MSC Macroeconomics G022, 2009

The Ramsey Model

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In this lecture

- The Ramsey Model
  - Assumptions
  - Analysis - the planning problem
  - First-order conditions and their interpretation
  - The steady-state: The modified golden rule
  - Dynamics and saddle path stability
- A tiny bit of dynamic programming
- Fiscal policy
- Investment theory

The Ramsey Model

- Infinite horizon model of optimal savings
- Features competitive agents, capital accumulation and production
- The economy is closed to foreign trade, there is no government, no labor supply decision (to begin with), no money
- There is no uncertainty - it is a perfect foresight economy
- These are all obviously very stark assumptions - it is a simplification
- But the insights are important!

Technology

Output is produced in this economy by a large number of competitive firms using input of capital only

- The production function is:

$$y_t = F(k_t) \tag{1}$$

where:

$$\begin{aligned} F(0) &= 0 \\ F'(k) &> 0 \\ F''(k) &< 0 \end{aligned}$$

and we impose Inada conditions:

$$\begin{aligned} \lim_{k \to 0+} F'(k) &= \infty \\ \lim_{k \to \infty} F'(k) &= 0 \end{aligned}$$

Capital Accumulation and Resource Constraint

Capital can be accumulated over time by investing in capital goods, but the capital stock depreciates

$$k_{t+1} = (1 - \delta)\, k_t + i_t \tag{2}$$

- This implies that the capital stock
  - rises over time when $i_t > \delta k_t$ (new purchases of capital goods exceed depreciation of existing capital goods - ie. net investment is positive)
  - and decreases when $i_t < \delta k_t$ (negative net investment)
- The good produced in the economy is used either for consumption or for investment
- Thus the resource constraint is given as:

$$y_t = c_t + i_t \tag{3}$$

The Golden Rule

Combining equations $(1) - (3)$ we get:

$$F(k_t) = c_t + k_{t+1} - (1 - \delta) k_t$$

or:

$$c_t = F(k_t) - (k_{t+1} - k_t) - \delta k_t \tag{4}$$

which defines consumption as output less resources spend on increasing the capital stock and resources spend on capital maintenance

Problem (Question)

What is the optimal level of consumption?

Two "naïve" answers would be

1. The maximum level of consumption that can be attained in the "short run" (myopic answer)
2. The maximum level of consumption that can be attained (sustained) in the "long run" (Golden Rule)

The Golden Rule

The maximum level attained in the short-run is given as:

$$c_t^{myopic} = F(k_t) + (1 - \delta) k_t$$

which is obtained by setting $k_{t+1} = 0$

- This cannot in any sense be optimal: Since $k_{t+1} = 0$, all future consumption will be equal to zero. Thus it is a party followed by starvation.

The maximum level of consumption that is sustainable in the long-run can be derived from imposing a steady-state condition on equation (4)

- Let us focus upon a situation where all variables are constant over time. Thus, equation (4) becomes:

$$c = F(k) - \delta k$$

The Golden Rule

- Maximizing the last expression wrt. the capital stock gives us:

$$F'\left(k^{GR}\right) = \delta \tag{5}$$

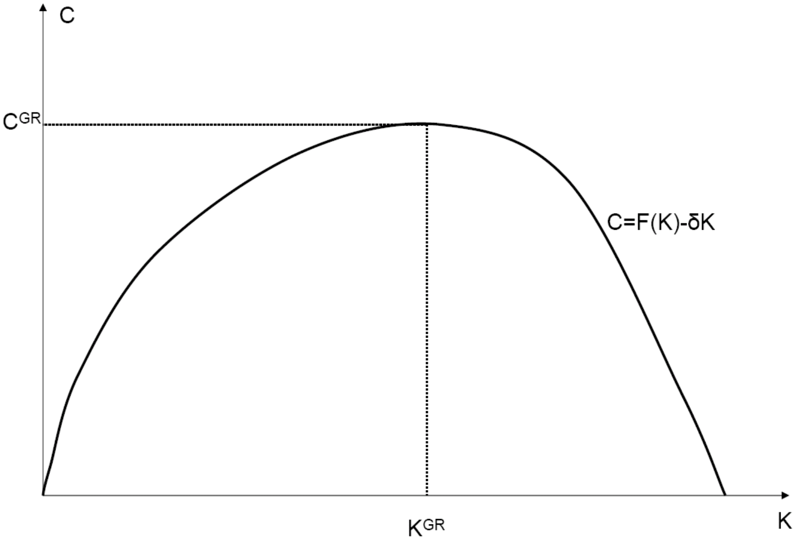
so that:

$$\begin{aligned} k^{GR} &= F'^{-1}(\delta) \\ c^{GR} &= F\left(k^{GR}\right) - \delta k^{GR} \end{aligned} \tag{6}$$

- Notice that this is indeed a maximum since the second order condition holds ($\partial^2 c/\partial k^2 = F''(k) \leq 0$ by assumption)
- The maximum level of consumption - the Golden Rule level of consumption - is attained when the marginal product of capital equals the depreciation rate
- Another property of this level of the capital stock is that it maximizes net output (output net of depreciation) in the steady-state:

$$y^{net} = F(k_t) - \delta k_t$$

The Golden Rule - Graphics



The Golden Rule - Optimality

But: Is the Golden Rule optimal?

- A capital stock above the Golden Rule certainly cannot be optimal - if we are to the right of k^{GR} we could eat some capital and increase consumption both in the long run and in the short run
- But might a capital stock below the Golden Rule be optimal? There is a trade-off:
  - Suppose we were initially at the Golden Rule: If we eat some of the capital stock we increase consumption in the short run but increase it in the long-run
  - Suppose we were initially below the Golden Rule: If we increase the capital stock we lower consumption in the short run but increase it in the long run
- So what's the best thing to do? It will depend on preferences - we need to find the optimal savings rate

Preferences

To address the quesion above, the Ramsey model introduces utility maximizing households: Assumes a large amount of identical and infinitely lived households that take all prices for given and have preferences

$$\begin{aligned} U_t &= \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \qquad (7) \\ u' &> 0,\ u'' < 0 \\ \lim_{c \to 0+} u'(c) &= \infty,\ \lim_{c \to \infty} u'(c) = 0 \\ \beta &= \frac{1}{1+\theta} \in (0,1),\ \theta > 0 \end{aligned}$$

- Since households are identical, we will work with a representative stand-in agent
- β is the subjective discount factor; θ is the rate of time discount
- Here we will derive the central planning solution which can also be interpreted in terms of a competitive equilibrium allocation due to the second fundamental welfare theorem

The Central Planner's Problem

The central planner is faced with the following intertemporal optimization problem:

$$\max_{(c_s, k_{s+1})_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

subject to:

$$\begin{aligned} c_s &= F(k_s) - (k_{s+1} - k_s) - \delta k_s,\ s = t, t+1, \ldots. \qquad (8) \\ k_t &> 0 \text{ given} \end{aligned}$$

- Since there is no uncertainty, this is a perfect foresight problem
- Recall that when agents optimize we need to impose to terminal condition that prevents them from building up too much debt
- The central planner faces a similar constraint - the Transversality Condition:

$$\lim_{s \to \infty} \beta^s u'(c_s) k_{s+1} = 0$$

The above maximization problem looks complicated: It is infinitely dimensional.

- Here I will first use a standard technique to analyze it and then introduce an alternative approach
- We can formulate the problem as a constrained maximization problem. Let me formulate the Lagrangean as:

$$\mathfrak{L} = \sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s\right) - \sum_{s=t}^{\infty} \beta^{s-t} \lambda_s \left(c_s - F\left(k_s\right) + \left(k_{s+1} - k_s\right) + \delta k_s\right)$$

- where k_0 is assumed given
- Notice that I have discounted the multipliers with β^{s-t}. This is simply to make things look nicer.

The Central Planner's Problem - First-Order Conditions

The first-order necessary conditions for this problem are given as:

$$\begin{aligned} c_s &: \beta^{s-t} u'(C_s) = \beta^{s-t} \lambda_s \ \forall s \geq t \\ k_{s+1} &: \beta^{s-t} \lambda_s = \beta^{s+1-t} \lambda_{s+1} \left(F'(k_{s+1}) + (1-\delta) \right) \ \forall s \geq t \\ \lambda_s &: c_s = F(k_s) - (k_{s+1} - k_s) - \delta k_s \ \forall s \geq t \end{aligned}$$

plus the transversality condition:

$$\lim_{s \to \infty} \beta^s u'(c_s) k_{s+1} = 0 \tag{9}$$

- We can make the first-order conditions for c_s and k_{s+1} slightly nicer:

$$c_s : u'(c_s) = \lambda_s \tag{10}$$

$$k_{s+1} : \lambda_s = \beta \lambda_{s+1} \left(F'(k_{s+1}) + (1-\delta) \right) \tag{11}$$

Interpretations of First-Order Conditions

- *The FOC for consumption*: In the optimum, the shadow price of increasing consumption in period s marginally $\left(\beta^{s-t}\lambda_s\right)$ equals the utility gain from a marginal increase in consumption $\left(\beta^{s-t}u'\left(c_s\right)\right)$
  - Suppose that $\beta^{s-t}u'\left(c_s\right) > \beta^{s-t}\lambda_s$: It would pay for the planner to increase c_s since goods in this period are "cheap" relative to consumers' valuation of consumption. Vice versa for $\beta^{s-t}u'\left(c_s\right) < \beta^{s-t}\lambda_s$
- *The FOC for capital*: In the optimum, the shadow price of increasing k_{s+1} marginally $\left(\beta^{s-t}\lambda_s\right)$ equals the gross marginal product of capital in period $s+1$ $\left(F'\left(k_{s+1}\right)+\left(1-\delta\right)\right)$ evaluated at the discounted shadow price $\left(\beta^{s+1-t}\lambda_{s+1}\right)$
  - Suppose that $\beta^{s-t}\lambda_s > \beta^{s+1-t}\lambda_{s+1}\left(F'\left(k_{s+1}\right)+\left(1-\delta\right)\right)$: It would pay for the planner to lower the k_{s+1} since the presented discounted value of the gross marginal product of capital tomorrow is below the price of increasing the capital stock slightly today. Vice versa for $\beta^{s-t}\lambda_s < \beta^{s+1-t}\lambda_{s+1}\left(F'\left(k_{s+1}\right)+\left(1-\delta\right)\right)$

The Euler Equation

We can combine the first-order conditions for consumption and capital to get:

$$u'(c_s) = \beta u'(c_{s+1}) \left(F'(k_{s+1}) + (1-\delta)\right) \tag{12}$$

Again it's intuitive:

- The left hand side is the cost of increasing the capital stock marginally in period $s+1$ measured in terms of the implied loss of marginal utility
- The right hand side is the benefit of a marginally higher capital stock in period $s+1$: Marginally more capital increases resources in period $s+1$ by $\left(F'(k_{s+1}) + (1-\delta)\right)$ which is translated into utility by multiplying by marginal utility $u'(c_{s+1})$ and discounted back to period t by multiplying by β

The Euler Equation - More Rigorously

Define the optimized objective function as:

$$V_t = \max \sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s\right)$$

and substitute the economy's resource constraint into it:

$$V_t = \max \sum_{s=t}^{\infty} \beta^{s-t} u\left(F\left(k_s\right) - \left(k_{s+1} - k_s\right) - \delta k_s\right)$$

Now examine the impact of increasing K_{s+1} marginally but keeping the capital stock fixed in all other periods:

$$dV_t = \underbrace{-\beta^{s-t} u'\left(c_s\right) dk_{s+1}}_{\substack{\text{utility loss in} \\ \text{period s disc.} \\ \text{to period t}}} + \underbrace{\beta^{s+1-t} u'\left(c_{s+1}\right)\left(F'\left(k_{s+1}\right) + (1-\delta)\right) dk_{s+1}}_{\substack{\text{utility gain in period s+1} \\ \text{discounted to period t}}}$$

The Euler Equation - More Rigorously

Since we are in an optimum, this change in utility obviously needs to equal zero, so we get:

$$\begin{aligned}
\beta^{s-t}u'(c_s)\,dk_{s+1} &= \beta^{s+1-t}u'(c_{s+1})\left(F'(k_{s+1})+(1-\delta)\right)dk_{s+1} \\
&\Rightarrow \\
u'(c_s) &= \beta u'(c_{s+1})\left(F'(k_{s+1})+(1-\delta)\right)
\end{aligned}$$

- This equation is a fundamental part of intertemporal macroeconomic models but also is key to may other parts of economics
  - Euler equations are key for consumption and investment theory
  - Euler equations are key for asset pricing
  - Euler equations are key for theories of optimal taxation
- Better get used to it!!

The Euler Equation - Last word ....

We can also express the Euler equation as:

$$1 = \frac{\beta u'\left(c_{s+1}\right)}{u'\left(c_s\right)}\left(F'\left(k_{s+1}\right) + (1-\delta)\right)$$

- The term $\frac{\beta u'(c_{s+1})}{u'(c_s)}$ is the slope of an indifference curve $\frac{dc_s}{dc_{s+1}}|_{dV=0}$ and it's inverse is the slope of an indifference curve $\frac{dc_{s+1}}{dc_s}|_{dV=0}$. These are intertemporal marginal rates of substitution between c_s and c_{s+1}
- Thus, in the optimum, the intertemporal marginal rate of substitution equals the gross marginal product of capital

Dynamics

We can summarize the optimality conditions as:

$$
\begin{aligned}
u'(c_s) &= \beta u'(c_{s+1})\left(F'(k_{s+1}) + (1-\delta)\right) \\
c_s &= F(k_s) - (k_{s+1} - k_s) - \delta k_s \\
\lim_{s\to\infty} \beta^s u'(c_s)\, k_{s+1} &= 0
\end{aligned}
$$

Definition (Question)

Suppose that we start with a capital stock k_0: How will the economy evolve over time?

In order to analyze this, I will use a phase diagram in the space of c and k. Two steps involved

1. Derive the steady-state relationships
2. Examine the dynamics of the variables away from the steady-state

The Resource constraint

The steady-state refers to the situation in which variables are constant over time (later we will look at models with growth in which we will look at balanced growth which is when the growth rates are constant over time)

- The resource constraint is:

$$c_s = F(k_s) - (k_{s+1} - k_s) - \delta k_s$$

- In steady-state this defines combinations of c and k such that k is constant over time:

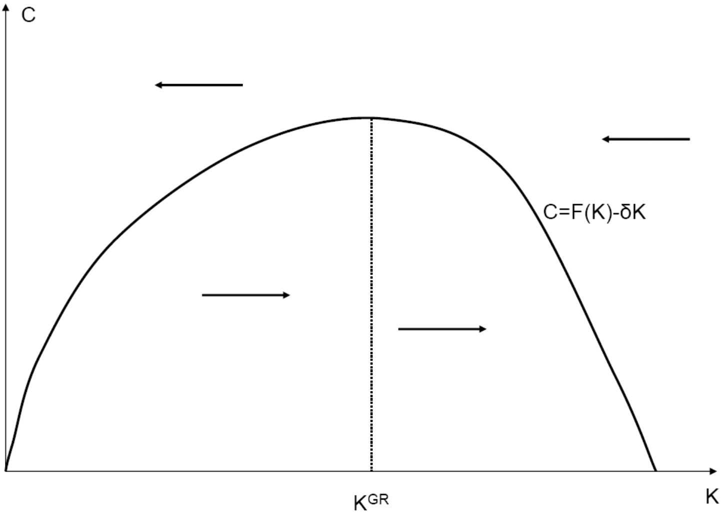
$$c = H(k) = F(k) - \delta k \tag{13}$$

1. This locus starts in the origin, is initially increasing (due to the Inada condition), and has a global maximum at the Golden Rule capital stock
2. When $c < H(k)$ the capital stock must be increasing since:

$$(k_{s+1} - k_s) = (F(k_s) - \delta k_s) - c_s$$

3. and vice versa for $c > H(k)$

The resource constraint - dynamics



The Modified Golden Rule

- The Euler equation evaluated in the steady-state

$$u'(c) = \beta u'(c)\left(F'(k) + (1-\delta)\right)$$

  or:

$$1 = \beta\left(F'(k) + (1-\delta)\right)$$

This implies that:

$$F'\left(k^{MGR}\right) = \frac{1}{\beta} - (1-\delta) = \theta + \delta \tag{14}$$

- This is called the Modified Golden Rule - it determines the optimal steady-state capital stock

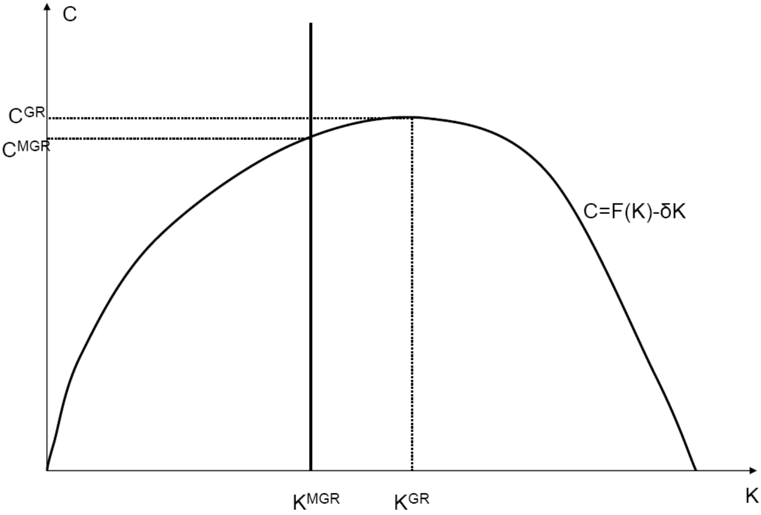
The Modified Golden Rule

- How does it compare to the Golden Rule? Compare the two conditions:

$$F'\left(k^{MGR}\right) = \frac{1}{\beta} - (1-\delta) = \theta + \delta$$
$$F'\left(k^{GR}\right) = \delta$$

- since $\beta < 1$, we have that $\frac{1}{\beta} - (1-\delta) > \delta$
- Therefore $k^{GR} > k^{MGR}$: The Golden Rule capital stock (the one that maximizes consumption in the steady-state) exceeds the Modified Golden Rule capital stock (the one that maximizes utility)
- Why? Agents are impatient - the cost of having low consumption for a long time to get to k^{GR} does not fully make up for the benefit.
- The more impatient are the consumers, the lower will be the optimal capital stock

The Modified Golden Rule - Graphics



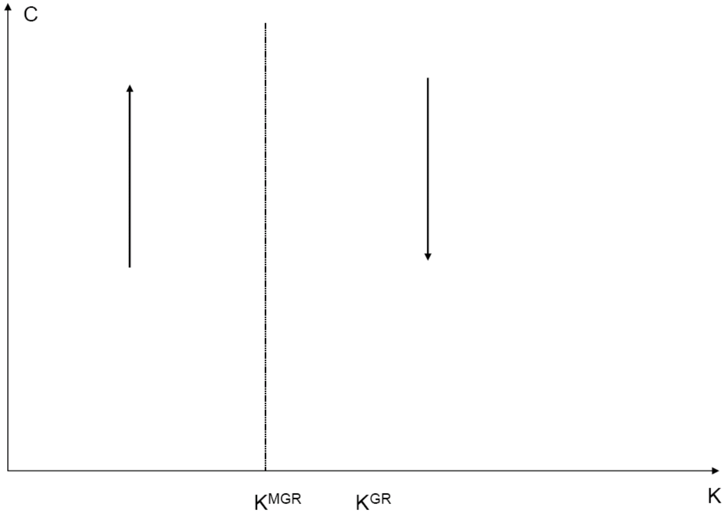
The Euler Equation: Dynamics

- The modified Golden Rule defines a vertical line in the (c, k) space for which consumption is constant
- What happens away from this? Recall the Euler equation:

$$\frac{u'(c_s)}{u'(c_{s+1})} = \beta\left(F'(k_{s+1}) + (1-\delta)\right)$$

1. When $k = k^{MGR}$ the right hand side of equals 1 since $\beta\left(F'\left(k^{MGR}\right) + (1-\delta)\right) = \beta\left(\frac{1}{\beta} - (1-\delta) + (1-\delta)\right) = 1$. This implies that $c_s = c_{s+1}$
2. When $k < k^{MGR}$ the right hand side exceeds 1 since $F'(k) > F'\left(k^{MGR}\right)$ for $k < k^{MGR}$. Thus marginal utility declines over time and consumption grows
3. When $k > k^{MGR}$ the right hand side is lower than 1 since $F'(k) < F'\left(k^{MGR}\right)$ for $k > k^{MGR}$. Thus marginal utility increases over time and consumption declines

The Euler Equation: Dynamics



Dynamics: Summary

We have that:

$$k \text{ dynamics} \quad : \quad \begin{pmatrix} \triangle\, k > 0 \text{ when } c < F(k) - \delta k \\ \triangle\, k = 0 \text{ when } c = F(k) - \delta k \\ \triangle\, k < 0 \text{ when } c > F(k) - \delta k \end{pmatrix}$$

$$c \text{ dynamics} \quad : \quad \begin{pmatrix} \triangle\, c > 0 \text{ when } k < k^{MGR} \\ \triangle\, c = 0 \text{ when } k = k^{MGR} \\ \triangle\, c < 0 \text{ when } k > k^{MGR} \end{pmatrix}$$

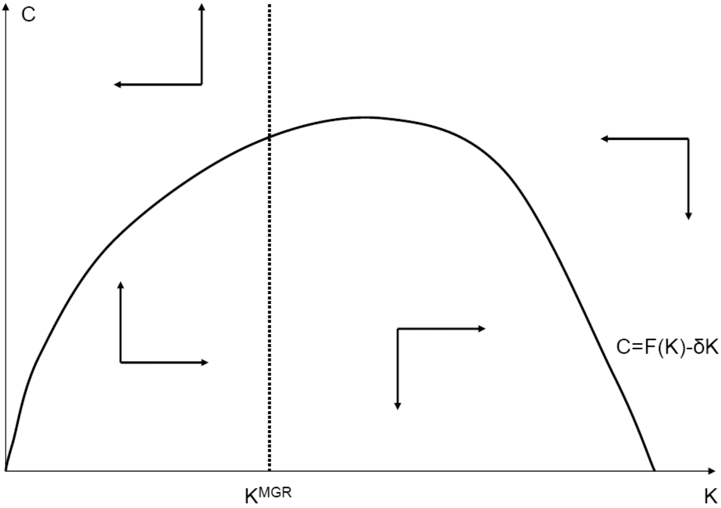
where $\triangle\, X$ denotes the change in X

- We also have the transversality condition:

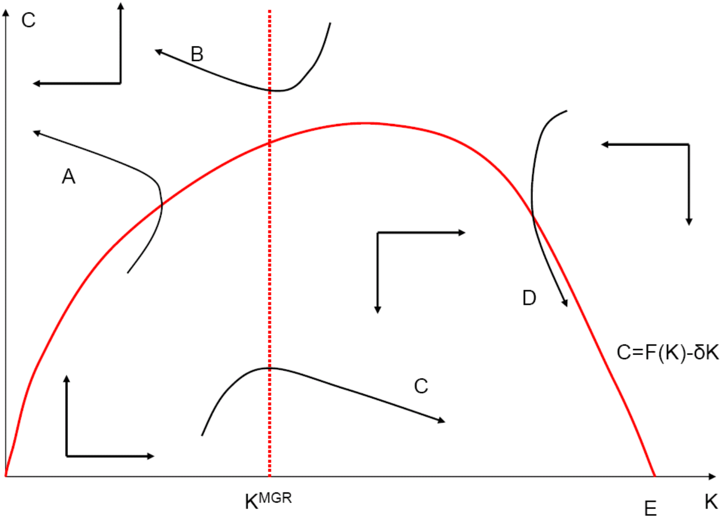
$$\lim_{s \to \infty} \beta^s u'(c_s)\, k_{s+1} = 0$$

- Finally, we notice that the capital stock is given at the beginning of the period - it is a predetermined variable also called a state variable. This means it cannot "jump"
- Consumption instead is a control variable than *can* jump

Dynamics: Graphically



Dynamics: Graphically



Dynamics:

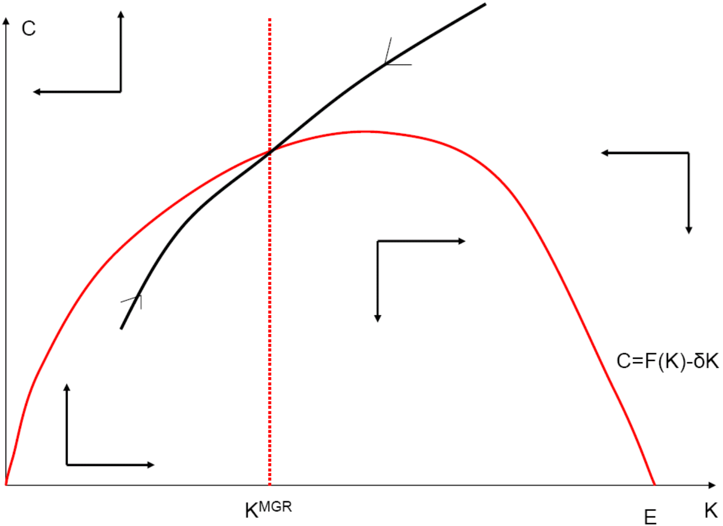
The trajectories A, B, C, and D cannot be optimal:

- Trajectories A and B will eventually lead to $k = 0$. At this point consumption will have to full suddenly from a positive number to zero. This cannot be optimal.
- Trajectories C and D will eventually lead to $c = 0$ as we get closer and closer to the point where the $H(k)$ locus hits the horizontal axis (indicated with an E). These paths cannot be optimal either. The reason is that along them, the transversality condition will be violated:

$$\lim_{(c,k)\to E} \beta^s u'(c)\, k = \infty$$

- Hence, the dynamics of (c, k) will be given by the saddle-path: There is a unique "dynamic" equilibrium and it will eventually take us to the steady-state where $k = k^{ss}$ and $c^{ss} = F(k^{ss}) - \delta k^{ss}$

Dynamics:



Local Dynamics Formally

If we constrain ourselves to local dynamics we can be even more precise

- The dynamics of the economy is described by the equations:

$$
\begin{aligned}
u'(c_s) &= \beta u'(c_{s+1})\left(F'(k_{s+1}) + (1-\delta)\right) \\
c_s &= F(k_s) - (k_{s+1} - k_s) - \delta k_s \\
\lim_{s\to\infty} \beta^s u'(c_s)\, k_{s+1} &= 0
\end{aligned}
$$

- This is a system of non-linear difference equations in the capital stock and consumption with a transversality condition
- The non-linearity makes it hard to solve. So we will linearize the system
- In order to linearize, we need to specify a point of approximation - we will choose this as the steady-state
- We then make a first-order Taylor approximation and I will use the notation $\widehat{c}_s = c_s - c^{ss}$ and $\widehat{k}_s = k_s - k^{MGR}$

Approximations

The Euler equation:

$$
\begin{aligned}
u'(c_s) &= \beta u'(c_{s+1})\left(F'(k_{s+1}) + (1-\delta)\right) \\
&\Rightarrow \\
u''(c^{ss})\,\widehat{c}_s &\simeq \beta\left(F'\left(k^{MGR}\right) + (1-\delta)\right) u''(c^{ss})\,\widehat{c}_{s+1} \\
&\quad +\beta u'(c^{ss})\,F''\left(k^{MGR}\right)\widehat{k}_{s+1} \\
&\Rightarrow \\
\widehat{c}_s &\simeq \widehat{c}_{s+1} + \beta\frac{u'(c^{ss})}{u''(c^{ss})}F''\left(k^{MGR}\right)\widehat{k}_{s+1}
\end{aligned}
$$

The Resource constraint:

$$
\begin{aligned}
c_s &= F(k_s) - (k_{s+1} - k_s) - \delta k_s \\
&\Rightarrow \\
\widehat{c}_s &\simeq \left[F'\left(k^{MGR}\right) + 1 - \delta\right]\widehat{k}_s - \widehat{k}_{s+1} \\
&= \frac{1}{\beta}\widehat{k}_s - \widehat{k}_{s+1}
\end{aligned}
$$

The Linearized Dynamics

We can formulate the dynamics close to the steady-state as:

$$\begin{bmatrix} 1, & \beta\frac{u'(c^{ss})}{u''(c^{ss})}F''\left(k^{MGR}\right) \\ 0, & 1 \end{bmatrix}\begin{bmatrix} \widehat{c}_{s+1} \\ \widehat{k}_{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1, & 0 \\ -1, & \frac{1}{\beta} \end{bmatrix}\begin{bmatrix} \widehat{c}_s \\ \widehat{k}_s \end{bmatrix}$$

which we can express as:

$$\begin{aligned} \begin{bmatrix} \widehat{c}_{s+1} \\ \widehat{k}_{s+1} \end{bmatrix} &= A\begin{bmatrix} \widehat{c}_s \\ \widehat{k}_s \end{bmatrix} \\ A &= \begin{bmatrix} 1+\beta\frac{u'(c^{ss})}{u''(c^{ss})}F''\left(k^{MGR}\right), & -\frac{u'(c^{ss})}{u''(c^{ss})}F''\left(k^{MGR}\right) \\ -1 & \frac{1}{\beta} \end{bmatrix} \end{aligned}$$

This system is saddle-path stable if the roots are real and if there are exactly one stable root (smaller than one in absolute value) and one unstable root (larger than one in absolute value)

The Roots

We find the roots from setting the determinant of $A - \lambda I$ equal to zero where λ is a scalar

$$|A - \lambda I| = \lambda^2 - \left(1 + \beta \frac{u'\left(c^{ss}\right)}{u''\left(c^{ss}\right)} F''\left(k^{MGR}\right) + \frac{1}{\beta}\right)\lambda + \frac{1}{\beta} = 0$$

The solution to this second order equation are

$$\begin{aligned} \lambda &= \frac{1}{2}\left(1 + \beta \frac{u'\left(c^{ss}\right)}{u''\left(c^{ss}\right)} F''\left(k^{MGR}\right) + \frac{1}{\beta}\right) \\ &\pm \frac{1}{2}\left[\left(1 + \beta \frac{u'\left(c^{ss}\right)}{u''\left(c^{ss}\right)} F''\left(k^{MGR}\right) + \frac{1}{\beta}\right)^2 - \frac{4}{\beta}\right]^{1/2} \end{aligned}$$

The Roots

Notice that the discriminant is positive:

$$
\begin{aligned}
D &= \left(1+\beta\frac{u'\left(c^{ss}\right)}{u''\left(c^{ss}\right)}F''\left(k^{MGR}\right)+\frac{1}{\beta}\right)^2-\frac{4}{\beta} \\
&= \left(1-\frac{1}{\beta}\right)^2+\left(\beta\frac{u'\left(c^{ss}\right)}{u''\left(c^{ss}\right)}F''\left(k^{MGR}\right)\right)^2 \\
& \quad 2\left(1+\frac{1}{\beta}\right)\beta\frac{u'\left(c^{ss}\right)}{u''\left(c^{ss}\right)}F''\left(k^{MGR}\right)
\end{aligned}
$$

which is positive

- Therefore the roots are real
- And it also follows that the first root exceeds 1 while the second root is smaller than 1.
- Thus, the system is saddle path stable

Some Experiments

How does the economy respond to:

1. A permanent decline in productivity
2. A permanent anticipated decline in productivity
3. A temporary decline in productivity

- We will think of productivity in terms of a shift in the production function. Suppose that we generalize the production function slightly:

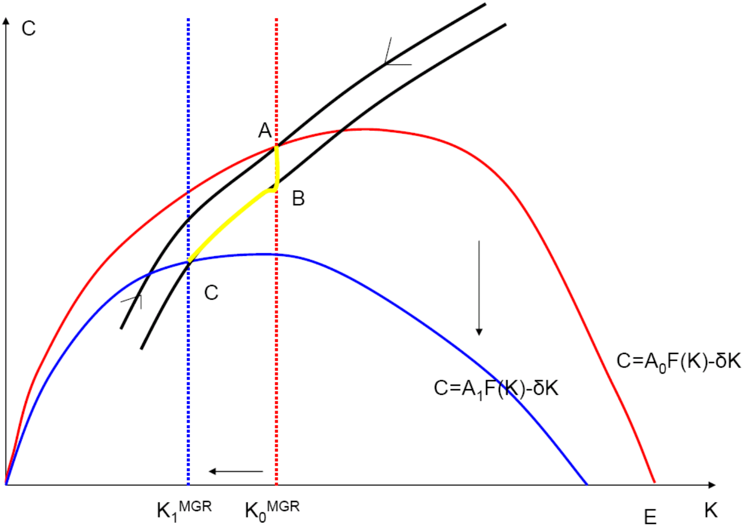
$$y_t = A_t F(k_t)$$

- then changes in A correspond to productivity shocks
- I will do 1. You will do 2 and 3 in the exercises.

A permanent decline in productivity

- A permanent decline in A leads to a decline in k^{MGR} as the marginal productivity of capital declines.
- It also shifts downwards the resource constraint
- Therefore, the saddle-path also shifts down
- These are shown in the diagram on the next page
- On the day that the decrease in productivity occurs:
  - Consumption declines suddenly. It declines exactly so much that we end up on the new saddle path. This is a jump from A to B
  - From then on, the economy moves along the new saddle path until we eventually end up in C. Thus, over the adjustment period we see a decline in consumption and in the capital stock (and therefore in investment). Output will also fall gradually.

A permanent decline in productivity



A Side-Remark on Dynamic Programming

When we analyzed the model earlier on we used a Lagrangean technique - but it's complicated since it's infinitely dimensional

- Are there other feasible approaches?
- Yes, dynamic programming. Recall that the planner's problem is:

$$\max_{(c_s, k_{s+1})_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

subject to:

$$\begin{aligned} c_s &= F(k_s) - (k_{s+1} - k_s) - \delta k_s \ \forall s \geq t \\ k_t &> 0 \text{ given} \end{aligned}$$

- I can also express the objective function as:

$$\max_{(C_s, K_{s+1})_{s=t}^{\infty}} \underbrace{u(C_s)}_{\text{today}} + \underbrace{\sum_{s=t+1}^{\infty} \beta^{s-t} u(C_s)}_{\text{future}}$$

- This is almost like a two-period problem

Dynamic Programming

- The problem has another nice feature - rewrite the resource constraints as:

$$\underbrace{c_s + k_{s+1}}_{\substack{\text{choices made} \\ \text{today}}} = \underbrace{F(k_s) + (1-\delta) k_s}_{\substack{\text{Choices made} \\ \text{in the past}}}$$

- Thus, the capital stock summarizes everything about the past that is relevant for today's choices
- These properties imply that the model is recursive - it can be unravelled over time - the past can be summarized by variables (in our case one variable) inherited from last period - no other aspects of the past are relevant

Dynamic Programming

Let me now exploit these features. First, let me define W_t as the maximum utility available from period t onwards:

$$W_t = \max_{(c_s, k_{s+1})_{s=t}^{\infty}} \left[u(c_s) + \sum_{s=t+1}^{\infty} \beta^{s-t} u(c_s) \right]$$

Notice then that:

$$W_{t+1} = \max_{(c_s, k_{s+1})_{s=t+1}^{\infty}} \left[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)} u(c_s) \right]$$

so, we can combine these two equations:

$$W_t = \max_{(c_s, k_{s+1})_{s=t}^{\infty}} \left[u(c_s) + \beta W_{t+1} \right]$$

subject to:

$$c_s + k_{s+1} = F(k_s) + (1-\delta) k_s$$

- Today's maximum utility stream is the utility we get today plus the maximum we can get from tomorrow onwards discounted back to today.

Dynamic Programming

- I noted earlier that the only aspect of the past that matters for day's choice is the inherited capital stock. Moreover, time as such doesn't really matter, so I can rewrite the problem as:

$$
\begin{aligned}
W(k) &= \max_{c,k'} \left[u(c) + \beta W'(k') \right] \\
& \quad s.t. \\
c + k' &= F(k) + (1-\delta) k
\end{aligned}
$$

- where k' denotes next period's value of k
- This equation is called a *Bellman equation* and the problem above is a *dynamic programming* problem and W is called a value function
- Since the optimization problem has infinite planning horizon, it seems plausible that the value function on the right hand side equals the value function on the right hand side

Dynamic Programming

- Indeed this is the case when $\beta < 1$ and under some further regularity conditions.
- In this case we have:

$$\begin{aligned} W(k) &= \max_{c,k'} \left[u(c) + \beta W(k') \right] \\ & \quad s.t. \\ c + k' &= F(k) + (1-\delta) k \end{aligned}$$

- This has a structure exactly like a two-period problem - there's today and tomorrow
- The reason why we can do this is that the Ramsey model is recursive

Dynamic Programming

- Now substitute the constraint into the objective:

$$W(k) = \max_{k'} \left[u\left(F(k) + (1-\delta)k - k'\right) + \beta W\left(k'\right) \right]$$

- First-order condition for k':

$$-\frac{\partial u(c)}{\partial c} + \beta \frac{\partial W(k')}{\partial k'} = 0$$

  and the derivative of the value function follows from the envelope condition:

$$\frac{\partial W(k)}{\partial k} = \frac{\partial u(c)}{\partial c} \left(\frac{\partial F(k)}{\partial k} + (1-\delta) \right)$$

- Combining these two equations gives us:

$$\frac{\partial u(c)}{\partial c} = \beta \frac{\partial u(c')}{\partial c'} \left(\frac{\partial F(k')}{\partial k'} + (1-\delta) \right)$$

- which is the Euler equation that we analyzed earlier but derived in a much simpler way from a two-period problem!!!

Fiscal Policy

Another nice illustration of the properties of the Ramsey model is to introduce fiscal policy

- Suppose the government purchases goods and services and after purchasing the goods, the government drops them in the ocean so they do not provide any services to the households
- Government purchases are financed through lump-sum taxes - or equivalently through debt issuance
- Due to lump sum taxation, we can still use the construct of the social planner (if taxes were distortionary, this would not be correct anymore)
- I will get back to this later when we look at the competitive solution

Fiscal Policy

The model is now:

$$\max_{(c_s, k_{s+1})_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

subject to:

$$\begin{aligned} c_s &= F(k_s) - (k_{s+1} - k_s) - \delta k_s - g_s \ \forall s \geq t \\ k_t &> 0 \text{ given} \end{aligned}$$

where g_s denotes government spending in period s.

- To make things simple, let me set $g_s = g$ for all s
- First-order conditions:

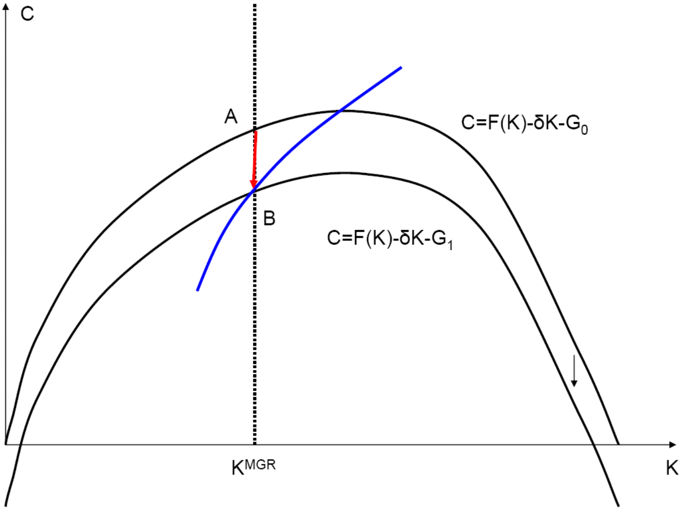
$$\begin{aligned} u'(c_s) &= \beta u'(c_{s+1}) \left(F'(k_{s+1}) + (1 - \delta) \right) \\ c_s &= F(k_s) - (k_{s+1} - k_s) - \delta k_s - g_s \\ \lim_{s \to \infty} \beta^s u'(c_s) k_{s+1} &= 0 \end{aligned}$$

Fiscal Policy

We see that the only thing that changes is the resource constraint - the Euler equation is unchanged

- Therefore, the Modified Golden Rule is still valid - the level of g will not affect the steady-state capital stock
- This implies that the level of government spending will not affect the level of output in the long run either
- So, if output and capital stocks do not depend upon level of g in the long run, the only effect of g is on consumption and it falls one-for-one with G
- Why? An increase in g has to be financed - taxes increase. This makes households poorer and they lower their consumption
- We will later see that these results do not generalize to models with labor supply, distortionary taxes etc.

Fiscal Policy - A Permanent Rise in G graphically



Investment and Adjustment Costs

The model so far has been one in which we can convert one unit of consumption goods into one unit of new capital with no added costs

- This is probably unrealistic - it takes time to build capital and it is costly
- We will now introduce investment adjustment costs and analyze how this impacts on the dynamics
- We will assume:

$$y_t = c_t + \left[1 + \frac{\phi}{2}\frac{i_t}{k_t}\right] i_t$$

- $\phi \geq 0$ parametrizes adjustment costs:
  - $\phi = 0$: Standard model with no adjustment costs
  - $\phi \rightarrow 0$: Investment becomes extremely costly
  - $\phi > 0$: Spread investment over time in order to save on adjustment costs

Adjustment Costs: The Central Planner's Problem

With adjustment costs, the central planner's problem is:

$$\max_{(c_s, k_{s+1}, i_s)_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

subject to:

$$\begin{aligned} c_s &= F(K_s) - \left[1 + \frac{\phi}{2} \frac{i_s}{k_s}\right] i_s \ \forall s \geq t \\ k_{s+1} &= (1-\delta) k_s + i_s, \ \forall s \geq t \\ k_t &> 0 \text{ given} \end{aligned}$$

and the Lagrangean becomes:

$$\begin{aligned} \mathcal{L} = \sum_{s=t}^{\infty} \beta^{s-t} \Big[& u(c_s) - \lambda_s \left(c_s - F(k_s) + \left[1 + \frac{\phi}{2} \frac{i_s}{k_s}\right] i_s \right) \\ & - \mu_s \left(k_{s+1} - (1-\delta) k_s - i_s \right) \Big] \end{aligned}$$

The Central Planner's Problem: Optimality Conditions

- μ_s is the shadow price of capital
- The first-order necessary conditions are:

$$
\begin{aligned}
c_s &: \quad u\left(c_s\right) = \lambda_s \\
i_s &: \quad \lambda_s \left\{1 + \phi \frac{i_s}{k_s}\right\} = \mu_s \\
k_{s+1} &: \quad \mu_s = \beta \left\{ \mu_{s+1}\left(1-\delta\right) + \lambda_{s+1}\left(F'\left(k_{s+1}\right) + \frac{\phi}{2}\left(\frac{i_{s+1}}{k_{s+1}}\right)^2 \right) \right\} \\
\lambda_s &: \quad c_s = F\left(k_s\right) - \left[1 + \frac{\phi}{2}\frac{i_s}{k_s}\right] i_s \\
\mu_s &: \quad k_{s+1} = \left(1-\delta\right) k_s + i_s \\
TVC &: \quad \lim_{s\to\infty} \beta^s u'\left(c_s\right) k_{s+1} = 0
\end{aligned}
$$

Investment and Adjustment Costs

- Impact of adjustment costs:
  - When $\phi = 0$ the investment condition implies that $\lambda_s = \mu_s$ and the first-order condition for k_{s+1} is identical to the standard model
  - When $\phi > 0$
    - the price of new capital exceeds the price of consumption (see condition for i_s)
    - the first-order condition for capital has an extra term which takes into account that increasing the capital stock saves on future adjustment costs
- From the first-order condition for investment we get that investment is determined as:

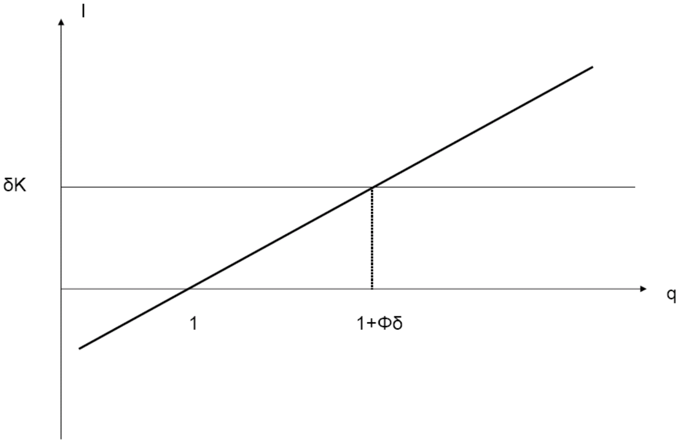
$$
\begin{aligned}
i_s &= \frac{1}{\phi}\left(\frac{\mu_s}{\lambda_s} - 1\right) k_s = \frac{1}{\phi}\left(q_s - 1\right) k_s \\
q_s &= \frac{\mu_s}{\lambda_s}
\end{aligned}
$$

Investment and Adjustment Costs

- q_s is the value of new capital relative to the value of consumption
- Thus, the condition from the previous slide implies that:
  - $q_s > 1$: Investment is positive
  - $q_s = 1$: Investment is zero
  - $q_s < 1$: Investment is negative
- Moreover, it follows from the capital accumulation equation that $i = \delta k$ in the steady-state
- Therefore, the steady-state value of q_s is:

$$\begin{aligned} \delta k &= \frac{1}{\phi}(q-1)k \\ &\Rightarrow \\ q^{ss} &= 1+\delta\phi \end{aligned}$$

Investment Graphics



The Capital Stock Dynamics

- Combining the first-order conditions for consumption, investment and capital we can write the latter conditions as:

$$F'(k_{s+1}) = q_s \frac{u'(c_s)}{\beta u'(c_{s+1})} - \frac{1}{2\phi}(q_{s+1}-1)^2 - q_{s+1}(1-\delta)$$

- Evaluating this at the steady-state implies:

$$\begin{aligned} F'(k^{ss}) &= \frac{q^{ss}}{\beta} - \frac{1}{2\phi}(q^{ss}-1)^2 - q^{ss}(1-\delta) \\ &= \left[\frac{1}{\beta} - (1-\delta)\right] + \delta\phi\left(\frac{1}{\beta} - 1 + \frac{\delta}{2}\right) \\ &> \left[\frac{1}{\beta} - (1-\delta)\right] \end{aligned}$$

- Hence, the steady-state capital stock is lower in this model than in the first model we examined with no adjustment costs

The Capital Stock Dynamics

- q can also be interpreted as the stock market valuation of the firm
- A first-order Taylor approximation to the quadratic term gives:

$$\begin{aligned}\frac{1}{2\phi}\left(q_{s+1}-1\right)^2 &\simeq \frac{1}{2\phi}\left(q^{ss}-1\right)^2+\frac{1}{2\phi}\left(q^{ss}-1\right)\left(q_{s+1}-q^{ss}\right) \\ &= \delta q_{s+1}-\frac{\phi\delta^2}{2}-\delta\end{aligned}$$

Using this and assuming that consumption is equal to its steady-state value gives:

$$\begin{aligned}F'\left(k_{s+1}\right) &= \frac{q_s}{\beta}+\frac{\phi\delta^2}{2}+\delta+q_{s+1} \\ &\Rightarrow \\ q_s-q^{ss} &= \beta\left(F'\left(k_{s+1}\right)-F'\left(k^{ss}\right)\right)+\beta\left(q_{s+1}-q^{ss}\right)\end{aligned}$$

- Iterating forwards gives:

$$q_s=q^{ss}+\sum_{h=1}^{\infty}\beta^h\left(F'\left(k_{h+1}\right)-F'\left(K^{ss}\right)\right)$$

Dynamics - Summing up

We have the following equations:

$$
\begin{aligned}
k_{s+1} &= (1-\delta)\, k_s + i_s \\
i_s &= \frac{1}{\phi}(q_s - 1)\, k_s \\
F'(k_{s+1}) &= q_s \frac{u'(c_s)}{\beta u'(c_{s+1})} - \frac{1}{2\phi}(q_{s+1}-1)^2 - q_{s+1}(1-\delta)
\end{aligned}
$$

I will now derive the local dynamics of k and q by approximating these close to the steady-state *and* holding constant consumption at its steady-state value

Dynamics - of the capital stock

Combining the first two equations gives us:

$$\begin{aligned} k_{s+1} &= (1-\delta) k_s + \frac{1}{\phi} (q_s - 1) k_s \\ \phi k_{s+1} &= \phi k_s + (q_s - 1 - \phi\delta) k_s \\ &\Rightarrow \\ \phi (k_{s+1} - k_s) &= (q_s - q^{ss}) k_s \end{aligned}$$

which when linearized around the steady-state implies:

$$\phi (k_{s+1} - k_s) \simeq k^{ss} (q_s - q^{ss})$$

(notice that the term with k_s on the right hand side drops out when linearizing)

- when $q_s > q^{ss}$ the capital stock grows and vice versa

Dynamics - of q

We have the expression:

$$F'(k_{s+1}) = q_s \frac{u'(c_s)}{\beta u'(c_{s+1})} - \frac{1}{2\phi}(q_{s+1} - 1)^2 - q_{s+1}(1 - \delta)$$

Which we can linearize (holding C constant) as (see appendix for derivation):

$$\left(\frac{1}{\beta} - 1\right)(q_s - q^{ss}) \simeq F''(k^{ss})(k_s - k^{ss}) + \delta(q_{s+1} - q_s)$$

So q is constant when:

$$\left(\frac{1}{\beta} - 1\right)(q_s - q^{ss}) \simeq F''(k^{ss})(k_s - k^{ss})$$

which is a negatively sloped line in a (k, q) diagram because $\frac{1}{\beta} - 1 > 0$ and $F''(k^{ss}) < 0$

Dynamics - summary

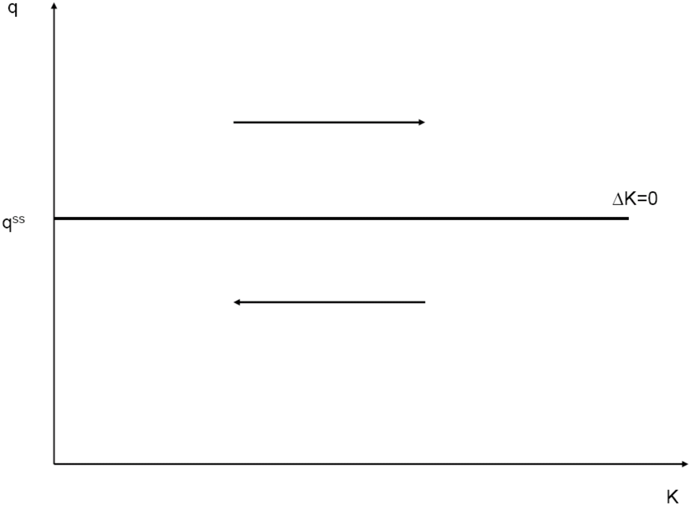
We have:

$$k \quad : \quad \begin{pmatrix} \triangle\, k > 0 \text{ when } q > q^{ss} = 1 + \delta\phi \\ \triangle\, k = 0 \text{ when } q = q^{ss} \\ \triangle\, k < 0 \text{ when } q < q^{ss} \end{pmatrix}$$

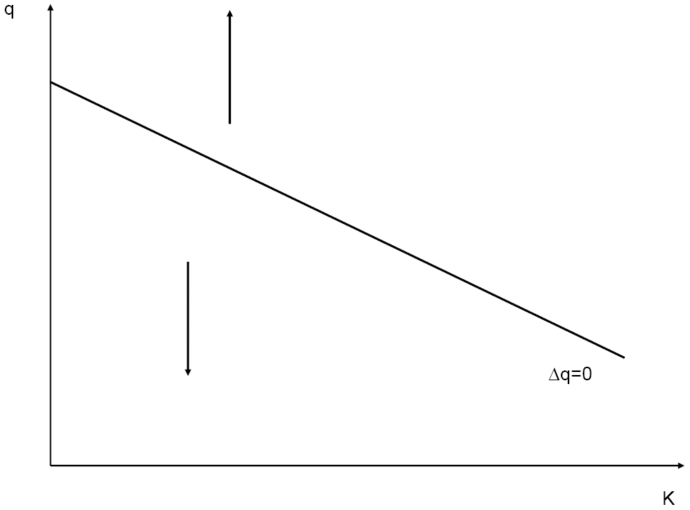
$$q \quad : \quad \begin{pmatrix} \triangle\, q > 0 \text{ when } \left(\frac{1}{\beta} - 1\right)(q_s - q^{ss}) > F''(k^{ss})(k_s - k^{ss}) \\ \triangle\, q = 0 \text{ when } \left(\frac{1}{\beta} - 1\right)(q_s - q^{ss}) = F''(k^{ss})(k_s - k^{ss}) \\ \triangle\, q < 0 \text{ when } \left(\frac{1}{\beta} - 1\right)(q_s - q^{ss}) < F''(k^{ss})(k_s - k^{ss}) \end{pmatrix}$$

- This system is saddle path stable

Graphics



Graphics



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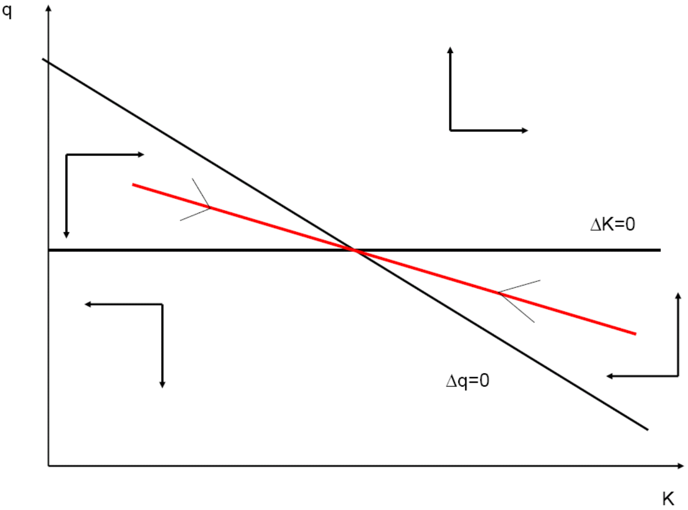
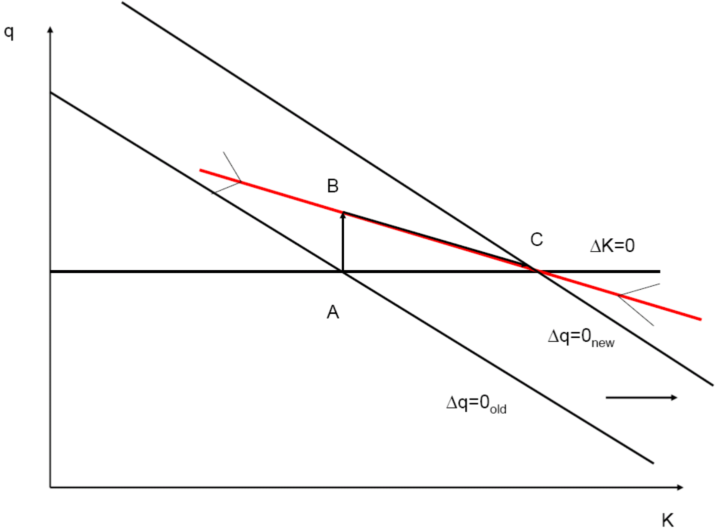


Experiment: Productivity Increases

Suppose productivity increases

- Shifts outwards the $\triangle q = 0$ locus because k^{ss} increases and it gets a bit steeper as well
- On the day that it happens, the stock market jumps from A to B
- Thereafter, the stock market remains in a boom until we get to C
- Essentially, the value of new capital increases after the productivity increase
- But because of adjustment costs, the adjustment of capital occurs gradually over time

Experiment: Productivity Increases



Appendix: The q dynamics

$$\begin{aligned}
F'(k_{s+1}) &= q_s \frac{u'(c_s)}{\beta u'(c_{s+1})} - \frac{1}{2\phi}(q_{s+1}-1)^2 - q_{s+1}(1-\delta) \\
&\text{Take 1'st order Taylor Expansion} \\
F''(k^{ss})(k_s - k^{ss}) &= \frac{1}{\beta}(q_s - q^{ss}) - \frac{1}{\phi}(q^{ss}-1)(q_{s+1}-q^{ss}) \\
&\quad -(1-\delta)(q_s - q^{ss}) \\
\text{Use that } q^{ss} &= 1+\delta\phi \\
F''(k^{ss})(k_s - k^{ss}) &= \frac{1}{\beta}(q_s - q^{ss}) - \frac{1}{\phi}(\delta\phi)(q_{s+1}-q^{ss}) \\
&\quad -(1-\delta)(q_s - q^{ss})
\end{aligned}$$

Appendix: The q dynamics

Simplify

$$F''(k^{ss})(k_s - k^{ss}) = \frac{1}{\beta}(q_s - q^{ss}) - \delta(q_{s+1} - q^{ss}) - (1-\delta)(q_s - q^{ss})$$

Add and subtract q_s on the right hand side

$$F''(k^{ss})(k_s - k^{ss}) = \left(\frac{1}{\beta} - 1\right)(q_s - q^{ss}) - \delta(q_{s+1} - q^{ss}) + \delta(q_s - q^{ss})$$

Rearrange

$$\left(\frac{1}{\beta} - 1\right)(q_s - q^{ss}) = F''(k^{ss})(k_s - k^{ss}) + \delta(q_{s+1} - q_s)$$