MSC Macroeconomics G022, 2009

Lecture 3: Growth and Development

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In this lecture

- Growth in the Solow Model
  - Catching up
  - Growth with technological progress
  - Convergence - theory and empirics
- Growth in the Ramsey Model
- Endogenous growth theories
- Growth and Levels Accounting

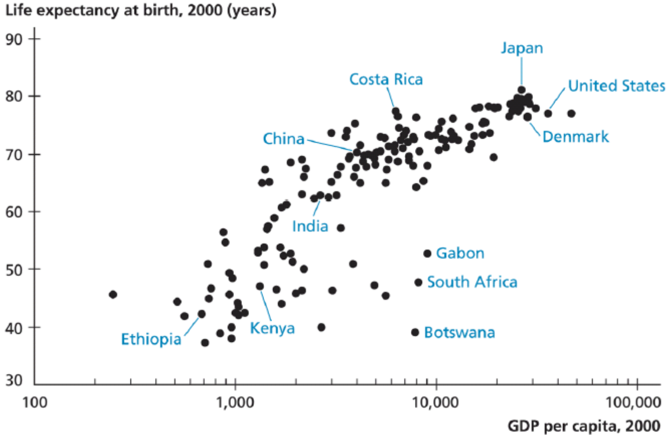
Economic Development

Understanding economic development and long-run growth is perhaps the most important issue in macroeconomics

- Differences in levels of income across countries are enormous
  - Implies large differences in the quality of life
  - Implies large differences in the outlook for children born in different countries
  - Differences in income also to health and life-expectancy implying even larger differences in welfare than just those directly reflected by income differences
- Differences in growth rates add up over time
  - can lead to enormous differences in the level of income over time

Some Graphics

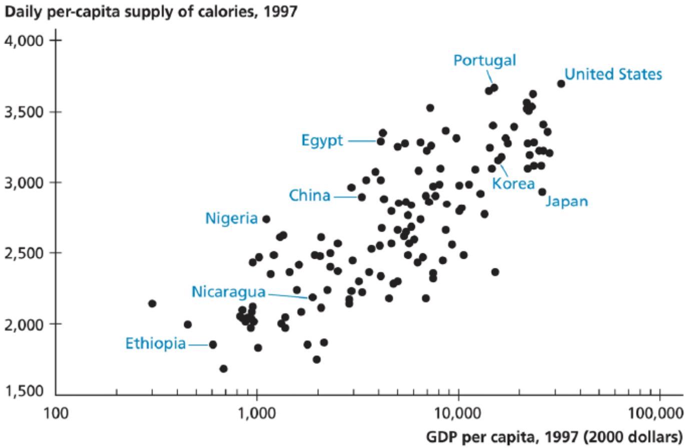
FIGURE 6.2
Life Expectancy Versus GDP per Capita



Source: Heston et al. (2002), World Bank (2003b).

Some Graphics

Figure 6.1
Nutrition Versus GDP per Capita



Source: FAOSTAT database, Heston et al. (2002).

Development and Welfare

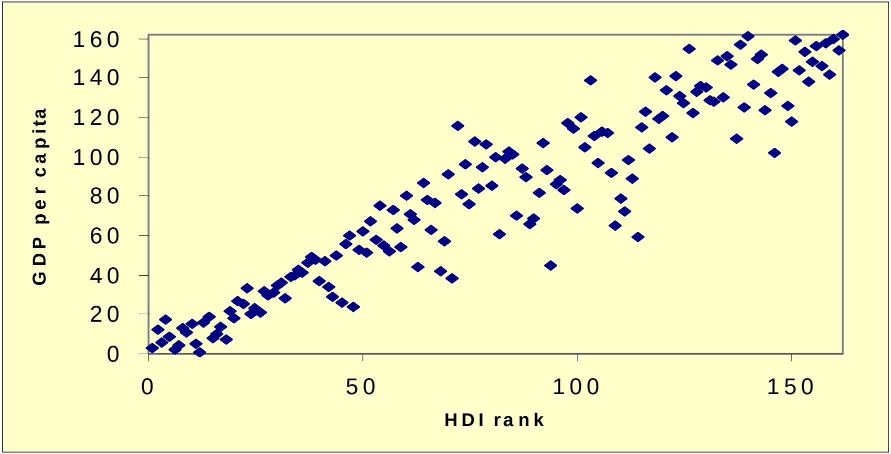
- We will think about economic welfare in terms of preferences that depend on consumption and possibly leisure - and often we will approximate this by real income
- Some think that this is wrong: "Gross National Happiness is more important than Gross National Product" (Bhutan's King Jigme Singye Wangchuk)
- United Nations therefore now produce a Human Development Index in their annual Human Development Report:

$$\begin{aligned}
\text{HDI} &= \frac{L+S+I}{3} \in (0,1) \\
\text{L} &= \frac{\text{life expectancy-25}}{85-25} \\
\text{S} &= \frac{2}{3}\text{Literacy rate} + \frac{1}{3}\text{school enrollment rate} \\
\text{I} &= \log \quad (\text{GDP per capita}) - \frac{\log 100}{\log 40000 - \log 100}
\end{aligned}$$

HDI 2009 (2006 Data)

Rank	Country	HDI	L	S	I
1.	Iceland	0.968	0.94	0.98	0.98
2.	Norway	0.968	0.92	0.99	1.00
3.	Canada	0.967	0.92	0.99	0.99
4.	Australia	0.965	0.93	0.99	0.97
76.	Turkey	0.798	0.78	0.82	0.79
94.	China	0.762	0.80	0.85	0.64
176.	Liberia	0.364	0.34	0.56	0.17
177.	Congo	0.361	0.35	0.56	0.17
178.	C.A.R.	0.352	0.32	0.42	0.32
179.	Sierra Leone	0.329	0.29	0.40	0.30

HDI vs. GDP



The key questions that we wish to address are:

1. What determines the level of income?
2. What determines the growth rate of the economy?

Kaldor's Stylized Facts of Growth

Kaldor, 1961, summarized a number of empirical regularities about long-run growth that since then have become key ingredients of many theories of economic development:

1. GDP per worker grows at a constant rate over time
2. Capital per worker grows at a constant rate
3. The Capital-Output ratio is constant
4. The return on capital is constant
5. The labor share of GDP is constant
6. There are large differences in growth rates across countries

Economic Development: The Solow Model Again

We formulated the Solow Model as:

$$\begin{aligned} Y_t &= F(K_t, N_t) \\ K_{t+1} &= (1-\delta) K_t + I_t \\ N_{t+1} &= (1+n) N_t \\ S_t &= sY_t \\ I_t &= S_t \end{aligned}$$

To make this consistent with Kaldor facts no. 1, 4-5 we must introduce restrictions on the production function

- Suppose output is produced by competitive firms that maximize profits:

$$\max Y_t - w_t N_t - r_t K_t$$

then the first-order conditions are:

$$\begin{aligned} F_K &= r \\ F_N &= w \end{aligned}$$

Technology

The factor income shares are then given as:

$$
\begin{aligned}
s_K &= \frac{rK}{Y} = \frac{F_K K}{Y} \\
s_N &= \frac{wN}{Y} = \frac{F_N N}{Y}
\end{aligned}
$$

When are these constant? Consider a CES specification:

$$
Y = \left(\alpha K^{1-1/\zeta} + (1-\alpha) N^{1-1/\zeta}\right)^{1/(1-1/\zeta)}
$$

which implies that:

$$
\begin{aligned}
s_K &= \frac{\alpha K^{1-1/\zeta}}{\alpha K^{1-1/\zeta} + (1-\alpha) N^{1-1/\zeta}} = \frac{\alpha}{\alpha + (1-\alpha)(K/N)^{1/\zeta-1}} \\
s_N &= \frac{1-\alpha}{\alpha (K/N)^{1-1/\zeta} + (1-\alpha)}
\end{aligned}
$$

Technology

If K/N changes over time, these expressions can only be constant if $1/\zeta = 1$, i.e. we need a Cobb-Douglas production function:

$$\begin{aligned} Y &= K^{\alpha} N^{1-\alpha} \\ s_K &= \alpha \\ s_N &= 1-\alpha \end{aligned}$$

- With this assumption, the model becomes

$$\begin{aligned} Y_t &= K_t^{\alpha} N_t^{1-\alpha} \\ K_{t+1} &= (1-\delta) K_t + I_t \\ N_{t+1} &= (1+n) N_t \\ S_t &= sY_t \\ I_t &= S_t \end{aligned}$$

which we can express as:

$$(1+n)(k_{t+1} - k_t) = sk_t^{\alpha} - (\delta + n) k_t$$

The Steady-State

The Solow model has a steady-state where $s\left(k^{ss}\right)^{\alpha-1} = (\delta + n)$ and the model has the following predictions:

1. Regardless of the initial condition, all countries converge to their steady-state as long as $k_0 > 0$
2. There is only transitional growth: The long run growth rate of per worker consumption, capital, investment and output is zero. But if $k_0 \neq 0$, the country will go through a period of transitional growth.
3. The lower is the initial capital stock per worker, the faster the country grows over the transitional path
4. There is conditional convergence: Countries that share the same steady-state should converge over time. This implies that amongst countries with identical steady-states, the transitional growth rate depends negatively on initial income.
5. The steady-state income level depends positively on the savings rate and negatively on the population growth rate and on the depreciation rate. But none of these affect the long-run growth rate

Technological Progress

Clearly, since we observe sustained growth in output per capita (and per worker) and other main macroeconomic aggregates, we need to consider sources of long-run growth

We will introduce this through growth in labor augmenting technology:

$$\begin{aligned} Y_t &= F\left(K_t, A_t N_t\right) \\ A_{t+1} &= \left(1+g\right) A_t \end{aligned}$$

- Growth in A_t represents growth in the efficiency with which we use the factors of production
- We can later think about what A really represents but for now we take it as manna from heaven
- Since A may be growing over time, increases in TFP can lead to sustained growth

The Solow Model with Technological Progress

Our model now becomes:

$$\begin{aligned} Y_t &= K_t^{\alpha} \left(A_t N_t\right)^{1-\alpha} \\ K_{t+1} &= (1-\delta) K_t + I_t \\ N_{t+1} &= (1+n) N_t \\ S_t &= sY_t \\ I_t &= S_t \end{aligned}$$

which implies:

$$\begin{aligned} K_{t+1} &= sK_t^{\alpha} \left(A_t N_t\right)^{1-\alpha} + (1-\delta) K_t \\ &\Rightarrow \\ \frac{K_{t+1}}{A_{t+1}N_{t+1}} \frac{A_{t+1}N_{t+1}}{A_t N_t} &= s\left(\frac{K}{A_t N_t}\right)^{\alpha} + (1-\delta)\frac{K}{A_t N_t} \end{aligned}$$

The Solow Model with Technological Progress

If we define $k_t^e = K_t / (N_t A_t)$ as the amount of capital per worker measured in efficiency units, we then get that:

$$
\begin{aligned}
k_{t+1}^e (1+g)(1+n) &= s (k_t^e)^\alpha + (1-\delta) k_t^e \\
&\Rightarrow \\
(1+g)(1+n)(k_{t+1}^e - k_t^e) &\simeq s (k_t^e)^\alpha - (\delta + n + g) k_t^e
\end{aligned}
$$

(with the approximation that $gn \approx 0$)

- This model has like the standard model with no technological progress a unique stable steady-state determined as:

$$
\begin{aligned}
\overline{k}^e &= \left(\frac{s}{\delta + n + g} \right)^{1/(1-\alpha)}, \ \overline{y}^e = \frac{Y}{AN} = \left(\overline{k}^e \right)^\alpha \\
\overline{i}^e &= \frac{I}{AN} = (\delta + n + g) \overline{k}^e, \ \overline{c}^e = \left(\overline{k}^e \right)^\alpha - (\delta + n + g) \overline{k}^e
\end{aligned}
$$

The Solow Model with Technological Progress

In this economy there exists a balanced growth path along which:

$$k^e, y^e, c^e, i^e \text{ are constant}$$

$$k, y, c, i \text{ grow at the rate of } g$$

The balanced growth path is consistent with Kaldor's growth facts 1-5:

1. GDP per worker grows at a constant rate over time - we derived this above
2. Capital per worker grows at a constant rate - we derived this above
3. The Capital-Output ratio is constant:

$$\frac{K}{Y} = \frac{\overline{k}^e}{\overline{y}^e} = \left(\frac{s}{\delta + n + g}\right)$$

4. The return on capital is constant:

$$r = F_K = \alpha \frac{Y}{K}$$

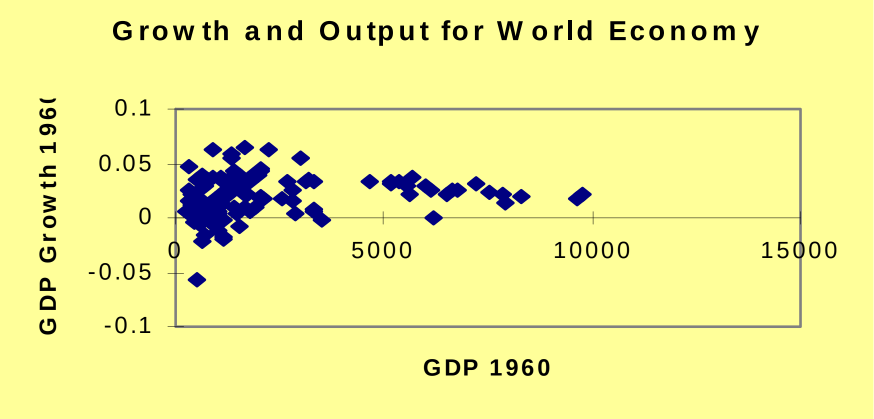
5. The labor share of GDP is constant because of Cobb-Douglas

The Solow Growth Model

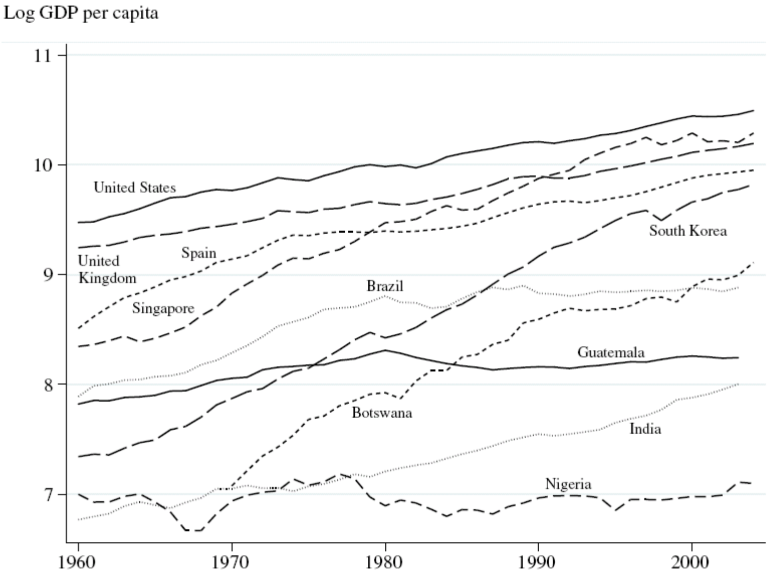
So let's check the predictions:

- Is there unconditional convergence?
- Is there conditional convergence?
- Does level of income depend on savings (or investment) rates but not the growth rates?

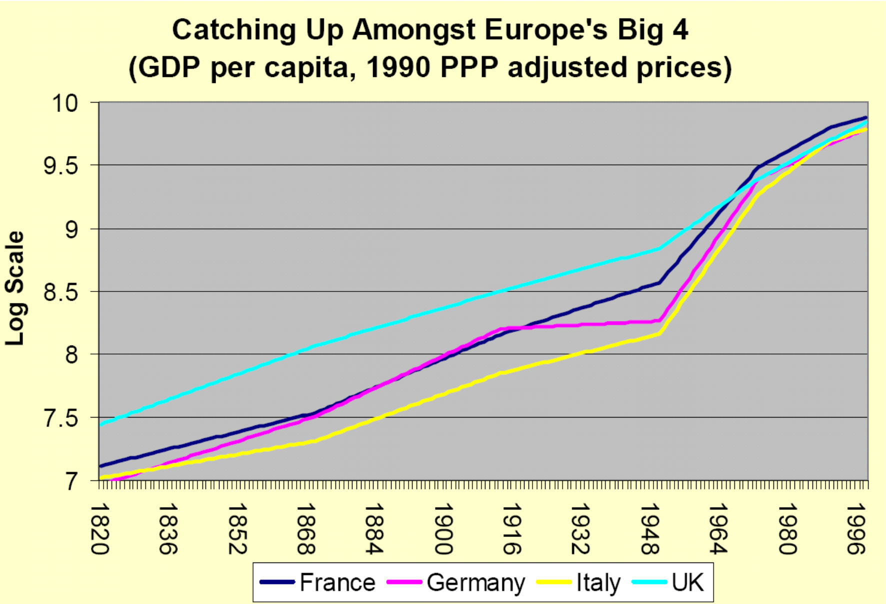
Unconditional Convergence?



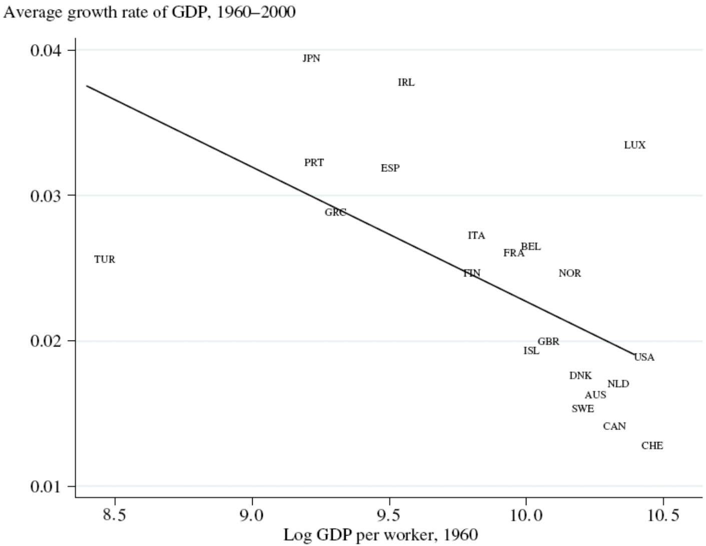
Unconditional Convergence? No



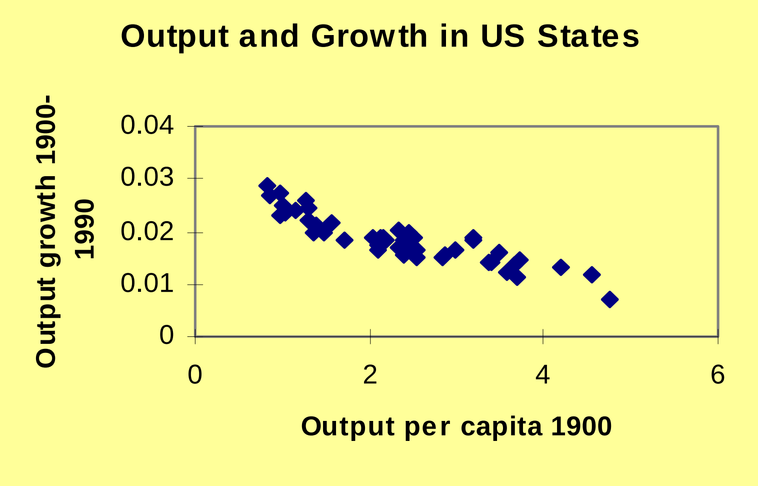
Conditional Convergence



Conditional Convergence



Conditional Convergence



The Speed of Convergence

A large literature has formally tested for conditional convergence using mostly cross-sectional tests. Two questions:

- Theoretically, how fast should countries converge to their steady-state?
- Empirically, how fast is (conditional convergence)?

Recall the expression:

$$
\begin{aligned}
(1+g)(1+n)\left(k_{t+1}^{e}-k_{t}^{e}\right) &= s\left(k_{t}^{e}\right)^{\alpha}-(\delta+n+g)\,k_{t}^{e} \\
&\Rightarrow \\
(1+g)(1+n)\,\gamma_{k^{e},t+1} &= s\left(k_{t}^{e}\right)^{\alpha-1}-(\delta+n+g) \\
\gamma_{k^{e},t+1} &= \frac{k_{t+1}^{e}-k_{t}^{e}}{k_{t}^{e}}
\end{aligned}
$$

The Speed of Convergence

We can log-linearize this last expression:

$$\gamma_{k^e,t+1} \simeq -(1-\alpha)(\delta+n+g)\log\left(\frac{k_t^e}{\overline{k}^e}\right)$$

(see appendix for derivation). This implies that the speed of convergence of the capital stock towards the balanced growth path is given as:

$$\beta = -(1-\alpha)(\delta+n+g)$$

This is also true for output:

$$\begin{aligned} y_t^e &= (k_t^e)^\alpha \\ &\Rightarrow \\ \gamma_{y^e,t+1} &= \alpha\gamma_{k^e,t+1} \text{ and } \log\left(\frac{y_t^e}{\overline{y}^e}\right) = \alpha\log\left(\frac{k_t^e}{\overline{k}^e}\right) \\ &\Rightarrow \\ \gamma_{y^e,t+1} &\simeq -(1-\alpha)(\delta+n+g)\log\left(\frac{y_t^e}{\overline{y}^e}\right) \end{aligned}$$

The Speed of Convergence

How big is β? Let one period be a year, then:

$$\begin{aligned}
\alpha &= \frac{rK}{Y} \in (0.25; 0.35) \\
\delta &\in (0.05; 0.15) \\
n &\simeq 0.01 \\
g &\simeq 0.02 \\
\beta &\in (0.052; 0.135)
\end{aligned}$$

It follows from this that convergence occurs quite fast - it takes between 5 and 12 years to eliminate 50 percent of any initial difference in income (the half-time can be computed as $T_{1/2} = \ln\left(\frac{1}{2}\right) / \ln(1-\beta)$

Convergence - Empirics

Barro and Sala-i-Martin and many others have investigated these issues empirically from cross-sectional regressions:

$$\frac{1}{T}\log\left(\frac{y_{i,T}}{y_{i,0}}\right) = a - b\log y_{i,0} + X_{i,0}^{T}\gamma + u_{i0,t}$$

where $X_{i,0}^{T}$ are other factors. b related to β as:

$$b = \frac{1}{T}\left(1 - e^{-\beta T}\right)$$

They find estimates for US state growth rates, European regions, countries that imply values of $\beta \simeq 0.02$ giving half-lives of approximately 35 years

- this estimate of β is much lower than the back of the envelope calculations above suggested
- this estimate would require a capital income share of close to 70 percent which seems counterfactual
- So - there is conditional convergence but it is slow

Why Labor Augmenting Technological Progress?

I assumed earlier on that technological progress was labor augmenting (Harrod neutral):

$$Y_t = F\left(K_t, A_t N_t\right)$$

Could I instead have assumed capital augmenting or Hicks neutral technological progress? In this case, the production function would have been:

$$Y_t = F\left(B_t K_t, A_t N_t\right)$$

Capital augmenting is the case when B grows but A is constant, Hicks neutral is when $A_t = B_t$. Suppose that:

$$\begin{aligned} B_{t+1} &= \left(1+\gamma_b\right) B_t \\ A_{t+1} &= \left(1+g\right) A_t \end{aligned}$$

Using that the production function is homogenous of degree 1, we can rewrite it as:

$$\frac{Y_t}{K_t} = B_t F\left(1, \frac{A_t}{B_t}\frac{N_t}{K_t}\right)$$

Why Labor Augmenting Technological Progress?

If there is a balanced growth path, then along this balanced growth path K_t must grow at a constant rate γ_K. Hence, along this path:

$$K_{t+1} = (1 + \gamma_K) K_t$$

Inserting in the above expression and normalizing $A_0 = B_0 = N_0 = K_0 = 1$ we can express it as:

$$\frac{Y}{K}|_{bg} = (1 + \gamma_b)^t F\left(1, \left[\frac{(1+g)}{(1+\gamma_b)}\frac{(1+n)}{(1+\gamma_K)}\right]^t\right) \tag{1}$$

Now recall the equation:

$$K_{t+1} = sY_t + (1 - \delta) K_t$$

which implies that along the balanced growth path, constancy of the capital stock growth rate requires constancy of the capital-output ratio:

$$\frac{K_{t+1}}{K_t}|_{bg} - 1 = s\frac{Y}{K}|_{bg} - \delta$$

Why Labor Augmenting Technological Progress?

Inspecting, equation (1), constancy os the capital-output ratio along the balanced growth path requires:

$$\begin{aligned}\text{either (a) } \gamma_b &= 0 \text{ and } \left(1+\gamma_K\right) = (1+g)\,(1+n) \\ \text{or (b) } \gamma_b &\neq 0 \text{ but } \left(1+\gamma_b\right)^t = 1/F\left(1, \left[\frac{(1+g)}{\left(1+\gamma_b\right)}\frac{(1+n)}{\left(1+\gamma_K\right)}\right]^t\right)\end{aligned}$$

Case (a) is the one that we looked at. When will case (b) hold? It will hold in the Cobb-Douglas case where the distinction between A and B is irrelevant

Productivity Growth in the Ramsey Model

- As we have seen above in the Solow model, long run growth is purely technologically driven
- So, in a sense, as we will see below, adding utility maximizing agents will not add too much in terms of insights
- I will make the following two assumptions:

$$\begin{aligned} Y_t &= K_t^{\alpha} \left(A_t N_t\right)^{1-\alpha} \\ u\left(c_t\right) &= c_t^{1-\sigma} / \left(1-\sigma\right) \end{aligned}$$

- The first of these maintains the Cobb-Douglas specification of the production function
- The second one assumes a utility function with constant intertemporal elasticity of substitution, $1/\sigma$
- Questions:
  - Is there a balanced growth path?
  - How does growth affect the economy?

Productivity Growth in the Ramsey Model

The model is summarized by the equations:

$$\begin{aligned} U_0 &= \sum_{t=0}^{\infty} \beta^t N_t \frac{c_t^{1-\sigma}}{1-\sigma} \\ y_t &= A_t \left(\frac{k_t}{A_t} \right)^{\alpha} \\ (1+n)\, k_{t+1} &= (1-\delta)\, k_t + i_t \\ y_t &= c_t + i_t \\ A_{t+1} &= (1+g)\, A_t \end{aligned}$$

where all variables are in per capita (per worker) terms

- N_t is included in objective in order to control for size of "dynasties"
- since A is growing over time, I will look for a steady-state in terms of efficiency units like in the Solow model

Productivity Growth in the Ramsey Model

Hence, I rewrite this model as:

$$\begin{aligned}
U_0 &= \sum_{t=0}^{\infty} (\beta(1+n))^t \frac{(A_t c_t^e)^{1-\sigma}}{1-\sigma} \\
y_t^e &= (k_t^e)^\alpha \\
(1+n)(1+g)k_{t+1}^e &= (1-\delta)k_t^e + i_t^e \\
y_t^e &= c_t^e + i_t^e \\
A_{t+1} &= (1+g)A_t \\
c_t^e &= \frac{c_t}{A_t},\ y_t^e = \frac{y_t}{A_t},\ i_t^e = \frac{i_t}{A_t},\ k_t^e = \frac{k_t}{A_t}
\end{aligned}$$

We see that A_t is present only in the first equation

- where I have normalized $N_0 = 1$ and used the approximation that $gn \simeq 0$

Productivity Growth in the Ramsey Model

But the first equation can be expressed as:

$$
\begin{aligned}
U_0 &= \sum_{t=0}^{\infty} (\beta (1+n))^t \frac{(A_t c_t^e)^{1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty} \beta (\beta (1+n))^t A_0^{1-\sigma} (1+g)^{(1-\sigma)t} \frac{(c_t^e)^{1-\sigma}}{1-\sigma} \\
&= A_0^{1-\sigma} \sum_{t=0}^{\infty} \left(\beta (1+n) (1+g)^{(1-\sigma)}\right)^t \frac{(c_t^e)^{1-\sigma}}{1-\sigma} \\
&\Rightarrow \\
U_0 &= A_0^{1-\sigma} \sum_{t=0}^{\infty} (\beta^* (1+n))^t \frac{(c_t^e)^{1-\sigma}}{1-\sigma} \\
\beta^* &= \beta (1+g)^{(1-\sigma)}
\end{aligned}
$$

Hence, we simply make a change in the definition of the discount factor

Productivity Growth in the Ramsey Model

Normalizing $A_0 = 1$ we can then express the Ramsey model with productivity growth as:

$$
\begin{aligned}
U_0 &= \sum_{t=0}^{\infty} \left(\beta^* (1+n)\right)^t \frac{\left(c_t^e\right)^{1-\sigma}}{1-\sigma} \\
y_t^e &= \left(k_t^e\right)^{\alpha} \\
(1+n)(1+g)\, k_{t+1}^e &= (1-\delta)\, k_t^e + i_t^e \\
y_t^e &= c_t^e + i_t^e
\end{aligned}
$$

with first-order conditions:

$$
\begin{aligned}
(1+n)(1+g)\left(c_t^e\right)^{-\sigma} &= \beta^* (1+n)\left(c_{t+1}^e\right)^{-\sigma}\left(\alpha\left(k_t^e\right)^{\alpha-1} + (1-\delta)\right) \\
(1+n)(1+g)\, k_{t+1}^e &= (1-\delta)\, k_t^e + \left(k_t^e\right)^{\alpha} - c_t^e
\end{aligned}
$$

Productivity Growth in the Ramsey Model

Going through the same arguments as in the last lecture, it follows that this model has a unique saddle path stable steady-state for (c^e, k^e) determined as:

$$
\begin{aligned}
(1+g) &= \beta^* \left(\alpha \left(\overline{k}^e \right)^{\alpha-1} + (1-\delta) \right) \\
\overline{c}^e &= \left(\overline{k}^e \right)^{\alpha} - (\delta + n + g)\, \overline{k}^e
\end{aligned}
$$

which is exactly like in the standard model apart from a small modification in the determination of the condition that determines the steady-state effective capital stock.

Productivity Growth in the Ramsey Model

- It then follows that there exists a balanced growth path in this model where per capita output, consumption, capital and investment grows at the rate of technological progress g. Thus, the model's long run growth properties are basically the same as the Solow model
- Notice also that the Ramsey model consistently with the Solow model implies a constant savings rate BUT ONLY ALONG THE BALANCED GROWTH PATH
- This savings rate is given as

$$
\begin{aligned}
s^{bg} &= \frac{S}{Y}\Big|_{bg} = 1 - \frac{C}{Y}\Big|_{bg} = 1 - \frac{\overline{c}^{e}}{\overline{y}^{e}} \\
&= 1 - (\delta + n + g)\frac{\overline{k}^{e}}{\overline{y}^{e}} = 1 - \frac{\alpha(\delta + n + g)}{(1+g)/\beta^{*} - (1-\delta)}
\end{aligned}
$$

The Ramsey Model: The Competitive Solution

So far we have computed the planning solution - let me know consider the competitive solution

- Large number of identical households that rent their capital stock and labor services to firms at prices r_t and w_t, respectively, taking all prices for given
- Large number of identical firms that produce output using capital and labor as inputs taking all prices for given
- There is free entry - hence, equilibrium profits must equal zero
- Since households and firms are all identical, I will look at the case with a representative agent and a representative firm

The Representative Household's Problem

The household is faced with the following maximization problem:

$$\max_{\left(c_t^e, k_{t+1}^e\right)} \sum_{t=0}^{\infty} \left(\beta^* \left(1+n\right)\right)^t \frac{\left(c_t^e\right)^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} c_t^e + i_t^e &= w_t^e + r_t k_t^e + \pi_t^e \\ \left(1+n\right)\left(1+g\right) k_{t+1}^e &= \left(1-\delta\right) k_t^e + i_t^e \end{aligned}$$

where $w_t^e = w_t / A_t$, that is the real wage per efficiency unit of labor and $\pi_t^e = \pi_t / \left(A_t N_t\right)$ where π_t denotes profits received from ownership of firms

The first-order conditions:

$$\begin{aligned} c_t^e &: \left(c_t^e\right)^{-\sigma} = \lambda_{c,t} \\ k_{t+1}^e &: \lambda_{c,t}\left(1+g\right) = \beta^* \lambda_{c,t}\left[r_{t+1} + \left(1-\delta\right)\right] \end{aligned}$$

plus a no-Ponzi game condition:

$$\lim_{n \to \infty} \frac{k_{t+n}^e}{\prod_{s=1}^{n}\left(1+r_{t+s}\right)} = 0$$

The Representative Firm's Problem

The representative firm is face with the maximization problem:

$$\max_{k_t^e} \pi_t^e = (k_t^e)^\alpha - r_t k_t^e - w_t^e$$

with the first-order condition:

$$r_t = \alpha (k_t^e)^{\alpha-1} = \alpha (K_t)^{\alpha-1} (N_t A_t)^{1-\alpha}$$

Hence, the rental price equals the marginal product of capital
Due to free entry and price taking behavior, equilibrium profits need to equal zero. Therefore:

$$\begin{aligned} w_t^e &= (k_t^e)^\alpha - r_t k_t^e = (k_t^e)^\alpha - \alpha (k_t^e)^\alpha \\ &= (1-\alpha)(k_t^e)^\alpha \end{aligned}$$

or:

$$w_t = A_t w_t^e = (1-\alpha) A_t (K_t)^\alpha (N_t A_t)^{-\alpha}$$

which is simply the marginal product of labor

The General Equilibrium

We can now impose the first-order conditions on the households problem. The budget constraint becomes:

$$
\begin{aligned}
c_t^e + i_t^e &= w_t^e + r_t k_t^e + \pi_t^e \\
&\Rightarrow \\
c_t^e + i_t^e &= (1-\alpha)\left(k_t^e\right)^{\alpha} + \alpha\left(k_t^e\right)^{\alpha} = y_t^e
\end{aligned}
$$

which is identical to the economy's resource constraint
The first-order conditions of the household become

$$
\left(c_t^e\right)^{-\sigma}(1+g) = \beta^*\left(c_{t+1}^e\right)^{-\sigma}\left[\alpha\left(k_{t+1}^e\right)^{\alpha} + (1-\delta)\right]
$$

which is identical to the first-order condition in the planning problem

One or Many Firms?

Can one aggregate across firms? Yes, under constant returns and competition in factor markets. A single firm's maximization problem:

$$\max F\left(K_i, AN_i\right) - rK_i - wN_i$$

with first-order conditions:

$$\begin{aligned} F_K\left(K_i, AN_i\right) &= r \\ F_N\left(K_i, AN_i\right) &= w \end{aligned}$$

Since F is assumed to be homogeneous of degree 1, its derivative is homogenous of degree 0. Therefore:

$$\begin{aligned} F_K\left(K_i, AN_i\right) &= F_K\left(\frac{K_i}{A_{N_i}}, 1\right) \\ F_N\left(K_i, AN_i\right) &= F_N\left(\frac{K_i}{A_{N_i}}, 1\right) \end{aligned}$$

This implies that, since all firms face the same factor prices, capital-effective labor ratios are equalized across firms

One or Many Firms?

So, from above we have that:

$$\frac{K_i}{A_{N_i}} = k^*$$

Again, because of constant returns, we can write:

$$\begin{aligned} Y_i &= F\left(K_i, AN_i\right) \\ &= AN_i F\left(\frac{K_i}{AN_i}, 1\right) \\ &= AN_i f\left(k^*\right) \end{aligned}$$

so total output in the economy is:

$$Y = \sum_i Y_i = ANf\left(k^*\right)$$

where N is total employment. This is simply the production function of the representative firm. QED

The neoclassical model: Summary

- Due to diminishing marginal returns, there exists a balanced growth path
- Choices and policies (and parameters such as population growth) can affect the long run level of income and the transitional growth rates
- But choices and policies cannot affect the long run growth rate of the economy
- Long run growth explained by technological progress which is not explained as such by the model
- Is this true?
    - Let's look at the impact of the investment rate on the level and growth rate of the economy

Investment Rates and the Level of Income

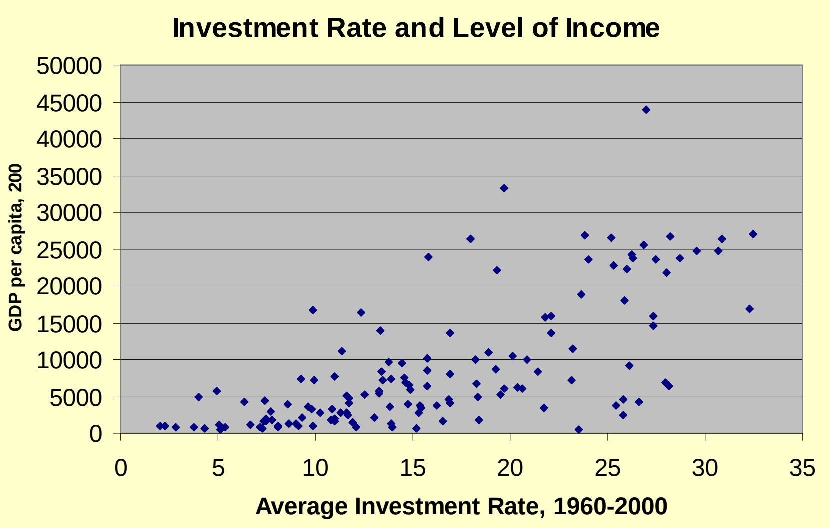


Figure: Source: PWT 6.1

Investment Rates and the Growth of Income

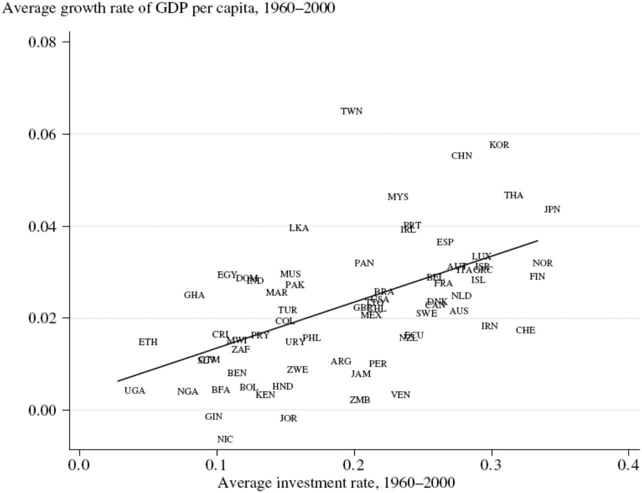
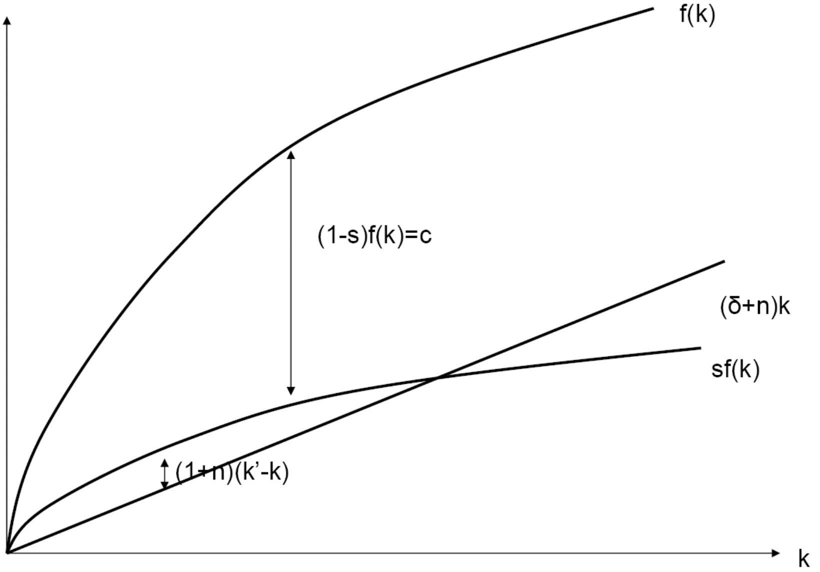


Figure: Source: Acemoglu

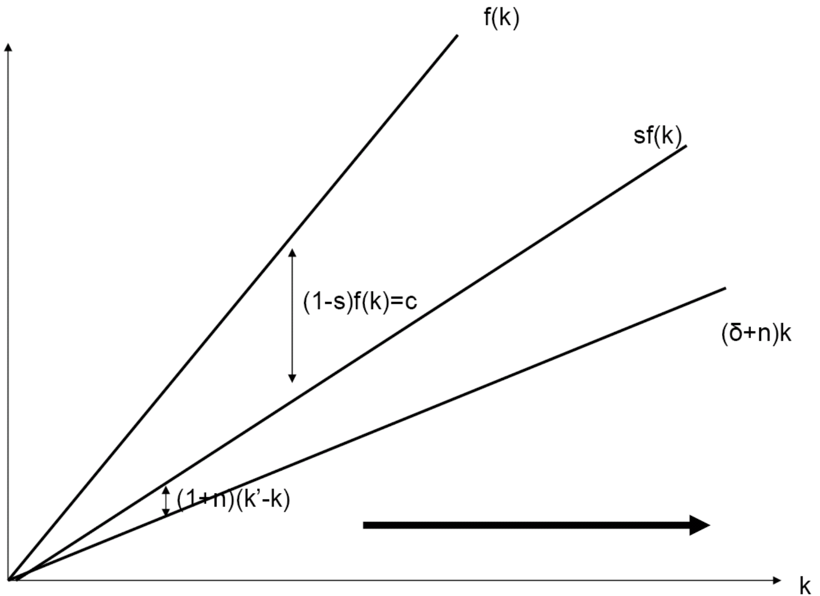
The two pictures above indicate:

- Higher investment rates definitely give rates to higher levels of income
- Also some graphical evidence that higher investment rates give rise to high growth rates
  - Evidence is not definitive for two reasons:
  - the period is relatively short - may be affected by transitional growth
  - the horizontal axis measures average investment rates - we should really be looking at initial investment rates
- The case is open but perhaps it's worth considering alternative growth theories: Endogenous growth

Diminishing Marginal Returns to Accumulable Factor



Constant Returns to the Accumulable Factor



Endogenous Growth

Thus:

- If there are constant (or even increasing) returns to factors of production that can be accumulated over time, the economy can grow forever through the accumulation of these factors as long as $sf(k) > (\delta + n)k$ (or as long as $f'(k) > (\delta + n)$ in the Ramsey model)
- And here growth rates are affected by the savings rate, policies etc.
- For that reason models with this property are called "endogenous growth models"

So how may we have constant returns to factors that can be accumulated?

- The "AK" model - here the production function is simply assumed to be linear in capital
- Models with human capital as well as physical capital
- Models with externalities across firms
- Models with R&D or other sources of growth

The AK model

In the AK model, the production function is assumed to be:

$$y_t = Ak_t$$

and therefore there are constant returns to capital - and capital can be accumulated

- As we have seen graphically above, this can lead to "endogenous growth" where changes in savings rates and other variables will impact on the long run growth rate
- What's the intuition?
- In the neoclassical model: The marginal product of capital falls when we accumulate capital - at some point, accumulating more capital is no longer profitable
- When there are constant returns to capital: The marginal product of capital remains constant - if it is profitable to accumulate more capital initially, it will remain to be profitable

Optimal Growth with AK Technology

In the AK model, the households' problem is (here I simplify and set $n = 0$)

$$\max_{(c_t, k_{t+1})} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} c_t + i_t &= r_t k_t + \pi_t \\ k_{t+1} &= (1-\delta) k_t + i_t \end{aligned}$$

and the Euler equation becomes:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left[r_{t+1} + 1 - \delta \right]$$

The firms' problem is:

$$\max \pi_t = Ak_t - rk_t$$

with the first-order condition:

$$r_t = A$$

Optimal Growth with AK Technology

Combining the first-order conditions, we get that:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} [A + 1 - \delta]$$

Thus, consumption will grow at the constant rate:

$$\gamma_{c,t+1} = \frac{c_{t+1}}{c_t} - 1 = \beta^{1/\sigma} [A + 1 - \delta]^{1/\sigma} - 1$$

which is positive if $[A - \delta] > 1/\beta - 1$.

Learning by Doing

The AK formulation is simple, perhaps too simple. An alternative perhaps more palatable specification is that the constant returns derives from learning-by-doing. Romer (1986) propose to model the production function of an individual firm as:

$$y_{i,t} = A\left(k_{i,t}\right)^{\gamma}\left(K_t\right)^{1-\gamma}, \ \gamma \in (0,1)$$

where $k_{i,t}$ denotes the capital stock held by firm i and K_t denotes the aggregate capital stock per firm.

- Since $\gamma \in (0,1)$ each individual firm faces diminishing marginal returns to capital because they take K_t for given
- But, in equilibrium all firms are identical, so there are aggregate constant returns
- This model features an externality - individual firms do not internalize the fact that their own capital stock affects the production of other firms. Due to the externality (which is positive) the welfare theorems do not hold and the competitive equilibrium will be inefficient (firms will accumulate too little capital)

Learning by Doing

The household's problem is the same as above. The firm maximizes:

$$\max_{k_t} \pi_t = A\left(k_{i,t}\right)^{\gamma}\left(K_t\right)^{1-\gamma} - rk_t$$

with the first-order condition:

$$r_t = \gamma A\left(k_{i,t}\right)^{\gamma-1}\left(K_t\right)^{1-\gamma}$$

In equilibrium, since firms are identical, we get that $k_{i,t} = K_t = k_t$. Imposing this on the first-order condition for the firms:

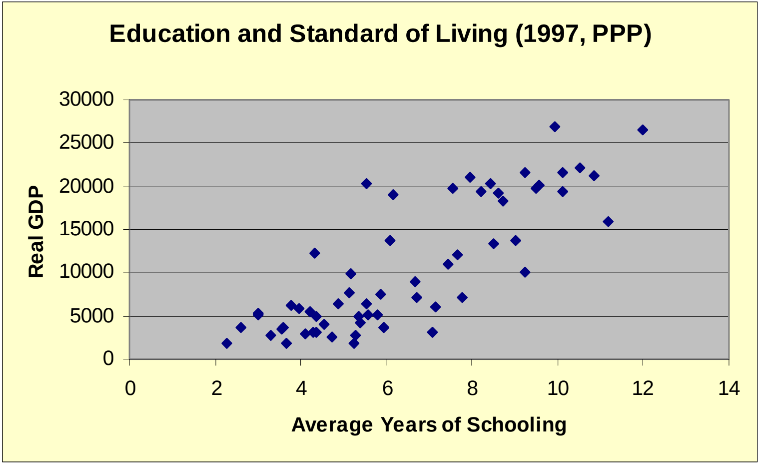
$$r_t = \gamma A$$

So in this model, the growth rate becomes:

$$\gamma_{c,t+1} = \frac{c_{t+1}}{c_t} - 1 = \beta^{1/\sigma}\left[\gamma A + 1 - \delta\right]^{1/\sigma} - 1$$

This, growth rate is inefficiently low because of the externality.

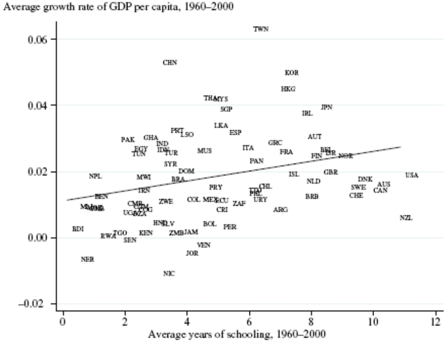
Human Capital Matters



Human Capital Matters

	Schooling (2000) (Years)	Enrolment Rates (%) (1995)			Gov. Spending per pupil (1995, US$)	
		Primary	Secondary	Tertiary	Primary	Secondary
US	12.25	102.3	93.1	75.2	2721	4181
UK	9.35	104.2	85.5	30.2	1967	3511
France	8.37	108.5	98.5	39.6	1664	3297
Germany	9.75	101.1	98.3	32.1	1722	3757
Brazil	4.56	106.3	38.4	11.2	364	-
China	5.75	125.2	48.7	3	146	375
India	4.77	97.2	44.4	6.1	138	166

Human Capital Matters



Human Capital I: One Sector Case

The two models above both assume aggregate constant returns to capital.

- But capital is not the only factor that can be accumulated over time
- While "raw labor" cannot be accumulated, human capital CAN
- The simplest formulation would be that the production technology is:

$$Y_t = AK_t^{\alpha} H_t^{1-\alpha}$$

where H_t denotes human capital. Both inputs can be accumulated by setting aside resources for investment:

$$\begin{aligned} K_{t+1} &= (1-\delta) K_t + I_t^k \\ H_{t+1} &= (1-\delta) H_t + I_t^h \\ C_t + I_t^k + I_t^h &= AK_t^{\alpha} H_t^{1-\alpha} \end{aligned}$$

- Hence, this induces constant returns to the factors that can be accumulated.

Human Capital II: Two Sector Case

The above formulation is a bit mechanical. A somewhat nicer formulation is by Lucas (1988).

- Households own physical and human capital.
- Output is produced as:

$$Y_t = AK_t^{\alpha} \left(u_t H_t\right)^{1-\alpha}$$

  where u_t is the fraction of human capital that is used for producing goods
- The capital accumulation equation is:

$$K_{t+1} = (1-\delta) K_t + I_t$$

- Human capital is produced as:

$$H_{t+1} = (1-\delta) H_t + B (1-u_t) H_t$$

- and the economy's resource constraint is:

$$Y_t = C_t + I_t$$

Endogenous Growth

There are many other types of growth models that are variants or alternatives to the models outlined above:

- Growth through productive government spending: Models with an externality like in the Romer model but where the externality comes from public expenditure (on infrastructure or education etc.)
- Growth through technological change:
  - expansion in varieties through R&D
  - development of lower cost technologies through R&D
  - These models build on imperfect competition and are therefore a bit different from those that we have looked at
- At the end of the day, all these models give rise to constant or increasing returns to accumulable factors but they stress different mechanisms

Growth Accounting

A large literature has carried out so-called growth accounting exercises: What has accounted for growth in the economy?

- TFP
- Capital accumulation
- Labor

Or it can be carried out in terms of output per worker: What has accounted for growth in output per worker:

- TFP
- Capital accumulation

Growth Accounting

Suppose that we assume a Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} (A_t N_t)^{1-\alpha}$$

- Then output per worker is given as:

$$y_t = k_t^{\alpha} A_t^{1-\alpha}$$

- We will then ask "what has accounted for changes in y_t"?
- Problem, A_t is unknown
- But, we can rearrange this equation as:

$$\begin{aligned} A_t^{1-\alpha} &= \frac{y_t}{k_t^{\alpha}} \\ &\Rightarrow \\ (1-\alpha)\log A_t &= \log y_t - \alpha \log k_t \end{aligned}$$

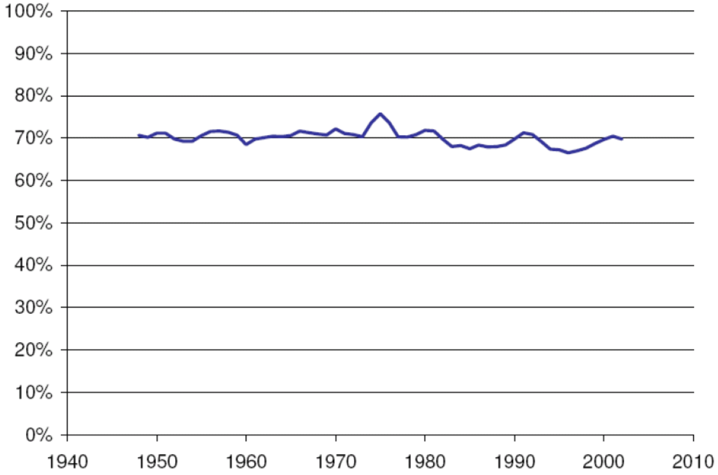
In this last equation:

-
    - Y_t is GDP which we can find in the national accounts
    - y_t is GDP per worker (or per hours worked) which we can find by dividing GDP with hours worked
    - K_t is the capital stock which we can either find in the national accounts or compute by iteration the equation

$$K_{t+1} = (1 - \delta) K_t + I_t$$

    - we just need an initial value of K_0 and a value of δ
    - k_t is K_t divided by hours worked
    - α is the capital share of income which we can compute from the national accounts
- Hence, we can find A_t as a residual

The UK Labor Income Share

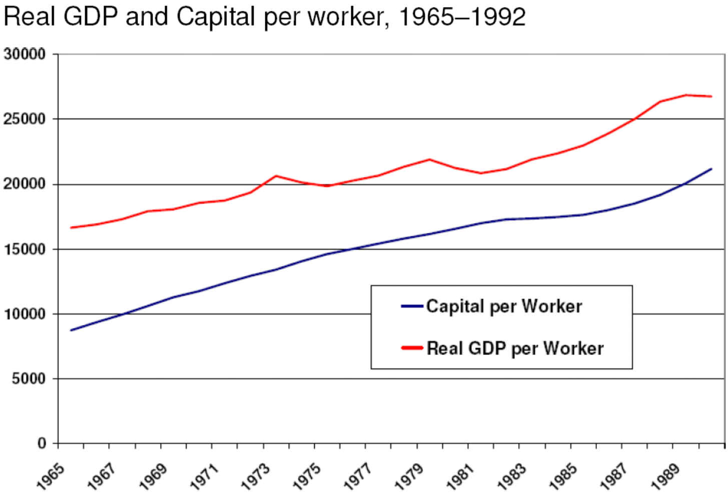


The UK Labor Income Share

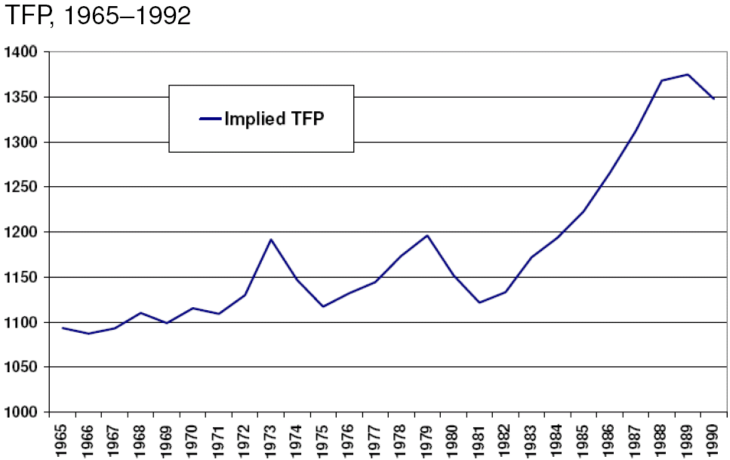
- It is quite stable with a mean of 70 percent
- Therefore the capital share is around 30 percent
- In other words, most of value added goes to pay labor
- With this number we can then compute time-series for $A_t^{1-\alpha}$, y_t and k_t
- We can also do this for growth rates since:

$$\log y_{t+1} - \log y_t = \alpha \left(\log k_{t+1} - \log k_t\right) + (1-\alpha)\left(\log A_{t+1} - \log A_t\right)$$

Time Series


Real GDP and Capital per worker, 1965–1992
30000
25000
20000
15000
10000
5000
0
1965
1967
1969
1971
1973
1975
1977
1979
1981
1983
1985
1987
1989
Capital per Worker
Real GDP per Worker

Time Series


TFP, 1965–1992
Implied TFP
1400
1350
1300
1250
1200
1150
1100
1050
1000
1965
1966
1967
1968
1969
1970
1971
1972
1973
1974
1975
1976
1977
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1979
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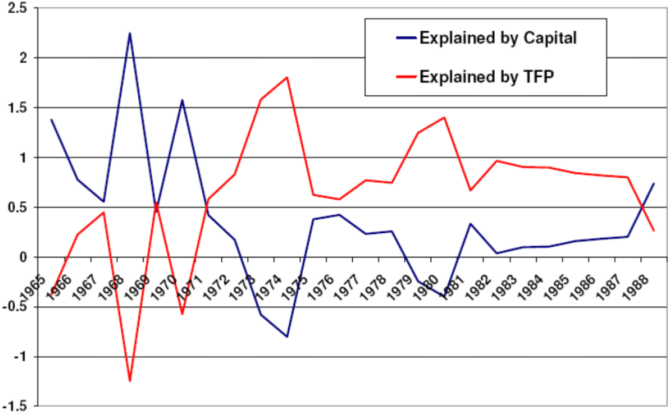
1982
1983
1984
1985
1986
1987
1988
1989
1990

GDP growth accounted for by capital and TFP



Conclusion:

- Capital accounts for most of the growth in labor productivity until early 1970's
- TFP seems to be the main force thereafter
- Several possible explanations for this:
  - IT
  - Skill-biased technological change
  - Deregulation
- But, growth accounting has several problems

Levels Accounting

Hall and Jones (1999, Quarterly Journal of Economics) derive instead a levels accounting exercise for output per worker

Suppose that we have the technology:

$$\begin{aligned} Y_i &= K_i^{\alpha} \left(A_i H_i\right)^{1-\alpha} \\ H_i &= e^{\phi(E_i)} N_i \end{aligned}$$

where:

- H_i is the amount of human-capital augmented labor
- N_i is "raw employment"
- $\phi\left(E_i\right)$ is a measure of efficiency of a unit of labor with E_i years of schooling

Levels Accounting

The production function can be expressed in terms of output per worker as:

$$y_i = \frac{Y_i}{N_i} = \left(\frac{K_i}{Y_i}\right)^{\alpha/(1-\alpha)} e^{\phi(E_i)} A_i$$

- Notice that $\left(\frac{K_i}{Y_i}\right)^{\alpha/(1-\alpha)}$ is constant along the balanced growth path which makes this formulation attractive

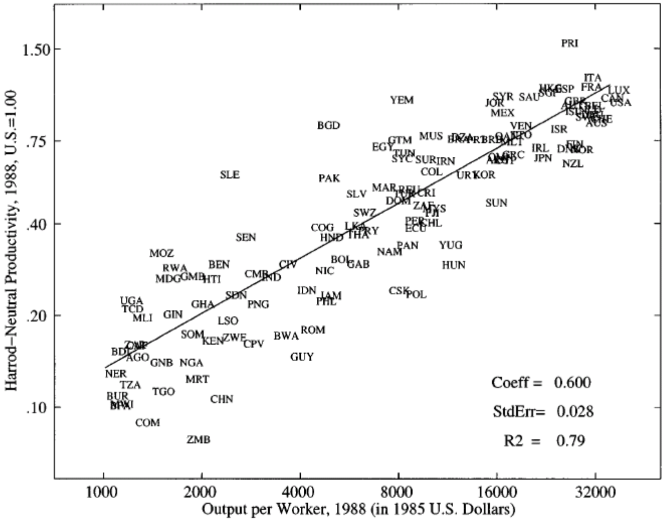
Hall and Jones measure the various quantities as:

- y_i : PWT measures of output per worker
- K_i: measured by summing investment levels over time and subtracting depreciation
- $\phi(E_i)$: measured using a Mincerian specification:

$$\phi(E_i) = \left\{ \begin{array}{cc} 13.4\% & \text{return on years 1-4} \\ 10.1\% & \text{return on years 5-8} \\ 6.8\% & \text{return per years above 9} \end{array} \right.$$

- A_i derived as residual assuming $\alpha = 1/3$

Output per Worker and TFP



Accounting for Levels Differences

Productivity Calculations: Ratios to U. S. Values

Country	Y/L	Contribution from $(K/Y)^{\alpha/(1-\alpha)}$	H/L	A
United States	1.000	1.000	1.000	1.000
Canada	0.941	1.002	0.908	1.034
Italy	0.834	1.063	0.650	1.207
West Germany	0.818	1.118	0.802	0.912
France	0.818	1.091	0.666	1.126
United Kingdom	0.727	0.891	0.808	1.011
Hong Kong	0.608	0.741	0.735	1.115
Singapore	0.606	1.031	0.545	1.078
Japan	0.587	1.119	0.797	0.658
Mexico	0.433	0.868	0.538	0.926
Argentina	0.418	0.953	0.676	0.648
U.S.S.R.	0.417	1.231	0.724	0.468
India	0.086	0.709	0.454	0.267
China	0.060	0.891	0.632	0.106
Kenya	0.056	0.747	0.457	0.165
Zaire	0.033	0.499	0.408	0.160
Average, 127 countries:	0.296	0.853	0.565	0.516
Standard deviation:	0.268	0.234	0.168	0.325
Correlation with Y/L (logs)	1.000	0.624	0.798	0.889
Correlation with A (logs)	0.889	0.248	0.522	1.000

The elements of this table are the empirical counterparts to the components of equation (3), all measured as ratios to the U. S. values. That is, the first column of data is the product of the other three columns.

Conclusions on levels accounting

- The rich vs. US: All the differences are accounted for by differences in human capital
  - Capital-output ratios and TFP often higher than in the US but differences are minimal
- The poor vs. US: All the factors are important but the single most important factor is TFP
  - Education also important, but TFP even more so
- So, the question is why TFP and human capital are so low in the poor countries

$$
\begin{aligned}
(1+g)(1+n)\gamma_{k^e,t+1} + (\delta+n+g) &= s\left(k_t^e\right)^{\alpha-1} \\
&\Rightarrow \\
\frac{(1+g)(1+n)\,d\gamma_{k^e,t+1}}{(1+g)(1+n)\overline{\gamma}_{k^e} + (\delta+n+g)} &= (\alpha-1)\frac{dk_t^e}{\overline{k}^e} \\
\text{Assume that } (1+g)(1+n) &\simeq 1 \Rightarrow \\
\frac{\gamma_{k^e,t+1} - \overline{\gamma}_{k^e}}{(\delta+n+g)} &= (\alpha-1)\frac{k_t^e - \overline{k}^e}{\overline{k}^e} \\
&\Rightarrow \\
\gamma_{k^e,t+1} &= -(1-\alpha)(\delta+n+g)\log\left(\frac{k_t^e}{\overline{k}^e}\right)
\end{aligned}
$$