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Lecture 4: The Decentralized Economy

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In this lecture

- Consumption theory
  - Aggregate tests and evidence
  - Households tests and evidence
- Labor supply
- Competitive equilibrium
- Solving Dynamic General Equilibrium models

Consumption

- Consumption makes up for around 60 percent of total aggregate spending
- Important to understand its determinants
- Fortunately, there is lots of data on consumption
  - Aggregate data
  - Household level data
- Tests based on Euler equations and tests based on consumption functions

A Life-Cycle Model

Suppose that we consider a single consumer - or household - that:

- lives for T periods
- is born with no assets
- must leave the end of period T without debt
- earns an exogenous but possibly stochastic income y_t in period t
- Assume also that the interest rate is exogenous and equal to r_t
- The household's problem

$$\max \mathrm{E}_t \sum_{s=t}^{T+t} \beta^{s-t} u\left(c_s\right)$$

$$W_{s+1} = \left(1+r_s\right) W_s + y_s - c_s, \; s = t, t+1, .., T+t$$

where $W_t = W_{T+t+1} = 0$

The Euler Equation

The household's problem implies the (by now well-known, I hope) Euler equation:

$$u'(c_t) = \mathrm{E}_t \beta (1 + r_{t+1}) u'(c_{t+1})$$

What does this imply for:

- evolution of consumption over time and over the life-cycle?
- relationship between consumption and income?
- relationship between consumption and real interest rates?

Clearly, to make things testable, we need to make assumptions on

- preferences and/or
- income process

The Random Walk Theory

Hall, 1978, derived a famous theory of consumption. Assume that

1. The real interest rate is constant and equal to r
2. $\beta = 1/(1+r)$
3. Quadratic preferences:

$$u(c_t) = ac_t - bc_t^2$$

- where $a > 0$, $b > 0$ and, strictly speaking we need to assume that:

$$c_t < \frac{a}{2b}$$

(in order to have positive marginal utility)

The Euler equation now becomes:

$$\begin{aligned} a - 2bc_t &= \mathrm{E}_t \beta (1+r)(a - 2bc_{t+1}) \\ &\Rightarrow \\ c_t &= \mathrm{E}_t c_{t+1} \end{aligned}$$

The Random Walk Theory

We can also evaluate the Euler equation for any future period:

$$c_{t+i} = \mathrm{E}_{t+i} c_{t+i+1}$$

Thus we get that:

$$c_t = \mathrm{E}_t c_{t+1} = \mathrm{E}_t \mathrm{E}_{t+1} c_{t+2} = .. = \mathrm{E}_t \mathrm{E}_{t+i} c_{t+i+1}$$

Since information accumulates over time, the law of iterated expectations implies that:

$$\mathrm{E}_t \mathrm{E}_{t+i} c_{t+i+1} = \mathrm{E}\left(\mathrm{E}\left(c_{t+i+1} | I_{t+i}\right) | I_t\right) = \mathrm{E}\left(c_{t+i+1} | I_t\right) = \mathrm{E}_t c_{t+i+1}$$

because $I_t \subseteq I_{t+i}$. Therefore:

$$c_t = \mathrm{E}_t c_{t+i} \text{ for all } i > 0$$

The Random Walk Theory

The above expression implies that:

$$\begin{aligned} c_{t+i} &= c_t + \upsilon_{t+i} \\ \mathrm{E}_t \upsilon_{t+i} &= 0 \end{aligned}$$

- This implies that consumption should behave like a random walk
- Given c_t, no other information available at date t should be helpful for forecasting future consumption
- Consumption changes only when new information arrives, but this is unforecastable

The Random Walk Theory

How could you test this? Consider the regression:

$$c_{t+1} = \gamma_1 c_t + \gamma_2 z_t + \varepsilon_{t+1}$$

where

- z_t is a vector of variables that should be in the information set at date t and ε_t is a regression error
- Test:
  - H_0: $\gamma_2 = 0$ (could also test whether $\gamma_1 = 1$)

Candidates for z_t considered by Hall:

- **Lagged consumption**, c_{t-s}: Little predictive power
- **Stock prices** at date t: Some predictive power
- **Lagged income**, y_{t-s}: Strong predictive power

Rule of Thumb Consumers

- One explanation for the significance of past income is that not all consumers behave like Permanent Income households.
- Suppose that a share μ behave like above while another share $(1-\mu)$ simply consume their income either because of liquidity constraints (can neither borrow nor save) or because they are irrational - use rule of thumbs. Suppose also that their income is given as:

$$c_{t+1}^{RT} = y_{t+1} = \rho y_t + e_{t+1}, \ \rho \in (0,1), \ \mathrm{E}_t e_{t+1} = 0$$

- Then aggregate consumption is:

$$\begin{aligned} c_{t+1}^{ag} &= \mu c_{t+1}^{PIH} + (1-\mu) c_{t+1}^{RT} = \mu c_t^{PIH} + \mu \varepsilon_{t+1} + (1-\mu) \rho y_t \\ &= \mu c_t^{PIH} + (1-\mu) c_t^{RT} + (1-\mu) \left(\rho y_t - c_t^{RT} \right) + \mu \varepsilon_{t+1} \\ &= c_t^{ag} + (1-\mu)(\rho - 1) y_t + \mu \varepsilon_{t+1} \end{aligned}$$

- Thus, finding a large coefficient on past income could be consistent with a large share of Rule-of-Thumb consumers but will depend on the persistence of their income

CRRA Preferences and the Impact of Interest Rates

Consider now a specification with CRRA preferences:

$$\begin{aligned} u(c_t) &= \frac{c_t^{1-\theta} - 1}{1-\theta}, \ \theta > 0, \neq 1 \\ &= \log c_t \text{ for } \theta = 1 \end{aligned}$$

This specification implies that

$$R_R = \frac{1}{IES} = -\frac{u'' c}{u'} = \theta$$

Assuming no uncertainty, the Euler equation becomes:

$$\begin{aligned} c_t^{-\theta} &= \beta c_{t+1}^{-\theta} (1 + r_{t+1}) \\ &\Rightarrow \\ \log c_{t+1} &= \frac{1}{\theta} \log \beta + \log c_t + \frac{1}{\theta} \log (1 + r_{t+1}) \\ &\Rightarrow \\ \log c_{t+1} - \log c_t &\simeq \frac{1}{\theta} \log \beta + \frac{1}{\theta} r_{t+1} \end{aligned}$$

CRRA Preferences and Interest Rates

$$\log c_{t+1} - \log c_t \simeq \frac{1}{\theta} \log \beta + \frac{1}{\theta} r_{t+1}$$

- The intertemporal elasticity of substitution determines the response of consumption growth to real interest rates
  - When $\frac{1}{\theta}$ is large, small changes in real interest rates have a large impact on consumption growth
  - IES and R_R are inversely related
- A large number of tests have estimated θ from such Euler equations. They most often find values imply that $\theta \in [0.5, 5]$, i.e. $IES = \frac{1}{\theta} \in [0.2; 2]$

Consumption Functions

- Testing Euler equations is nice because one tests theories directly and does not require one to specify the income process
- The main problem is that the outcome of tests can often be difficult to interpret - if rejected, what is the counter-hypothesis?
- Therefore, it may sometimes be more insightful to estimate consumption functions - this requires one to specify the income process but the resulting test is very clear

Quadratic Preference Example

Recall from above that in the quadratic preference example:

$$\begin{aligned} c_{t+i} &= c_t + \upsilon_{t+i} \\ \mathrm{E}_t \upsilon_{t+i} &= 0 \end{aligned}$$

The budget constraint is given as:

$$\begin{aligned} W_{t+1} &= (1+r) W_t + y_t - c_t \\ &\Rightarrow \\ W_t &= \frac{W_{t+1}}{1+r} - \frac{y_t - c_t}{1+r} = -\sum_{s=t}^{T+t} \left(\frac{1}{1+r} \right)^{s-t+1} (y_s - c_s) \end{aligned}$$

$$\Rightarrow$$

$$\sum_{s=t}^{T+t} \left(\frac{1}{1+r} \right)^{s-t+1} c_s = W_t + \sum_{s=t}^{T+t} \left(\frac{1}{1+r} \right)^{s-t+1} y_s$$

Quadratic Preference Example

Now take expectations on both sides, let $T \to \infty$, and use the Euler equation

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t+1} \mathrm{E}_t c_s = W_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t+1} \mathrm{E}_t y_s$$

$$\Rightarrow$$

$$\left(\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t+1}\right) c_t = W_t + \sum_{s=t}^{T} \left(\frac{1}{1+r}\right)^{s-t+1} \mathrm{E}_t y_s$$

$$\Rightarrow$$

$$\left(\frac{1}{1-\frac{1}{1+r}} - 1\right) c_t = W_t + \sum_{s=t}^{T} \left(\frac{1}{1+r}\right)^{s-t+1} \mathrm{E}_t y_s$$

where I have used that:

$$\left(\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t+1}\right) = \left(\frac{1}{1-\frac{1}{1+r}} - 1\right) = \frac{1}{r}$$

Quadratic Preference Example

Thus, we get that the level of consumption is given as

$$c_t = r \left[W_t + \sum_{s=t}^{T} \left(\frac{1}{1+r} \right)^{s-t+1} \mathrm{E}_t y_s \right]$$

- This is the annuity value of the wealth - and is called permanent income
- It is the maximum constant level of consumption that can be sustained under the constraint that wealth remains constant
- It corresponds to consuming the interest on current financial assets plus the present value of all future income
- Does this correspond to your level of consumption?

Quadratic Preference Example

Now assume that the income process is given as:

$$\begin{aligned} y_{t+1} &= \rho y_t + \eta_{t+1} \\ \mathrm{E}_t \eta_{t+1} &= 1 \end{aligned}$$

We can then express permanent income as:

$$\begin{aligned} y_t^p &= W_t + \frac{y_t}{1+r} + \frac{\mathrm{E}_t y_{t+1}}{(1+r)^2} + + \frac{\mathrm{E}_t y_{t+2}}{(1+r)^3}.. \\ &= W_t + \frac{y_t}{1+r} + \frac{\rho y_t}{(1+r)^2} + \frac{\rho^2 y_t}{(1+r)^3}.. \\ &= W_t + \frac{1}{1+r}\frac{y_t}{1-\rho/(1+r)} = W_t + \frac{y_t}{1+r-\rho} \end{aligned}$$

Thus, we end up with the following consumption function:

$$c_t = rW_t + \frac{r}{1+r-\rho} y_t$$

Quadratic Preference Example

We could attempt testing this consumption function directly:

$$c_t = rW_t + \frac{r}{1+r-\rho} y_t$$

- But, wealth (W_t) is very hard to measure in practise
- For that reason, let us express the consumption function in terms of observables. We can express c_{t-1} as:

$$\begin{aligned} c_{t-1} &= rW_{t-1} + \frac{r}{1+r-\rho} y_{t-1} \\ &= r\left(\frac{W_t}{1+r} + \frac{c_{t-1} - y_{t-1}}{1+r}\right) + \frac{r}{1+r-\rho} y_{t-1} \\ &\Rightarrow \\ c_{t-1} &= rW_t + r\left(\frac{1+r}{1+r-\rho} - 1\right) y_{t-1} \end{aligned}$$

- subtracting this from the expression for c_t gives us:

$$c_t - c_{t-1} = \frac{r}{1+r-\rho} \left(y_t - \rho y_{t-1}\right)$$

The Permanent Income Hypothesis

So from above we get that:

$$\triangle\ c_t = \frac{r}{1+r-\rho}\left(y_t - \rho y_{t-1}\right)$$

and we recall that

$$y_t = \rho y_{t-1} + \upsilon_t$$

thus:

$$\triangle\ c_t = \frac{r}{1+r-\rho}\upsilon_t$$

According to the PIH, consumption changes only because of unexpected income shocks. The persistence of changes in income matters:

- When income is a random walk ($\rho = 1$): $\triangle\ c_t = \upsilon_t$, ie. the change in current and future consumption is of the same size as the change in current income
- When income has no persistence (is white noise, $\rho = 0$): $\triangle\ c_t = \frac{r}{1+r}\upsilon_t$ which means that the increase in current income is spread out over the infinite future

Testing PIH

According to PIH we have the two equations:

$$\begin{aligned} c_t &= c_{t-1} + \frac{r}{1+r-\rho}\left(y_t - \rho y_{t-1}\right) + e_t \\ y_t &= \rho y_{t-1} + v_t \end{aligned}$$

where e_t is a regression residual. This implies the reduced form:

$$\begin{aligned} c_t &= a_1 c_{t-1} + a_2 y_t + a_3 y_{t-1} + e_t \\ y_t &= b_1 y_{t-1} + v_t \end{aligned}$$

Thus, there are 4 coefficients to estimate - but there are only 2 structural parameters - therefore the model is over-identified.

Aggregate Evidence on PIH

1. Majorie Flavin, Journal of Political Economy, 1981, found excess sensitivity:

- Coefficient on lagged income higher than what is consistent with economic theory

2. Campbell and Mankiw, 1989, showed that this may be due to a significant share of rule-of-thumb consumers
3. Campbell and Deaton, 1989 instead found excess smoothness: They estimated a variance specification of PIH:

$$var\left(\triangle\ c_t\right) = \frac{r}{1+r-\rho} var\left(\upsilon_t\right)$$

- They argued that income is close to a random walk (so that the variance of consumption growth should be the same as the variance of income growth) but showed that the variance of consumption is too high

4. Poterba, AER, 1988, showed that aggregate consumption reacts little to predictable changes in income

- Changes in taxes in the US often pre-announced
- Thus, consumers have prior knowledge of changes in their income
- He estimated:

$$c_t = c_{t-1} + \frac{r}{1 + r - \rho}\left(y_t - \rho y_{t-1} + \kappa_{t+1}\right) + e_t$$

  where κ_{t+1} are changes in taxes in period $t+1$ that the agents are informed about at date t

- He found that κ_{t+1} did not seem to affect c_t but affect c_{t+1}

PIH: Summary of the Aggregate Evidence and Verdict

Taken together, the evidence is quite negative

- Perhaps a significant amount of consumers are liquidity constrained?
- Perhaps lack of forward looking behavior

But, it builds on many restrictive assumptions:

- Quadratic utility - people act as if there was no uncertainty (certainty equivalence)
- No choice of hours worked, no impact of house worked on marginal utility of consumption
- Exogenous and constant interest rates
- No taste shocks

Smoothing over the Life-Cycle

The main idea here is to use consumption theory to investigate individual households' smoothing of consumption over time - i.e. over the life-cycle

- household data can be more informative than aggregate data
  - much more powerful data available
  - can take into account observable differences between households
  - can attempt at evaluating the relevance of liquidity constraints
- Household consumption models have become very important tools for macroeconomists and for policy makers

Smoothing of Household's over Time

Smoothing for a household implies that marginal utility growth of household consumption should be related to the real interest rate
Consumption itself may not necessarily be smooth:

- constant consumption expenditure requires quadratic preferences and no changes in taste
- If prices vary, consumption expenditure may vary substantially
- Households may have periods with high marginal utility (for example when you have kids)
- There may be times of the year when marginal utility is high - summer and Xmas

So we could think of

- Smoothing within the year
- Smoothing over time (business cycles)
- Smoothing over the life-cycle (the working-retirement cycle)

Smoothing within the Year

- It wouldn't be very interesting to show that consumption expenditure is higher at Xmas or around summer holidays, but it would be interesting to examine whether household consumption is smoothed in response to predictable changes in income
- Thus, the idea in part of this literature is to look for instances where income changes are very predictable and then test whether consumption is smoothed

1. Souleles (1999), Parker (1999): Most US tax-payers pay their social security contributions during Jan-August, thus after tax income tends to be higher during the last months of the year

- consumption tends also to be higher during the last months of the year
- those that increase consumption during these months do not seem to be liquidity constrained

Smoothing within the Year

2. Hsieh (2000): Reproduces the above result for Alaska but also shows that households do not overreact to payments from the Alaskan Permanent Fund that are pre-announced and paid in October
3. Browning and Collado (2001): Investigates household consumption in Spain - Many Spanish workers receive double salary in June and December

- Consumption profiles of bonus and non-bonus receivers are indistinguishable

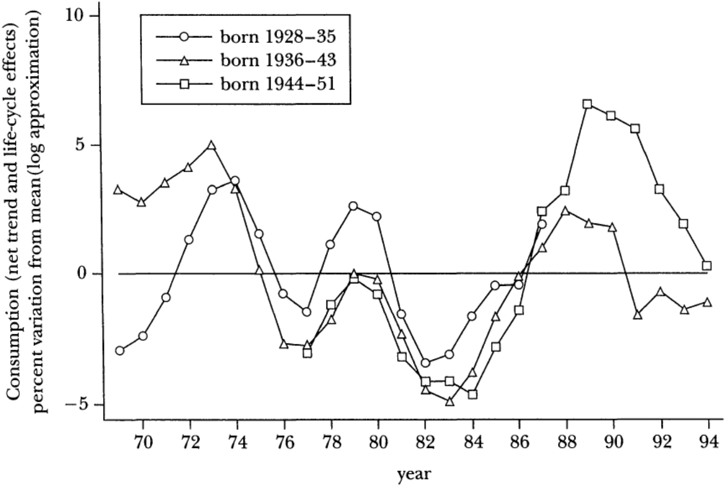
Smoothing within the Year

How can the evidence above possibly be right? Is it not contradictory?

- Possibly due to differences in the size of the income changes
- The extra payments in Spain are quite big (you get 1/14 of annual salary in "normal" months, 2/14 in June and December) - this corresponds to a 7 percent change in June and September - implies that it is worthwhile to spend the income wisely
- The payments from the Alaskan Permanent Fund are also big - the average October payment is $1648 corresponds to little less than 3 weeks salary
- The change in after tax income due to social security payments are instead very small

Smoothing Over the Business Cycle

Business-Cycle Patterns of Consumption



Smoothing over the Business Cycle

Consumption varies over the business cycle

- But there are also unexpected changes in income
- But still evidence on smoothing in response to "shocks"

Income and Expenditure After Unemployment in Canada

	Income Change Percentile			
	10th	25th	50th	75th
Income Change	-$1500	-$800	-$400	0
Expenditure Change	-$700	-$300	0	$25

- Expenditure is smoothed relative to income change

Smoothing over the Life-Cycle

Life-Cycle Patterns of Income and Consumption

Smoothing over the Life-Cycle

- Both income and consumption are bell-shaped
  - income is rising during early parts of the work life
  - income flattens out during early 50's and then starts falling
- But, shouldn't consumption be smoothed?

1. Labor supply also rises along with wages - thus, if consumption and work are complements, consumption will to some extent track income
2. Children follow bell shape as well: This affects the total household size and therefore consumption
3. Liquidity constraints, especially during early part of life cycle (hard to borrow on the promise of later professional success - although sometimes possible)
4. Precautionary motives: There are substantial risks for individuals - therefore you might want to be cautious in setting initial consumption too high

Labor Supply

Another important aspect of the economy is the labor market

- While "raw" labor does not contribute much to long run growth, fluctuations in output at short and medium term frequencies are mainly due to fluctuations in the labor input
- In the models we have looked at so far, labor supply was exogenous
- So now we will consider the consequences of introducing endogenous labor supply
- What determines a household's labor supply?

Labor supply

Labor supply enters the household's optimization problem in two distinct ways:

- hours worked affect utility - if that was not the case, everyone should be willing to work 24/7
- working generates income
- Preferences - two ways of doing this
- **Utility of leisure**:

$$u_{it} = u\left(c_{it}, l_{it}\right) = u\left(c_{it}, T - h_{it}\right), \ \frac{\partial u}{\partial l} > 0, \ \frac{\partial^2 u}{\partial l^2} \leq 0$$

  where l_{it} denotes the household's consumption of leisure, T is the time-endowment, and h_{it} denotes hours worked

- **Disutility of working**:

$$u_{it} = u\left(c_{it}, h_{it}\right), \ \frac{\partial u}{\partial h} < 0, \ \frac{\partial^2 u}{\partial h^2} \leq 0$$

Labor Supply

The household's optimization with labor supply is:

$$
\begin{aligned}
& \max \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, h_t\right) \\
a_{t+1} &= \left(1+r_t\right) a_t + w_t h_t - c_t, \; t \geq 0
\end{aligned}
$$

with first-order conditions:

$$
\begin{aligned}
c_t &: \quad u_c\left(c_t, h_t\right) = \lambda_{c,t} \\
h_t &: \quad -u_h\left(c_t, h_t\right) = w_t \lambda_{c,t} \\
a_{t+1} &: \quad \lambda_{c,t} = \beta \lambda_{c,t+1}\left(1+r_{t+1}\right)
\end{aligned}
$$

which imply

$$
\begin{aligned}
-\frac{u_h\left(c_t, h_t\right)}{u_c\left(c_t, h_t\right)} &= w_t \\
u_h\left(c_t, h_t\right) &= \beta u_h\left(c_{t+1}, h_{t+1}\right)\left(1+r_{t+1}\right)\frac{w_t}{w_{t+1}} \\
u_c\left(c_t, h_t\right) &= \beta u_c\left(c_{t+1}, h_{t+1}\right)\left(1+r_{t+1}\right)
\end{aligned}
$$

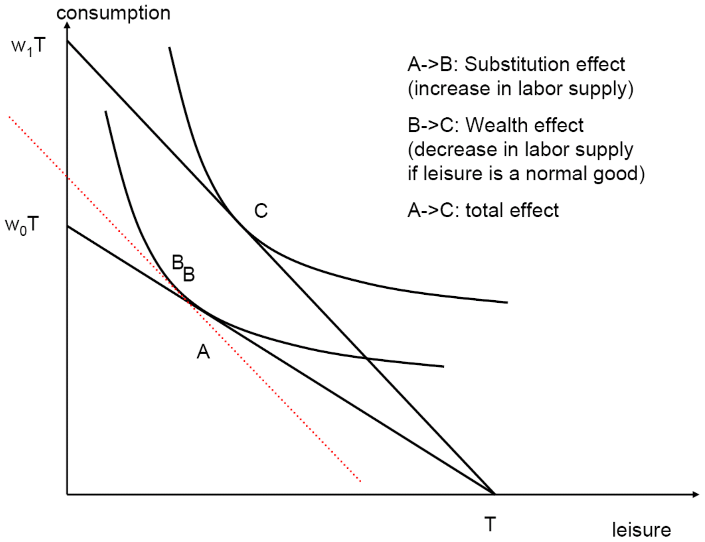
Labor Supply

Thus, in the optimum the household will set the marginal rate of substitution between consumption and work equal to the real wage, and the intertemporal marginal rate of substitution between work today and tomorrow equal to the inverse of wage growth in present value terms (note that $(1 + r_{t+1}) \frac{w_t}{w_{t+1}} = \frac{w_t}{w_{t+1}/(1+r_{t+1})}$)

How will an increase in the real wage affect labor supply?

- **Substitution effect**: An increase in wage makes leisure more expensive to the agent will work harder
- **Wealth effect**: Higher wage means - for unchanged labor supply - higher income. If consumption and leisure are both normal goods, labor supply must fall
- Thus, the overall impact depends on the relative strength of substitution and wealth effects

Labor Supply - Graphically



A->B: Substitution effect
(increase in labor supply)

B->C: Wealth effect
(decrease in labor supply
if leisure is a normal good)

A->C: total effect

Labor Supply

The Frisch elasticity: An important determinant for the behavior of labor supply is the Frisch labor supply elasticity which is defined as the elasticity of labor supply for a constant level marginal utility of wealth. This is the labor supply elasticity that enters the first-order condition for labor supply:

$$
\begin{aligned}
-u_h(c_t, h_t) &= \lambda_{c,t} w_t \\
\zeta^h &= \frac{dh_t / h_t}{dw_t / w_t}\Big|_{\lambda_{c,t}} = \frac{u_h(c_t, h_t)}{h_t u_{hh}(c_t, h_t)}
\end{aligned}
$$

- This parameter determines, for given wealth, the elasticity of the labor supply response to changes in wages and is a key parameter in many macroeconomic theories
- Unfortunately, macroeconomists and microeconomists disagree fundamentally on the appropriate value of this parameter
  - macroeconomists: This elasticity is high (perhaps even infinite)
  - microeconomists: This elasticity is low

Why this disagreement?

- **macroeconomists** find that to account for size of fluctuations in aggregate per capita hours worked, the elasticity must be large. They therefore think about the combined impact of:
  - *the intensive margin*: Hours per worker changes
  - *the extensive margin*: Changes in number of households that work
- **microeconomists** when estimating individual labor supply responses find small elasticities. The extensive margin (mainly for females) also appears inelastic.
- Who is right? Don't know ... but in macroeconomic models perhaps the high elasticity is more relevant

Some Parametric Examples

$$
\begin{aligned}
u^1(c_t, h_t) &= \frac{\left(c_t^{\theta}(T-h_t)^{1-\theta}\right)^{1-\sigma} - 1}{1-\sigma}, \;\; \sigma, \theta \geq 0 \\
u^2(c_t, h_t) &= \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \frac{\phi}{1+\kappa} h_t^{1+\kappa}, \;\; \sigma, \phi, \kappa \geq 0 \\
u^3(c_t, h_t) &= \frac{\left(c_t - \frac{\phi}{1+\kappa} h_t^{1+\kappa}\right)^{1-\sigma}}{1-\sigma} \;\; \sigma, \phi, \kappa \geq 0
\end{aligned}
$$

For these three specifications, the first-order condition becomes:

$$
\begin{aligned}
u^1 &: \;\; (1-\theta)\, c_t^{\theta(1-\sigma)} (T-h_t)^{(1-\theta)(1-\sigma)-1} = \lambda_{c,t} w_t \\
u^2 &: \;\; \phi h_t^{\kappa} = \lambda_{c,t} w_t \\
u^3 &: \;\; \phi h_t^{\kappa} \left(c_t - \frac{\phi}{1+\kappa} h_t^{1+\kappa}\right)^{-\sigma} = \lambda_{c,t} w_t
\end{aligned}
$$

Thus, the Frisch elasticities for these three examples, evaluated at the steady-state are given as:

$$\zeta_1^h = \frac{1}{1-(1-\theta)(1-\sigma)}\frac{T-h}{h}$$

$$\zeta_2^h = \frac{1}{\kappa}$$

$$\zeta_3^h = \frac{1}{\kappa+\sigma\phi h^{1+\kappa}/\left(c-\frac{\phi}{1+\kappa}h^{1+\kappa}\right)}$$

Some Parametric Examples

When combining the first-order conditions for consumption and labor, we get for the three examples that:

$$\begin{aligned} u^1 &: \frac{1-\theta}{\theta}\frac{c_t}{T-h_t} = w_t \\ u^2 &: \frac{\phi h_t^{\kappa}}{c_t^{-\sigma}} = w_t \\ u^3 &: \phi h_t^{\kappa} = w_t \end{aligned}$$

- for u^1: The ratio is spending on consumption and leisure is constant - this is because of the Cobb-Douglas specification. The Frisch elasticity depends on (a) the spending share θ, (b) the curvature σ, and the steady-state level of hours worked
- for u^2: Nice simple expression for the Frisch elasticity. The spending share is in general not constant
- for u^3: Complicated Frisch elasticity, but the implied labor supply does not depend on wealth - only real wages matter. In other words there is no wealth effect on labor supply

A Long Run Perspective

Which of these specifications are consistent with balanced growth? Suppose that firms produce output with Cobb-Douglas production functions and that there is Harrod Neutral technological progress. In this case, the first-order condition for labor demand is:

$$\begin{aligned} w_t &= (1-\alpha) A_t K_t^{\alpha} (A_t N_t)^{-\alpha} \\ &= (1-\alpha) A_t \left(\frac{K_t}{A_t N_t} \right)^{\alpha} \end{aligned}$$

- In the last lecture we showed that $\frac{K_t}{A_t N_t}$ is constant along the balanced growth path
- Therefore real wages must be growing at the same rate as A_t
- Total hours worked per capita do not display a consistent upward or downward trend. To a first approximation, hours worked per capita is trendless

A Long Run Perspective

How can we have trendless hours per capita by growing real wages?

- This is inconsistent with u^3 since a growth in real wages necessarily will have to be associated with an increase in hours worked.
- It is consistent with u^1 since c_t will also grow at the same rate as A_t
- It is consistent with u^2 under the restriction that $\sigma = 1$

What is the implication?

- The wealth and substitution effects of **permanent** changes in wages cancel out under these restrictions
- Thus, in all the relevant cases, since the wealth effect of temporary changes must be smaller than the wealth effect of permanent changes while the substitution effect is the same, temporary wage changes must be associated with a non-negative impact on labor supply
- This is an important example that will become important for us

The Competitive Equilibrium

With labor supply we can formulate the general equilibrium model as:

- There is a large number of identical households that rent out capital and labor to the firms
- There is a large number of identical firms that rent capital and labor from the households
- A competitive equilibrium consists of allocations $(c_t, k_{t+1}, h_t, i_t)_{t=0}^{\infty}$ and prices $(r_t, w_t)_{t=0}^{\infty}$ such that (i) households maximize their utility subject to their budget constraint taking all prices for given, (ii) firms maximize profits taking all prices for given, and (iii) input and output markets clear

The Competitive Equilibrium

Since firms are all identical and households are all identical, we can examine the optimization problem of the representative firm and the representative household.
The representative household's problem is:

$$\max \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, h_t\right)$$

$$\begin{aligned} c_t + i_t &= r_t k_t + w_t h_t + \pi_t,\ t \geq 0 \\ k_{t+1} &= \left(1-\delta\right) k_t + i_t,\ \ t \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

plus the no-Ponzi game restriction:

$$\lim_{T\to\infty} \frac{k_T}{\Pi_{t=1}^{T}\left(1+r_{t+1}\right)} = 0$$

The Household's problem

The first-order conditions are:

$$\begin{aligned} c_t &: \quad u_c\left(c_t, h_t\right) = \lambda_{c,t} \\ h_t &: \quad -u_h\left(c_t, h_t\right) = w_t \lambda_{c,t} \\ k_{t+1} &: \quad \lambda_{c,t} = \beta \lambda_{c,t+1} r_{t+1} + \beta \lambda_{k,t+1}\left(1-\delta\right) \\ i_t &: \quad \lambda_{c,t} = \lambda_{k,t} \end{aligned}$$

where $\lambda_{c,t}$ is the (discounted) multiplier on the budget constraint in period t and $\lambda_{k,t}$ is the (discounted) multiplier on the capital accumulation equation in period t.

- Since there are no adjustment costs or any other frictions, the price of capital equals the price of investment, and therefore the first order condition for investment implies that the two multipliers are the same

The Household's problem

Combining these equations we get that we can summarize the household's optimality conditions as:

$$
\begin{aligned}
-\frac{u_h\left(c_t, h_t\right)}{u_c\left(c_t, h_t\right)} &= w_t \\
u_h\left(c_t, h_t\right) &= \beta u_h\left(c_{t+1}, h_{t+1}\right)\left(1 + r_{t+1} - \delta\right)\frac{w_t}{w_{t+1}} \\
u_c\left(c_t, h_t\right) &= \beta u_c\left(c_{t+1}, h_{t+1}\right)\left(1 + r_{t+1} - \delta\right)
\end{aligned}
$$

which set marginal rates of substitution equal to relative prices

The Firm's Problem

The representative firm's problem is:

$$\max \pi_t = f\left(k_t, h_t\right) - r_t k_t - w_t n_t$$

where we assume $f\left(k_t, h_t\right)$ displays constant returns to scale. For this reason, due to competitive behavior, equilibrium profits are zero.
The first-order conditions are:

$$\begin{aligned} f_k\left(k_t, h_t\right) &= r_t \\ f_h\left(k_t, h_t\right) &= w_t \end{aligned}$$

These conditions say that factor demand functions are simply given by marginal products.

The Competitive Equilibrium

The competitive equilibrium then is the allocation and price system such that (i) the household's first-order conditions and budget constraints are satisfied, (ii) the firm's first-order conditions are satisfied, and (iii) labor, capital, and goods markets clear.
As we have seen a few times, due to competitive behavior and lack of externalities etc., the allocation that solves the competitive equilibrium will correspond to the planning solution.

- We can see this by substituting the firm's first-order conditions into the household's optimality conditions and their budget constraint.
- Effectively this means that we clear labor and capital markets.
- Due to Walras' law this also leads to goods market clearing.

The Competitive Equilibrium

We get:

$$
\begin{aligned}
-\frac{u_h(c_t, h_t)}{u_c(c_t, h_t)} &= f_h(k_t, h_t) \\
u_h(c_t, h_t) &= \beta u_h(c_{t+1}, h_{t+1})(1 + f_k(k_{t+1}, h_{t+1}) - \delta) \frac{f_h(k_t, h_t)}{f_h(k_{t+1}, h_{t+1})} \\
u_c(c_t, h_t) &= \beta u_c(c_{t+1}, h_{t+1})(1 + f_k(k_{t+1}, h_{t+1}) - \delta)
\end{aligned}
$$

which set marginal rates of substitution equal to marginal rates of transformation. Moreover, substituting the factor demands into the household's budget constraint we get:

$$
\begin{aligned}
c_t + i_t &= r_t k_t + w_t h_t + \pi_t \\
&= f_k(k_t, h_t) k_t + f_h(k_t, h_t) h_t \\
&= y_t
\end{aligned}
$$

which is simply the resource constraint. This establishes equivalence between the two allocations.

How do we solve Dynamic General Equilibrium Models?

Above we described how to formulate the model. But how can we say more about the dynamics.
In general it is hard to write down explicit solutions and one therefore often needs to somehow approximate the model. This most often done by log-linearizations. But before turning to this, let me consider a special case which three assumptions:

1. Cobb-Douglas Production Function:
$$y_t = k_t^{\alpha} h_t^{1-\alpha}$$
2. Log-log utility function:
$$u(c_t, h_t) = \theta \log c_t + (1-\theta) \log (T - h_t)$$
3. Complete depreciation:
$$k_{t+1} = i_t$$

How do we solve Dynamic General Equilibrium Models?

I will now show three different ways of solving the model with the three assumptions above

- Consider first solving directly for the decision rules using the first-order conditions
- With the special assumptions, the first-order conditions are given as:

$$\begin{aligned}
\frac{(1-\theta)\,c_t}{\theta\,(T-h_t)} &= (1-\alpha)\,k_t^{\alpha}h_t^{-\alpha} = (1-\alpha)\,\frac{y_t}{h_t} \\
\frac{1}{c_t} &= \beta\frac{1}{c_{t+1}}\alpha k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha} = \beta\alpha\frac{y_{t+1}}{k_{t+1}}\frac{1}{c_{t+1}} \\
c_t + k_{t+1} &= k_t^{\alpha}h_t^{1-\alpha}
\end{aligned}$$

- Let me make an informed guess about the solution. I will then verify the guess. The guess:

$$\begin{aligned}
h_t &= \bar{h} \\
c_t &= \gamma y_t \\
k_{t+1} &= (1-\gamma)\,y_t
\end{aligned}$$

Method 1

- Inserting the guess in the two first first-order conditions:

$$
\begin{aligned}
\frac{(1-\theta)\,\gamma y_t}{\theta\left(T-\overline{h}\right)} &= (1-\alpha)\,\frac{y_t}{\overline{h}} \\
\frac{1}{\gamma y_t} &= \beta\alpha\frac{y_{t+1}}{(1-\gamma)\,y_t}\frac{1}{\gamma y_{t+1}} \\
&\Rightarrow \\
\frac{T-\overline{h}}{\overline{h}} &= \frac{1-\theta}{\theta}\frac{\gamma}{(1-\alpha)} \\
1-\gamma &= \beta\alpha
\end{aligned}
$$

Method 1

- These equations confirm the guess and we get that:

$$
\begin{aligned}
h_t &= \overline{h} = \frac{T}{1 + \frac{1-\theta}{\theta}\frac{\gamma}{(1-\alpha)}} \\
c_t &= (1 - \beta\alpha)\, y_t \\
k_{t+1} &= \beta\alpha y_t \\
y_t &= k_t^{\alpha}\overline{h}^{1-\alpha}
\end{aligned}
$$

- Thus, hours worked are constant - why? For this specification, substitution and wealth effects on labor supply cancel out not only in the long run but also in the short run

Method 1

- Notice, that we can also represent the solution in a "recursive" form as a function of the capital stock only. Taking logs, we see:

$$
\begin{aligned}
\log c_t &= \mu_c + \alpha \log k_t \\
\log k_{t+1} &= \mu_k + \alpha \log k_t \\
\log y_t &= \mu_y + \alpha \log k_t \\
\mu_c &= \log(1 - \beta\alpha) + (1 - \alpha)\log \overline{h} \\
\mu_k &= \log \beta\alpha + (1 - \alpha)\log \overline{h} \\
\mu_y &= (1 - \alpha)\log \overline{h}
\end{aligned}
$$

Method 2

Method 2, uses the insight from above. Here we will

1. log-linearize the first-order conditions around the steady-state
2. Make a guess on the solutions
3. Verify the guess

- The system of first-order conditions are:

$$\begin{aligned} \frac{(1-\theta)\,c_t}{\theta\,(T-h_t)} &= (1-\alpha)\,k_t^{\alpha} h_t^{-\alpha} \\ \frac{1}{c_t} &= \beta \frac{1}{c_{t+1}} \alpha k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} \\ c_t + k_{t+1} &= k_t^{\alpha} h_t^{1-\alpha} \end{aligned}$$

Method 2

- We can log-linearize (take logs, totally differentiate and evaluate at the steady-state) to get:

$$
\begin{aligned}
\widehat{c}_t + \frac{\overline{h}}{T-\overline{h}}\widehat{h}_t &= \alpha\widehat{k}_t - \alpha\widehat{h}_t \\
-\widehat{c}_t &= -\widehat{c}_{t+1} + (\alpha-1)\,\widehat{k}_{t+1} + (1-\alpha)\,\widehat{h}_{t+1} \\
\frac{\overline{c}}{\overline{y}}c_t + \frac{\overline{k}}{\overline{y}}k_{t+1} &= \alpha\widehat{k}_{t+1} + (1-\alpha)\,\widehat{h}_{t+1}
\end{aligned}
$$

where $\widehat{x}_t = \log\left(\frac{x_t}{\overline{x}}\right) = \frac{dx_t}{\overline{x}}$

- Let me make the following guess on the solution:

$$
\begin{aligned}
\widehat{c}_t &= \phi_c\widehat{k}_t \\
\widehat{h}_t &= \phi_h\widehat{k}_t \\
\widehat{k}_{t+1} &= \phi_k\widehat{k}_t
\end{aligned}
$$

Method 2

- Let's insert the guess:

$$\begin{aligned}
\phi_c \widehat{k}_t + \frac{\overline{h}}{T - \overline{h}} \phi_h \widehat{k}_t &= \alpha \widehat{k}_t - \alpha \phi_h \widehat{k}_t \\
-\phi_c \widehat{k}_t &= -\phi_c \widehat{k}_{t+1} + (\alpha - 1)\, \phi_k \widehat{k}_t + (1 - \alpha)\, \phi_h \widehat{k}_t \\
&= -\phi_c \phi_k \widehat{k}_t + (\alpha - 1)\, \phi_k \widehat{k}_t + (1 - \alpha)\, \phi_h \widehat{k}_t \\
\frac{\overline{c}}{\overline{y}} \phi_c \widehat{k}_t + \frac{\overline{k}}{\overline{y}} \phi_k \widehat{k}_t &= \alpha \widehat{k}_t + (1 - \alpha)\, \phi_h \widehat{k}_t
\end{aligned}$$

- It follows immediately from these (since $\frac{\overline{c}}{\overline{y}} + \frac{\overline{k}}{\overline{y}} = 1$) that the solution is:

$$\begin{aligned}
\phi_h &= 0 \\
\phi_k &= \phi_c = \alpha
\end{aligned}$$

Method 2

Therefore the solution is that:

$$\begin{aligned}\log h_t &= \log \overline{h} \\ \log c_t &= \left(\log \overline{c} - \alpha \log \overline{k}\right) + \alpha \log k_t \\ \log k_{t+1} &= (1-\alpha) \log \overline{k} + \alpha \log k_t\end{aligned}$$

which are identical to those that we derived exactly. But, this is a special case though.

Method 3: Dynamic Programming

The third way of solving the model uses dynamic programming
Let me write down Bellman's equation for the social planner's problem:

$$\begin{aligned} V(k_t) &= \max_{c_t, k_{t+1}, h_t} \left(\theta \log c_t + (1-\theta)\log(T - h_t) + \beta V(k_{t+1})\right) \\ c_t + k_{t+1} &= k_t^{\alpha} h_t^{1-\alpha} \end{aligned}$$

I can substitute the resource constraint into the Bellman equation to get:

$$V(k_t) = \max_{k_{t+1}, h_t} \left(\theta \log\left(k_t^{\alpha} h_t^{1-\alpha} - k_{t+1}\right) + (1-\theta)\log(T - h_t) + \beta V(k_{t+1})\right)$$

Method 3: Dynamic Programming

I could think about writing the Bellman equation as:

$$
\begin{aligned}
& V_{i+1}\left(k_t\right) \\
= \quad & \max_{k_{t+1},h_t} \left(\theta \log\left(k_t^{\alpha} h_t^{1-\alpha} - k_{t+1}\right) + (1-\theta)\log\left(T - h_t\right) + \beta V_i\left(k_{t+1}\right)\right)
\end{aligned}
$$

I could then iterate on the Bellman equation as follows:

1. Make a guess on V_0
2. Given the guess solve the maximization problem.
3. Find V_1
4. Return to step 1 unless $V_1 = V_0$

Method 3: Dynamic Programming

Let me make the following guess on V_0 :

$$V_0 = a + b \log k_t$$

where a and b are constants that we wish to find. I could have made all sorts of other guess, this is just a good one ...

Given the guess, Bellman's equation is:

$$\begin{aligned} V_1(k_t) &= \max_{k_{t+1}, h_t} \left(\theta \log\left(k_t^\alpha h_t^{1-\alpha} - k_{t+1}\right)\right. \\ &\quad \left. + (1-\theta)\log(T - h_t) + \beta(a + b \log k_{t+1})\right) \end{aligned}$$

The first-order conditions are:

$$\begin{aligned} k_{t+1} &: \theta \frac{1}{c_t} = \beta b \frac{1}{k_{t+1}} \\ h_t &: \theta \frac{(1-\alpha) k_t^\alpha h_t^{-\alpha}}{c_t} = (1-\theta)\frac{1}{T - h_t} \end{aligned}$$

Method 3: Dynamic Programming

From the first one of these we get that:

$$
\begin{aligned}
k_{t+1} &= \frac{\beta b}{\theta} c_t = \frac{\beta b}{\theta}\left(k_t^{\alpha} h_t^{1-\alpha} - k_{t+1}\right) \\
&\Rightarrow \\
k_{t+1} &= \frac{1}{1+\frac{\beta b}{\theta}} k_t^{\alpha} h_t^{1-\alpha} = \frac{\frac{\beta b}{\theta}}{1+\frac{\beta b}{\theta}} y_t
\end{aligned}
$$

From the second one we get that:

$$
\begin{aligned}
T - h_t &= \frac{(1-\theta)}{\theta(1-\alpha)} \frac{c_t}{y_t} h_t \\
\frac{c_t}{y_t} &= 1 - \frac{k_{t+1}}{y_t} = \frac{1}{1+\frac{\beta b}{\theta}} \Rightarrow \\
h_t &= \bar{h} = \frac{T}{1 + \frac{(1-\theta)}{\theta(1-\alpha)} \frac{1}{1+\frac{\beta b}{\theta}}}
\end{aligned}
$$

To summarize:

$$
\begin{aligned}
h_t &= \bar{h} = \frac{T}{1 + \frac{(1-\theta)}{\theta(1-\alpha)} \frac{1}{1+\frac{\beta b}{\theta}}} \\
k_{t+1} &= \frac{\frac{\beta b}{\theta}}{1 + \frac{\beta b}{\theta}} y_t \\
c_t &= \frac{1}{1 + \frac{\beta b}{\theta}} y_t
\end{aligned}
$$

Now let us insert into Bellman's equation:

$$
\begin{aligned}
a + b \log k_t \quad = \quad & \theta \log \frac{\frac{\beta b}{\theta}}{1 + \frac{\beta b}{\theta}} + \theta\alpha \log k_t + \theta (1 - \alpha) \log \overline{h} + \\
& (1 - \theta) \log \left(T - \overline{h} \right) + \beta (a + b \log \frac{1}{1 + \frac{\beta b}{\theta}} \\
& + b\alpha \log k_t + b\theta (1 - \alpha) \log \overline{h})
\end{aligned}
$$

If we equate the left hand and right hand side coefficients on the terms involving the capital stock we get:

$$b = \theta\alpha + \beta b\alpha \Rightarrow b = \frac{\theta\alpha}{(1 - \beta\alpha)}$$

and all the other terms just involves constants (which we can solve for a. We have therefore verified the guess in one step. We can also substitute the solution for b into the decision rule above to get that:

$$\begin{aligned} k_{t+1} &= \frac{\frac{\beta b}{\theta}}{1 + \frac{\beta b}{\theta}} y_t = \frac{\beta\alpha / (1 - \beta\alpha)}{1 + \beta\alpha / (1 - \beta\alpha)} y_t \\ &= \beta\alpha y_t \\ c_t &= (1 - \beta\alpha) y_t \end{aligned}$$

which exactly the same as we got using the other two methods

Solving Models

Three methods were proposed:

1. Solving directly for decision rules
2. Log-linearizing and solving for decision rules
3. Guess the Bellman equation and iterate

- The first method is not generally applicable and requires luck.
- The second method will work but relies upon an approximation. But, if we are willing to accept the log-linearization, it is a very powerful way to solve models that can be used even in very complicated settings. Most macroeconomists uses methods like this.
- The third method is powerful - it should always work. But it can be slow, and it's not for sure that you can solve for an explicit function. But the computer can solve it if there is a solution.