**ECON3021 Urban Economics**
**Winter 2009**
**Assignment 3: Transportation**
**Solutions**

1. All transport in a city is provided by two tube lines, the Northern Line and the Victoria Line. The trains are owned and operated by the government. Both lines suffer from congestion during the morning peak travel period. During this period, the per passenger cost (to the consumer) of travel on the Northern Line is $c_n = f_n + b_n p_n$ where $c_n$ is the per passenger cost of travel (to the consumer) and $p_n$ is the number of passengers on the line during the period. Similarly, the per passenger cost of travel (to the consumer) on the Victoria Line is $c_v = f_v + b_v p_v$ where $c_v$ is the per passenger cost of travel (to the consumer) and $p_v$ is the number of passengers on the line during the period. On both lines, the per passenger cost to the consumer has two components, a fare component $(f_n, f_v)$ and a time cost component $(b_n p_n, b_v p_v)$. The time cost component increases with the number of passengers. The social cost of travel on the Northern line is $s_n = k_n + b_n p_n$ and the social cost of travel on the Victoria line is $s_v = k_v + b_v p_v$. In general, it may be the case that $k_n \neq f_n$ and $k_v \neq f_v$. Assume there are $N$ commuters in the city all of whom make one trip during the morning peak period if the cost of travel is less than or equal to $c_0$. Each commuter uses the lowest cost tube line.

   (a) Assuming $c_0$ is large so that all consumers travel, what is the equilibrium number of passengers on each line and the equilibrium social cost of travel on each line? What is the equilibrium revenue? Explain.

      - **Solution:** Assuming all passengers travel, an equilibrium will be determined when the cost of travel on the two lines is equal. That is when

$$\begin{aligned} f_v + b_v p_v &= f_n + b_n p_n \\ p_v + p_n &= N. \end{aligned}$$

The equilbrium numbers are

$$\begin{aligned} p_n &= \frac{b_v N + f_v - f_n}{b_n + b_v} \\ p_v &= \frac{b_n N + f_n - f_v}{b_n + b_v}. \end{aligned}$$

Each passenger incurs a cost of

$$c_v = c_n = \frac{b_v f_n + b_n f_v + b_n b_v N}{b_n + b_v}.$$

This is the private cost. The social cost is

$$\begin{aligned} s_n &= \left(k_n + b_n \frac{b_v N + f_v - f_n}{b_n + b_v}\right) p_n \\ s_v &= \left(k_v + b_v \frac{b_n N + f_n - f_v}{b_n + b_v}\right) p_n. \end{aligned}$$

(b) What is the efficient number of passengers on each line?

- **Solution:** The efficient number solves

$$\min_{\{p_n\}} \left\{ (k_n + b_n p_n) p_n + (k_v + b_v (N - p_n)) (N - p_n) \right\}$$

with first order conditions

$$\frac{\partial SC}{\partial p_n} = k_n + 2b_n p_n - (k_v + 2b_v (N - p_n)) = 0$$

and so

$$\begin{aligned} p_n &= \frac{k_v - k_n + 2b_v N}{2(b_n + b_v)} \\ p_v &= \frac{2Nb_n + k_n - k_v}{2(b_n + b_v)}. \end{aligned}$$

(c) Assuming that initially $f_n = f_v = 0$, is the initial equilibrium efficient. What is the impact of increasing the fares to $f_n = f_v = 1$? What level of fares would yield an efficient outcome? Who benefits and who loses if the government increases fares to the efficient level?

- **Solution:** If $f_n = f_v = 0$, then the equilibrium satisfies

$$\begin{aligned} p_n^e &= \frac{b_v N}{b_n + b_v} \\ p_v^e &= \frac{b_n N}{b_n + b_v}. \end{aligned}$$

This is efficient if $k_n = k_v$ because when $k_n = k_v$

$$\begin{aligned} p_n^* &= \frac{b_v N}{b_n + b_v} \\ p_v^* &= \frac{b_n N}{b_n + b_v}, \end{aligned}$$

the efficient solution, equals the equilibrium solution. Increasing fares to $f_n = f_v = 1$ has no impact on the number of passengers but it does result in a transfer of resources from the consumers to the government. The fares would be efficient if $p_n^* = p_n^e$ and $p_v^* = p_v^e$. This will occur if

$$f_v - f_n = \frac{k_v - k_n}{2}.$$

If the government charges fares to achieve the efficient outcome, then consumers are worse off and government revenues increase. The gain in government revenue is larger than the loss in consumer welfare. The consumers will only be made better off if the increase in government revenue is redistributed to consumers in some manner such as a lump sum transfer, a reduction in other taxes, or an increase in spending on public goods.

(d) Under what circumstances is it efficient to set $f_v = f_n$?

- **Solution:** This is efficient if $k_v = k_n$.

2. Utility functions for bus and train travel are

$$u_{bi} = \beta_0 + \beta_1 w_i t_b - (\beta_2 + \beta_3 w_i)\, p_b + \varepsilon_{ib}$$

$$u_{ti} = \beta_1 w_i (t_{t0} + t_{t1} d_i) - (\beta_2 + \beta_3 w_i)\, p_t$$

where $w_i$ is the wage (£ per hour) of person $i$, $t_b$ is the bus transit time, $p_b$ is the bus fare, $t_{t0} + t_{t1} d_i$ is the train transit time for person $i$, $p_t$ is the train fare, $d_i$ is the distance from the home of person $i$ to the train station, and $\varepsilon_{ib}$ is heterogeneity in other factors that affect the utility of bus travel. Suppose $t_b = 1$, $t_{t0} = 0.5$, $t_{t1} = 1$, $p_b = 1$, $p_t = 3$ and $w_i = 1$. Further suppose $\beta_0 = -12$, $\beta_1 = -4$, $\beta_2 = 5$, and $\beta_3 = -1$. Finally, assume that $d_i$ is uniformly distributed between 0 and 1 and that $\varepsilon_{ib}$ is uniformly distributed between 0 and 10.

(a) What is the marginal value of time in this model? Why?

- **Solution:** The marginal utility of time is $\frac{\partial u_{bi}}{\partial t_b} = \frac{\partial u_{ti}}{\partial t_t} = \beta_1 w_i$. The marginal utility of money is $\frac{\partial u_{bi}}{\partial p_b} = \frac{\partial u_{ti}}{\partial p_t} = \beta_2 + \beta_3 w_i$. The marginal value of time is the monetary value of a unit of time, or the marginal rate of substitution between money and time, or the marginal amount an individual would be willing to pay to save a marginal amount of time. It is

$$-\frac{\frac{\partial u_{bi}}{\partial t_b}}{\frac{\partial u_{bi}}{\partial p_b}} = \frac{\beta_1 w_i}{\beta_2 + \beta_3 w_i}.$$

In this case this expression equals

$$-\frac{\frac{\partial u_{bi}}{\partial t_b}}{\frac{\partial u_{bi}}{\partial p_b}} = -\frac{4 w_i}{5 - 1 w_i} = -1.$$

That is, an individual would be willing to pay £1 for a 1 hour reduction in travel time (assuming $t$ is measured in hours.)

(b) If everyone makes 1 trip, what fraction of the population travels by bus and what fraction by train?

- **Solution:** Plugging in the numbers given and dropping the $i$ and $b$ subscripts when not needed, we have

$$u_b = -20 + \varepsilon$$

$$u_t = -14 - 4d$$

with $\varepsilon \in [0, 10]$ and $d \in [0, 1]$. A person uses the bus if

$$u_b \geq u_t,$$

that is if

$$\varepsilon \geq 6 - 4d.$$

Since both $\varepsilon$ and $d$ are uniform, the fraction who use the bus is

$$\begin{aligned} \Pr(\varepsilon \geq 6 - 4d) &= \int_0^1 \int_{6-4d}^{10} f_\varepsilon(\varepsilon) f_d(d)\, \partial\varepsilon \partial d \qquad (1) \\ &= \int_0^1 f_d(d) \left( \int_{6-4d}^{10} \frac{1}{10} \partial\varepsilon \right) \partial d \\ &= \int_0^1 f_d(d) \left( \frac{[10 - (6 - 4d)]}{10} \right) \partial d \\ &= \frac{2}{5} + \int_0^1 \frac{2d}{5} \cdot \partial d \\ &= \frac{3}{5}. \end{aligned}$$

Another way to see this is the following. Those consumers with $d = 0$ use the bus if $\varepsilon \geq 6$. 40% have $\varepsilon \geq 6$. Those with $d = 1$ use the bus if $\varepsilon \geq 2$. 80% have $\varepsilon \geq 2$. Since $d$ is distributed uniformly between 0 and 1, then the total fraction who use the bus is the average of 40% and 80%. That is, it is 60% which equals $\frac{3}{5}$. Another way to see this is to draw a picture with $d$ on the $x - axis$ and $\varepsilon$ on the $y - axis$. Draw the line $\varepsilon = 6 - 4d$. Everyone with $\varepsilon \leq 6-4d$ uses the train. Everyone with $\varepsilon \geq 6-4d$ uses the bus. The integral (1) calculates the area above the line.

(c) Assume that all bus and train fares are paid to the government. What would be the impact on travel choices, utilities, and government revenue of an increase in the train fare from £3 to £4 pounds? Is this a good policy?

- **Solution:** If the train fare increases to £4, then the calculations in b. yield

$$u_b = -20 + \varepsilon$$

$$u_t = -18 - 4d.$$

People now choose to use the bus if

$$\begin{aligned} -20 + \varepsilon &\geq -18 - 4d \\ \varepsilon &\geq 2 - 4d. \end{aligned}$$

Notice that if $d = 0$, 80% use the bus while if $d \geq 0.5$, everyone uses the bus. The population with $d \geq 0.5$ is 50% of the population. For the population with $d \leq 0.5$, the fraction who use the bus is $\frac{1}{2}0.8 + \frac{1}{2}1.0 = 0.9$. So 90% of those with $d \leq 0.5$ use the bus. The total population who use the bus is 0.5+0.9*0.5=0.95. 95% use the bus. Government revenue from train fares falls from 0.4*3 to 0.05*4. Government revenue from bus fare increases from 0.6*1 to 0.95*1. The total change in revenue is

$$\begin{aligned} \Delta R &= \Delta R_t + \Delta R_b \\ &= (0.05 * 4 - 0.4 * 3) + (0.95 - 0.6) \\ &= -1 + 0.35 = -0.65. \end{aligned}$$

The utility change can be calculated by focusing on 3 groups.

i. Hard core bus users $(\varepsilon \geq 6 - 4d \text{ and } \varepsilon \geq 2 - 4d)$. This represents 60% of the population. They experience no change in welfare.

ii. Hard core train users $(\varepsilon \leq 6 - 4d \text{ and } \varepsilon \leq 2 - 4d)$. This represents 5% of the population. Their change in utility is

$$\begin{aligned} \Delta u_{ii} &= -(\beta_2 + \beta_3 w_i)\left(p_t^{new} - p_t^{old}\right) \\ &= -4. \end{aligned}$$

iii. Train to bus switchers $(\varepsilon \leq 6 - 4d \text{ and } \varepsilon \geq 2 - 4d)$. This represents 35% of the population. For a member of this group, the change in utility is

$$\begin{aligned} \Delta u_{iii} &= u_b^{new} - u_t^{old} \\ &= -6 + \varepsilon + 4d. \end{aligned}$$

Each member of this group has a different value. The total change in utility for this group is

$$\begin{aligned} \Delta u_{iii}^{total} &= \int_0^{0.5} \int_{2-4d}^{6-4d} \frac{(-6 + \varepsilon + 4d)}{10} \partial\varepsilon \partial d \\ &+ \int_{0.5}^{1} \int_{0}^{6-4d} \frac{(-6 + \varepsilon + 4d)}{10} \partial\varepsilon \partial d. \end{aligned}$$

The computation has to be broken up into two subintervals because $2 - 4d \leq 0$ when $d \geq 0.5$. This integral is

$$\begin{aligned}\Delta u_{iii}^{total} &= -6\,(0.65) + \int_{0}^{0.5}\int_{2-4d}^{6-4d} \frac{(\varepsilon + 4d)}{10} \partial\varepsilon\partial d \\ &\quad + \int_{0.5}^{1}\int_{0}^{6-4d} \frac{(\varepsilon + 4d)}{10} \partial\varepsilon\partial d\end{aligned}$$

which is equivalent to

$$\Delta u_{iii}^{total} = -2\,(0.65) + \frac{4}{5} + \frac{2}{3}.$$