ECON3021 - Urban Economics
Winter 2009
Assignment 2: More locational equilibrium
Solutions

Question 1A

Only firm 1 will operate in the city if firm 1 always bids more for land. That is, if

$$b_1(x) \geq \max\{b_2(x), b_c(x)\} \text{ for all } x. \tag{1}$$

Consider $b_1(x)$. When $x = 0$, $b_1(0) = A_1 p_1^2$. Moreover, $b_1(x)$ reaches its minimum value of zero when $x = \frac{p_1}{t_1}$. Similarly, $b_2(0) = A_2 p_2$ and b_2 reaches its minimum value at $x_2 = \frac{p_2}{t_2}$. Finally, $b_c(0) = \frac{I_c^2}{4p_c V_c^2}$ and b_c reaches its minimum value at $x_c = \frac{I_c}{t_c}$. Each of the bid rent functions is quadratic. Thus, (1) is the equilibrium if

$$A_1 p_1^2 \geq A_2 p_2^2 \text{ and } \frac{p_1}{t_1} \geq \frac{p_2}{t_2}$$

$$A_1 p_1^2 \geq \frac{I_c^2}{4p_c V_c^2} \text{ and } \frac{p_1}{t_1} \geq \frac{I_c}{t_c}.$$

You should draw the picture to make this clear. The condition requires either that firm 1 is the more productive firm ($A_1 > A_2$) or that it produces a more valueable product ($p_1 > p_2$) or both. The condition also requires that the ratio of price to marginal transport cost of firm 1 is bigger either because firm 1 sells a more valuable product or because it has lower transport costs. Similarly, the condition requires that the value of the city is higher to firm 1 then to the consumer. This will be true if productivity or output price or both are high and if income is low or consumption prices are high, or if reservation utility is high.
The equilibrium rent will equal $b_1(x)$ for all x such that $b_1(x) \geq r_A$ and will equal r_A for all x such that $b_1(x) \leq r_A$. Let x_B satisfy

$$\begin{aligned} b_1(x_B) &= r_A \\ A_1(p_1 - tx_B)^2 &= r_A \end{aligned}$$

so that

$$x_B = \frac{p_1 - \sqrt{\frac{r_A}{A_1}}}{t_1}.$$

Then x_B is the boundary of the city and the equilibrium rent function is

$$r(x) = \left\{ \begin{array}{cc} A_1(p_1 - t_1 x)^2 & x \leq x_B \\ r_A & x \geq x_B \end{array} \right\}.$$

Question 1B

In this case, firm 1 is willing to pay the most for land close to the centre only. It must be the case that

$$b_1(0) \geq \max\{b_2(0), b_c(0)\}$$

or

$$\begin{aligned} A_1 p_1^2 &\geq A_2 p_2^2 \\ A_1 p_1^2 &\geq \frac{I_c^2}{4 p_c V_c^2}. \end{aligned}$$

That is, the combination of price and productivity for firm 1 must be bigger than for firm 2. Also, the combination of household income, reservation utility and prices must be small enough.

But, it is also the case, that firm 2 is willing to pay the most for land satisfying $x_1 \leq x \leq x_2$. Thus it must be the case that

$$\frac{p_1}{t_1} < \frac{p_2}{t_2}$$

and

$$A_1 (p_1 - t_1 x_1)^2 = A_2 (p_2 - t_2 x_1)^2 > \frac{(I_c - t_c x_1)^2}{4 p_c V_c^2}.$$

The ratio of price to transport cost must be smaller for firm 1 than for firm 2.

This implies that

$$x_1 = \frac{p_2 - \sqrt{\frac{A_1}{A_2}} p_1}{t_2 - \sqrt{\frac{A_1}{A_2}} t_1}.$$

The boundary between firm 1 locations and firm 2 locations is determined by the difference in prices, weighted by productivities and by the difference in transport costs weighted by productivity.

Finally, it must be the case that

$$\frac{p_2}{t_2} < \frac{I_c}{t_c}$$

and that

$$A_2 (p_2 - t_2 x_2)^2 = \frac{(I_c - t_c x_2)^2}{4 p_c V_c^2}.$$

This implies that

$$x_2 = \frac{I_c - 2 V_c p_2 \sqrt{p_c A_2}}{t_c - 2 V_c \sqrt{p_c A_2} t_2}.$$

The boundary between firm 2 locations and household locations is determined by household income, household utility, firm 2 prices and productivity, and by transport cost differences.

Finally the boundary of the city satisfies

$$\frac{(I_c - t_c x_B)^2}{4 p_c V_c^2} = r_A$$

or

$$x_B = \frac{I_c - 2V_c\sqrt{p_c r_A}}{t_c}.$$

The boundary of the city is determined by household income, household utility, retail prices, agricultural rent, and transport costs.

Question 2A

For consumer F, the budget constraint is

$$I_F - t_F x = pC + r(x) L.$$

For consumer S, the budget constraint is

$$I_S - t_S x = pC + r(x) L.$$

Question 2B

For consumer F, the first order conditions are

$$\begin{aligned} 0.5C^{-0.5}(L-2)^{0.5} &= \lambda p \\ 0.5C^{-0.5}(L-2)^{-0.5} &= \lambda r(x) \\ \lambda\left(-t_F - \frac{\partial r(x)}{\partial x} L\right) &= 0 \\ I_F - t_F x - pC - r(x) L &= 0. \end{aligned}$$

The conditional demand functions are

$$\begin{aligned} C^* &= 0.5\left(\frac{I_F - t_F x - 2r(x)}{p}\right) \\ L^* &= 0.5\left(\frac{I_F - t_F x + 2r(x)}{r(x)}\right). \end{aligned}$$

Note that with these conditional demand function, the utility obtained by type F at location x within city A is

$$\begin{aligned} v_F(x) &= u_F(C^*, L^*) \\ &= 0.5(I_F - t_F x - 2r(x)) p^{-0.5} (r(x))^{-0.5} \end{aligned} \tag{2}$$

For the single types, the conditional demand functions are

$$\begin{aligned} C^* &= 0.5\left(\frac{I_S - t_S x}{p}\right) \\ L^* &= 0.5\left(\frac{I_S - t_S x}{r(x)}\right) \end{aligned}$$

and the utility obtained by type S at location x in city A is

$$\begin{aligned} v_S(x) &= u_S(C^*, L^*) \\ &= 0.5(I_S - t_S x) p^{-0.5} (r(x))^{-0.5} \end{aligned} \tag{3}$$

Question 2C

The bid rent functions describe how much each type would be willing to pay or bid to live at each location. Since type F obtains V_F if they live in city B, the most they would be willing to pay to live at any location x in city A is the amount that would enable them to obtain at least this same utility. For example, using formula (2), if type F were to live at location x and pay rent $r(x)$ then their utility would be

$$v_F(x) = 0.5\left(I_F - 2r(x) - t_F x\right) p^{-0.5} r(x)^{-0.5}.$$

The bid rent of type F at location x, is the amount $r_F(x)$ such that

$$V_F = v_F(x) = 0.5\left(I_F - 2r_F(x) - t_F x\right) p^{-0.5} r_F(x)^{-0.5}. \tag{4}$$

We can solve this equation for $r_F(x)$ as follows. First square both sides of (4)

$$V_F^2 = 4\left(I_F - 2r_F(x) - t_F x\right)^2 p^{-1} r_F(x)^{-1}.$$

Then multiply by $r_F(x)$ to get

$$V_F^2 r_F(x) = 4\left(I_F - 2r_F(x) - t_F x\right)^2 p^{-1}.$$

This is equivalent to

$$r_F(x)^2 - \left(\frac{pV_F^2 + 16}{16}\right) r_F(x) + \frac{\left(I_F - t_F x\right)^2}{4} = 0.$$

This has solution

$$r_F(x) = \frac{1 + \frac{pV_F^2}{16} - \sqrt{\left(\frac{pV_F^2}{16} + 1\right)^2 - \left(I_F - t_F x\right)^2}}{2}$$

Alternatively, we can recall that every location will provide the same utility only if the locational equilibrium condition is met so that

$$\frac{\partial r_F(x)}{\partial x} = \frac{-t_F}{0.5\left(\frac{I_F - t_F x + 2r_F(x)}{r_F(x)}\right)}.$$

We can do similar calculations for the single type. Since type S obtains V_S if they live in city B, the most they would be willing to pay to live at any location x in city A is the amount that would enable them to obtain at least this same utility. For example, using formula (3), if type S were to live at location x and pay rent $r_S(x)$ then their utility would be

$$v_S(x) = 0.5\left(I_S - t_S x\right) p^{-0.5} r_S(x)^{-0.5}.$$

The bid rent of type S at location x, is the amount $r_S(x)$ such that

$$V_S = v_S(x) = 0.5\left(I_S - t_S x\right) p^{-0.5} r_S(x)^{-0.5}.$$

That is,

$$r_S(x) = \frac{\left(I_S - t_S x\right)^2}{4V_S^2 p}.$$

In this problem, there are four factors that determine which type will live closer to the centre. Firstly, we must consider income. The two types may have different levels of income. Everything else equal, the type with higher income will live further away. Second, we must consider preferences for land relative to consumption. The two types have different preferences in this dimension. The Family type must be able to afford the minimum consumption of land of 2 units. This will tend to lead the Family type to locate farther from the centre. Third, the two types have different preferences for city A versus city B. The higher is V_F relative to V_S the more likely that F will have a lower bid rent function. Finally, we also must consider transport costs differences. In this problem, if $t_F > t_S$ we might expect that, everything else equal, the family type would live closer to the centre.