ECON3021 Urban Economics
Winter 2009
Assignment 1: Solutions

Reading assignment: Lecture notes 1, 2.

1. In a circular city, all consumers commute to the centre. Transportation costs for a consumer living at distance x are tx. All consumers have identical incomes I and identical preferences over consumption of b (bread) and h (housing). p the price of b is the same everywhere in the city while $r(x)$ the price of housing varies depending on distance from the centre. Let the utility function be $u(b,h) = (b - b_L)^{0.5} h^{0.5}$.

   (a) What are the first-order conditions characterising the consumers' optimal choices?

   - Introduce the new variable λ. The Lagrangian for this problem is

     $$\mathcal{L}(b,h,\lambda,x) = (b - b_L)^{0.5} h^{0.5} + \lambda (I - tx - pb - r(x) h)$$

     where λ is the Lagrange multiplier. The consumer chooses (b,h,λ,x) to maximise this function. The first order conditions are

     $$\begin{aligned} \frac{\partial \mathcal{L}}{\partial b} &= 0.5 (b - b_L)^{-0.5} h^{0.5} - \lambda p = 0 && (1) \\ \frac{\partial \mathcal{L}}{\partial h} &= 0.5 (b - b_L)^{0.5} h^{-0.5} - \lambda r(x) = 0 && (2) \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= I - tx - pb - r(x) h = 0 && (3) \\ \frac{\partial \mathcal{L}}{\partial x} &= \lambda \left(-t - \frac{\partial r(x)}{\partial x} h \right) = 0 && (4) \end{aligned}$$

     If (b,h,λ) maximise $\mathcal{L}$, then they must satisfy these equations. That is, at the optimum the marginal benefit of changing any of the variables, equals that marginal cost. In particular, the third condition ensures that at the optimiser, the budget constraint is satisfied.

   (b) For fixed location x, solve for the optimal choices of b and h as functions of x and the other variables.

   - First rewrite the first and second first order conditions as

     $$\begin{aligned} 0.5 (b - b_L)^{-0.5} h^{0.5} &= \lambda p && (5) \\ 0.5 (b - b_L)^{0.5} h^{-0.5} &= \lambda r(x). && (6) \end{aligned}$$

     To eliminate λ from the expressions, divide equation (5) by (6) to obtain

     $$\frac{0.5 (b - b_L)^{-0.5} h^{0.5}}{0.5 (b - b_L)^{0.5} h^{-0.5}} = \frac{\lambda p}{\lambda r(x)}$$

     which can be simplified to

     $$\frac{h}{b - b_L} = \frac{p}{r(x)}.$$

The left side is the marginal rate of substitution between b and h. The right side is the price ratio. After solving for b, this equation implies

$$b = b_L + \frac{r(x)\,h}{p}. \tag{7}$$

This expression can be substituted into the budget constraint (3) to produce

$$h^*(I - tx, p, r(x)) = \frac{1}{2}\left(\frac{I - tx - pb_L}{r(x)}\right). \tag{8}$$

Then, using (7) we obtain

$$b^*(I - tx, p, r(x)) = b_L + \frac{1}{2}\left(\frac{I - tx - pb_L}{p}\right) \tag{9}$$

(c) Solve for the differential equation describing the slope of the equilibrium land rent function.

- If all locations yield equal utility, then the slope of the rent function must satisfy

  $$-t - \frac{\partial r(x)}{\partial x} h^*(I - tx, p, r(x)) = 0$$

  or

  $$\frac{\partial r(x)}{\partial x} = \frac{-t}{h^*(I - tx, p, r(x))}$$

  for all $x \leq x_B$ where x_B is the boundary of the city. In spatial equilibrium with identical agents living at all locations $x \leq x_B$, this equation must be satisfied to ensure equal utilities across locations.

(d) What are the equilibrium conditions in this model? Draw a graph showing the approximate shape of the equilibrium price function.

- Condition 1: All consumers maximise utility.
- Condition 2: Spatial equilibrium. No one wants to move and land goes to the highest willingness to pay user.
- Condition 3: Market clearing. Supply equals demand for housing and bread at every location.
- The rent function will be downward sloping and convex. The boundary will be fixed at some point x_B. The rent at the centre will be r_0.

(e) In a full equilibrium of this city, how is the radius of the city and the population determined (Hint: What assumptions were made in the lecture to complete the model?)

- Assume total population is fixed at N.
- Assume the rent at the boundary of the city is fixed at r_A.
- Assume the supply of land at every location x is $S_L(x) = 2\pi x$.

- Under these assumptions, the population of the city is fixed. Then the radius of the city and the utility level in the city are determined by adjusting the rent function until

$$r_A = r_0 - \int_0^{x_B} \frac{t}{h^*(I - tx', p, r(x'))} dx'$$

$$N = \int_0^{x_B} N(x')\, dx'$$

where

$$N(x) = \frac{2\pi x}{h^*(I - tx, p, r(x))}.$$

- Alternatively, we could assume:
    1. N is fixed and x_B are fixed and (r_0, r_A) adjust until an equilibrium is obtained. (Fixed population, fixed radius model)
    2. Utility is fixed at V_0 and r_A is fixed and (r_0, N) adjust until an equilibrium is reached (Free migration, fixed boundary rent model
    3. Utility is fixed and x_B is fixed.
- For details, see Lecture notes 3.

2. Consider a circular city with *free migration* and *fixed boundary rent* of r_b per unit of land. Initially, the city is in spatial equilibrium with equilibrium rent function $r^0(x)$. That is, $r^0(x)$ is the rent per unit of land at a distance of x miles from the centre. Initially all land within the city is available for housing and the supply of land at distance x from the centre is $2\pi x$. The city has initial equilibrium population N^0 and equilibrium boundary of $x_b^0 > 1$ miles. People are free to move into and out of the city and obtain reservation utility V_A if they leave. Since the city is in equilibrium all residents initially obtain utility level $V^0 = V_A$. Each of the residents of the city is identical. Each obtains utility from C (a consumption good) and L (land) and has income I. A household living x miles from the centre of the city, faces prices for C and L of p and $r^0(x)$ respectively, and must commute to the centre at cost of t per mile. Suppose a hurricane destroys all the housing within 1 mile of the centre so that after the hurricane the supply of land available for housing at each distance $x \leq 1$ is zero. The supply of land at other locations is unaffected. Explain using words and graphs how a new equilibrium will be determined, what it will look like, and how it will compare to the initial equilibrium.

    - Solution: Before the hurricane the equilibrium rent function is a downward sloping convex function with $r^0(0) = r_0$ and $r^0(x_b^0) = r_b$. All residents obtain utility $V^0 = V_A$. After the hurricane, all residential land at $x \leq 1$ is destroyed. Everyone still commutes to the centre. If the rent function remains unchanged at locations $1 \leq x \leq x_b$, then everyone living at these locations can continue living without any changes. Before the hurricane they obtain $V^0 = V_A$. After the hurricane, if the rent function is the same, they continue to obtain $V^0 = V_A$. What about those who live at locations $x \leq 1$ before the hurricane. They cannot remain where they are. They can either move within the city to some locations $x \geq 1$ or they can move out of the city and obtain utility V_A. If they move to a location $x \geq 1$, then they must bid some amount $r^1(x) > r^0(x)$ to obtain a house. Since the land in these locations is fully occupied prior to the hurricane, that is the only option. However, if they rent a house at one of these locations for such a high rent, their utility will be less than V_A. Why? Therefore, all these people will leave the city. The impact of the hurricane will be the loss of population at all locations $x \leq 1$. No other changes will occur in this example.