Lecture 4− Locational Equilibrium Continued

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1 Introductory remarks

- Last lecture solved the consumer choice problem.
- Computed conditional demand functions: $C^* (I - tx, p, r(x))$ and $L^* (I - tx, p, r(x))$.
- Also, derived expression for optimal location choice:

$$-t - \frac{\partial r(x)}{\partial x} L^* (I - tx, p, r(x)) = 0$$

- Also, computed utility as a function of location:

$$L(C^*, L^*, \lambda^*, x) = u(C^*(x), L^*(x))$$

- Use these to study locational equilibrium
- Definition of locational equilibrium
  1. All consumers maximise utility.
  2. Locational equilibrium
     (a) No one wants to move.
     (b) Land goes to highest bidder.
  3. Equilibrium in land market.

1.1 Equilibrium condition 1: consumer's optimise

1. This determines consumer demand for C and L and consumer location choice.

2. Conditional demand functions. If a consumer chooses location x, then his demand for C and L conditional on that choice of x can be expressed as: $C^*(p, r(x), I - tx)$ and $L^*(p, r(x), I - tx)$

3. These are called conditional demand functions, because they express the demand conditional on choice of location.

4. Given the conditional demand functions, the consumer's location choice satsifies
$$L^*(I - tx, p, r(x)) \frac{\partial (x)}{\partial x} + t = 0. \tag{1}$$
   (a) Each person chooses optimal location x so that marginal cost of moving equals marginal benefit.
   (b) This determines consumer's optimal location choice x^*.
   (c) This does not determine $r(x)$. Need additional conditions to determine $r(x)$.

2 Equilibrium condition 2: locational equilibrium

1. Locational equilibrium condition.
   (a) In equilibrium, no one wants to move.
   (b) An equilibrium rent function satisfies the condition that no one wants to move.

2. Implications.
   (a) N identical people.

(b) If all identical, and none want to move, then all locations that are inhabited must yield the same utility. Otherwise, if some live at location 1 and some live at location 2 but location 1 provides lower utility, then all the people at location 1 would want to move to location 2.

(c) In an equilibrium in which all people are identical and some people live at all locations $x \leq x_B$, all locations $x \leq x_B$ yield identical utility.

(d) People spread out across locations $x \leq x_B$.

(e) For this to be true, it must be the case that (1) holds at every location $x \leq x_B$. This means that

$$\frac{\partial r(x)}{\partial x} = -\frac{t}{L^*(I - tx, p, r(x))} \text{ for all } x \in [0, x_B].$$

(f) Locational equilibrium condition results in a condition on the *slope* of the rent function.

(g) x_B is the boundary of the city. The value of x_B is yet to be determined.

3. Equilibrium condition is a differential equation

$$\frac{\partial r(x)}{\partial x} = \frac{-t}{L^*(I - tx, p, r(x))}. \tag{2}$$

(a) Under standard conditions, a solution exists. Usually must be calculated using a computer.

(b) The slope of the rent function depends on transport costs and on the demand for land.

i. If tranport costs are *high*, the slope will be steep.

ii. If demand for land is *low*, the slope will be steep.

4. Let r_0 be the rent at the centre. A solution to (2) will be of the form

$$\begin{aligned} r(x, r_0) &= r_0 + \int_0^x \frac{\partial dr(s)}{\partial x} ds \\ &= r_0 - \int_0^x \frac{t}{L^*(I - ts, p, r(s, r_0))} ds. \end{aligned}$$

This equation states that the rent at a location x is equal to the rent at the centre plus the change in rent between 0 and x. Because the change in rent is negative, $r(x) < r_0$ for all x. The rent function will depend on r_0. The variable s is a dummy integration variable that takes on values between 0 and x.

5. Further implication of locational equilibrium.

   (a) Recall that x_B is the boundary of the city.

   (b) Locational equilibrium implies that the level of rent at the boundary equals r_A. Why?

   (c) Since the urban boundary is x_B. The rent at the boundary must satisfy $r(x_B) = r_A$.

$$r_A = r_0 - \int_0^{x_B} \frac{t}{L^*(I - ts, p, r(s, r_0))} ds.$$

6. In these equations, p, t, I and r_A are *known parameters* whose values are fixed outside the model. The function L^* is a *known function* derived from the solution of the consumer's maximisation problem. The variable r_0 is an *unknown variable* whose value is to be determined in equilibrium. Once r_0 is known, the function $r(x, r_0)$ can be calculated using a computer. In this class we will ask qualitative questions like: 1) How do changes in (p, t, I, r_A, N) affect $r(x)$ and r_0? 2) Given values for (p, t, I, r_A, N), what is the welfare that consumers obtain?

2.1 What we have so far.

1. Locational equilibrium among workers.

   (a) $\frac{\partial r}{\partial x} = -\frac{t}{L^*(I-tx,p,r(x))}$.

   (b) $r = r(x, r_0) = r_0 - \int_0^{x} \frac{t}{L^*(I-ts,p,r(s,r_0))} ds$.

   (c) $r_A = r(x, r_0)$.

2. Consumer choices (maximisation).

(a) Demand for food and land.

i. $L^* \left(I - tx, p, r\left(x, r_0\right)\right), C^* \left(I - tx', p, r\left(x, r_0\right)\right).$

ii. L^* and C^* can be shown on the standard picture of consumer maximisation subject to a budget constraint.

iii. Draw picture showing budget constraint, tangency of indifference curve, and optimal choice of C and L.

3. Review.

(a) Each consumer maximises.

i. Consumer chooses (C, L, x) to maximise utility subject to budget constraint.

ii. Conditional demand functions

A. $C^* \left(I - tx^*, p, r\left(x^*\right)\right).$

B. $L^* \left(I - tx^*, p, r\left(x^*\right)\right).$

iii. Optimal location choice: $x^* \left(I, t, p, r\left(x\right)\right).$

(b) Equilibrium in urban spatial economy.

i. Level and slope of rent function.

ii. Number of people in city or level of welfare.

iii. Radius of city or level of rent.

(c) Locational equilibrium.

i. All *identical* consumers obtain same utility regardless of where they choose to live

$$-t - \frac{dr\left(x\right)}{dx} L^* \left(I - tx, p, r\left(x\right)\right) = 0$$

$$\frac{dr\left(x\right)}{dx} = -\frac{t}{L^* \left(I - tx, p, r\left(x\right)\right)}.$$

ii. Land is used in highest value use.

A. Land goes to highest bidder

$$r\left(x\right) = r_0 - \int_0^x \frac{t}{L^* \left(I - ts, p, r\left(s, r_0\right)\right)} ds.$$

B. $r(x, r_0) > r_A$ for $x < x_B$, urban (residential and commuting).

C. $r(x, r_0) = r_A$ for $x \geq x_B$, rural (farming). In equilibrium $r(x, r_0) \geq r_A$ for all x. Why?

D. $r(x) > r_A$ land goes to housing, $r(x) = r_A$ land goes to farming.

E. Hence, the value of r_0 is determined by

$$r_A = r_0 - \int_0^{x_B} \frac{t}{L^*(I - ts, p, r(s, r_0))} ds.$$

iii. What's missing?

A. What determines x_B?

B. Supply and demand for land must be equated.

3 Equilibrium condition 3: supply and demand for land are equal

1. Equilibrium in land market.

(a) Supply of land at distance x is $S_L(x) = 2\pi x$, draw picture.

(b) How many people live at distance x? $N(x)$

(c) Total demand for land in housing at distance x is $D_L(x) = N(x) \cdot L^*(I - tx, p, r(x, r_0))$ if $x \leq x_B$.

(d) $N(x)$ is unknown but if supply equals demand at location x then $N(x)$ is determined by

$$N(x) = \frac{2\pi x}{L^*(I - tx, p, r(x, r_0))}$$

for all $x \leq x_B$.

(e) x_B is still unknown. But, we know that the total population in the city is N.

(f) In equilibrium, everyone must live somewhere. Therefore,

$$\begin{aligned} N &= \int_0^{x_B} N(s)\, ds \qquad (3) \\ &= \int_0^{x_B} \frac{2\pi s}{L^*(I - ts, p, r(s, r_0))} ds. \end{aligned}$$

(g) This final equation pins down x_B. The variable x_B must satisfy equation (3)

(h) To compute an equilibrium:

i. Guess a value for x_B.

ii. Compute the value of the right side of equation (3).

iii. If the right side of (3) is less than the left side, increase x_B. Why?

iv. If the right side of (3) is larger than the left side, reduce x_B. Why?

v. If the right side of equation (3) equals the left side, then x_B is an equilibrium value.

vi. Repeat steps i.-, iv. until v. is true.

2. Equilibrium in consumption good market trivial.

(a) Demand for the consumption good is: $D_C = \int_0^{x_B} N(s) \cdot C^*(I - ts, p, r(s, r_0))\, ds.$

(b) Supply of consumption good is infinitely elastic at price p. That is supply of C adjusts so that supply equals demand at price p.

(c) What would the equilibrium condition be if the supply were not infinitely elastic?

3. Equilibrium equations.

(a) $r(x, r_0) = r_0 - \int_0^{x} \frac{t}{L^*(I - ts, p, r(s, r_0))} ds.$

(b) $r_A = r_0 - \int_0^{x_B} \frac{t}{L^*(I - ts, p, r(s, r_0))} ds.$

(c) $N = \int_0^{x_B} N(s)\,ds = \int_0^{x_B} \frac{2\pi s}{L^*(I-ts,p,r(s,,r_0))} ds.$

3.1 Reprise of equilibrium conditions

1. Equilibrium.

   (a) Condition 1: consumer maximizes utility.

   (b) Condition 2: locational equilibrium.

      i. For *identical* people, changes in rent across space compensate exactly for changes in transport costs

$$\frac{dr(x)}{dx} = \frac{-t}{L^*(I - tx, p, r(x))}.$$

      ii. For *non-identical* people, land goes to the highest bidder.

         A. If $r(x) > r_A$, consumers bid more.

         B. If $r_A >$ willingness to pay of consumers, farmers bid more.

         C. $r(x) = r_A$, urban boundary.

   (c) Condition 3: land market equilibrium: At every location x, supply equals demand for land.

   (d) Condition 4: consumption good market equilibrium: Supply equals demand for the consumption good.

4 Summarise

1. Equilibrium.

   (a) Inputs to equilibrium: $t, N,\ r_A,\ U(\cdot,\cdot),\ I,\ p$. These are the *parameters* of the problem. They are fixed, specified outside the model.

   (b) Outputs: $C^*,\ L^*,\ r(\cdot),\ r_0,\ x_B,\ V^* = U(C^*, L^*)$. These are determined in equilibrium.

   (c) By choosing different values for the parameters, we can analyse how the outputs of the problem vary.