Instrumental Variables

(Wooldridge, chapter 5)

1 Introduction

We consider the model,

$$y = \mathbf{x}\beta + e \tag{1}$$

where the subscript for observation has been withdrawn for ease of notation and will be used only when necessary for clarity. y is the dependent variable, $\mathbf{x}$ is a vector of explanatory variables of size $1 * k$, β is a vector of size $k * 1$ containing the unknown coefficients and e is the unobservable component of y.
The OLS estimator is based on two assumptions,

OLS1. $E(\mathbf{x}'e) = 0$

OLS2. $E(\mathbf{x}'\mathbf{x}) = M_{xx}$ where M_{xx} is positive definite (with rank equal to k).

In this note we study the solution to the problem of estimating β when OLS1 fails - we say there is an *endogeneity* problem.

2 When does OLS1 fail to hold?

2.1 Omitted variables

Consider a Mincer model of returns to schooling,

$$y = \mathbf{x}\beta + s\alpha + e \tag{2}$$

$$s = \mathbf{z}v + u \tag{3}$$

where y is earnings, s is schooling, $\mathbf{x}$ is a vector of $k-1$ observable variables explaining earnings (excluding education, s) and $\mathbf{z}$ is a vector of variables explaining education achievement. $\mathbf{x}$ and $\mathbf{z}$ may or not be the same set of variables. We separate s from the other regressors $\mathbf{x}$ to emphasise that endogeneity problems arise from s. (e, u) are the unobservable components of the model.
In this model we might expect s to be correlated with e. This may happen through some correlation between e and some variable(s) in $\mathbf{z}$ not included in $\mathbf{x}$ or through some correlation between u and e.

The first case can be solved by enlarging $\mathbf{x}$ with the observable variables that cause the problem. The second case requires more care. It is caused by some unobservable component that simultaneously determines earnings and schooling. In this example we typically think of it as ability.

2.2 Measurement error

Suppose our model is,

$$y = \mathbf{x}\beta + s\alpha + e \tag{4}$$

but s is observed with error, so that what we observe is,

$$s^* = s - v \tag{5}$$

We may think of using the observable measure of s instead of the true s to estimate the unknown parameters of model (4). The new equation is,

$$y = \mathbf{x}\beta + s^*\alpha + v\alpha + e \tag{6}$$

where $v\alpha + e$ is now the unobserved component of the model.
However, even if the OLS assumptions hold when applied to equation (4), this is generally no longer true with respect to equation (6) since v determines s^*. Such violation occurs independently of whether or not the measurement error v is related to s. In fact, it is generally assumed that v is independent of s, $\mathbf{x}$ and e.

2.3 Serially correlated residuals in the presence of lagged dependent variable

Suppose we have a time series model,

$$y_t = \mathbf{x}_t\beta + y_{t-1}\alpha + e_t \tag{7}$$

where e_t is serially correlated and the subscript t denotes the time period. At time $t-1$, the value of the dependent variable y is partly determined by the unobservable component at time $t-1$,

$$y_{t-1} = \mathbf{x}_{t-1}\beta + y_{t-2}\alpha + e_{t-1}$$

But since e_t and e_{t-1} are correlated, we expect y_{t-1} to be correlated with e_t through its dependence on e_{t-1}.

3 What do we do when OLS1 does not hold?

The two most commonly used approaches are:

1. Instrumental Variable (IV) estimation.

2. Maximum Likelihood Estimation (MLE).

In this note we look at IV estimation.

4 Why does OLS fail when OLS1 does not hold?

Intuitively, the problem is that the failure of OLS1 precludes changes in $\mathbf{x}$ keeping everything else constant, particularly e, and this will confound the effect of $\mathbf{x}$.
To see this more precisely, notice that the OLS estimator, b^{OLS}, is defined as a solution to the system:

$$\begin{aligned} 0 &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i' e_i\left(b^{OLS}\right) \qquad (8) \\ &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i'\left(y_i - \mathbf{x}_i b^{OLS}\right) \end{aligned}$$

which is a set of k equations.
Condition (8) is the sample analog of assumption OLS1, which states that

$$0 = E\left(\mathbf{x}_i' e_i\left(b^{OLS}\right)\right) \qquad (9)$$

Under OLS2, the unique solution to (9) is $b^{OLS} = \beta$, the true set of parameters. However, if (9) does not hold then β will not solve (9) and the OLS estimator will now be based on a condition that does not hold for the true β. As a consequence, it will yield inconsistent estimates of the parameters β:

$$\begin{aligned} b^{OLS} &= \left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\sum_{i=1}^{N}\mathbf{x}_i' y_i \\ &= \left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\left(\mathbf{x}_i\beta + e_i\right) \\ &= \beta + \left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\sum_{i=1}^{N}\mathbf{x}_i' e_i \\ &= \beta + \left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i' e_i \\ &\xrightarrow{p} \beta + M_{xx}^{-1}E\left(\mathbf{x}_i' e_i\right) \end{aligned}$$

which is different from β since $E\left(\mathbf{x}_i' e_i\right) \neq 0$.

5 The IV solution

Consider model (1) when condition OLS1 fails to hold. In general, let $\mathbf{x} = (x_1, ...x_{k-1}, x_k)$ and assume x_k is the only endogenous regressor. Now consider an alternative k-dimensional random vector, $\mathbf{z} = (z_1, \ldots, z_{k-1}, z_k) = (x_1, ...x_{k-1}, z_k)$ where $\mathbf{z}$ is called the vector of instruments under some assumptions to be introduced below. In the present case, both $\mathbf{x}$ and $\mathbf{z}$ are of size $1 * k$ but we will later see the possibility of having more instruments than regressors.
For $\mathbf{z}$ to be a valid set of instruments, a condition similar to OLS1 must hold with respect to z:

$$E(\mathbf{z}'e) = 0 \tag{10}$$

This means that all variables in $\mathbf{z}$ are exogenous in the equation for y, including the regressors $x_1 \ldots, x_{k-1}$ and the additional instrument z_k. Assumption (10) implies that $\mathbf{z}$ has no explanatory power of y conditional on the x's.
We can now use assumption (10) within model (1) to derive:

$$\begin{aligned} \mathbf{z}'y &= \mathbf{z}'\mathbf{x}\beta + \mathbf{z}'e \text{ implying that} \\ E(\mathbf{z}'y) &= E(\mathbf{z}'\mathbf{x})\beta \end{aligned}$$

To be able to solve for β we need an additional assumption guaranteing that z_k does indeed explain x_k conditional on $(x_1, ..., x_{k-1})$. As will be made clear in the next section, this is equivalent to assume that

$$\operatorname{rank}\left(E(\mathbf{z}'\mathbf{x})\right) \neq k \tag{11}$$

which implies that $E(\mathbf{z}'\mathbf{x})$ is invertible. This is known as the *rank condition*.
We can now use assumption (11) to write

$$\beta = E(\mathbf{z}'\mathbf{x})^{-1}E(\mathbf{z}'y)$$

The expectations $E(\mathbf{z}'\mathbf{x})$ and $E(\mathbf{z}'y)$ can be consistently estimated, yielding the IV estimator,

$$b^{IV} = (Z'X)^{-1}(Z'\mathbf{y})$$

In the returns to education example, $\mathbf{z}$ is formed by the exogenous regressors in the earnings equation (2) and one additional regressor in the educational attainment equation (3). Taken together, assumptions (10) and (11) require the existence of (at least) one element in $\mathbf{z}$ not in $(\mathbf{x}, s)$. For this reason, taken together these conditions are typically known as the *exclusion restriction*: for each endogenous regressor in (2) we need at least one variable explaining such regressor that does not affect the dependent variable y otherwise.

6 Two-Stage Least Squares

6.1 Same number of instruments as explanatory variables

The IV estimator can be derived as the two-stage least squares (2SLS) estimator. To show this, take the model as detailed above, where x_k is endogenous but all the other explanatory variables are exogenous:

$$y = \beta_1 x_1 + ... + \beta_{k-1} x_{k-1} + \beta_k x_k + e \tag{12}$$

As above, suppose there exists an additional variable z_k that is correlated with x_k conditionally on all other explanatory variables, $x_1, ..., x_{k-1}$ but does not explain y conditional on $x_1, ..., x_k$. The later means that z_k and e are uncorrelated. The former implies that if we write a reduced form model for x_k as follows,

$$x_k = \alpha_1 x_1 + ... + \alpha_{k-1} xk - 1 + \alpha_k z_k + u \tag{13}$$

the coefficient α_k is different from zero.
By construction u has mean zero and is uncorrelated with $z = (x_1, ..., x_{k-1}, z_k)$. Notice that (13) is a reduced form equation for x_k and we have no intention of recovering structural parameters here. So we can run an OLS regression to estimate the parameters in (13) and obtain the *fitted values*,

$$\widehat{x}_k = \widehat{\alpha}_1 x_1 + ... + \widehat{\alpha}_{k-1} xk - 1 + \widehat{\alpha}_k z_k$$

Since all variables in $\mathbf{z}$ are uncorrelated with e, so is $\widehat{x}_k$. So we can replace x_k with $\widehat{x}_k$ in (14) and estimate the transformed model

$$y = \beta_1 x_1 + ... + \beta_{k-1} x_{k-1} + \beta_k \widehat{x}_k + e \tag{14}$$

using OLS. This is the 2SLS and in the present case (equal number of explanatory variables and instruments), *it is the same as the IV estimator we presented above.*

To see why, define as $\widehat{\alpha}$ the following $k*k$ matrix of coefficients

$$\widehat{\alpha} = \begin{bmatrix} 1 & 0 & \dots & \widehat{\alpha}_1 \\ 0 & 1 & \dots & \widehat{\alpha}_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \widehat{\alpha}_k \end{bmatrix}$$

We can now write

$$\begin{aligned} \widehat{X} &= Z\widehat{\alpha} \\ &= Z(Z'Z)^{-1}Z'X \end{aligned}$$

where X is the $N*k$ matrix of regressors, $\widehat{X}$ is the $N*k$ matrix of fitted regressors and Z is the $N*k$ matrix of instruments (for each individual i, $\mathbf{x}_i = (x_{i1},...,x_{ik-1},x_{ik})$, $\widehat{\mathbf{x}}_i = (x_{i1},...,x_{ik-1},\widehat{x}_{ik})$ and $\mathbf{z}_i = (x_{i1},...,x_{ik-1},z_{ik})$).
The OLS estimator on the second stage is,

$$\begin{aligned} b^{2SLS} &= \left(\widehat{X}'\widehat{X}\right)^{-1}\widehat{X}'\mathbf{y} \\ &= \left[X'Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'\mathbf{y} \\ &= \left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'\mathbf{y} \\ &= (Z'X)^{-1}Z'Z(X'Z)^{-1}X'Z(Z'Z)^{-1}Z'\mathbf{y} \\ &= (Z'X)^{-1}Z'\mathbf{y} = b^{IV} \end{aligned}$$

which is precisely the IV estimator as presented before. That is, the two approaches are the same in this case.
Thus, the IV estimator can be implemented as a 2 steps procedure (2SLS) as follows,

1. regress x_k on $(x_1,...,x_{k-1},z_k)$ and get $\widehat{x}_k$;

2. regress y on $(x_1,...,x_{k-1},\widehat{x}_k)$ to obtain b^{2SLS}.

6.2 What to do when we have more than k instruments?

Although the standard IV procedure above cannot be applied when we have more instruments than regressors ($Z'X$ would not be a square matrix and, therefore, would not be invertible), the 2SLS estimator is easily extensible to this case. Suppose we have $l=(k-1)+m$ instruments (the $k-1$ exogenous regressors plus some additional m variables explaining x_k but excluded from the equation for y conditional on $x_1,...,x_k$. The two stages are now,

1. regress x_k on $(x_1, ..., x_{k-1}, z_k, ..., z_m)$ and get $\widehat{x}_k$;

2. regress y on $(x_1, ..., x_{k-1}, \widehat{x}_k)$ to obtain b^{2SLS}.

Notice that 2SLS can still be applied as a simple IV estimator by noticing that, for as long as the number of instruments is at least as large as the number of regressors, then,

$$\begin{aligned}
b^{2SLS} &= \left(\widehat{X}'\widehat{X}\right)^{-1}\widehat{X}'\mathbf{y} \\
&= \left[X'Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'\mathbf{y} \\
&= \left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'\mathbf{y} \\
&= (\widehat{X}'X)^{-1}\widehat{X}'\mathbf{y}
\end{aligned}$$

which is the IV estimator using $\widehat{X}$ as instruments.
The 2 steps procedure described above provides consistent estimates of the parameters β. However, one should be careful applying it since the outcome from the second stage regression will not report the correct standard errors for the estimates, b^{2SLS}. This is because it will not take into account that the instruments, $\widehat{X}$, have themselves been estimated. Instead, one should directly apply a 2SLS routine, which takes into account the whole procedure. In what follows we derive some properties of the 2SLS estimator and it will be clear that the OLS standard errors are not the correct ones.

6.3 Properties of 2SLS

Consider the estimation of model (1) under the assumption $E(\mathbf{x}'e) \neq 0$. Suppose there exists a set of l variables, $\mathbf{z}$, with $l \geqslant k$ where k is the number of regressors in $\mathbf{x}$. The 2SLS assumptions are,

2SLS1. $E(\mathbf{z}'e) = 0$

2SLS2. $\text{rank}\,(E(\mathbf{z}'\mathbf{x})) = k$

2SLS3. $\text{rank}\,(E(\mathbf{z}'\mathbf{z})) = l$

2SLS4. $E(e^2\mathbf{z}'\mathbf{z}) = E(e^2)E(\mathbf{z}'\mathbf{z}) = \sigma_e^2 E(\mathbf{z}'\mathbf{z})$

Bias The 2SLS estimator is,

$$\begin{aligned}
b^{2SLS} &= \beta + \left[X'Z\,(Z'Z)^{-1}\,Z'X\right]^{-1}X'Z\,(Z'Z)^{-1}\,Z'\mathbf{e} \\
&= \beta + \left[\left(\frac{1}{N}\sum_{i=1}^N \mathbf{x}_i'\mathbf{z}_i\right)\left(\frac{1}{N}\sum_{i=1}^N \mathbf{z}_i'\mathbf{z}_i\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \mathbf{z}_i'\mathbf{x}_i\right)\right]^{-1}\left(\frac{1}{N}\sum_{i=1}^N \mathbf{x}_i'\mathbf{z}_i\right)\left(\frac{1}{N}\sum_{i=1}^N \mathbf{z}_i'\mathbf{z}_i\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \mathbf{z}_i'e_i\right)
\end{aligned}$$

with expected value,

$$E\left[b^{2SLS}|X,Z\right] = \beta + \left[X'Z(Z'Z)^{-1}Z'X\right] X'Z(Z'Z)^{-1}Z'E[\mathbf{e}|X,Z]$$

where $E[\mathbf{e}|X,Z] \neq 0$ at least for some (X,Z) since $E[\mathbf{e}|X] \neq 0$. Thus, the estimator is biased,

$$E\left[b^{2SLS}|X,Z\right] \neq \beta$$

Consistency Assuming the sample is iid from a sample with finite variances:

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{z}_i \xrightarrow{p} E\left[\mathbf{x}_i'\mathbf{z}_i\right]$$
$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{z}_i'\mathbf{z}_i \xrightarrow{p} E\left[\mathbf{z}_i'\mathbf{z}_i\right]$$

and under (2SLS1)

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{z}_i'e_i \xrightarrow{p} E\left[\mathbf{z}_i'e_i\right] = 0$$

Then under (2SLS2)-(2SLS3), $b^{2SLS} \xrightarrow{p} \beta$.

Distribution From the above expressions

$$\sqrt{N}\left(b^{2SLS}-\beta\right) =$$
$$\left[\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{z}_i\right)\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{z}_i'\mathbf{z}_i\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{z}_i'\mathbf{x}_i\right)\right]^{-1}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{z}_i\right)\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{z}_i'\mathbf{z}_i\right)^{-1}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{z}_i'e_i\right)$$

and from the CLT under (2SLS1) and (2SLS4),

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{z}_i'e_i \xrightarrow{d} \mathcal{N}\left(0, \sigma_e^2 E(\mathbf{z}_i\mathbf{z}_i')\right)$$

But then, under (2SLS2) and (2SLS3),

$$\sqrt{N}\left(b^{2SLS}-\beta\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_e^2\left[E(\mathbf{x}\mathbf{z}')E(\mathbf{z}\mathbf{z}')^{-1}E(\mathbf{z}\mathbf{x}')\right]^{-1}\right)$$

Exercise: Can you derive the properties for the standard IV estimator?