Lecture 3 - Locational Equilibrium Continued

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1 Introduction

1. Review.
2. Properties of equilibrium.
3. 4 types of equilibrium.
4. Multiple types of consumer.

2 Review

1. Locational equilibrium.
   (a) Inputs to equilibrium: $t, N, r_A, U(C, L), I, p$. These are the *parameters* of the problem. They are fixed, specified by us, determined outside the model.
   (b) Equilibrium conditions:
      i. Consumers maximise.
      ii. Locational equilibrium.
      iii. Land market equilibrium.
      iv. Consumption good market equilibrium.
   (c) Outputs.
      i. Conditional demand functions: C^*, L^* solve consumer first order conditions holding x fixed. Express demand for consumption and land as functions of prices and income.

ii. Rent function: $r(x, r_0), r_0$. Expresses rent per unit of land as function of distance from centre.
iii. City size: x_B.
iv. Welfare: $V^* = U(C^*, L^*)$.
v. These are determined in equilibrium.

(d) By choosing different values for the parameters, we can analyse how the outputs of the problem vary.

3 Properties of equilibrium

1. When locational equilibrium condition holds, no change in utility when moving from x_1 to x_2. Change in transport costs exactly compensates for change in price of land. Pure substitution effect. Draw graph.

2. Move from x_1 to $x_2 > x_1$.

   (a) $r(x_1) > r(x_2)$

   (b) $L^*(p, r(x_1), I - tx_1) < L^*(p, r(x_2), I - tx_2)$ because of substitution effect

   (c) The slope of rent function is

   $$\frac{dr}{dx} = -\frac{t}{L^*(p_F, r(x), x)}.$$

   (d) Hence, $\frac{-t}{L^*(x_1)} < \frac{-t}{L^*(x_2)}$ and so $\left|\frac{dr(x_1)}{dx}\right| > \left|\frac{dr(x_2)}{dx}\right|$.

   (e) Why do rents increase so dramatically as move to centre?

      i. Higher rents balance lower transport costs. But total rent is $r(x) * L^*$. As rents increase, people near centre consume less land. In order to maintain equilibrium, rent must increase more than proportional to distance.
      ii. Change in rate of increase depends on ability to substitute into C.
      iii. If all consumers need to consume same L^*, impossible to substitute for land, rate of increase is constant.

iv. If eating a little bit more in a much smaller flat maintains same utility than change in slope of rent very fast.

v. Other reasons rate of increase so dramatic.

A. We'll see later, that if different types of people (families with children, people who like a lot of space, different incomes) live in the city who have relatively high demands for land at the centre of the city, then the rate at which rents increase as we move toward the centre will increase.

B. Other demands for land at center, business, transport, retail, government also generate upward pressure on central land rents.

C. Congestion can lead to upward pressure on central land rents.

4 Comparative statics

4.1 Change in transport cost

One type of question we often ask is, if one of the parameters of the model changes, how will the equilibrium change?

As an example consider a city with fixed a fixed population N of identical consumers. Each consumer has utility function $u(c,l)$, income I, and pays transport cost t per mile to commute to the city centre. Suppose we start off in an equilibrium with boundary rent r_A. Suppose the initial level of transport costs is t_0 and the initial equilibrium rent function is $r_0(x) = r_0 - \int_0^x \frac{t_0}{L^*(I - t_0 x', p, r_0(x'))} dx'$. Further, assume that the initial utility level attained is $v_0 = u(c^*, l^*)$.

If transport costs fall to $t_1 < t_0$, what will a new equilibrium look like? How will it compare to the old equilibrium?

First of all if transport costs, fall, then everyone in the city must be better off. Everything else equal they can spend less on transport and still consume what they consumed in the initial equilibrium. So, the fall in transport cost, everything else equal will lead to an increased demand for c and l. At the same time, if transport costs fall, then locations far from the centre will become more valuable while those close to the centre will become less valuable. Everything else equal, you would expect demand to shift away from the centre and toward the outskirts of the city. So in the new equilibrium, the aggregate demand for land and consumption must be higher than in the initial equilibrium.

Consider first the location $x = 0$. At this location, the rent in the old equilibrium is $r_0(0) = r_0$. Consumers living at this location pay no transport cost and in the initial equilibrium obtain utility level v_0. In the new equilibrium, consumers living at $x = 0$ still pay no transport costs. Thus, if there utility is higher than v_0 then it must be that $r_1(0) < r_0(0)$. That is, rent at the centre must fall. In moving from the old equilibrium to the new equilibrium, some residents at $x = 0$ move to locations $x > 0$ and some remain at $x = 0$ and increase c and l. The level of rent at $x = 0$ falls.

Next consider the boundary of the city. Initially it is $x = x_b^0$. But now, since aggregate demand for land has risen, the supply of land for housing must increase so the new boundary must be larger. That is, $x_b^1 > x_b^0$. The only way this can happen is if rents at locations x near x_b^0 rise relative to r_A. In particular, $r_1(x_b^0) > r_A$ and $r_1(x_b^1) = r_A$.

As a result, the rent function in the new equilibrium must be larger than $r_0(x)$ for large x and smaller than $r_0(x)$ for small x and there must be some point, x_c where the rent functions cross so that

$$\begin{aligned} r_1(x) &< r_0(x) \text{ for } x < x_c \\ r_1(x) &= r_0(x) \text{ for } x = x_c \\ r_1(x) &> r_0(x) \text{ for } x > x_c \end{aligned}$$

In summary, utility or welfare rises, aggregate demand for c and l rise. The population at locations near the centre falls while the population farther out rises. Rents near the centre fall. Rents farther out from the centre rise.

4.2 Changes in other parameters

1. Impact of increase in population on equilibrium

    (a) Increase in population, everything else equal, increases demand for land.

    (b) This creates upward pressure on rents and creates pressure to increase land supply.

(c) City adjusts to new equilibrium with higher rents, and larger area (increase in x_b).

(d) Increase in rent lowers demand for land at individual level, in part counteracting increased demand for land.

(e) Utility levels fall as individual level demand for land falls due to higher rents.

(f) Draw graph of equilibrium rent function before the population increase and after.

(g) New equilibrium rents are higher than old at all locations x.

2. Impact of increase in income

(a) Increase in income raises utility.

(b) Everything else equal, increases demand for both land and consumption good.

(c) This creates pressure to increase rents, to increase supply of land, and for some people to move farther away from the centre. If the final effect is small, rents at all locations (including the centre) will rise. If the final effect is large enough however, it is possible that rents at the centre actually fall while rents farther out rise.

(d) On average, rents will rise (they may fall close to centre but rise farther out) and the boundary will move farther out, partially counteracting the increased demand for land induced by the increase in income.

(e) Draw a graph of the change in the equilibrium rent under the assumptions that a) rents rise at all locations, b) rents fall close to the centre and rise farther out.

5 Alternative equilibrium assumptions about city

1. Model 1: closed city, free boundary.

(a) Population fixed at N, urban boundary free x_B, boundary rent fixed r_A.

(b) This is the example we already studied.

(c) This is the model of a closed city with a free boundary. The city is closed because no migration is allowed. The boundary is free because land can be added to or subtracted from the city by taking land from rural production or giving up land for rural production.

(d) x_B and V adjust until equilibrium is attained.

2. Model 2: closed city, fixed boundary.

   (a) Population N fixed, urban boundary fixed x_B, boundary rent free.

   (b) This is a model with a closed city and a fixed boundary. The city is closed because migration is not allowed. The boundary is fixed because the total amount of land is fixed perhaps because the city is located on an island. The boundary rent may adjust and is not fixed at a pre-specified level.

   (c) In an equilibirum in this city, r_A and V adjust until equilibrium is attained.

   (d) How should the equilibrium conditions obtained for model 1 be rewritten to characterize equilibrium in this city?

3. Model 3: open city, free boundary.

   (a) Population N is variable, the boundary x_B is free, and the utility level is fixed at V. The rural rent is fixed at r_A.

   (b) The is a model of an open city with a free boundary. The city is open because migration is allowed. The boundary is free because land can be freely added to or taken away from the city.

   (c) People migrate to the city if the utility obtained in the city is higher than V, the utility obtainable elsewhere.

   (d) People leave the city if the utility obtained elswehere, V, is higher than the utility obtainable in the city.

   (e) In equilibrium, the utility obtained in the city is $U(C^*(x), L^*(x))$. In equilibrium, the utility obtained in the city must equal V.

   (f) The population, N, and the boundary x_B adjust until equilibrium is attained.

(g) How should the equilibrium conditions obtained in model 1 be adjusted to characterize equilibrium in model 3?

4. Model 4: open city, fixed boundary.

   (a) Population N is variable as is the boundary rent r_A. The utility level V is fixed as is the boundary x_B.

   (b) This is a model of an open city with a fixed boundary. The city is open because migration is allowed. The boundary is fixed because the total quantity of land available is fixed.

   (c) In equilibrium, the utility obtained in the city is $U(C^*(x), L^*(x))$. In equilibrium, the utility obtained in the city must equal V.

   (d) The population, N, and the boundary rent r_A adjust until equilibrium is attained.

   (e) How should the equilibrium conditions obtained in model 1 be adjusted to characterize equilibrium in model 4?

6 Equilibrium with multiple consumer types

1. A consumer's "type" is determined by income, preferences, and transport costs. In an equilibrium with a single type of consumer (everyone is identical), consumers lives at different locations but all consumers attain the same utility. All locations provide the same utility.

2. In an equilibrium with multiple consumer types, different types will, in general, live at different locations. The city will segregated into sectors, each sector inhabited by a single type. Everyone of the same type will attain the same utility level but different types will in general attain different utility levels. Comparing the different types, those who value land close to the centre the most will live in a sector close to the centre. Those who value land close to the center the most are those who are willing to "bid" the most or pay the most for land close to the centre. They are the ones who have the highest "bid rent" or the highest "willingness to pay". For consumers, three factors determine who values land close to the centre the most: 1) Income, 2) transport costs, 3) preferences.

3. Everything else equal, if land is a normal good, those with more income will want to consume more land and will bid less to live close to the centre. They will bid less close to the centre because land is cheaper farther away from the centre and they will want to consume more land. This assumes that those with high income have the same transport costs and the same preferences as those with less income.

4. Everything else equal, those with higher transport costs will bid more for land close to the centre. They will bid more for land close to the centre because it is relatively more costly for them to commute longer distances. This assumes those that have higher transport costs have the same income and the same preferences as those with low transport costs.

5. Everything else equal, those with preferences such that it is relatively easy to maintain a fixed utility level by substituting consumption of other goods for consumption of land will bid more for land closer to the centre. They will bid more for land closer to the centre because they can maintain a constant utility level near the centre by consuming less land despite paying a higher price.

6. In general, in determining whether one type of consumer will live closer to the centre than another, one must consider all 3 factors. For instance, in an economy with 2 types of people, type 1 may have higher income, higher transport costs, and different preferences than type 2. In this case, to determine whether type 1 lives closer to the centre or farther, one must consider all 3 factors.

6.1 A simple example: Bid rent functions

1. There are two types of consumers with incomes, $I_1 < I_2$. The two types have identical transport cost per mile t and identical preferences. Assume land is a normal good. Who lives closer to centre, type 1 or type 2?

2. Suppose type 1 obtains utility level v_1 in equilibrium and type 2 obtains utility level v_2 in equilibrium.

(a) How much would type 1 be willing to pay to live at location x? They would pay any amount $b(x)$ such that

$$v(I_1 - tx, p, b(x)) = u(c^*(I_1 - tx, p, b(x)), L^*(I_1 - tx, p, b(x))) \geq v_1.$$

(b) The function v is the indirect utility function. It describes the utility obtained by a consumer as a function of net income $(I_1 - tx)$, the price of consumption p and the price of land at x(in this case $b(x)$).

(c) The bid rent or willingness to pay of type 1 at location x is the value of $b_1(x)$ such that

$$v(I_1 - tx, p, b_1(x)) = v_1 \tag{1}$$

for all x. If $r(x)$, the rent at location x, is higher than $b_1(x)$ than the consumer will refuse to pay it. If the rent $r(x)$ is less than or equal to $b_1(x)$, the consumer will be willing to pay it. The bid rent is the amount that holds utility constant as the household moves across locations. In particular at $x = 0$, it satisfies

$$v(I_1, p, b_1(0)) = v_1.$$

So, $b_1(0)$ is the amount that type 1 is willing to pay to live at the centre. It depends on the equilibrium utility level v_1

3. Suppose type 1 were willing to bid $b_1(0)$ to live at the centre and type 2 were willing to bid $b_2(0)$. Equation (1) is one way to describe the bid rent function at every location x. We can differentiate this equation to get a condition on the slope of the bid rent function

$$\frac{\partial b_i(x)}{\partial x} = -t\left(\frac{-\frac{\partial v}{\partial I}}{\frac{\partial v}{\partial b}}\right). \tag{2}$$

This is similar to the condition on the slope of the equilibrium rent function that we derived for the city with a single type. In fact, this condition on the slope of the bid rent function is equivalent to the condition derived previously. That is, an equivalent way to express (2) is that the bid rent function for each type must satisfy

$$\frac{db_i(x)}{dx} = \frac{-t}{L^*(I_i - tx, p, b_i(x))}. \tag{3}$$

Why? These two equations are equivalent because of a theorem known as Roy's identity. Roy's identity states that if $v(I,p,b)$ is an indirect utility function and b is the price of land, then

$$L^*(I,p,b) = \frac{-\frac{\partial v}{\partial b}}{\frac{\partial v}{\partial I}}.$$

That is the demand function can be calculated by differentiating the indirect utility function and taking the ratio of derivatives. Intuitively, (2) and (3) are equivalent because both conditions impose that utility is held constant when the consumer moves across locations. They imply that the bid rent funciton for type i is given by

$$\begin{aligned} b_i(x) &= b_i(0) + \int_0^x \frac{\partial b_i(s)}{\partial x} ds \\ &= b_i(0) - \int_0^x \frac{t}{L^*(I_i - ts, p, b_i(s))} ds. \end{aligned}$$

4. The functions $b_1(x)$ and $b_2(x)$ are *bid rent* functions. $b_1(x)$ expresses how much type 1 would be willing to pay in rent (or how much they would bid) to live at location x assuming they were willing to bid $b_1(0)$ at the centre (recall this amount depends on v_1 the utility level attained by type 1 in equilibrium). That is, if type 1 lived at the centre and paid rent $b_1(0)$, then type 1 would obtain the same utility living at x if and only if the rent at x equaled $b_1(x)$. Similarly, if type 2 lived at the centre and paid rent $b_2(0)$, then type 2 would obtain the same utility living at x if and only if the rent at x equaled $b_2(x)$.

5. Type 1 will be willing to live at location x if and only if $r(x) \le b_1(x)$.

6. Type 2 will be willing to live at location x if and only if $r(x) \le b_2(x)$.

7. Suppose the equilbrium rent at location x_1 is $r_e(x_1)$ and in equilibrium both type 1 and type 2 live at location x_1. That is $r_e(x_1) = b_1(x_1) = b_2(x_1)$. Which is larger in magnitude $\frac{\partial b_1(x_1)}{\partial x}$ or $\frac{\partial b_2(x_1)}{\partial x}$? Since L is a normal good, $\frac{\partial L^*(I - tx_1, p, r_e(x_1))}{\partial I} > 0$. Thus, $L^*(I_1 - tx_1, p, r_e(x_1)) <$

$L^*(I_2 - tx_1, p, r_e(x_1))$. Therefore,

$$\frac{-t}{L^*(I_1 - tx_1, p, r_e(x_1))} < \frac{-t}{L^*(I_2 - tx_1, p, r_e(x_1))}.$$

Type 1's bid rent function is steeper than type 2's at location x_1.

(a) If both live at same location x_1, then type 1 has steeper slope at location x_1.

(b) The equilibrium rent function is equal to the maximum of the two bid rent functions

$$r_e(x) = \max\{b_1(x), b_2(x)\}.$$

8. Compute equilibrium for two consumer types.

   (a) Assume populations of types 1 and 2 equal N_1 and N_2 and assume the boundary rent $r(x_b) = r_A$.

   (b) Guess values for bid rents at centre: $b_1(0)$ and $b_2(0)$.

   (c) Compute the bid rent functions:

$$b_1(x) = b_1(0) - \int_0^x \frac{t}{L^*(I_1 - ts, p, b_1(s))} ds$$

$$b_2(x) = b_2(0) - \int_0^x \frac{t}{L^*(I_2 - ts, p, b_2(s))} ds.$$

   (d) Set $r(x) = \max\{b_1(x), b_2(x)\}$. Set equilibrium rent equal to the highest bid.

   (e) Compute x_b :

$$r_A = \int_0^{x_b} r(s)\, ds.$$

   (f) Set

$$N_1(x) = \left\{ \begin{array}{cc} \frac{2\pi x}{L(I_1 - tx, p, r(x))} & \text{if } b_1(x) > b_2(x) \\ 0 & \text{if } b_1(x) < b_2(x) \\ \frac{1}{2}\left(\frac{2\pi x}{L(I_1 - tx, p, r(x))}\right) & \text{if } b_1(x) = b_2(x) \end{array} \right\}$$

and

$$N_2(x) = \left\{ \begin{array}{cc} \frac{2\pi x}{L(I_2 - tx, p, r(x))} & \text{if } b_2(x) > b_1(x) \\ 0 & \text{if } b_2(x) < b_1(x) \\ \frac{1}{2}\left(\frac{2\pi x}{L(I_2 - tx, p, r(x))}\right) & \text{if } b_2(x) = b_1(x) \end{array} \right\}.$$

This requires supply of land to equal demand for land at every location and requires demand for land of type 1 at location x to be zero if 1 is not the highest bidder at location x. It also requires demand for land of type 2 to be zero at location x if 2 is not the higher bidder at location x.

(g) Check whether

$$N_1 = \int_0^{x_b} N_1(s)\,ds \tag{4}$$

and

$$N_2 = \int_0^{x_b} N_2(s)\,ds. \tag{5}$$

(h) If the right side of (4) is larger than N_1, increase $b_1(0)$. If it is less than N_1, decrease $b_1(0)$.

(i) If the right side of (5) is larger than N_2, increase $b_2(0)$. If it is less than N_2, decrease $b_2(0)$.

(j) In equilibrium, equations (4) and (5) are satisfied.

(k) In equilibrium, in this example type 1, the poor people, will live closer to the centre and type 2, the rich will live farther away. See graph.

(l) How could you change the model to change this conclusion?

9. Equilibrium rules are similar to those in the model with one type of consumer.

(a) Consumers maximise.

(b) Identical people who live at different locations in equilibirum, obtain the same utility

(c) Each plot of land goes to the highest bidder.

(d) Supply equals demand in every market.