforts of the other students as summarized by the effort distribution $F_E(\cdot)$, then it must be that $e_i \geq e_{nr}(c_i)$, where $e_{nr}(c_i)$ solves the first-order condition in the absence of rank concerns, i.e., $V_1 a(\mu)|_{e=e_{nr}} = -V_2\frac{\partial q}{\partial e}|_{e_{nr}}$. This is because if $e < e_{nr}(c_i)$, then $F_E(e)+\phi < F_E(e_n)+\phi$ and $V\left(y(e(c)),q(e(c),c)\right) < V\left(y(e_{nr}(c)),q(e_{nr}(c),c)\right)$. Therefore, $V\left(y(e),q(e,c)\right)\left(F_E(e)+\phi\right) < V\left(y(e_{nr}),q(e_{nr}(c),c)\right)\left(F_E(e_{nr})+\phi\right)$, i.e., any level of effort below the no-rank-concerns level is strictly dominated by the no-rank-concerns level. Suppose that equality holds, so $e_i = e_{nr}(c_i)$. Then $e^*(\cdot)$ is decreasing because $e_{nr}(c_i)$ is decreasing. This follows from the assumptions on $V(\cdot)$ that $V_{11} = 0$, $V_{22} = 0$, $V_{ij} \leq 0$ for $i \neq j$, and from the assumptions on the cost of effort function that $\frac{\partial q}{\partial c} > 0$, $\frac{\partial q}{\partial e} > 0$, $\frac{\partial^2 q}{\partial^2 e} > 0$ and $\frac{\partial^2 q}{\partial e \partial c} \geq 0$. To see why, let $FOC(e,c) = V_1 a(\mu) + V_2 q_1$ and notice that by the Implicit Function Theorem:

$$\frac{de_{nr}}{dc} = -\frac{\partial FOC/\partial c}{\partial FOC/\partial e}.$$

The numerator is:

$$\frac{\partial FOC}{\partial c} = a(\mu)V_{12}\frac{\partial q}{\partial c} + V_{22}\frac{\partial q}{\partial e}\frac{\partial q}{\partial c} + V_2\frac{\partial^2 q}{\partial e \partial c} \leq 0.$$

The denominator is:

$$\frac{\partial FOC}{\partial e} = a(\mu)^2 V_{11} + a(\mu) V_{12} \frac{\partial q}{\partial e} + \left(a(\mu) V_{21} + V_{22} \frac{\partial q}{\partial e} \right) \frac{\partial q}{\partial e} + V_2 \frac{\partial^2 q}{\partial^2 e} \leq 0.$$

As a result, $e^*(\cdot)$ is decreasing in c when it is equal to optimally chosen effort in the absence of rank concerns, because $\frac{de_{nr}}{dc} \leq 0$.

If equality does not hold, we want to show that if e_i is a best-response and $e_i > e_{nr}(c_i)$, then it is still the case that e_i is decreasing in c_i. First, I show that for any other choice $\tilde{e}_i \in (e_{nr}(c_i), e_i)$,

$$\frac{\partial V}{\partial c_i} \left(y(e_i), q(e_i, c_i) \right) \left(F_E(e_i) + \phi \right) < \frac{\partial V}{\partial c_i} \left(y(\tilde{e}_i), q(\tilde{e}_i, c_i) \right) \left(F_E(\tilde{e}_i) + \phi \right). \tag{3}$$

Rewrite the left-hand side as:

$$\frac{\partial V}{\partial c_i} \left(y(e_i), q(e_i, c_i) \right) \left(F_E(\tilde{e}_i) + \phi \right) + \frac{\partial V}{\partial c_i} \left(y(e_i), q(e_i, c_i) \right) \left(F_E(e_i) - F_E(\tilde{e}_i) \right).$$

The first term is smaller or equal to the right-hand side of equation 3, because $\frac{\partial V}{\partial c}$ is decreasing in e by the assumptions that $V_{21} \leq 0, V_{22} = 0, \frac{\partial q}{\partial c} > 0, V_2 < 0$, and $\frac{\partial^2 q}{\partial c \partial e} \geq 0$. To see why, notice that $\frac{\partial^2 V}{\partial c \partial e} = \left(V_{21} a(\mu) + V_{22} \frac{\partial q}{\partial e} \right) \frac{\partial q}{\partial c} + V_2 \frac{\partial q}{\partial c \partial e} \leq 0$. The second term is strictly negative, because first, $\frac{\partial V}{\partial c_i}$ is strictly negative by virtue of the assumptions that $V_2 < 0$ and $\frac{\partial q}{\partial c} > 0$, and second, $(F_E(e_i) - F_E(\tilde{e}_i)) > 0$. To see why the latter is true, notice that for $e > e_{nr}$, $V\left(y(e), q(e, c)\right)$ is decreasing in e. Therefore, if e is a best-response, it must be the case that $F_E(e_i) > F_E(\tilde{e}_i)$, otherwise a student could lower effort and obtain a higher utility, while not lowering her status. This establishes the inequality in 3, so that at e_i, the overall marginal utility with respect to c ($\frac{\partial}{\partial c}(V(y, q)(F_E(e) + \phi))$) is strictly decreasing in e. This implies that an increase in type c leads to a decrease in the marginal return to e, therefore, the optimal choice of effort e must decrease.

To show that if an effort function is an equilibrium strategy, then it must be continuous, suppose not. That is, suppose that that there was a jump downwards in the equilibrium effort function $e^*(c)$ at $\tilde{c}$, so that $\lim_{c \to \tilde{c}^+} e^*(c) = \tilde{e} < e^*(\tilde{c})$. Then, there would exist an $\epsilon > 0$ small enough, such that the student of type $\tilde{c} - \epsilon$ can reduce her effort to $\tilde{e}$, which is below $e^*(\tilde{c} - \epsilon)$, and obtain a discrete increase in utility because of the lower effort, while her rank would decrease by less, by continuity of the rank function $S(\cdot)$ at $\tilde{c}$. Therefore, there exists a student with an incentive to deviate, and such discontinuous $e^*(c)$ function cannot be an equilibrium strategy.[7]

Uniqueness of the solution to the differential equation in 2, and therefore uniqueness of the equilibrium, follows from the fundamental theorem of differential equations. The boundary condition pins down the unique solution.

Intuition for equilibrium uniqueness. Intuitively, uniqueness of the equilibrium follows from two key assumptions: achievement gives utility per se, i.e., irrespectively of

[7]The remaining part of the proof, showing that the equilibrium strategy is strictly decreasing and differentiable, is a straightforward adaptation of the lengthy proofs in Hopkins and Kornienko (2004), and it is available from the author upon request.

the status (rank utility) it provides, and individuals have different costs of producing achievement. A common type of multiplicity of equilibria in this class of games is when all individuals exert the same amount of effort. If this were an equilibrium, there would be an infinite number of equilibria. However, all students playing the same level of effort e^* is not an equilibrium, because students with a high enough cost (i.e., with a cost above a certain cutoff that depends on e^*, i.e., $c > cutoff(e^*)$), have an incentive to reduce effort, obtain zero rank, and enjoy their private utility from achievement. Therefore, the classical problem of multiplicity of equilibria in coordination games does not arise.

A.3 Comparative Statics

Definition Two distributions G_A, G_B with support on $[\underline{c}, \bar{c}]$ satisfy the Unimodal Likelihood Ratio (ULR) order, $G_A \succ_{ULR} G_B$, if the ratio of their densities $L(c) = g_A(c)/g_B(c)$ is strictly increasing for $c < \tilde{c}$ and strictly decreasing for $c > \tilde{c}$ for some $\tilde{c} \in [\underline{c}, \bar{c})$ and if $\mu_A \geq \mu_B$.

In particular, if B has the same mean but higher variance than A, then $G_A \succ_{ULR} G_B$. Define the cutoffs $\hat{c}^-$ and $\hat{c}^+$ as the extremal points of the ratio $(1-G_A(c)+\phi)/(1-G_B(c)+\phi)$ when $G_A \succ_{ULR} G_B$. It can be shown that these cutoffs are such that $\underline{c} < \hat{c}^- < \hat{c}^+ \leq \bar{c}$, and they can be conveniently interpreted as cutoffs that separate type categories.[8] Low c students are those with $c \in [\underline{c}, \hat{c}^-)$, middle c students as those with $c \in (\hat{c}^-, \hat{c}^+)$, and high c students as those with $c \in (\hat{c}^+, \bar{c}]$. The model has the following prediction:

Proposition A.1 (Adapted from Proposition 4 in Hopkins and Kornienko (2004)). *Suppose $e_A(c)$ and $e_B(c)$ are the equilibrium choices of effort for distributions G_A and G_B. If $G_A \succ_{ULR} G_B$ and $\mu_A = \mu_B$, then:*

- *$y(e_A(c))$ crosses $y(e_B(c))$ at most twice. Moreover, $y(e_A(c)) < y(e_B(c))$ for all $c \in [\hat{c}^+, \bar{c})$ with a crossing in $(\tilde{c}, \hat{c}^+)$ so that $y(e_A(c)) > y(e_B(c))$ for all $c \in [\hat{c}^-, \tilde{c}]$, with a possible crossing on $[\underline{c}, \hat{c}^-)$.*

Proof The proof is a straightforward adaptation of the lengthy proof in Hopkins and Kornienko (2004). It is available from the author upon request.

[8] The proof is available upon request from the author. It is a modification of the lengthy proof in Hopkins and Kornienko (2004), there the c.d.f. functions, rather than their complement, appear in the ratio.