ECON3021
Urban Economics
Lecture 3 – Residential Location Theory

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1 Introductory remarks

1. Outline of today.

   (a) Individual's optimal location choice: city, where to live within city.

      i. How people decide where to live within city and across cities?

      ii. What fraction of income do they spend on housing? What fraction on other goods?

2 Residential location choice model

1. We assume that a city is a large flat plain with an export hub at the centre. Land is to be divided up between business, housing, and agriculture. Business is at the centre, housing is distributed throughout the city and agriculture is outside the city.

2. Business requires no land (or a very very small amount). It locates next to export hub.

   (a) We do not *explicitly* model business location choice in this simple model. *Implicitly*, we assume that there is some form of increasing returns to scale either in production or in transport of output for export.

   (b) In different interpretations of this model, the centre of the city could be a central business district, a market, a factory, a port, an airport, a tube station, a train station, etc.

   (c) Later we will consider a model in which business choose locations other than the centre.

3. Agriculture uses land outside the city and earns rent r_A per unit land.

4. There is a fixed population of N of identical people. Each earns income I, works at the centre of the city, and purchases land somewhere within the city for a house in which to live.

5. All consumers/workers commute to the centre of the city to work at cost t per mile. If a consumer lives x miles from the centre, then total transport costs are tx.

6. Within the city, consumers must pay a rent per unit land of $r(x)$. Locations near the centre charge higher rents because locations near the centre are more valuable. They are more valuable because residents of locations near the centre incur smaller transport costs.

7. Consumers must choose how far away from centre to live x and how much money to spend on C, a consumption good, and L land. The price of the consumption good is p per unit.

8. The supply of land at distance x from the centre is $S_L(x) = 2\pi x$.

9. Preferences are $u(C, L)$.

3 Goals of analysis

1. For each consumer, optimal choice of C^*, L^*, and x^*.

   (a) Analyse how (C^*, L^*) depend on income $(I - tx)$, prices $(p, r(x))$, location x, and the utility function u.

   (b) Analyse how optimal location choice x^* depends on income $(I - tx)$, transport costs t, the price of C, and the rent function $r(x)$.

2. Use the results to analyse a spatial equilibrium

(a) Determinants of equilibrium $r(x)$, x_B (the radius of the city), and the level of utility $u(C^*, L^*)$ that is obtained.

(b) $r(x)$ is equilibrium rent function.

(c) x_B is the equilibrium boundary of the city.

(d) $u(C^*, L^*)$ is the equilibrium utility obtained by a resident in the city. In equilibrium, all identical people will obtain the same utility.

(e) How do these variables depend on

   i. income (I)
   ii. transport costs (t)
   iii. population (N)
   iv. the utility function $(u(C, L))$
   v. the supply of land $S_L(x)$
   vi. the price of consumption (p)
   vii. agricultural rent (r_A)

4 Solving the consumer problem

First, we solve the consumer problem, choose (C, L, x) to maximise utility. There are several more or less equivalent ways to analyse this problem. In the lecture, I used a 2 stage method. I first studied the optimal choice of (C, L) conditional on x and then studied the optimal choice of x. That is detailed in section 4.1. An alternative method is to analyse the simultaneous choice of (C, L, x). This is detailed in section 4.2.

4.1 Method 1: 2 stage method

Suppose the consumer has already chosen a location x. Given x or conditional on x, the choice problem is to maximise

$$\begin{array}{cc} u(C, L) & \text{Utility function} \\ \text{subject to} & \\ pC + r(x) L = I - tx & \text{Budget constraint} \end{array}.$$

This is a standard problem from a first year microeconomic class. One way to solve this **constrained maximisation problem** is to introduce a new variable λ and form the Lagrangian function

$$\mathcal{L}(C, L, \lambda) = u(C, L) + \lambda (I - tx - pC - r(x) L) \tag{1}$$

that depends on (C, L, λ).

Under standard conditions, the optimiser of this function maximises utility. The maximiser of (1) satisfies the first order conditions

$$u_C(C, L) - \lambda p = 0 \tag{2}$$

$$u_L(C, L) - \lambda r(x) = 0 \tag{3}$$

$$I - tx = pC + r(x) L. \tag{4}$$

In other words, conditional on x, an optimal choice of (C, L, λ):

1. Equates the marginal utility of consumption to its marginal cost (equation (2)).

2. Equates the marginal utility of land consumption to its marginal cost (equation (3).

3. Equates total income (net of tranport costs) to total expenditure (equation (4)).

These first order conditions are a set of three equations with three unknown variables (C, L, λ). To solve the consumer's problem conditional on x, solve the three equations for the three unknowns.

The consumer treats the rent function as a known function. The solution to the consumer's problem can be expressed as three *functions* describing how the optimal choices depend on income and prices. One way to write these functions is:

$$\begin{aligned} C &= C^*(I - tx, p, r(x)) \\ L &= L^*(I - tx, p, r(x)) \\ \lambda &= \lambda^*(I - tx, p, r(x)). \end{aligned}$$

The first line expresses for example the idea that the optimal choice of C depends on income minus tranport costs, the price of the consumption good, and the rent at location x.

In the utility maximisation problem, the optimal Lagrange multiplier λ^* is interpreted as the marginal utility of income because the derivative $\frac{\partial \mathcal{L}(C^*, L^*, \lambda^*)}{\partial I} = \lambda^*$. The variable λ^* measures how much utility increases when income increases.

From these expressions, we can also calculate the level of utility or welfare obtained at location x. It is

$$\mathcal{L}\left(C^*, L^*, \lambda^*\right) = u\left(C^*, L^*\right) + \lambda^*\left(I - tx - pC^* - r\left(x\right)L^*\right). \tag{5}$$

4.1.1 Example

In the Cobb-Douglass utility case where

$$u\left(C, L\right) = C^{\alpha} L^{1-\alpha}$$

equations $(2) - (4)$ become

$$\alpha C^{\alpha-1} L^{1-\alpha} - \lambda p = 0 \tag{6}$$

$$\left(1-\alpha\right) C^{\alpha} L^{-\alpha} - \lambda r\left(x\right) = 0 \tag{7}$$

$$I - tx = pC + r\left(x\right) L. \tag{8}$$

Divide (7) by (8) to obtain

$$\frac{\left(1-\alpha\right) C^{\alpha} L^{-\alpha}}{\alpha C^{\alpha-1} L^{1-\alpha}} = \frac{r\left(x\right)}{p}$$

which is equivalent to

$$\frac{\left(1-\alpha\right) C}{\alpha L} = \frac{r\left(x\right)}{p}$$

or

$$C = \frac{\alpha}{1-\alpha} \frac{r\left(x\right) L}{p}.$$

Substitute this into the budget constraint (8) and solve for L to obtain

$$\begin{aligned} L^* &= \left(1-\alpha\right)\left(\frac{I - tx}{r\left(x\right)}\right) \\ C^* &= \alpha\left(\frac{I - tx}{p}\right) \\ \lambda^* &= \frac{\alpha\left(C^*\right)^{\alpha-1}\left(L^*\right)^{1-\alpha}}{p}. \end{aligned}$$

4.1.2 Stage 2: Optimal location choice

In stage 1 we derived conditional demand functions and utility as a function of x : $\mathcal{L}(C^*(x), L^*(x), \lambda^*(x), x)$. We can now solve the location choice problem.

The consumer chooses x to maximise utility. The utility obtained at location x depends on x. To make the dependence clear, we rewrite (5) as

$$\mathcal{L}(C^*, L^*, \lambda^*, x) = u(C^*, L^*) + \lambda^*(I - tx - pC^* - r(x)L^*). \tag{9}$$

When the consumer increases x, there are four impacts on $\mathcal{L}$. There are indirect impacts through changes in C^*, L^*, and λ^* and there is a direct impact through the role of x in the budget constraint.

An optimal choice of x must equate the marginal benefit of increasing x to the marginal cost. The derivate of $\mathcal{L}$ with respect to x can be calculated using the chain rule. It is

$$\frac{d\mathcal{L}(C^*, L^*, \lambda^*, x)}{dx} = \frac{\partial \mathcal{L}}{\partial C}\frac{\partial C^*}{\partial x} + \frac{\partial \mathcal{L}}{\partial L}\frac{\partial L^*}{\partial x} + \frac{\partial \mathcal{L}}{\partial \lambda}\frac{\partial \lambda^*}{\partial x} + \frac{\partial \mathcal{L}}{\partial x}. \tag{10}$$

This expression captures the idea that changes in x affect $\mathcal{L}$ through 4 channels. The nice feature of this expression is that it will turn out that the first 3 effects will all equal zero.

To see this compute

$$\begin{aligned} \frac{d\mathcal{L}(C^*, L^*, \lambda^*, x)}{dC^*}\frac{\partial C^*}{\partial x} &= \left(\frac{\partial u(C^*, L^*)}{\partial C} - \lambda^* p\right)\frac{\partial C^*}{\partial x} \\ &= 0. \end{aligned}$$

Since C^* is optimal, equation (2) implies that

$$\frac{\partial u(C^*, L^*)}{\partial C} - \lambda^* p = 0,$$

that is, the marginal utility of consumption equals the marginal cost when C^* is optimal. Similarly,

$$\begin{aligned} \frac{d\mathcal{L}(C^*, L^*, \lambda^*, x)}{dL}\frac{\partial L^*}{\partial x} &= \left(\frac{\partial u(C^*, L^*)}{\partial L} - \lambda^* r(x)\right)\frac{\partial L^*}{\partial x} \\ &= 0. \end{aligned}$$

When L^* is optimal, then the marginal utility of land consumtion equals the marginal cost. Thirdly,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \lambda}\frac{\partial \lambda^*}{\partial x} &= \left(I - tx - pC^* - r(x)L^*\right)\frac{\partial \lambda^*}{\partial x} \\ &= 0\end{aligned}$$

since the term in parentheses is the budget constraint.

Since the first 3 terms in (10) equal zero, we conclude that

$$\begin{aligned}\frac{d\mathcal{L}(C^*, L^*, \lambda^*, x)}{dx} &= \frac{\partial \mathcal{L}}{\partial x} \\ &= \lambda^*\left(-t - \frac{\partial r(x)}{\partial x}L^*\right).\end{aligned}$$

The total derivative of $\mathcal{L}$ with respect to x (the left side of this equation) equals the partial derivative (the right side). The total effect on utility of increasing x is to reduce utility by $-t\lambda^*$ (due to increased transport costs) and to increase utility by $-\lambda^*\frac{\partial r(x)}{\partial x}L^*$ (due to reduced expenditure on land. This result is an example of the "envelope theorem."

An optimal location choice will satisfy the equation

$$\frac{d\mathcal{L}}{dx} = \lambda^*\left(-t - \frac{\partial r(x)}{\partial x}L^*\right) = 0.$$

An optimal choice of location will equate the marginal costs of increasing x to the marginal benefits.

4.2 Method 2: 1 stage method

The consumer choice problem can also be solved all at once. In this case, the consumer maximises

$$\begin{array}{cc} u(C, L) & \text{Utility function} \\ \text{subject to} & \\ pC + r(x)L + tx = I & \text{Budget constraint} \end{array}$$

or

$$\max_{(C,L,x)} \left\{u(C, L) + \lambda\left(I - tx - pC - r(x)L\right)\right\}$$

or equivalently

$$\max_{(L,x)} u\left(\frac{I - tx - r(x)L}{p}, L\right).$$

The variable λ is the Lagrange multiplier. The first order conditions are

$$u_C(C, L) - \lambda p = 0 \tag{11}$$

$$u_L(C, L) - \lambda r(x) = 0 \tag{12}$$

$$L\frac{dr(x)}{dx} + t = 0 \tag{13}$$

$$I = pC + r(x)L + tx. \tag{14}$$

Now there are 4 equations. An optimal choice:

1. adjusts the amount of C until marginal utility of consumption equals marginal cost (in utility terms).

2. adjusts land consumption until marginal utility of land equals marginal cost.

3. adjusts location until marginal benefit of moving farther away equals marginal cost.

4. satisfies the budget constraint

The consumer problem has four equations (equations (11) – (14)) in four unknown variables (C, L, x, λ).

1. (a) If (C, L, x, λ) are maximising choices then they must satisfy these 4 equations.

   (b) To solve the consumers problem, solve the four equations for the four unknowns.

   (c) The consumer treats the rent function as a known function.

   (d) The solution to the consumer's problem can be expressed as four functions describing the optimal choices as functions of income, prices, and transport costs. One way to write these functions is:

$$\begin{aligned} C &= C^*(I - tx^*, p, r(x^*)) \\ L &= L^*(I - tx^*, p, r(x^*)) \\ x &= x^*(I, p, t, r(x^*)) \\ \lambda &= \lambda^*(I, p, t, r(x^*)). \end{aligned}$$

The first line expresses for example the idea that the optimal choice of C depends on income minus tranport costs (at the optimal location), the price of the consumption good, and the rent at location x^*.