Lecture 7 - Locational equilibrium continued

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1 Review

1. Constant returns to scale (CRS) production function

$$\begin{aligned} y &= f(K, L) \\ &= K^{\alpha} L^{1-\alpha}. \end{aligned} \tag{1}$$

2. Profits are

$$(p - tx) \cdot K^{\alpha} L^{1-\alpha} - qK - r(x) L.$$

   Businesses hire land and capital to produce y. The price of the output net of transport costs is $p - tx$. The price of capital is q. The price of land at location x is $r(x)$.

3. Optimal capital-land ratio $\frac{K^*}{L^*}$

$$\frac{K^*}{L^*} = \frac{\alpha}{1-\alpha} \cdot \frac{r(x)}{q}.$$

2 Business location choice continued

1. Since the production function is a CRS function we cannot uniquely define the optimal choice of L^*. If $L^* = L_1$ is an optimal choice then so is $L^* = 2 \cdot L_1$.

2. However, if the firm earns zero profits, then every value of L is optimal. We can choose to focus on the equilibrium outcome in which all firms earn zero profits, one firm chooses to locate in every location, and the

supply of land equals the demand for land. If the supply of land must equal demand, the supply of land is equal to $2\pi x$, and there is 1 firm at every location earning zero profits, then $L^*(x) = 2\pi x$ is an optimal choice for the firm that is consistent with equilibrium.

3. Each firm increases production until all available land is used up. Then combining this fact with the optimal capital-land ratio above implies that

$$K^*(x) = \frac{\alpha}{1-\alpha} \cdot \frac{r(x)}{q} \cdot 2\pi x$$
$$L^*(x) = 2\pi x$$

and

$$\begin{aligned} y^*(x) &= K^*(x)^{\alpha} L^*(x)^{1-\alpha} \\ &= y^*(x) = 2\pi x \left(\frac{\alpha}{1-\alpha} \cdot \frac{r(x)}{q} \right)^{\alpha}. \end{aligned}$$

4. $K^*(x), L^*(x)$ demand for land and labor at every location and output $y^*(x)$.

5. Note if $r(x_1) > r(x_2)$, then the optimal capital land ratio will be higher at x_1 than at x_2.

3 Locational equilibrium condition for firms

1. Locational equilibrium: all locations earn zero profits

$$(p - tx) \cdot K^*(x)^{\alpha} L^*(x)^{1-\alpha} - qK^*(x) - r(x) L^*(x) = 0. \qquad (2)$$

2. Let $\pi(x)$ be the profit function in (2). Equation (2) states that $\pi(x) = 0$ for all x. In order for this to be true at all locations, it must be the case that $\frac{\partial \pi}{\partial x} = 0$ at all locations. Differentiating $\pi(x)$ with respect to

x, we obtain

$$\begin{aligned}
&\frac{\partial \pi(x)}{\partial x} = \\
&-t \cdot K^*(x)^{\alpha} L^*(x)^{1-\alpha} - \frac{\partial r(x)}{\partial x} L^*(x) \\
&+ \left(\alpha (p - tx) K^*(x)^{\alpha-1} L^*(x)^{\alpha} - q\right) \left(\frac{\partial K^*(x)}{\partial x}\right) \\
&+ \left((1-\alpha)(p - tx) K^*(x)^{\alpha} L^*(x)^{-\alpha} - r(x)\right) \left(\frac{\partial L^*(x)}{\partial x}\right).
\end{aligned}$$

If we look closely at the final two lines in this expression, we see that the term multiplying $\frac{\partial K^*(x)}{\partial x}$ is identically equal to zero. This term is equal to zero because at the optimum, the firm sets the marginal product of capital $\left(\alpha (p - tx) K^*(x)^{\alpha-1} L^*(x)^{\alpha}\right)$ equal to the marginal cost (q). We also see that the term multiplying $\frac{\partial L^*(x)}{\partial x}$ is identically equal to zero. This term equals zero because the firm chooses the optimal ratio of land L^* and capital K^* so that the marginal product of land $\left((1-\alpha)(p - tx) K^*(x)^{\alpha} L^*(x)^{-\alpha} - r(x)\right)$ equals the marginal cost $(r(x))$. Hence, the final two lines in the expression equal zero. This is an application of the envelope theorem. As a result the complete expression for $\frac{\partial \pi(x)}{\partial x}$ can be simplified to

$$\frac{\partial \pi(x)}{\partial x} = -t \cdot K^*(x)^{\alpha} L^*(x)^{1-\alpha} - \frac{\partial r(x)}{\partial x} L^*(x).$$

An incremental increase in distance from the centre reduces profits by an amount equal to the incremental increase in transport costs and increases profits by an amount equal to the incremental reduction in rent. In equilibrium this incremental change must equal zero. Setting $\frac{\partial \pi(x)}{\partial x} = 0$, we have

$$\begin{aligned}
\frac{\partial r(x)}{\partial x} &= \frac{-t \cdot K^*(x)^{\alpha} L^*(x)^{1-\alpha}}{L^*(x)} \qquad (3) \\
&= -t \cdot \left(\frac{K^*(x)}{L^*(x)}\right)^{\alpha} \\
&= -t \cdot \left(\frac{\alpha}{1-\alpha} \cdot \frac{r(x)}{q}\right)^{\alpha}.
\end{aligned}$$

An equilibrium rent function must satisfy this differential equation.

3. This implies that $\frac{\partial r(x)}{\partial x} < 0$ at every location that has at least one firm.

4. It also implies $\frac{\partial^2 r}{\partial x^2} > 0$ if there is input substitution.

   (a) As one moves toward the centre, tranport costs fall, the price of land rises, and firms substitute toward capital. The capital-land ratio rises toward the centre, and the slope of the rent function becomes steeper. That is, if $x_1 < x_2$, $\frac{\partial r(x_1)}{\partial x} < \frac{\partial r(x_2)}{\partial x}$.

4 Equilibrium conditions

1. Given, $L^*(x)$ firms choose $K^*(x)$ and $y^*(x)$ to maximize profits.

2. Free entry.

   (a) Profits are zero.

3. Equilibrium in land market and assuming one firm per location.

   (a) $L^*(x) = 2\pi x$.

4. Locational equilibrium conditions ensures that every location earns the same profits.

   (a) Hence, the slope of the rent function must satisfy (3).

   $$\frac{\partial r(x)}{\partial x} = -t \cdot \left(\frac{\alpha}{1-\alpha} \cdot \frac{r(x)}{q} \right)^{\alpha}.$$

   (b) Let rent at centre equal r_0.

   (c) Then the rent function satisfies

   $$r(x) = r_0 + \int_0^x \frac{\partial r(s)}{\partial x} ds$$

(d) At the urban boundary x_b the rent must equal the agricultural rent r_A. Hence,

$$r_A = r_0 + \int_0^{x_b} \frac{\partial r(s)}{\partial x} ds \tag{4}$$

This condition is obtained from the condition that firms earn the same profits at all locations. In this equation r_A is known while x_b and r_0 are unknown.

(e) When we also impose that firms earn zero profits, we can determine x_b. If firms earn zero profits, the firm choosing $x = x_b$ must earn zero profits. Since $y^*(x_b) = 2\pi x_b \left(\frac{\alpha r_b}{(1-\alpha)q}\right)^\alpha$, $K^*(x_b) = \left(\frac{\alpha}{1-\alpha}\right)\frac{r_A}{r_K} 2\pi x_b$, and $L^*(x_b) = 2\pi x_b$, profits at the boundary must satisfy

$$p - tx_b 2\pi x_b \left(\frac{\alpha r_A}{(1-\alpha)q}\right)^\alpha - q\left(\frac{\alpha}{1-\alpha}\right)\frac{r_A}{q} 2\pi x_b - r_A 2\pi x_b = 0.$$

Dividing both sides by $2\pi x_b$ we obtain

$$(p - tx_b)\left(\frac{\alpha r_A}{(1-\alpha)q}\right)^\alpha - \left(\frac{\alpha}{1-\alpha}\right) r_A - r_A = 0$$

or

$$(p - tx_b)\left(\frac{\alpha r_A}{(1-\alpha)q}\right)^\alpha = r_A\left(\frac{1}{1-\alpha}\right).$$

This is equivalent to

$$p - tx_b = r_A^{1-\alpha}\left(\frac{1}{1-\alpha}\right)\left(\frac{(1-\alpha)q}{\alpha}\right)^\alpha.$$

When solved for x_b this becomes

$$x_b = \frac{p - r_A^{1-\alpha} q^\alpha (1-\alpha)^{\alpha-1}\alpha^{-\alpha}}{t} \tag{5}$$

(f) Once we know x_b, we can determine r_0 from (4).

5. Supply equals demand for output

$$\int_0^{x_b} y^* (x, r(x))\, dx = D(p)$$
$$\int_0^{x_b} y^* (x, r(x))\, dx = D(p)$$

The function $D(p)$ is the demand for output when price equals p. In the case, of perfectly elastic demand, this means that the price is fixed at p and demand adjusts so that demand equals supply regardless of the quantuty supplied.

   (a) Land goes to highest bidder
      i. $r(x) \geq r_A$ for all $x \leq x_b$ with equality at $x = x_b$.
      ii. Edge of city induces zero profits.

6. Summary
   (a) (Identical) firms get zero profits at every location
   (b) $\frac{\partial r}{\partial x} < 0$
   (c) $\frac{\partial^2 r}{\partial x^2} > 0$
   (d) $r(x)$ solves (3)
   (e) r depends on t, $f(K, L)$, r_A, q, $D(p)$
   (f) Slope depends on how easy it is to substitute capital for land
   (g) $\frac{K}{L}$ decreases with x, $\frac{y}{L}$ decreases with x

7. Comparative statics
   (a) Bigger t smaller city
   (b) Steeper rent
   (c) Offset by more output being produced closer to center. If capital intensive output technology can offset.
   (d) Increase demand for product, increase city size
   (e) Increase cost of capital, makes it harder to substitute capital, increases costs, tends to reduce city size

8. What if $y = K^{\alpha}L^{1-\alpha}$ and α increases (invention of new technology)?

   (a) A bit more complicated to work out
   (b) Technology becomes more capital intensive
   (c) Relatively easy to shift into capital and maintain output
   (d) If capital costs are small fraction of total costs, costs should decline, output increase, x_b could increase or decrease, total land in city should become more valuable
   (e) If capital costs are a large fraction of total costs, costs should increase, total land value should fall

9. In the city with only a business sector, what would be the effect on welfare of a £1 billion investment in transportation that reduced t by 20%?

   (a) There are three effects to consider: the effects on businesses, the effects on landowners, and the effects on final output prices and hence on consumers.
   (b) Since all firms earn zero profits, there is no effect on business profits.
   (c) The effect on landowners equals the change in rents. Let t_1 be the transport cost prior to the investment, let x_{b1} be the boundary of the city prior to the investment and let $r(x, t_1)$ be the rent function prior to the investment. Similarly, let t_2 be the transport cost after the investment, let x_{b2} be the boundary of the city after the investment, and let $r(x, t_2)$ be the rent function after the investment. The total change in land rents in the city is

$$\Delta r = \int_0^{x_{b2}} r(x, t_2)\, dx - \int_0^{x_{b1}} r(x, t_1)\, dx.$$

   Draw a graph. This is the total value of land in the city after the investment minus the total value of land in the city before the investment. This measures the total change in the welfare of landowners. In this case, the total change is likely to be positive since total the total costs of production in the city have fallen (because transport costs have fallen). Since the value of land in the city is determined in this example by its value in production, the fall in transport costs make the land more valuable overall. It is possible that land rents decline at some locations.

(d) Finally, if demand for the output of the city is not perfectly elastic, there will also be an effect on prices of the output. Total supply will increase in response to the investment. This total increase in supply will lower prices of the final output good. This will increase consumer surplus in the economy. If CS_2 is the consumer surplus after the price change and CS_1 is the consumer surplus before the price change. The total change in consumer welfare is $\Delta CS = CS_2 - CS_1$. This will be positive if demand is not perfectly elastic.

(e) The total benefit of the investment is $\Delta r + \Delta CS$

(f) The total cost of the investment is £1 billion plus any deadweight loss associated with raising the revenue required to pay for the investment.