MSC Macroeconomics G022, 2009

Introduction

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Course Instructors

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- Office: Drayton House, office 231
- Second part taught by *Guy Laroque*
- Classes taught by:
  - *Andreas Uthemann*, also does the practical
  - *Alejandro Tumola*
  - *James Cloyne*

Where and When:

- Lectures: Tuesdays 14-15:30 Harrie Massie LT, 16:30-18:00 Fleming LT
- Tutorials: check the timetable

What should you do?

- Read the references
  - main text is *Michael R. Wickens*, “Macroeconomic Theory: A Dynamic General Equilibrium Approach”, Princeton University Press
  - a close substitute is *David Romer*, “Advanced Macroeconomics”, McGraw-Hill, 3rd Edition, 2006.
- Come to the lectures
- Do the exercises
- Come to the practical and the tutorials

Today:

1. Introduction and motivation
2. Recap on Solow Model
3. Recap on Two-Period Model with and without Production
4. Welfare Theorems
5. Uncertainty and Expectations

What will we do?

- Modern macroeconomics
    - Dynamics models
    - Based on welfare based choice models - microfounded models
    - General equilibrium
    - stochastic
- sometimes we might deviate slightly from these principles, but they will be maintained in most of the analysis that we will undertake

Topics

- The Ramsey Model
- Long Run Growth
- Business Cycles
- Fiscal Policy (time permitting)

What is it all about?

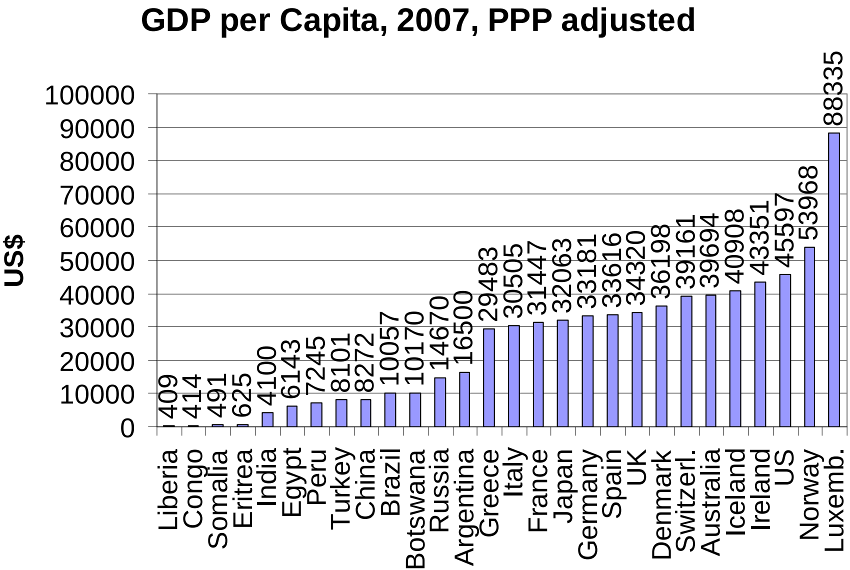
The aims of the course are:

- learn some standard models of modern macroeconomics
- obtain a toolbox that enables you to think about macroeconomics in a systematic manner
- acquire research skills
- get used to the idea that models are wrong but extremely helpful - models are vast simplifications of reality since there is no way we can capture absolutely everything

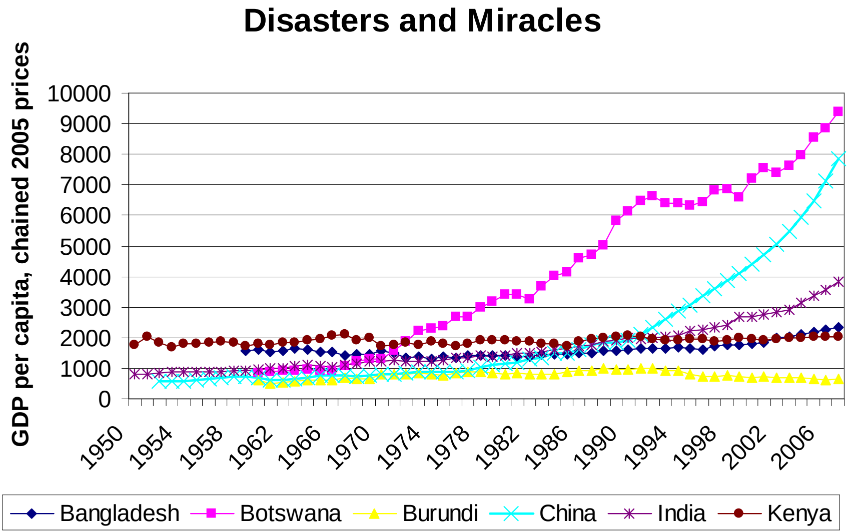
Macroeconomic Questions:

- What determines the level of income in the economy?
- What determines the level of unemployment?
- How does economic policy impact on the economy?
- Optimal policy responses
- What determines long run growth rates?
- What leads to fluctuations in the economy over time?
- What typically happens during booms and recessions?
- Etc.

Level of Income Comparison, 2007 (PWT data)



Growth Experiences (chained GDP per capita)



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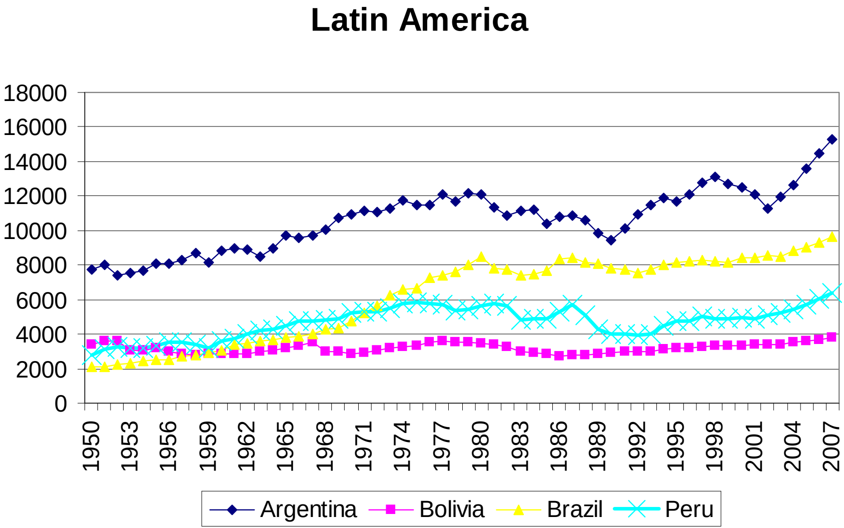
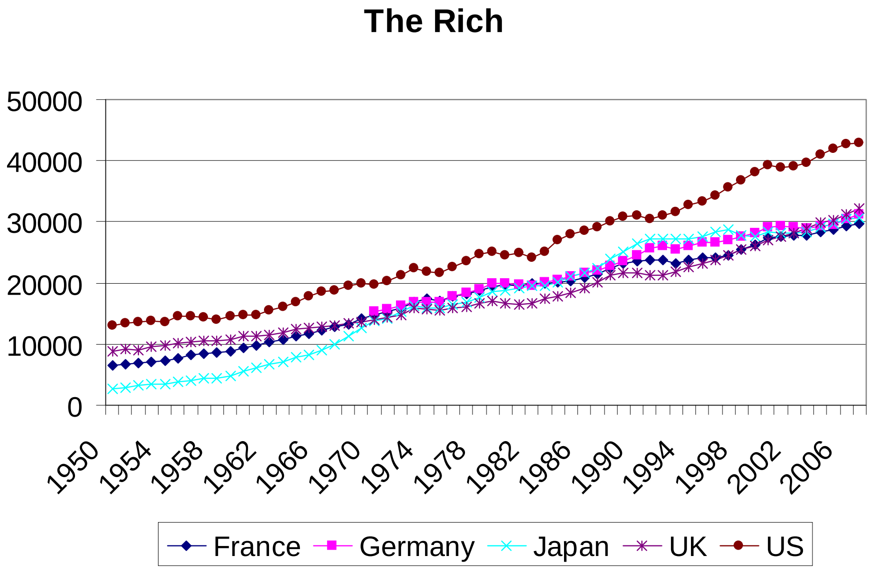
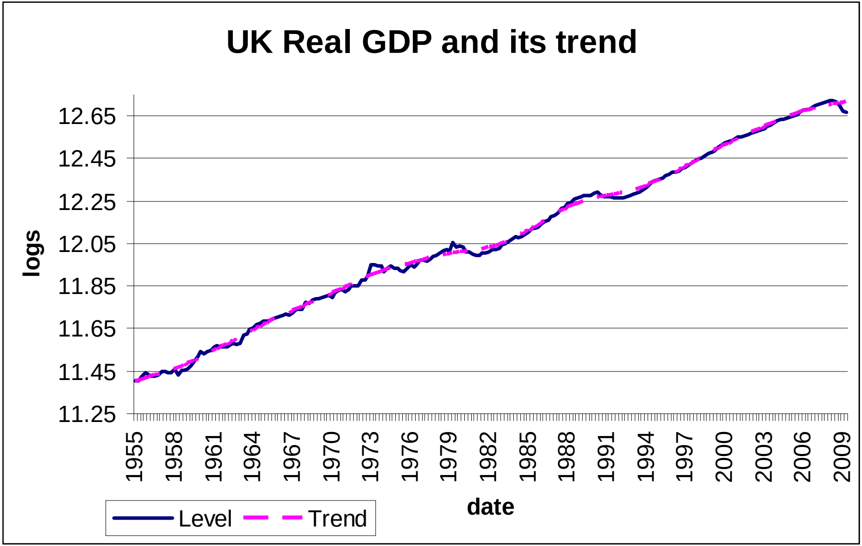


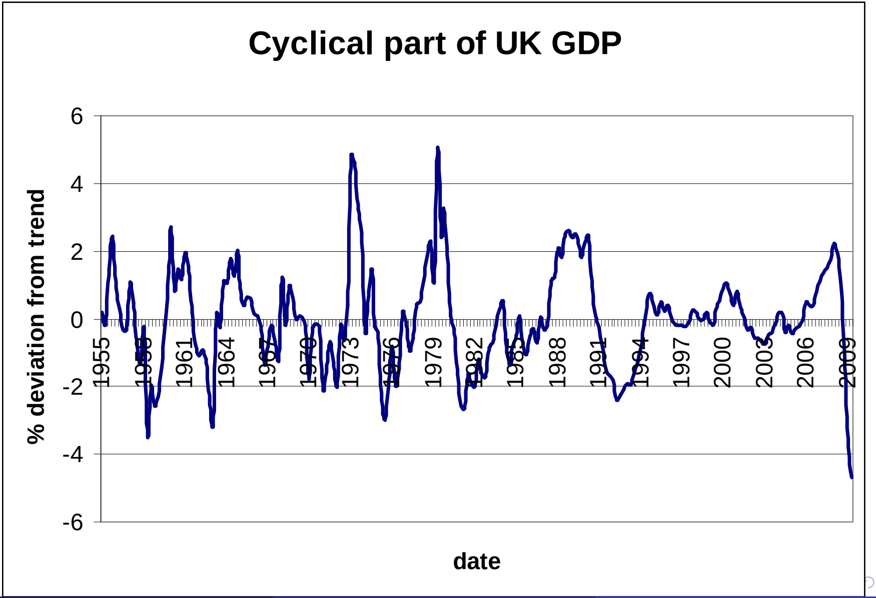
Growth Experiences (chained GDP per capita)



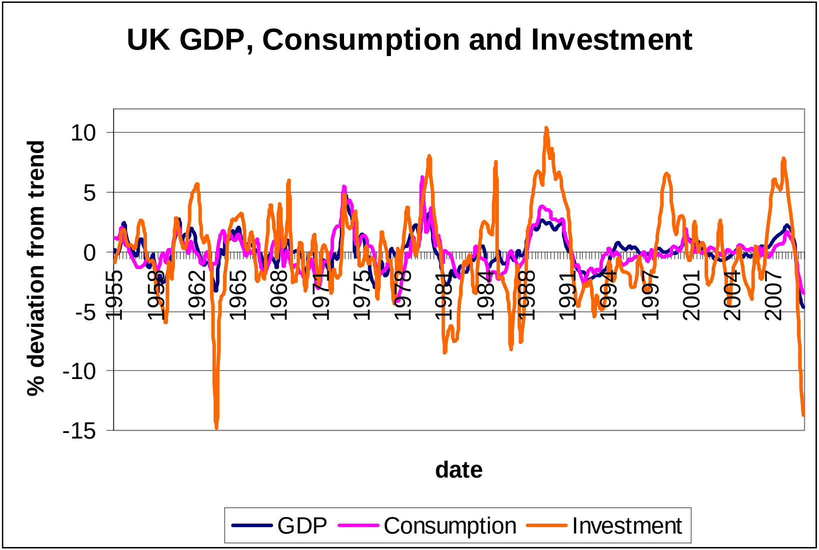
The UK economy 1955-2009



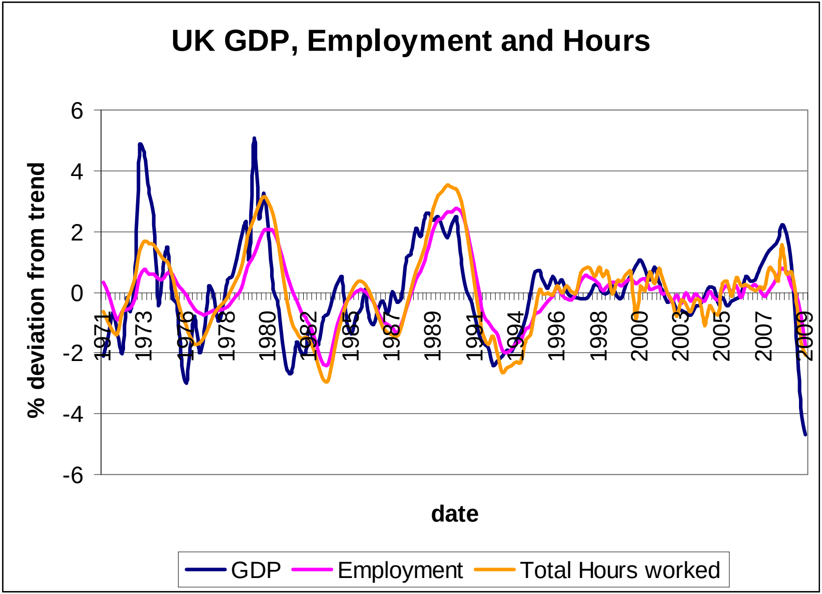
The UK Business Cycle



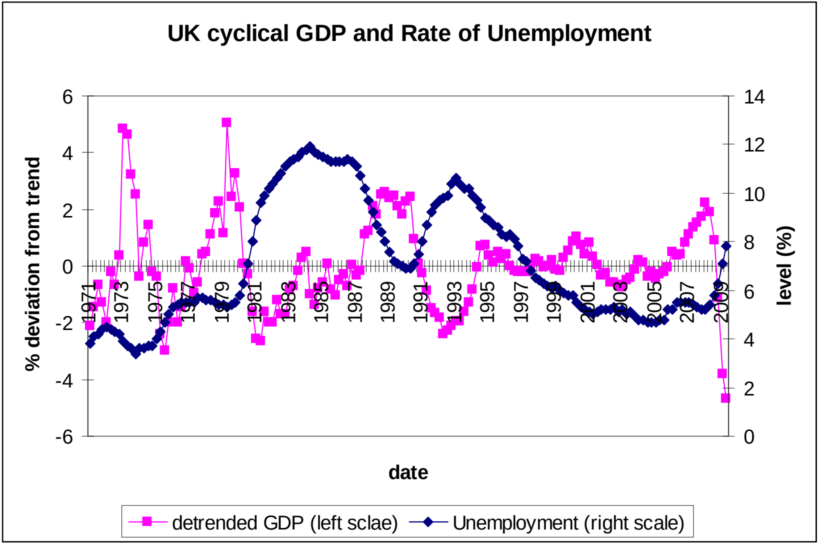
Business Cycle Comovement: Expenditures



Business Cycle Comovement: The Labor Market



Business Cycle Comovement: The Labor Market



Many interesting issues ....

- We won't be able to address them all
- But: to think about them we need models and theories
- A theory:
  - It is a simplification
  - It is false
  - But: If it's a good theory, it will be helpful
- We will think about micro-founded, dynamic (sometimes stochastic), general equilibrium theories
  - micro-founded: We can address welfare, we can address the Lucas critique
  - dynamic: We can say something about adjustment and growth
  - general equilibrium: The models are internally consistent

A quick review of the Solow growth model

The Solow Model:

- dynamic model of capital accumulation
- it features production and capital accumulation
- and it incorporates one key assumption: Diminishing marginal returns to the accumulable factor (capital)
- but very little behavior - in particular, we will assume an exogenous savings function
- understanding well this model will turn out to be very useful for all kinds of stuff in this course

The Solow growth model

- Technology: A neoclassical production function:

$$Y_t = F(\underbrace{K_t}_{\text{capital}}, \underbrace{N_t}_{\text{labor}})$$

where we assume:

$$\begin{aligned}
A &: F(K_t, 0) = F(0, N_t) = 0 \\
B &: F_i'(K_t, N_t) \geq 0 \text{ for } i = K, N \\
C &: F_{ii}''(K_t, N_t) < 0 \text{ for } i = K, N \\
D &: F_K(K_t, N_t) K_t + F_N(K_t, N_t) N_t = F(K_t, N_t)
\end{aligned}$$

Technology con't

- The latter of these assumptions implies that the production function displays constant returns to scale
- We can therefore express per worker output as:

$$
\begin{aligned}
y_t &= \frac{Y_t}{N_t} = F\left(\frac{K_t}{N_t}, 1\right) = f(k_t) \\
k_t &= \frac{K_t}{N_t}
\end{aligned}
$$

- We will also be assuming that:

$$
\begin{aligned}
\lim_{k \to 0_+} f'(k) &= \infty \\
\lim_{k \to \infty} f'(k) &= 0
\end{aligned}
$$

- which mean that the production function becomes vertical when capital per worker approaches zero and completely flat when capital per worker becomes very large

Capital Accumulation

- The fundamental capital accumulation equation is given as:

$$\underbrace{K_{t+1}}_{\substack{\text{capital at}\\ \text{beginning of}\\ \text{period } t+1}} = \underbrace{K_t}_{\substack{\text{capital at}\\ \text{beginning of}\\ \text{period } t}} - \underbrace{\delta K_t}_{\substack{\text{depreciation}\\ \text{during period}\\ t}} + \underbrace{I_t}_{\substack{\text{investment}\\ \text{during period}\\ t}}$$

- which obviously can be expressed as:

$$K_{t+1} = (1-\delta)\, K_t + I_t$$

- $\delta \in (0,1]$ is the rate of depreciation of the capital stock per period

Labor Market and Population Dynamics

- We will assume that labor supply is exogenous and that the labor market is competitive
- For that reason employment equals population and we will let N_t denote population as well as employment
- Population grows at a constant rate n :

$$N_{t+1} = (1+n) N_t$$

- We can therefore express the capital accumulation equation in per capita terms as:

$$\begin{aligned} \frac{K_{t+1}}{N_t} &= \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = (1-\delta) \frac{K_t}{N_t} + \frac{I_t}{N_t} \\ &\Rightarrow \\ k_{t+1}(1+n) &= (1-\delta) k_t + i_t \end{aligned}$$

- Population growth effectively depreciates the capital stock

Savings

- In the Solow model, savings are assumed to be given as a fixed fraction of output (income):

$$S_t = sY_t$$

or in per capita terms:

$$s_t = sy_t$$

- where $s \in (0, 1)$ is the exogenously given savings rate
- The more advanced models that we will see later effectively endogenize the savings rate
- The assumption that this rate is constant is obviously strong but it simplifies the analysis a lot
- Nevertheless, the simplifying assumption also comes with a big cost which is that we cannot say much about welfare - that's why we later have to work hard to endogenize savings

General Equilibrium

- So far we have cleared the labor market: But we also need to clear the goods market
- We will assume that the economy is closed to trade in goods and assets
- This assumption implies that the good market clearing condition is that national savings equal national investment:

$$\begin{aligned} S_t &= I_t \Rightarrow \\ s_t &= i_t \end{aligned}$$

This condition follows from the national accounts identity that output must equal its final use:

$$\begin{aligned} Y_t &= C_t + I_t \\ &\Rightarrow \\ &\quad \underbrace{Y_t - C_t}_{\text{savings}} = I_t \end{aligned}$$

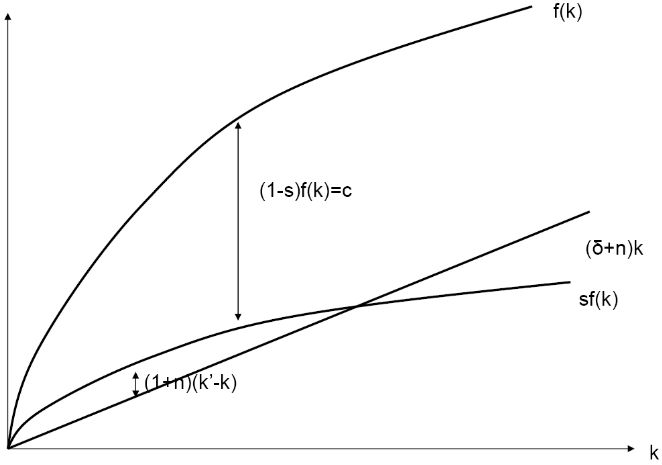
Dynamics

- Combining the relationships, we then get that the general equilibrium is given by:

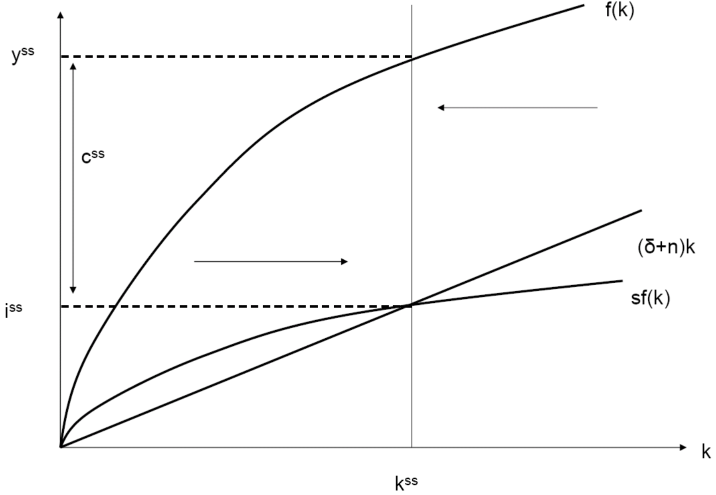
$$
\begin{aligned}
i_t &= sy_t \\
&\Rightarrow \\
(1+n)\,k_{t+1} - (1-\delta)\,k_t &= sf\,(k_t) \\
&\Rightarrow \\
(1+n)\,(k_{t+1} - k_t) &= sf\,(k_t) - (\delta+n)\,k_t
\end{aligned}
$$

- which is a non-linear deterministic difference equation in k_{t+1}

Dynamics - Graphically



Dynamics - Graphically



The Steady-State

The economy has two rest points:

1. $k^* = 0$: This is an unstable rest point - if there is no capital to begin with, the economy gets stuck in a poverty trap. But, this situation is unstable: Whenever we have a little bit of capital to begin with, we will move away from this point
2. $k^* = k^{ss}$: This is a stable rest point. Whenever $k_0 > 0$, we will in the long run always converge to this level of the per capita capital stock. We will call this for the steady-state of the economy.

- We can then summarize the dynamics of the model as:

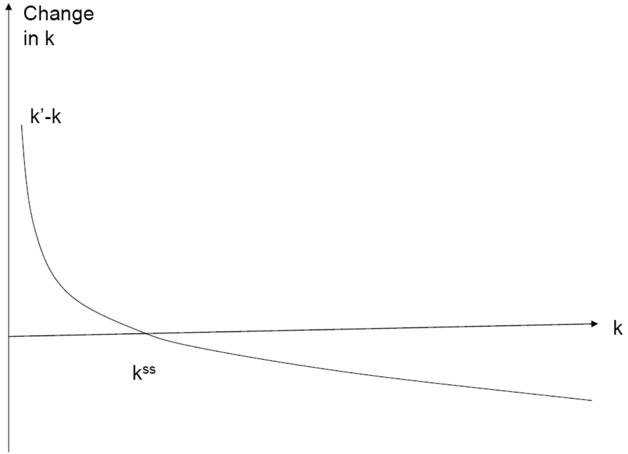
$$\begin{aligned} 0 &< k_0 < k^{ss} : k_{t+1} - k_t > 0 \\ k_0 &> k^{ss} : k_{t+1} - k_t < 0 \\ k_0 &= 0 : k_{t+1} = k_t = 0 \\ k_0 &= k^{ss} : k_{t+1} = k_t = k^{ss} \end{aligned}$$

The dynamics again

- Intuitively, the dynamics are explained as:
  - When the initial capital stock is low (positive but below k^{ss}), the marginal product of capital is high relative to the effective depreciation rate. Therefore, the economy will be growing.
  - When the initial capital stock is very large (above k^{ss}), the marginal product of capital falls below the effective depreciation rate. Therefore, the capital stock will diminish over time and the economy will be shrinking.
  - The closer we get to the steady-state from below k^{ss}, the slower the economy will grow
- The economy has a unique stable steady with a positive capital stock
  - This implies that the economy cannot forever grow from capital accumulation - eventually it settles down in the steady-state
  - moreover, long run growth rate is independent of the savings rate - a higher savings rate means a higher steady-state level of income, and an increase in the savings rate increases the economy's growth rate temporarily. But the long run growth rate is not affected.

The growth diagram

We can also illustrate the dynamics of the model in this diagram:



The Golden Rule

- Notice that consumption is simply given as:

$$c_t = (1 - s) y_t$$

- We could then ask: "What is the optimal savings rate?"
- The question is imprecise because we have not specified preferences
- However, it would make some sense to derive the savings rate that maximizes steady-state consumption
- We can find this from:

$$\begin{aligned} \max_s c^{ss}(s) &= y^{ss}(s) - s y^{ss}(s) \\ &= f\left(k^{ss}(s)\right) - (\delta + n) k^{ss}(s) \end{aligned}$$

- which has the first-order condition:

$$f'\left(k^{ss}\left(s^{opt}\right)\right) = (\delta + n)$$

The Golden Rule

- The savings rate that maximizes steady-state consumption is the one that leads the marginal product of capital being equal to the gross depreciation rate of capital - this is called the *Golden Rule savings rate*
- Why is this interesting?
- A savings rate higher than the Golden Rule can never be optimal:
  - Suppose we were at the steady-state associated with the Golden Rule: An increase in the savings rate will decrease both short run and long run consumption - this cannot be a good idea if we prefer more consumption to less
- A savings rate lower than the Golden Rule might be optimal:
  - Repeat the experiment above: While consumption falls in the long-run, it increases in the short run.

A Two Period Model of Utility Maximization

In this course, we will see a lot of dynamic models of optimization and much of it in infinite horizon models

- Here, I will consider a two-period model in order to set the scene for what is to come
- A consumer lives for two periods. Her name is 'i'
- She receives income x_1^i in period 1 and income x_2^i is period 2
- She wishes to consume in both periods, c_1^i and c_2^i and can borrow or lend at the interest rate $r > 0$
- The good is itself non-storable so the consumer cannot herself carry it from one period to another
- The consumer starts off her life with no assets and must leave with no debt

Preferences

The consumer values two goods, c_1^i and c_2^i, and we can therefore express her preferences as:

$$U^i = U\left(c_1^i, c_2^i\right)$$

We will assume that this utility function is increasing and concave in each of its arguments:

$$\begin{aligned} U_j'\left(c_1^i, c_2^i\right) &> 0 \text{ for } j = c_1^i, c_2^i \\ U_{jj}''\left(c_1^i, c_2^i\right) &< 0 \text{ for } j = c_1^i, c_2^i \end{aligned}$$

and we will most often also assume that:

$$\lim_{c_j^i \to 0} U_j'\left(c_1^i, c_2^i\right) = \infty$$

- this last assumption reassures that consumption remains positive in both periods

Preferences - the Intertemporal Aspect

The preferences above are fairly general and parallel to preferences that we typically would assume in models with choice over multiple commodities However, in dynamic settings, it is typical to specialize the preferences

- The most standard is to assume that preferences are intertemporally separable:

$$U^i = U\left(c_1^i, c_2^i\right) = u\left(c_1^i\right) + \beta u\left(c_2^i\right)$$

- where
    - β is a subjective discount factor. We will in most cases require that agents are impatient which implies that $0 < \beta < 1$
    - The lower is β, the more the agent discounts the future (the agent is impatient)
    - Some times we will also refer to $\beta = 1/\left(1+\rho\right)$ where ρ is the discount rate and impatience implies $\rho > 0$
- Note that, apart from discounting, the utility function is the same in both periods

Preferences - the Intertemporal Aspect

- We could consider more general specifications:
- **Habit persistence**: Agents derive utility today from current consumption relative to a "habit". If the habit is simply last period's consumption, this corresponds to:

$$U^i = U\left(c_1^i, c_2^i\right) = u\left(c_1^i - \phi c_0^i\right) + \beta u\left(c_2^i - \phi c_1^i\right)$$

- $\phi > 0$ determines the strength of the habit. Thus, utility depends on the "quasi" difference of consumption
- Catching-up-with-the-Joneses case (or external habits):

$$U^i = U\left(c_1^i, c_2^i\right) = u\left(c_1^i - \phi \overline{c}_0\right) + \beta u\left(c_2^i - \phi \overline{c}_1\right)$$

  where $\overline{c}_t$ denotes the consumption level of some reference group (the Joneses) in period t
- In both cases, utility is no longer intertemporally separable - utility today depends directly on past variables

Preferences - the Intertemporal Aspect

- **Hyperbolic discounting**: Suppose that we have a three period model. In the standard case we would write:

$$U^i = U\left(c_1^i, c_2^i, c_3^i\right) = u\left(c_1^i\right) + \beta u\left(c_2^i\right) + \beta^2 u\left(c_3^i\right)$$

  which means constant discounting

- In experiments, agents often appears to behave according to the following specification:

$$U^i = U\left(c_1^i, c_2^i, c_3^i\right) = u\left(c_1^i\right) + \theta\beta u\left(c_2^i\right) + \theta\beta^2 u\left(c_3^i\right), \ \theta \in (0, 1)$$

- where the discounting between periods 1 and 2 is $\theta\beta$ while that between 2 and 3 is β
- In this case, agents behave in a time-inconsistent manner: They value today over tomorrow more strongly than tomorrow over the day after that. However, when tomorrow comes they will value tomorrow more highly over the day after tomorrow than they were planning to today.

Budget constraints

- The agent starts life without assets and must leave the economy without debt. The budget constraints are therefore:

$$\begin{aligned} c_1^i + b_1^i &= x_1^i && (1) \\ c_2^i + b_2^i &= (1+r)\, b_1^i + x_2^i && (2) \\ b_2^i &\geq 0 && (3) \end{aligned}$$

- where (1) is the agent's budget constraint in period 1, and (2) is the agent's budget constraint in period 2
- b_1^i denotes the agent's purchases of assets in period 1 and b_2^i denotes the agent's purchases of assets in period 2
- if $b_1^i > 0$ the agent saves in period 1 while $b_1^i < 0$ means that the agent issues debt and is a borrower
- The interest rate on the asset is fixed at r and the asset can therefore be thought of as a debt contract (a bond)

The budget constraint

- The condition in equation (3) is a borrowing constraint. It says that the agent cannot leave the end of period 2 with any debt
- Since the agent values consumption, it also cannot be the case that $b_2^i > 0$ since this would lead to a waste of resources
- Therefore, we will have that $b_2^i = 0$
- In this case, we can combine the two budget constraints to obtain a single lifetime budget constraint:

$$\underbrace{c_1^i + \frac{c_2^i}{1+r}}_{\substack{\text{present discounted value of} \\ \text{consumption expenditure}}} = \underbrace{x_1^i + \frac{x_2^i}{1+r}}_{\substack{\text{present discounted value} \\ \text{of endowment stream}}}$$

The optimization problem

- We can now formulate the optimization problem as:

$$\max_{c_1^i, c_2^i \in \Gamma} u\left(c_1^i\right) + \beta u\left(c_2^i\right)$$

where Γ is the budget set

$$\Gamma = \left\{ \left(c_1^i, c_2^i\right) | c_1^i + \frac{c_2^i}{1+r} = x_1^i + \frac{x_2^i}{1+r} \right\}$$

- This is a standard constrained maximization problem of a concave function subject to a closed and convex budget set

Solving the maximization problem

- There are different ways of solving this problem.
- Here I will use a Lagrangian method which is the standard way of solving constrained optimization problems
- The Lagrangean is given as:

$$\mathcal{L} = u\left(c_1^i\right) + \beta u\left(c_2^i\right) - \lambda\left(c_1^i + \frac{c_2^i}{1+r} - x_1^i - \frac{x_2^i}{1+r}\right)$$

- The first-order necessary conditions are:

$$\begin{aligned}
c_1^i &: \quad u'\left(c_1^i\right) - \lambda = 0 \\
c_2^i &: \quad \beta u'\left(c_2^i\right) - \lambda \frac{1}{1+r} = 0 \\
\lambda &: \quad c_1^i + \frac{c_2^i}{1+r} = x_1^i + \frac{x_2^i}{1+r}
\end{aligned}$$

The Euler equation

- Combining the first order conditions for c_1^i and c_2^i implies that:

$$u'\left(c_1^i\right) = (1+r)\,\beta u'\left(c_2^i\right) \tag{4}$$

or if we rearrange:

$$\frac{\beta u'\left(c_2^i\right)}{u'\left(c_1^i\right)} = \frac{1}{(1+r)}$$

- The left hand side of this is the (negative of) *the intertemporal marginal rate of substitution* between consumption in periods 1 and 2 - it is the slope of the indifference curve:

$$\frac{dc_1^i}{dc_2^i}\Big|_{dU=0} = -\frac{\beta u'\left(c_2^i\right)}{u'\left(c_1^i\right)}$$

- The right hand side is instead the price of period 2 consumption relative to period 1 consumption - to buy one unit of period 2 consumption, the agent needs to give up $1/\left(1+r\right)$ units of period 1 consumption

The Euler equation

- Hence the Euler equation implies that the growth in marginal utility is inversely related to the interest rate
- This implies that:
  - If $\beta > \frac{1}{(1+r)}$, the agent wishes to "backload" consumption, i.e. $c_2^i > c_1^i$
  - If $\beta = \frac{1}{(1+r)}$, the agent wishes to equalize consumption across periods
  - If $\beta < \frac{1}{(1+r)}$, the agent wishes to frontload consumption, i.e. $c_1^i > c_2^i$
- We can then deduce the agent's behavior - her optimal choices - from the Euler equation and the budget constraint:

$$\begin{aligned} \frac{\beta u'\left(c_2^i\right)}{u'\left(c_1^i\right)} &= \frac{1}{(1+r)} \\ c_1^i + \frac{c_2^i}{1+r} &= x_1^i + \frac{x_2^i}{1+r} \\ s_1^i &= x_1^i - c_1^i \end{aligned}$$

General Equilibrium

- Now assume that there are many (a continuum of mass 1) consumers exactly like the one outlined above
- Assume also for now that the consumers are *identical* and that $x_1^i = x_2^i = x$ so that endowments are constant over time and agents
- Since agents are identical they will make the same choices so that $c_1^i = c_1$, $c_2^i = c_2$ and $b_1^i = b_1$
- We then need to clear goods and asset markets

$$\begin{aligned} c_1 &= x \\ c_2 &= x \\ b_1 &= 0 \end{aligned}$$

- where the two first of these are the goods market clearing conditions and the last one is the asset market clearing condition

General Equilibrium

- These conditions determine the equilibrium real interest rate
- We have from the Euler equation that:

$$\frac{\beta u'\left(c_2^i\right)}{u'\left(c_1^i\right)} = \frac{1}{(1+r)}$$

- We can now evaluate this in equilibrium:

$$\begin{aligned} \frac{\beta u'(x)}{u'(x)} &= \frac{1}{(1+r)} \\ &\Rightarrow \\ \beta &= \frac{1}{(1+r)} \\ &\Rightarrow \\ r &= \frac{1}{\beta} - 1 > 0 \end{aligned}$$

- So in equilibrium, the interest rate implements that consumption equals endowments. Since agents are impatient, this requires that the real interest rate is positive

General Equilibrium

- A slightly different formulation of the problem is to express the budget constraint as:

$$p_1 c_1^i + p_2 c_2^i = p_1 x_1^i + p_2$$

- where p_1 is the price of goods in period 1 and p_2 is the price of goods in period 2. The Euler equation becomes:

$$\frac{\beta u'\left(c_2^i\right)}{u'\left(c_1^i\right)} = \frac{p_2}{p_1}$$

- Hence, the gross interest rate is simply:

$$1 + r = \frac{p_1}{p_2}$$

- Since the real interest rate is positive, this implies that prices must be falling over time: If prices were constant, the agent would prefer to consume today rather than tomorrow and we couldn't have that in equilibrium $c_1 = c_2$

General Equilibrium

- Somewhat more formally, what we have found here is the competitive equilibrium

Definition

A Competitive Equilibrium is a price system (p_1, p_2) (or r) and an allocation (c_1^I, c_2^I) such that (i) Households maximize their utility subject to their budget constraints (utility maximization), and (ii) Goods and asset markets clear (feasibility)

- The idea is that we find the competitive equilibrium by first deriving individual decision rules and then letting prices clear the markets
- This approach - which we applied above - can sometimes be cumbersome, and we would also like to know something about its properties

Pareto Optimum

An alternative allocation that would be of interest is the Pareto optimum:

Definition

A Pareto Optimal allocation is an allocation $\left(c_1^{PO}, c_2^{PO}\right)$ such that the allocation maximizes utility subject to the economy's resource constraint

- In a Pareto optimum there is no way of making one (or more) consumer better off without making at least one consumer worse off
- This allocation therefore has an efficiency property: We cannot improve upon such an allocation without harming someone
- It is a cornerstone to which we will often compare competitive equilibria

- A PO allocation can be derived from a social planner's problem:

$$\max_{c^i} \sum_i \Omega_i U\left(c_1^i, c_2^i\right)$$

subject to:

$$\begin{aligned} \sum_i c_1^i &= \sum_i x_1^i \\ \sum_i c_2^i &= \sum_i x_2^i \end{aligned}$$

where $\Omega_i \geq 0$ is a welfare weight associated to household i

- Notice that the social planner maximizes only subject to resource constraints - prices do not enter this problem. However, shadow prices can be derived from the multipliers associated with the resource constraints
- In the representative agent case that we have looked at, computation is even simpler as we can simply maximize utility of a single stand-in consumer subject to the resource constraints.

Welfare Theorems

Theorem

The First Fundamental Welfare Theorem: If every good is traded at publicly known prices, and if all agents act competitively taking all prices for given, then the market outcome is Pareto optimal.

Theorem

The Second Fundamental Welfare Theorem: In convex economies (economies with convex preferences and production sets), any Pareto optimal allocation can be achieved as competitive equilibrium subject to appropriate lump-sum transfers of wealth . The associated competitive equilibrium requires that all agents take prices for given and that every good is traded at publicly known prices.

- The first of these results is great to know. The second result is extremely useful since it implies that we can compute allocations from central planning problems which may be a lot easier than computing competitive equilibria directly.

Production in the Two Period Model

Let me now introduce production into the two period model

- The consumers receive an endowments x_1^i period 1
- This endowment can either be consumed or turned into capital k_1
- k_1 can be rented out to firms that produce output according to a production function $f\left(k_1\right)$ in period 2
- Capital depreciated at the rate $\delta > 0$
- Households own firms

Production in the Two Period Model

Let us compute the solution to the central planners problem assuming as before that all agents are identical

- The central planner's problem is:

$$\max_{c_1, c_2, k_1} u(c_1) + \beta u(c_2)$$

- subject to:

$$\begin{aligned} c_1 + k_1 &= x \\ c_2 &= f(k_1) + (1-\delta) k_1 \end{aligned}$$

- Let the multipliers be given as λ_1 and λ_2 respectively so the Lagrangean is:

$$\begin{aligned} \mathcal{L} &= u(c_1) + \beta u(c_2) - \lambda_1 (c_1 + k_1 - x) \\ &\quad - \lambda_2 (c_2 - f(k_1) - (1-\delta) k_1) \end{aligned}$$

The Planning Problem

The first order necessary conditions are:

$$\begin{aligned}
c_1 &: \quad u'(c_1) = \lambda_1 \\
c_2 &: \quad \beta u'(c_2) = \lambda_2 \\
k_1 &: \quad \lambda_1 = \lambda_2 \left(f'(k_1) + (1-\delta) \right) \\
\lambda_1 &: \quad c_1 + k_1 = x \\
\lambda_2 &: \quad c_2 = f(k_1) + (1-\delta) k_1
\end{aligned}$$

which we can combine to get:

$$u'(c_1) = \beta u'(c_2) \left[f'(k_1) + (1-\delta) \right]$$

The Optimality Condition

- To understand this condition, let us consider the set of feasible allocations (c_1, c_2) which is given by:

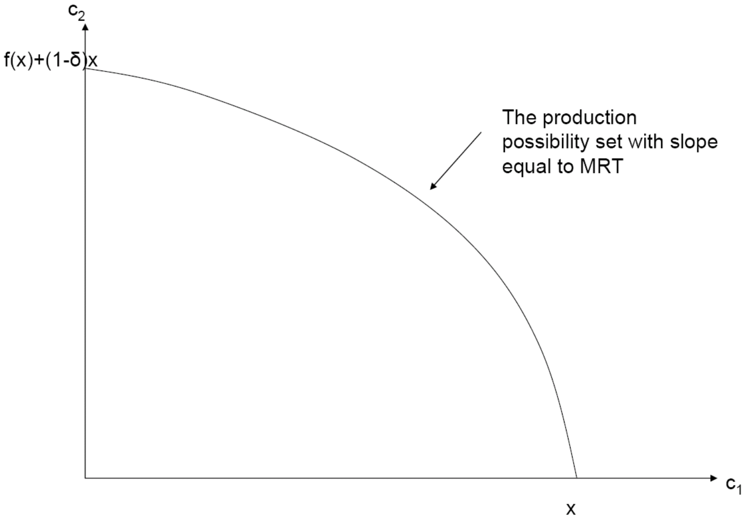
$$\begin{aligned} c_2 &= f(k_1) + (1-\delta) k_1 \\ &= f(x - c_1) + (1-\delta)(x - c_1) \end{aligned}$$

- The slope of this set is the marginal rate of transformation:

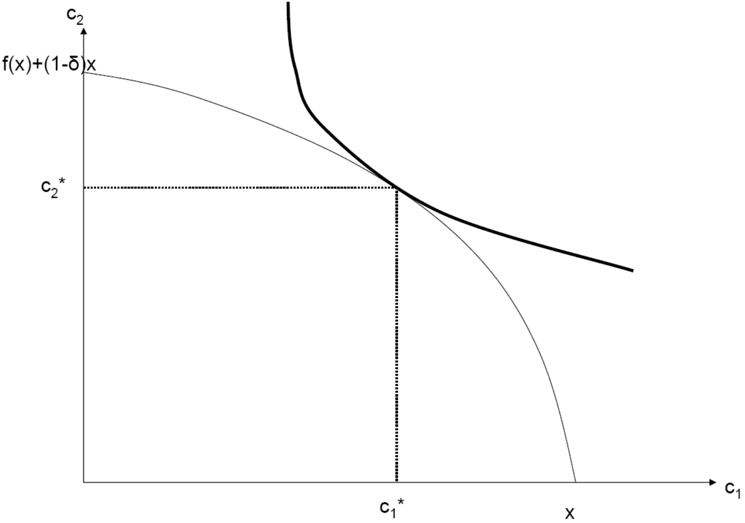
$$\frac{\partial c_2}{\partial c_1} = -\left[f'(k_1) + (1-\delta)\right]$$

- Thus, the Euler equation above implies that the marginal rate of substitution equals the marginal rate of transformation

The production economy graphically:



The production economy graphically:



The production economy - Competitive equilibrium

We could instead have formulated the problem as a competitive equilibrium

- Assume that firms are competitive and that they rent capital from the households at the price R^k
- Firms are owned by the households and profits therefore belong to the households
- The representative household's problem is then:

$$\max_{c_1, c_2, k_1} u(c_1) + \beta u(c_2)$$

subject to:

$$\begin{aligned} p_1 c_1 + p_1 k_1 &= p_1 x \\ p_2 c_2 &= R^k k_1 + \Pi + (1-\delta) p_2 k_1 \end{aligned}$$

where Π denotes profits received from firms, p_1 is the price of goods in period 1, p_2 is the price of goods in period 2

The households' problem

The Lagrangean is given as:

$$\begin{aligned}\mathcal{L} &= u(c_1) + \beta u(c_2) - \lambda_1 (p_1 c_1 + p_1 k_1 - p_1 x) \\ &\quad - \lambda_2 \left(p_2 c_2 - R^k k_1 + \Pi + (1-\delta) p_2 k_1 \right)\end{aligned}$$

and the first-order necessary conditions are:

$$\begin{aligned} c_1 &: \quad u'(c_1) = \lambda_1 p_1 \\ c_2 &: \quad \beta u'(c_2) = \lambda_2 p_2 \\ k_1 &: \quad \lambda_1 p_1 = \lambda_2 \left(R^k + (1-\delta) p_2 \right) \end{aligned}$$

which imply that:

$$u'(c_1) = \beta u'(c_2) \left[\frac{R^k}{p_2} + (1-\delta) \right]$$

The firms

- There is a large number of identical price-taking firms. Their maximization problem is:

$$\begin{aligned} \max \Pi &= p_2 c_2 - R_k k_1 \\ c_2 &= f(k_1) \end{aligned}$$

- Substituting the constraint into the profit expression and taking the first-order condition implies:

$$p_2 f'(k_1) = R^k$$

- Thus: The value of the marginal product of capital must equal the cost of capital

The Competitive equilibrium

We define the competitive equilibrium as:

Definition

A competitive equilibrium is a price system $\left(p_1, p_2, R^k\right)$ and an allocation $\left(c_1, c_2, k_1\right)$ such that (i) households maximize utility taking all price for given, (ii) firms maximize profits taking all prices for given, and (iii) goods and capital markets clear, i.e. $c_1 + k_1 = x$, $c_2 = f\left(k_1\right) + \left(1 - \delta\right) k_1$

Combining the first-order conditions for households and firms, we see that this implies that:

$$u'\left(c_1\right) = \beta u'\left(c_2\right)\left[f'\left(k_1\right) + \left(1 - \delta\right)\right]$$

- Hence, we get the same condition as in the planning problem that equalizes marginal rates of substitution and marginal rates of transformation - this is basically the invisible hand at work

The Competitive equilibrium

- Moreover, by substituting the equilibrium profits into the household's budget constraint, we get that:

$$\begin{aligned} p_2 c_2 &= R^k k_1 + \Pi + (1-\delta)\, p_2 k_1 \\ &= p_2 f'(k_1)\, k_1 + p_2 f(k_1) - k_1 p_2 f'(k_1) + (1-\delta)\, p_2 k_1 \\ &= p_2 f(k_1) + (1-\delta)\, p_2 k_1 \end{aligned}$$

- Hence, the consumer's budget constraints correspond to the resource constraints in equilibrium (notice that we can eliminate p_1 in the constraint for period 1 and p_2 in the constraint for period 2)
- This establishes the equivalence between the two allocations

Uncertainty

We shall often be dealing with models in which there is uncertainty due to stochastic shocks hitting the economy. Two issues need to be addressed when we have uncertainty:

- Choice under uncertainty: Consumers no longer face choices under certainty
- Expectations: Stochastic shocks make variables stochastic and agents need to form expectations

Choice under uncertainty: Consumers need to rank uncertain bundles

- Let there be finite set of states $s \in \{1, 2, , .., S\}$ of the world where S is finite. These are stochastic events. If these occur over time (later) then we will also talk about histories $s^t = \left(s^{t-1}, s_t\right)$
- Probabilities $\pi_s^i \in (0, 1)$, $\sum_s \pi_s^i = 1$ that consumer i associates with state s

Expected utility

- Let there be M commodities and let $c^i_{s,m}$ denote consumer i's consumption of good m in state s
- $c^i = \left(c^i_{1,1}, .., c^i_{S,M}\right)' \in R^{SM}_+$ is consumer i's consumption vector which belongs to the commodity space R^{SM}_+. "$+$" denotes that consumption levels are non-negative (each $c^i_{s.m} \geq 0$)
- Each consumer has a preference ordering which is represented by a utility function:

$$u_i : R^{SM}_+ \longrightarrow R$$

- The preference ordering is complete and we will assume that the utility function is (weakly) increasing and concave.
- When we can write this as:

$$u_i\left(c^i\right) = \sum_{s=1}^{S} \pi^i_s U_i\left(c^i_{s,1}, .., c_{s,M}\right)$$

- where π^i_s denotes the probability of state s, then we say that consumer i has **expected utility** and we note that it is additive over states of nature

Expectations

When we work with stochastic models, we also need to model how agents form expectations

- Much of modern macro relies on the assumption that agents have *Rational Expectations*: Subjective probability distributions coincide with objective probability distributions - this implies that the expectations are model consistent. This corresponds to modeling expectations as the true conditional expectations operator
- This does not mean that agents can perfectly predict - it simply means that they do their best when predicting
- It is a strong assumption but it is a cornerstone
- We do not necessarily want to believe that agents are so rational, but we want at least first to understand how things work in a rational expectations environment

An example

- In the two period endowment economy example, suppose that the first period endowment is known and given as x_1
- Second period endowment is instead given as x_2^h with probability q and as x_2^l with probability $1-q$ where $x_2^h > x_2^l$ and $q \in (0,1)$
- The interest rate is instead supposed to be constant and equal to r
- Let me also assume that preferences are logarithmic
- The consumer's problem is then:

$$\max EU = \log c_1 + \beta E_1 \log c_2$$

subject to:

$$\begin{aligned} c_1 + s_1 &= y_1 \\ c_2^i &= (1+r)\, s_1 + x_2^i, \;\; i = l, h \end{aligned}$$

- We can here express expected utility as:

$$EU = \log c_1 + \beta q \log c_2^h + \beta (1-q) \log c_2^l$$

Savings under uncertainty

The first order conditions here become:

$$
\begin{aligned}
c_1 &: \quad \frac{1}{c_1} = \lambda_1 \\
c_2^h &: \quad \beta \frac{q}{c_2^h} = \lambda_2^h \\
c_2^l &: \quad \beta \frac{1-q}{c_2^l} = \lambda_2^l \\
s_1 &: \quad \lambda_1 = (1+r)\left(\lambda_2^h + \lambda_2^l\right)
\end{aligned}
$$

- Combining these we get that:

$$\frac{1}{c_1} = \beta (1+r) \left(\frac{q}{c_2^h} + \frac{1-q}{c_2^l} \right)$$

which can also be expressed as:

$$u'(c_1) = \beta (1+r) E_1 u'(c_2)$$

Savings under uncertainty

In the specific example, since the interest rate is not stochastic, we can express the Euler equation as:

$$\frac{1}{(1+r)} = \beta c_1 E_1 \frac{1}{c_2}$$

where the right hand side is the expected intertemporal marginal rate of substitution

- Due to Jensen's inequality it follows that:

$$\beta c_1 E_1 \frac{1}{c_2} > \beta c_1 \frac{1}{E_1 c_2}$$

- Therefore, the equilibrium interest rate will be lower in the stochastic savings model than in the deterministic savings model
- Intuition: Since preferences are concave, the consumer is risk averse. He will therefore engage in precautionary savings which will lower the equilibrium return on savings.