ECON3021 Urban Economics
Lecture 2
Residential Location Theory (2)

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1 Introductory remarks

1. Review
   (a) Existence of a city requires IRS
      i. E.g. natural advantage plus transport costs
      ii. IRS in production
      iii. agglomeration economy
   (b) Simple city model
      i. N_I live in the city
      ii. Locational equilibrium condition

$$-t - \frac{\partial r(x)}{\partial x} = 0$$

      iii. Income maximisation
      iv. Land equilibrium
      v. Landowners maximise profits
      vi. Equilibrium

$$\begin{aligned} r(x) &= r_0 - tx \\ r(x_B) &= r_0 - tx_B = r_A \\ y_I(x) &= y - r_0 = 0 \\ N_I &= \pi x_B^2 \end{aligned}$$

2. Outline of today.
   (a) Limitations of previous solution
      i. Land rent function is a straight line.
         A. In reality, it is not.
      ii. Demand for land by each individual is one unit.
         A. Apartments in central london are larger than those farther out.
         B. Population density at cetnre is higher?
      iii. No consumption.
      iv. Welfare of workers is completely independent of (t, y, r_A).
         A. Perfectly elastic supply of workers.
   (b) What can we do?
      i. Combine individual's optimal location choice with utility maximisation over consumption and land.
      ii. Perfectly inelastic supply of workers.
   (c) Definition of locational equilibrium (more formal)
      i. Workers maximise utility
         A. Optimal C and L
         B. Optimal location choice
         C. No one wants to move
      ii. Landowners maximise profits
      iii. Market equilibrium conditions
         A. Supply equals demand for land at every location x.
         B. Supply equals demand for consumption.

2 Residential location choice model

1. We assume that a city is a large flat plain with an export hub at the centre. Land is to be divided up between business, housing, and agriculture. Business is at the centre, housing is distributed throughout the city and agriculture is outside the city.

2. Business requires no land (or a very very small amount). It locates next to export hub.

    (a) We do not *explicitly* model business location choice in this simple model. *Implicitly*, we assume that there is some form of increasing returns to scale either in production or in transport of output for export.

    (b) In different interpretations of this model, the centre of the city could be a central business district, a market, a factory, a port, an airport, a tube station, a train station, etc.

    (c) Later we will consider a model in which business choose locations other than the centre.

3. Agriculture uses land outside the city and earns rent r_A per unit land.

4. There is a fixed population of N of identical people. Each earns income y, works at the centre of the city, and purchases land somewhere within the city for a house in which to live.

5. All consumers/workers commute to the centre of the city to work at cost t per mile. If a consumer lives x miles from the centre, then total transport costs are tx.

6. Within the city, consumers must pay a rent per unit land of $r(x)$. Locations near the centre charge higher rents because locations near the centre are more valuable. They are more valuable because residents of locations near the centre incur smaller transport costs.

7. Consumers must choose how far away from centre to live x and how much money to spend on C, a consumption good, and L land. The price of the consumption good is p per unit.

8. The supply of land at distance x from the centre is $S_L(x) = 2\pi x$.

9. Preferences are $u(C, L)$.

3 Goals of analysis

1. For each consumer, optimal choice of C^*, L^*, and x^*.

   (a) Analyse how (C^*, L^*) depend on income $(y - tx)$, prices $(p, r(x))$, location x, and the utility function u.

   (b) Analyse how optimal location choice x^* depends on income $(y - tx)$, transport costs t, the price of C, and the rent function $r(x)$.

2. Use the results to analyse a spatial equilibrium

   (a) Determinants of equilibrium $r(x)$, x_B (the radius of the city), and the level of utility $u(C^*, L^*)$ that is obtained.

   (b) $r(x)$ is equilibrium rent function.

   (c) x_B is the equilibrium boundary of the city.

   (d) $u(C^*, L^*)$ is the equilibrium utility obtained by a resident in the city. In equilibrium, all identical people will obtain the same utility.

   (e) How do these variables depend on

      i. income (y)
      ii. transport costs (t)
      iii. population (N)
      iv. the utility function $(u(C, L))$
      v. the supply of land $S_L(x)$
      vi. the price of consumption (p)
      vii. agricultural rent (r_A)

4 Solving the consumer problem

First, we solve the consumer problem, choose (C, L, x) to maximise utility. The consumer maximises

$$\begin{array}{cc} u(C, L) & \text{Utility function} \\ \text{subject to} & \\ pC + r(x)L + tx = y & \text{Budget constraint} \end{array}$$

or

$$\max_{(C,L,x)} \{u(C,L) + \lambda(y - tx - pC - r(x)L)\}$$

or equivalently

$$\max_{(L,x)} u\left(\frac{y - tx - r(x)L}{p}, L\right).$$

The variable λ is the Lagrange multiplier. The first order conditions are

$$u_C(C,L) - \lambda p = 0 \tag{1}$$

$$u_L(C,L) - \lambda r(x) = 0 \tag{2}$$

$$L\frac{dr(x)}{dx} + t = 0 \tag{3}$$

$$y = pC + r(x)L + tx. \tag{4}$$

There are 4 equations and 4 unknown variables. An optimal choice:

1. adjusts the amount of C until marginal utility of consumption equals marginal cost (in utility terms).

2. adjusts land consumption until marginal utility of land equals marginal cost.

3. adjusts location until marginal benefit of moving farther away equals marginal cost.

4. satisfies the budget constraint

The consumer problem has four equations (equations $(1) - (4)$) in four unknown variables (C, L, x, λ).

- If (C, L, x, λ) are maximising choices then they must satisfy these 4 equations.

- To solve the consumer's problem, solve the four equations for the four unknowns.

- The consumer treats the rent function as a known function.

- The solution to the consumer's problem can be expressed as four functions describing the optimal choices as functions of income, prices, and transport costs. One way to write these functions is:

$$\begin{aligned} C &= C^* \left(y - tx^*, p, r\left(x^*\right)\right) \\ L &= L^* \left(y - tx^*, p, r\left(x^*\right)\right) \\ x &= x^* \left(y, p, t, r\left(x^*\right)\right) \\ \lambda &= \lambda^* \left(y, p, t, r\left(x^*\right)\right). \end{aligned}$$

  The first line expresses for example the idea that the optimal choice of C depends on income minus tranport costs (at the optimal location), the price of the consumption good, and the rent at location x^*.

4.0.1 Example: Compute demand functions for (C, L) conditional on x

In the Cobb-Douglas utility case where

$$u(C, L) = C^{\alpha} L^{1-\alpha}$$

equations $(1) - (4)$ become

$$\alpha C^{\alpha-1} L^{1-\alpha} - \lambda p = 0 \tag{5}$$

$$(1-\alpha) C^{\alpha} L^{-\alpha} - \lambda r(x) = 0 \tag{6}$$

$$L \frac{dr(x)}{dx} + t = 0 \tag{7}$$

$$y - tx = pC + r(x) L. \tag{8}$$

First treat x as fixed and solve for the optimal choices of (C, L, λ) conditional on x.

Divide (6) by (5) to obtain

$$\frac{(1-\alpha) C^{\alpha} L^{-\alpha}}{\alpha C^{\alpha-1} L^{1-\alpha}} = \frac{r(x)}{p}$$

which is equivalent to

$$\frac{(1-\alpha) C}{\alpha L} = \frac{r(x)}{p}.$$

The marginal rate of substitution between C and L equals the ratio of prices. This can be solved for C as a function of L to obtain

$$C = \frac{\alpha}{1-\alpha}\frac{r(x)L}{p}.$$

Substitute this into the budget constraint (8) and solve for L to obtain

$$\begin{aligned} L^* &= (1-\alpha)\left(\frac{y-tx}{r(x)}\right) \\ C^* &= \alpha\left(\frac{y-tx}{p}\right) \\ \lambda^* &= \left(\frac{\alpha}{p}\right)^{\alpha}\left(\frac{1-\alpha}{r(x)}\right)^{1-\alpha}. \end{aligned}$$

These are the demand functions for (C, L) conditional on location choice x.

5 Optimal location choice

We now have the optimal demand functions condiotnal on location choice x.

Question: Is L^* increasing or decreasing in x?

Now we can study the optimal location choice problem. Consider the location choice first order condition (7). It is

$$-t - \frac{\partial r(x)}{\partial x}L^*(y-tx, p, r(x)) = 0. \tag{9}$$

This is similar to the optimal lcoation choice equaiton in Lecture 1. The optimal location choice equates the amrginal benefit of moving farther away form teh centre to the marginal cost. The marginal benefit of movning farhter away is determined by the reduction in rent. The marginal cost is the increase in transport cost. This location choice ewquatio can be rewritten as

$$\frac{\partial r(x)}{\partial x} = \frac{-t}{L^*(y-tx, p, r(x))}. \tag{10}$$

In contrast to the location choice in the income maximizatoin economy, this equaiton implies that $\frac{\partial r(x)}{\partial x}$ is not a constant. If $x > 0$ is optimal then equation (10) must be satistifed at x. In fact, if all $x \leq x_B$ are optimal then equation (10) must be satisfied for all $x \leq x_B$. We will look for an equilibrium where all workers are indifferent between all locations $x \leq x_B$. We call equation (10), the **location equilibrium condition**.

6 Definition of locational equilibrium

1. All consumers maximise utility in terms of (C, L)
   (a) Optimal choices of (C, L).
2. Locational equilibrium condition
   (a) Optimal location choice.
   (b) No one wants to move.
   (c) All optimal locations yield the same utility.
3. Landlords maximize profits
   (a) Land goes to highest bidder.
4. Equilibrium in land market.
5. Equilibrium in consumption market.

6.1 Equilibrium condition 1: consumer's optimise (C, L)

1. This determines consumer demand for C and L and consumer location choice.
2. Conditional demand functions. If a consumer chooses location x, then his demand for C and L conditional on that choice of x can be expressed as: $C^*(p, r(x), I - tx)$ and $L^*(p, r(x), I - tx)$
3. These are called conditional demand functions, because they express the demand conditional on choice of location.

7 Equilibrium condition 2: locational equilibrium

1. Given the conditional demand functions, the consumer's location choice satisfies
$$L^*(I - tx, p, r(x)) \frac{\partial(x)}{\partial x} + t = 0. \tag{11}$$

   (a) Each person chooses optimal locations x so that marginal cost of moving equals marginal benefit.
   (b) This determines consumer's optimal location choices x^*.

2. If all N consumers identical, all locations with positive population must yield same utility.

3. Guess that all locations $x \leq x_B$ have positive population.

4. Then equation (11) must hold for all $x \leq x_B$.

5. In other words,
$$\frac{\partial r(x)}{\partial x} = -\frac{t}{L^*(I - tx, p, r(x))} \text{ for all } x \in [0, x_B].$$

6. Locational equilibrium condition results in a condition on the *slope* of the rent function.

   (a) x_B is the boundary of the city. The value of x_B is yet to be determined.

7. Locational equilibrium condition generates a differential equation
$$\frac{\partial r(x)}{\partial x} = \frac{-t}{L^*(I - tx, p, r(x))}. \tag{12}$$

   (a) Under standard conditions, a solution exists. Usually must be calculated using a computer.
   (b) The slope of the rent function depends on transport costs and on the demand for land.
      i. If tranport costs are *high*, the slope will be steep.
      ii. If demand for land is *low*, the slope will be steep.

8. Let r_0 be the rent at the centre. A solution to (12) will be of the form
$$\begin{aligned} r(x) &= r_0 + \int_0^x \frac{\partial dr(s)}{\partial x} ds \\ &= r_0 - \int_0^x \frac{t}{L^*(I - ts, p, r(s))} ds. \end{aligned}$$

This equation states that the rent at a location x is equal to the rent at the centre plus the change in rent between 0 and x. Because the change in rent is negative, $r(x) < r_0$ for all x. The rent function will depend on r_0. The variable s is a dummy integration variable that takes on values between 0 and x.

8 Equilibrium condition 3: Landowners maximise profits

1. For $x \geq x_B$, $r(x) = r_A$.
2. For $x \leq x_B$, $r(x) \geq r_A$.
3. For $x = x_B$, $r(x) = r_A$.
   (a) Recall that x_B is the boundary of the city.
   (b) Combine this point with point 8. above to get

$$r_A = r_0 - \int_0^{x_B} \frac{t}{L^*(I - ts, p, r(s, r_0))} ds.$$

8.1 What we have so far

In these equations, p, t, I and r_A are *known parameters* whose values are fixed outside the model. The function L^* is a *known function* derived from the solution of the consumer's maximisation problem. The variable r_0 is an *unknown variable* whose value is to be determined in equilibrium. Once r_0 is known, the function $r(x, r_0)$ can be calculated using a computer. In this class we will ask qualitative questions like: 1) How do changes in (p, t, I, r_A, N) affect $r(x)$ and r_0? 2) Given values for (p, t, I, r_A, N), what is the welfare that consumers obtain?

Summary:

1. Locational equilibrium among workers.
   (a) $\frac{\partial r}{\partial x} = -\frac{t}{L^*(I - tx, p, r(x))}$.

(b) $r = r(x, r_0) = r_0 - \int\limits_0^x \frac{t}{L^*(I-ts,p,r(s,r_0))} ds.$

(c) $r_A = r(x, r_0)$.

2. Consumer choices (maximisation).

   (a) Demand for food and land.

      i. $L^*(I - tx, p, r(x, r_0))$, $C^*(I - tx', p, r(x, r_0))$.
      ii. L^* and C^* can be shown on the standard picture of consumer maximisation subject to a budget constraint.
      iii. Draw picture showing budget constraint, tangency of indifference curve, and optimal choice of C and L.

3. Review.

   (a) Each consumer maximises.

      i. Consumer chooses (C, L, x) to maximise utility subject to budget constraint.
      ii. Conditional demand functions
         A. $C^*(I - tx^*, p, r(x^*))$.
         B. $L^*(I - tx^*, p, r(x^*))$.
      iii. Optimal location choice: $x^*(I, t, p, r(x))$.

   (b) Equilibrium in urban spatial economy.

      i. Level and slope of rent function.
      ii. Number of people in city or level of welfare.
      iii. Radius of city or level of rent.

   (c) Locational equilibrium.

      i. All *identical* consumers obtain same utility regardless of where they choose to live

$$-t - \frac{dr(x)}{dx} L^*(I - tx, p, r(x)) = 0$$

$$\frac{dr(x)}{dx} = -\frac{t}{L^*(I - tx, p, r(x))}.$$

ii. Land is used in highest value use.
   A. Land goes to highest bidder

$$r(x) = r_0 - \int_0^x \frac{t}{L^*(I - ts, p, r(s, r_0))} ds.$$

   B. $r(x, r_0) > r_A$ for $x < x_B$, urban (residential and commuting).
   C. $r(x, r_0) = r_A$ for $x \geq x_B$, rural (farming). In equilibrium $r(x, r_0) \geq r_A$ for all x. Why?
   D. $r(x) > r_A$ land goes to housing, $r(x) = r_A$ land goes to farming.
   E. Hence, the value of r_0 is determined by

$$r_A = r_0 - \int_0^{x_B} \frac{t}{L^*(I - ts, p, r(s, r_0))} ds.$$

iii. What's missing?
   A. What determines x_B?
   B. Supply and demand for land must be equated.

9 Equilibrium condition 4: supply and demand for land are equal

1. Equilibrium in land market.

   (a) Supply of land at distance x is $S_L(x) = 2\pi x$, draw picture.
   (b) How many people live at distance x? $N(x)$
   (c) Total demand for land in housing at distance x is $D_L(x) = N(x) \cdot L^*(I - tx, p, r(x, r_0))$ if $x \leq x_B$.
   (d) $N(x)$ is unknown but if supply equals demand at location x then $N(x)$ is determined by

$$N(x) = \frac{2\pi x}{L^*(I - tx, p, r(x, r_0))}$$

   for all $x \leq x_B$.

(e) x_B is still unknown. But, we know that the total population in the city is N.

(f) In equilibrium, everyone must live somewhere. Therefore,

$$\begin{aligned} N &= \int_0^{x_B} N(s)\, ds \qquad (13) \\ &= \int_0^{x_B} \frac{2\pi s}{L^*(I - ts, p, r(s, r_0))} ds. \end{aligned}$$

(g) This final equation pins down x_B. The variable x_B must satisfy equation (13)

(h) To compute an equilibrium:

i. Guess a value for x_B.

ii. Compute the value of the right side of equation (13).

iii. If the right side of (13) is less than the left side, increase x_B. Why?

iv. If the right side of (13) is larger than the left side, reduce x_B. Why?

v. If the right side of equation (13) equals the left side, then x_B is an equilibrium value.

vi. Repeat steps i.-, iv. until v. is true.

2. Equilibrium in consumption good market trivial.

(a) Demand for the consumption good is: $D_C = \int_0^{x_B} N(s) \cdot C^*(I - ts, p, r(s, r_0))\, ds$.

(b) Supply of consumption good is infinitely elastic at price p. That is supply of C adjusts so that supply equals demand at price p.

(c) What would the equilibrium condition be if the supply were not infinitely elastic?

3. Equilibrium equations.

(a) $r(x, r_0) = r_0 - \int_0^{x} \frac{t}{L^*(I - ts, p, r(s, r_0))} ds$.

(b) $r_A = r_0 - \int\limits_0^{x_B} \frac{t}{L^*(I-ts,p,r(s,r_0))} ds.$

(c) $N = \int\limits_0^{x_B} N(s)\, ds = \int\limits_0^{x_B} \frac{2\pi s}{L^*(I-ts,p,r(s,,r_0))} ds.$

9.1 Reprise of equilibrium conditions

1. Equilibrium.

   (a) Condition 1: consumer maximizes utility.

   (b) Condition 2: locational equilibrium.

      i. For *identical* people, changes in rent across space compensate exactly for changes in transport costs

$$\frac{dr(x)}{dx} = \frac{-t}{L^*(I - tx, p, r(x))}.$$

      ii. For *non-identical* people, land goes to the highest bidder.

         A. If $r(x) > r_A$, consumers bid more.

         B. If $r_A >$ willingness to pay of consumers, farmers bid more.

         C. $r(x) = r_A$, urban boundary.

   (c) Condition 3: land market equilibrium: At every location x, supply equals demand for land.

   (d) Condition 4: consumption good market equilibrium: Supply equals demand for the consumption good.

10 Summarise

1. Equilibrium.

   (a) Inputs to equilibrium: $t, N,\ r_A,\ U(\cdot,\cdot),\ I,\ p$. These are the *parameters* of the problem. They are fixed, specified outside the model.

   (b) Outputs: $C^*,\ L^*,\ r(\cdot),\ r_0,\ x_B,\ V^* = U(C^*, L^*)$. These are determined in equilibrium.

(c) By choosing different values for the parameters, we can analyse how the outputs of the problem vary.

2. Hint for Assignment 1

   (a) In Lecture 1, the population could freely move between farming and industry. That is there was free migration between farming and agriculture. Workers in the city would leave the city if net income was less than zero.

   (b) In Lecture 2, the population was fixed at N. Migration was not allowed; presumably because it was too costly.

   (c) In Assignment 1, problem 2 there is free migration. People will leave the city if utility in the city is less than V_R. People will migrate to the city if utility in the city is greater than V_R. In equlibrium, the utility in the city will equal V_R. That is, the population of the city will adjust until $V_{city} = V_R$.