Panel Data Models - part III

1 Testing for Fixed Effects Under Strict Exogeneity: Wu/Hausman test

De-Min Wu, Econometrica, 1973. Hausman, Econometrica, 1978.

- We want to test the hypothesis H_0 against H_1 where

  $H_0:$ no fixed effect (random effect), meaning that $E[f_i x_{it}] = 0$ for any t

  $H_1:$ fixed effect

- We maintain the hypothesis of strict exogeneity.

- β^{GLS} is consistent and efficient under the null but inconsistent under the alternative hypothesis. β^{WG} is consistent under both hypothesis but less efficient than β^{GLS} under the null.

- *Result from statistics:* Consider two consistent estimators β^0 and β^1 which are both asymptotically normally distributed,

$$\begin{aligned}\sqrt{N}\left(\beta^0-\beta\right) &\sim \mathcal{N}(0,V^0)\\ \sqrt{N}\left(\beta^1-\beta\right) &\sim \mathcal{N}(0,V^1)\end{aligned}$$

  If β^0 is consistent and more efficient than β^1, then

$$\text{avar}\left(\sqrt{N}\left(\beta^1-\beta^0\right)\right) = V^1 - V^0$$

  and,

$$\sqrt{N}\left(\beta^1-\beta^0\right) \overset{a}{\sim} \mathcal{N}\left(0, V^1-V^0\right)$$

- Applying this result to the comparison between GLS and WG yields

$$\left(\beta^{WG}-\beta^{GLS}\right)'\left(V^{WG}-V^{GLS}\right)^{-1}\left(\beta^{WG}-\beta^{GLS}\right) \overset{a}{\sim} \chi^2_k$$

  where $V^{WG} = \text{avar}\left(\beta^{WG}\right)$ and $V^{GLS} = \text{avar}\left(\beta^{GLS}\right)$ and the comparison is over the k parameters that are identified by both GLS and WG.

- Note that $\left(\widehat{V}^{WG}-\widehat{V}^{FGLS}\right)$ may not be positive definite. In large samples it will be positive definite. In small samples it could turn out to be indefinite.

2 Testing for Fixed Effects in Dynamic Models

- Same principle but we must carefully choose the instruments under the null and under the alternative.
- Consider the following model,
$$y_{it} = \alpha y_{it-1} + f_i + u_{it}$$
under the assumption of serially uncorrelated errors.
- The hypothesis are,

  $H_0:$ no fixed effects (and no random effects since these cannot exist in a dynamic model)

  $H_1:$ fixed effects
- Under H_0, the model can be estimated using OLS. Under H_1 need to use GMM using y_{it-2} as an instrument on the first differences model. Then apply Wu/Hausman test.
- But suppose u_{it} is $MA(1)$. Then under H_0 should use GMM on levels with y_{it-2} as the most recent instrument. Under H_1, notice that Δu_{it} is a MA(2) process. Then need to use GMM on first differences with y_{it-3} as the most recent instrument.

3 The Sargan test of over-identifying restrictions

Hansen, 1982 - *Econometrica*

- Suppose we have $L > K$ instruments where K is the number of parameters we need to identify.
- We want to test whether in fact the additional $r = L - K$ instruments are valid and provide additional identifying assumptions.
- The test of hypothesis is

  $H_0:$ the $L > K$ instruments are valid, so that $E(Z_i' u_i) = 0$

  $H_1:$ some instruments are invalid: $E(Z_i' u_i) = 0$

  Note we assume the instruments satisfy the rank condition, which is easy to test as we have discussed before.

- Under the null, we can apply the CLT to ensure that,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} Z_i' u_i \overset{a}{\sim} \mathcal{N}\left(0, E\left(Z_i' u_i u_i' Z_i\right)\right)$$

- We can use consistent estimates to construct the Sargan statistics. Using the GMM estimator we obtain

$$S\left(\beta^{GMM}\right) = \widehat{u}' Z\left(Z'\widehat{u}\widehat{u}' Z\right)^{-1} Z'\widehat{u} \overset{d}{\longrightarrow} \chi^2_r$$

where $\widehat{u}_i = y_i - X_i\beta^{GMM}$. Notice that the degrees of freedom for this test equals the number of over-identifying assumptions.

- To see that this is true, notice that the estimated residuals can be written as,

$$\begin{aligned}
\widehat{u} &= y - X\beta^{GMM} \\
&= y - X(X'Z\Omega^{-1}Z'X)^{-1}X'Z\Omega^{-1}Z'y \\
&= \left[I - X(X'Z\Omega^{-1}Z'X)^{-1}X'Z\Omega^{-1}Z'\right] y \\
&= \left[I - X(X'Z\Omega^{-1}Z'X)^{-1}X'Z\Omega^{-1}Z'\right] u
\end{aligned}$$

where $\Omega = E\left(Z_i' u_i u_i' Z_i\right)$.

- But then,

$$Z'\widehat{u} = \left[I - Z'X(X'Z\Omega^{-1}Z'X)^{-1}X'Z\Omega^{-1}\right] Z'u$$

- Define R such that,

$$\Omega^{-1} = \operatorname*{plim}_{N\to\infty}\left(\frac{Z'uu'Z}{N}\right)^{-1} = RR'$$

We can always do this because Ω^{-1} is a symmetric positive definite matrix.

- Then we can write,

$$\begin{aligned}
R'\frac{Z'\widehat{u}}{\sqrt{N}} &= \left[I - R'Z'X(X'ZRR'Z'X)^{-1}X'ZR\right] R'\frac{Z'u}{\sqrt{N}} \\
&= \left[I - P_R\right] R'\frac{Z'u}{\sqrt{N}}
\end{aligned}$$

- We now notice that

$$\begin{aligned} S\left(\beta^{GMM}\right) &= \frac{u'Z}{\sqrt{N}} R(I-P_R)'(I-P_R)R' \frac{Z'u}{\sqrt{N}} \\ &= \frac{\widehat{u}'Z}{\sqrt{N}} RR' \frac{Z'\widehat{u}}{\sqrt{N}} \\ &= \frac{\widehat{u}'Z}{\sqrt{N}} \Omega^{-1} \frac{Z'\widehat{u}}{\sqrt{N}} \end{aligned}$$

- Under the null $RR' = \Omega^{-1} = \text{plim}_{N\to\infty} \left(\frac{Z'\widehat{u}\widehat{u}'Z}{N}\right)^{-1}$ and we can write

$$R' \frac{Z'u}{\sqrt{N}} \xrightarrow{d} \mathcal{N}(0, I_L)$$

where I_L is the identity matrix of dimension L, the total number of instruments.

- *General result:* if $q \sim \mathcal{N}(0,I)$ and Q is idempotent, then $q'Qq \sim \chi^2_{\text{rank}(Q)}$.

- Choose $Q = I_L - P_R$ (which is idempotent) and $q = R'\frac{Z'u}{\sqrt{N}}$. We just need to establish the rank of $I_L - P_R$.

- Notice that since $I_L - P_R$ is idempotent, its trace equals its rank. Thus we can as well compute its trace,

$$\begin{aligned} \text{trace}(I_L - P_R) &= \text{trace}(I_L) - \text{trace}(P_R) \\ &= L - \text{trace}(P_R) \end{aligned}$$

where,

$$\begin{aligned} \text{trace}(P_R) &= \text{trace}\left(R'Z'X(X'ZRR'Z'X)^{-1}X'ZR\right) \\ &= \text{trace}\left((X'ZRR'Z'X)^{-1}X'ZRR'Z'X\right) \\ &= \text{trace}\left(I_K\right) = K \end{aligned}$$

- Thus,

$$\begin{aligned} S\left(\beta^{GMM}\right) &= \frac{u'Z}{\sqrt{N}} R(I-P_R)R' \frac{Z'u}{\sqrt{N}} \\ &= \frac{\widehat{u}'Z}{\sqrt{N}} RR' \frac{Z'\widehat{u}}{\sqrt{N}} \xrightarrow{d} \chi^2_{L-K} \end{aligned}$$

where $L - K$ is the number of over-identifying restrictions.

- In practice, we use $\left(\frac{Z'\widehat{u}\widehat{u}'Z}{N}\right)^{-1}$ to replace $\Omega^{-1} = RR' = \text{plim}_{N\to\infty}\left(\frac{Z'uu'Z}{N}\right)^{-1}$, which is asymptotically equivalent under the null since

$$\Omega^{-1} = \text{plim}_{N\to\infty}\left(\frac{Z'uu'Z}{N}\right)^{-1} \quad = \quad \text{plim}_{N\to\infty}\left(\frac{Z'\widehat{u}\widehat{u}'Z}{N}\right)^{-1}$$

- Notice that in the first differences model we should replace u by Δu and y and X by the respective first differences as well.

4 Testing for the Validity of a Subset of Instruments

- We need to consider two Sargan test Statistics,
  - S_1: all instruments, r_1 degrees of freedom.
  - S_0: a subset of instruments, $r_0 < r_1$ degrees of freedom.
- The latter is the unrestricted model. Using more instruments is equivalent to imposing restrictions because we are imposing more moments conditions (more assumptions).
- The statistic of the test is,

$$S_1 - S_0 \xrightarrow{d} \chi^2_{r_1 - r_0}$$