Lecture 6 - Locational equilibrium continued

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1 Introduction

1. Multiple types of consumer.
2. Business location choice.

2 Equilibrium with multiple consumer types

1. A consumer's "type" is determined by income, preferences, and transport costs. In an equilibrium with a single type of consumer (everyone is identical), consumers lives at different locations but all consumers attain the same utility. All locations provide the same utility.

2. In an equilibrium with multiple consumer types, different types will, in general, live at different locations. The city will segregated into sectors, each sector inhabited by a single type. Everyone of the same type will attain the same utility level but different types will in general attain different utility levels. Comparing the different types, those who value land close to the centre the most will live in a sector close to the centre. Those who value land close to the center the most are those who are willing to "bid" the most or pay the most for land close to the centre. They are the ones who have the highest "bid rent" or the highest "willingness to pay". For consumers, three factors determine who values land close to the centre the most: 1) Income, 2) transport costs, 3) preferences.

3. Everything else equal, if land is a normal good, those with more income will want to consume more land and will bid less to live close to the centre. They will bid less close to the centre because land is cheaper farther away from the centre and they will want to consume more land. This assumes that those with high income have the same transport costs and the same preferences as those with less income.

4. Everything else equal, those with higher transport costs will bid more for land close to the centre. They will bid more for land close to the centre because it is relatively more costly for them to commute longer distances. This assumes those that have higher transport costs have the same income and the same preferences as those with low transport costs.

5. Everything else equal, those with preferences such that it is relatively easy to maintain a fixed utility level by substituting consumption of other goods for consumption of land will bid more for land closer to the centre. They will bid more for land closer to the centre because they can maintain a constant utility level near the centre by consuming less land despite paying a higher price.

6. In general, in determining whether one type of consumer will live closer to the centre than another, one must consider all 3 factors. For instance, in an economy with 2 types of people, type 1 may have higher income, higher transport costs, and different preferences than type 2. In this case, to determine whether type 1 lives closer to the centre or farther, one must consider all 3 factors.

3 A simple example: Bid rent functions

1. There are two types of consumers with incomes, $I_1 < I_2$. The two types have identical transport cost per mile t and identical preferences. Assume land is a normal good. Who lives closer to centre, type 1 or type 2?

2. Suppose type 1 obtains utility level v_1 in equilibrium and type 2 obtains utility level v_2 in equilibrium.

(a) How much would type 1 be willing to pay to live at location x? They would pay any amount $b(x)$ such that

$$v(I_1 - tx, p, b(x)) = u(c^*(I_1 - tx, p, b(x)), L^*(I_1 - tx, p, b(x))) \geq v_1.$$

(b) The function v is the indirect utility function. It describes the utility obtained by a consumer as a function of net income $(I_1 - tx)$, the price of consumption p and the price of land at x(in this case $b(x)$).

(c) The bid rent or willingness to pay of type 1 at location x is the value of $b_1(x)$ such that

$$v(I_1 - tx, p, b_1(x)) = v_1 \qquad (1)$$

for all x. If $r(x)$, the rent at location x, is higher than $b_1(x)$ than the consumer will refuse to pay it. If the rent $r(x)$ is less than or equal to $b_1(x)$, the consumer will be willing to pay it. The bid rent is the amount that holds utility constant as the household moves across locations. In particular at $x = 0$, it satisfies

$$v(I_1, p, b_1(0)) = v_1.$$

So, $b_1(0)$ is the amount that type 1 is willing to pay to live at the centre. It depends on the equilibrium utility level v_1

3. Suppose type 1 were willing to bid $b_1(0)$ to live at the centre and type 2 were willing to bid $b_2(0)$. Equation (1) is one way to describe the bid rent function at every location x. We can differentiate this equation to get a condition on the slope of the bid rent function

$$\frac{\partial b_i(x)}{\partial x} = -t\left(\frac{-\frac{\partial v}{\partial I}}{\frac{\partial v}{\partial b}}\right). \qquad (2)$$

This is similar to the condition on the slope of the equilibrium rent function that we derived for the city with a single type. In fact, this condition on the slope of the bid rent function is equivalent to the condition derived previously. That is, an equivalent way to express (2) is that the bid rent function for each type must satisfy

$$\frac{db_i(x)}{dx} = \frac{-t}{L^*(I_i - tx, p, b_i(x))}. \qquad (3)$$

Why? These two equations are equivalent because of a theorem known as Roy's identity. Roy's identity states that if $v(I,p,b)$ is an indirect utility function and b is the price of land, then

$$L^*(I,p,b) = \frac{-\frac{\partial v}{\partial b}}{\frac{\partial v}{\partial I}}.$$

That is the demand function can be calculated by differentiating the indirect utility function and taking the ratio of derivatives. Intuitively, (2) and (3) are equivalent because both conditions impose that utility is held constant when the consumer moves across locations. They imply that the bid rent funciton for type i is given by

$$\begin{aligned} b_i(x) &= b_i(0) + \int_0^x \frac{\partial b_i(s)}{\partial x} ds \\ &= b_i(0) - \int_0^x \frac{t}{L^*(I_i - ts, p, b_i(s))} ds. \end{aligned}$$

4. The functions $b_1(x)$ and $b_2(x)$ are *bid rent* functions. $b_1(x)$ expresses how much type 1 would be willing to pay in rent (or how much they would bid) to live at location x assuming they were willing to bid $b_1(0)$ at the centre (recall this amount depends on v_1 the utility level attained by type 1 in equilibrium). That is, if type 1 lived at the centre and paid rent $b_1(0)$, then type 1 would obtain the same utility living at x if and only if the rent at x equaled $b_1(x)$. Similarly, if type 2 lived at the centre and paid rent $b_2(0)$, then type 2 would obtain the same utility living at x if and only if the rent at x equaled $b_2(x)$.

5. Type 1 will be willing to live at location x if and only if $r(x) \leq b_1(x)$.

6. Type 2 will be willing to live at location x if and only if $r(x) \leq b_2(x)$.

7. Suppose the equilbrium rent at location x_1 is $r_e(x_1)$ and in equilibrium both type 1 and type 2 live at location x_1. That is $r_e(x_1) = b_1(x_1) = b_2(x_1)$. Which is larger in magnitude $\frac{\partial b_1(x_1)}{\partial x}$ or $\frac{\partial b_2(x_1)}{\partial x}$? Since L is a normal good, $\frac{\partial L^*(I - tx_1, p, r_e(x_1))}{\partial I} > 0$. Thus, $L^*(I_1 - tx_1, p, r_e(x_1)) <$

$L^*(I_2 - tx_1, p, r_e(x_1))$. Therefore,

$$\frac{-t}{L^*(I_1 - tx_1, p, r_e(x_1))} < \frac{-t}{L^*(I_2 - tx_1, p, r_e(x_1))}.$$

Type 1's bid rent function is steeper than type 2's at location x_1.

(a) If both live at same location x_1, then type 1 has steeper slope at location x_1.

(b) The equilibrium rent function is equal to the maximum of the two bid rent functions

$$r_e(x) = \max\{b_1(x), b_2(x)\}.$$

8. Compute equilibrium for two consumer types.

   (a) Assume populations of types 1 and 2 equal N_1 and N_2 and assume the boundary rent $r(x_b) = r_A$.

   (b) Guess values for bid rents at centre: $b_1(0)$ and $b_2(0)$.

   (c) Compute the bid rent functions:

$$b_1(x) = b_1(0) - \int_0^x \frac{t}{L^*(I_1 - ts, p, b_1(s))} ds$$

$$b_2(x) = b_2(0) - \int_0^x \frac{t}{L^*(I_2 - ts, p, b_2(s))} ds.$$

   (d) Set $r(x) = \max\{b_1(x), b_2(x)\}$. Set equilibrium rent equal to the highest bid.

   (e) Compute x_b :

$$r_A = \int_0^{x_b} r(s) ds.$$

   (f) Set

$$N_1(x) = \left\{ \begin{array}{cc} \frac{2\pi x}{L(I_1 - tx, p, r(x))} & \text{if } b_1(x) > b_2(x) \\ 0 & \text{if } b_1(x) < b_2(x) \\ \frac{1}{2}\left(\frac{2\pi x}{L(I_1 - tx, p, r(x))}\right) & \text{if } b_1(x) = b_2(x) \end{array} \right\}$$

and

$$N_2(x) = \left\{ \begin{array}{cc} \frac{2\pi x}{L(I_2 - tx, p, r(x))} & \text{if } b_2(x) > b_1(x) \\ 0 & \text{if } b_2(x) < b_1(x) \\ \frac{1}{2}\left(\frac{2\pi x}{L(I_2 - tx, p, r(x))}\right) & \text{if } b_2(x) = b_1(x) \end{array} \right\}.$$

This requires supply of land to equal demand for land at every location and requires demand for land of type 1 at location x to be zero if 1 is not the highest bidder at location x. It also requires demand for land of type 2 to be zero at location x if 2 is not the higher bidder at location x.

(g) Check whether

$$N_1 = \int_0^{x_b} N_1(s)\, ds \tag{4}$$

and

$$N_2 = \int_0^{x_b} N_2(s)\, ds. \tag{5}$$

(h) If the right side of (4) is larger than N_1, increase $b_1(0)$. If it is less than N_1, decrease $b_1(0)$.

(i) If the right side of (5) is larger than N_2, increase $b_2(0)$. If it is less than N_2, decrease $b_2(0)$.

(j) In equilibrium, equations (4) and (5) are satisfied.

(k) In equilibrium, in this example type 1, the poor people, will live closer to the centre and type 2, the rich will live farther away. See graph.

(l) How could you change the model to change this conclusion?

9. Equilibrium rules are similar to those in the model with one type of consumer.

   (a) Consumers maximise.

   (b) Identical people who live at different locations in equilibirum, obtain the same utility

   (c) Each plot of land goes to the highest bidder.

   (d) Supply equals demand in every market.

4 Business location choice

1. Now, we will analyse equilibrium in a model with no consumers but with a business sector.

2. As before, suppose the city is a circle.

   (a) The supply of land in every **ring** at distance x is $S_L(x) = 2\pi x$.

3. What is the business demand for land? That is what we want to determine.

4. Assume businesses export output from the transport hub at the city center. (There is increasing returns to scale in export. That is why there is a city in the first place.)

5. Output y is produced with land L and capital K. Assume a constant returns to scale (CRS) production function

$$\begin{aligned} y &= f(K, L) \\ &= K^{\alpha} L^{1-\alpha} \end{aligned} \tag{6}$$

6. Profits of each business are

$$(p - tx) \cdot K^{\alpha} L^{1-\alpha} - r_K K - r(x) L$$

   Businesses hire land and capital to produce y. The price of the output net of transport costs is $p - tx$. The price of capital is r_K. The price of land at location x is $r(x)$.

7. Firm at location x maximises profits. The first order conditions of their maximisation problem are

$$\alpha (p - tx) K^{\alpha-1} L^{1-\alpha} = r_K \tag{7}$$

$$(1-\alpha)(p - tx) K^{\alpha} L^{-\alpha} = r(x) \tag{8}$$

8. These equations can be used to determine the firms' optimal choices of the capital-land ratio $\frac{K^*}{L^*}$. Dividing the left and right sides of equations (7) and (8) we have

$$\frac{\alpha (p - tx) K^{\alpha-1} L^{1-\alpha}}{(1-\alpha)(p - tx) K^{\alpha} L^{-\alpha}} = \frac{r_K}{r(x)}$$

$$\frac{\alpha}{1-\alpha} \cdot \frac{L}{K} = \frac{r_K}{r(x)}$$

$$\frac{K^*}{L^*} = \frac{\alpha}{1-\alpha} \cdot \frac{r(x)}{r_K}$$

9. Since the production function is a CRS function we cannot uniquely define the optimal choice of L^*. If $L^* = L_1$ is an optimal choice then so is $L^* = 2 \cdot L_1$.

10. However, if the firm earns zero profits, then every value of L is optimal. We can choose to focus on the equilibrium outcome in which all firms earn zero profits, one firm chooses to locate in every location, and the supply of land equals the demand for land. If the supply of land must equal demand, the supply of land is equal to $2\pi x$, and there is 1 firm at every location earning zero profits, then $L^*(x) = 2\pi x$ is an optimal choice for the firm that is consistent with equilibrium.

11. Each firm increases production until all available land is used up. Then combining this fact with the optimal capital-land ratio above implies that $K^*(x) = \frac{\alpha}{1-\alpha} \cdot \frac{r(x)}{r_K} \cdot 2\pi x$ and $y^*(x) = K^*(x)^{\alpha} L^*(x)^{1-\alpha}$.

    (a) Note that $y^*(x) = 2\pi x \left(\frac{\alpha}{1-\alpha} \cdot \frac{r(x)}{r_K}\right)^{\alpha}$

12. $K^*(x), L^*(x)$ demand for land and labor at every location and output $y^*(x)$.

13. Note if $r(x_1) > r(x_2)$, then the optimal capital land ratio will be higher at x_1 than at x_2.