Auctions with Limited Commitment*

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Abstract

We study the role of limited commitment in a standard auction environment. In each period, the seller can commit to an auction with a reserve price but not future auctions. We characterize the set of equilibrium profits attainable for the seller as her commitment power vanishes. If the number of buyers exceeds a distribution-specific cutoff, an efficient auction is the unique limit of equilibrium outcomes; otherwise, profits above the efficient auction profit are achievable. We give exact conditions under which the maximal profit is attained through an initial auction with a reserve price, followed by a continuously decreasing price path.

1 Introduction

It is well understood that in standard auctions such as first-price or second-price auctions, the seller can increase her profit by imposing a minimal bid (or reserve price) (Myerson, 1981; Riley and Samuelson, 1981). A reserve price leads to inefficient exclusion of low-valued

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buyers which helps extract higher payments from high-valued buyers. If no bidder bids above the reserve price, the seller has to commit to not auctioning the object again, even though there is common knowledge of unrealized gains from trade with the excluded buyers.

This aspect of full commitment is not entirely satisfactory in many applications. In practice, buyers may not find it credible that the seller can withhold an unsold object from the market forever, as aborted auctions are common and unsold objects are frequently re-auctioned or offered for sale later. Economists have long recognized the importance of understanding the role of commitment in the auction setting. Milgrom (1987) and McAfee and Vincent (1997) consider a seller with limited commitment in the sense that in each period she can commit to an auction with a reserve price but not future auctions. They show that efficient auctions are the only stationary equilibrium outcome because limited commitment will force the reserve price to drop to her reservation value.[1] However, their analysis and stark result crucially rely on stationarity, leaving open the question of the exact equilibrium implication of limited commitment. A better understanding of the role of limited commitment in auctions not only is economically relevant but also enriches the existing auction theory.

We adopt the modeling approach of Milgrom-McAfee-Vincent to limited commitment. There are one seller with a single indivisible object and multiple buyers whose values are drawn independently from a common distribution. In each period until the object is sold, the seller posts a reserve price and holds an auction. For simplicity, we restrict the exposition to second-price auctions, but our results do not change if the seller can choose from a larger class of auctions in each period. Each buyer can either wait for the next auction, or submit a bid no smaller than the reserve price. Waiting is costly and both the buyers and the seller discount at the same rate. Within a period, the seller commits to the rules of the auction and the announced reserve price. The seller cannot, however, commit to future reserve prices.

This framework is sufficiently rich to investigate the role of commitment. The seller's commitment power varies with the period length (or effectively with the discount factor). If the period length is infinite, the seller has full commitment power. As the period length shrinks, the seller's commitment power diminishes. We adopt the solution concept of perfect Bayesian equilibrium, which is well-defined for the discrete-time game, and restrict attention to buyer-symmetric equilibria. Within the framework, we analyze the continuous-time limit at which the seller's commitment power vanishes. We ask the following questions: what is the set of equilibrium payoffs that is attainable by the seller? What is the equilibrium selling

[1] Milgrom constructs a buyer-symmetric, stationary equilibrium directly in continuous time, while McAfee and Vincent focus on a discrete-time model.

strategy that attains the maximal payoff? When can the seller credibly use reserve prices above her reservation value to increase her profit?

We obtain the following results. First, the full commitment profit cannot be achieved under limited commitment. In order to attain the full commitment profit, the seller would have to maintain a constant reserve price above her reservation value (Myerson, 1981). This is not sequentially rational. Once the initial auction fails, the seller can deviate and end the game with a positive profit by running an efficient auction—that is, by setting a reserve price equal to her reservation value. Second, if the number of bidders exceeds a distribution-specific cutoff, an efficient auction maximizes the seller's profit and implements the unique limit of the equilibrium outcomes. For many widely used distributions, the temptation to gather the profit immediately through a zero reserve price from a small number of buyers is sufficient to induce the seller to give up screening completely. For instance, if the type distribution has a finite density, then an efficient auction is revenue-maximizing if there are more than two buyers. Third, if the number of bidders falls short of the aforementioned cutoff, strictly positive reserve prices can arise in equilibrium and the efficient auction is not optimal. Finally, under the assumption that the monopoly profit function is concave, we obtain an ordinary differential equation that describes the optimal limit outcome if the efficient auction is not optimal. We characterize the exact maximal revenue and show that it can be attained through an initial auction with a strictly positive reserve price followed by a sequence of continuously declining reserve prices.

It is worthwhile to compare our results to those in Milgrom (1987) and McAfee and Vincent (1997) whose analysis relies on the assumption of stationarity. The logic behind our results is entirely different. We show that the optimality of the efficient auction has nothing to do with stationarity, and we give precise conditions under which the efficient auction is and is not revenue-maximizing without stationarity assumption. This helps clarify the role of limited commitment in the auction setting. Our results are also related to Bulow and Klemperer (1996) who show that running the efficient auction with one more bidder, though not optimal, is more beneficial to the seller with full commitment than setting the reserve price optimally. We show that if the number of bidders is not too small, the efficient auction is in fact revenue-maximizing for a seller with limited commitment.

Since the full commitment profit is not attainable, we first have to identify the maximal attainable profit. Moreover, with an infinite horizon we cannot use backward induction to identify equilibria, and with limited commitment we cannot rely on the revelation principle.[2]

[2] Bester and Strausz (2001) develop a version of the revelation principle with limited commitment for environments with one agent and a finite number of periods. It does not apply to our setting because our

Therefore, we need a new method to characterize the set of the seller's equilibrium profits.

The key idea we employ is to translate the limited-commitment problem into an auxiliary mechanism design problem with full commitment, but with a crucial extra constraint. The extra constraint is intended to capture limited commitment. In the original limited-commitment problem, at any stage of the game, the seller can always run an efficient auction to end the game, so her continuation value in any equilibrium must be bounded below by the payoff from an efficient auction for the corresponding posterior belief.[3] This equilibrium payoff restriction, which we refer to as the "payoff floor" constraint, is a necessary implication of the seller's sequential rationality. Hence, the value of the auxiliary problem provides an upper bound for the equilibrium payoffs in the original game (in the continuous-time limit). We proceed to solve the auxiliary problem and show that its value and its solution can be approximated by a sequence of equilibrium outcomes of the original game. Therefore, the value of the auxiliary problem is precisely the maximal attainable equilibrium payoff in our original problem, and the solution to the auxiliary problem is precisely the limiting selling strategy that attains this maximal payoff.

As in Milgrom (1987) and McAfee and Vincent (1997), there is no definitive last period in our model when the seller can fully commit. Milgrom (1987) restricts attention to stationary equilibria, while McAfee and Vincent (1997) assume that the seller's valuation is strictly below the support of the buyers' valuations, which leads to an endogenous finite horizon, allowing them to use backward induction to characterize equilibria. A general mechanism design framework with a finite horizon is developed by Skreta (2006, 2016) who shows by backward induction that the optimal mechanism is a sequence of standard auctions with reserve prices.[4] In contrast, we restrict attention to auction mechanisms in each period and characterize the full set of equilibrium profits as the commitment power vanishes.

An alternative approach to modeling limited commitment is to assume that the seller cannot commit to trading rules even for the present period. McAdams and Schwarz (2007) consider an extensive form game in which the seller can solicit multiple rounds of offers from buyers. Their paper shows that if the cost of soliciting another round of offers is large, the seller can credibly commit to a first-price auction, and if the cost is small, the equilibrium

model has multiple buyers (Bester and Strausz, 2000).

[3]For a given auxiliary mechanism, the seller knows exactly which set of types are left at each moment in time, if the mechanism is carried out. Consequently, she can compute the posterior beliefs as well as her continuation payoff from the given mechanism.

[4]Hörner and Samuelson (2011) and Chen (2012) analyze the dynamics of posted prices under limited commitment in a finite horizon model. They assume that the winner is selected randomly when multiple buyers accept the posted price.

outcome approximates that of an English auction. In Vartiainen (2013), a mechanism is a pure communication device that permits the seller to receive messages from bidders. The seller cannot commit to any action after receiving the messages, and there is no discounting. Vartiainen shows that the only credible mechanism is an English auction. In contrast to these papers, we posit that the seller cannot renege on the agreed terms of the trade in the current period. For example, this might be enforced by the legal environment.

A special case of our setup is the model of bilateral bargaining or durable goods monopoly, in which an uninformed seller makes price offers to a single privately informed buyer, as already recognized by Milgrom (1987) and McAfee and Vincent (1997). In his seminal paper, Coase (1972) argues that a price-setting monopolist completely loses her monopoly power and prices drop quickly to her marginal cost if she can revise prices frequently. Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986) confirm that stationary equilibria satisfy the Coase conjecture. If the seller's reservation value lies in the support of the single buyer's valuation, however, Ausubel and Deneckere (1989) show that in addition to the stationary Coasian equilibria, there is a continuum of non-stationary "reputational equilibria". In these equilibria, the price sequence posted by the seller may start with some arbitrary price which decreases over time, and any seller deviation from the equilibrium price path is deterred by the threat to switch to the low-profit Coasian equilibrium path. In the limit as the period length diminishes, these trigger-strategy equilibria allow the seller to achieve any profit between zero and the monopoly profit.[5]

Different from Ausubel and Deneckere (1989), our model has multiple buyers. A natural idea is to replicate their trigger strategy equilibrium with efficient auctions as off-path punishment. However, this type of reputation equilibrium does not always exist (e.g. when there are three bidders with uniform distribution) and when it does, the equilibrium profit is still strictly lower than the full commitment profit. It opens the question of whether strategies more complicated than the simple trigger strategy can yield a better profit. Therefore, an extension of Coase conjecture argument and the equilibrium construction of Ausubel and Deneckere (1989) is not useful for us to characterize the maximal profit. We propose the auxiliary mechanism design problem with full commitment to characterize the maximal profit attainable in all equilibria of the original problem with limited commitment, and show that the simple trigger strategy is optimal among all strategies which attain the maximal profit.

The paper is organized as follows. In the next section, we formally introduce the model.

[5] Wolitzky (2010) analyzes a Coasian bargaining model in which the seller cannot commit to delivery. In his model, the full commitment profit is achievable even in discrete time because there is always a no-trade equilibrium which yields zero profit.

Section 3 uses a uniform example to illustrate how to construct a particular class of equilibria and why equilibrium construction alone does not yield our main results. Section 4 states the formal results. Section 5 presents our methodological approach. In Section 6 we comment on alternative modeling assumptions. Unless noted otherwise, proofs can be found in Appendix A. Omitted proofs can be found in the Supplemental Material.

2 Model

We consider the standard auction environment where a seller (she) wants to sell an indivisible object to n potential buyers (he). Buyer i privately observes his own valuation for the object $v^i \in [0,1]$. We use $(v^i, v^{-i}) \in [0,1]^n$ to denote the vector of the n buyers' valuations, and $v \in [0,1]$ to denote a generic buyer's valuation. Each v^i is drawn independently from a common distribution with full support, c.d.f. $F(\cdot)$, and a continuously differentiable density $f(\cdot)$ such that $f(v) > 0$ for all $v \in (0,1)$. The highest order statistic of the n valuations (v^i, v^{-i}) is denoted by $v^{(n)}$, its c.d.f. by $F^{(n)}$, and the density by $f^{(n)}$. The seller's reservation value for the object is constant over time and we normalize it to zero.[6]

Time is discrete and the period length is denoted by Δ. In each period $t = 0, \Delta, 2\Delta, \ldots,$ the seller runs a second-price auction (SPA) with a reserve price. To simplify notation, we often do not explicitly specify the dependence of the game on Δ. The timing within period t is as follows. First, the seller publicly announces a reserve price p_t for the auction run in period t, and invites all buyers to submit a valid bid, which is restricted to the interval $[p_t, 1]$. After observing p_t, all buyers decide simultaneously either to bid or to wait. If at least one valid bid is submitted, the winner and the payment are determined according to the rules of the second-price auction and the game ends. If no valid bid is submitted, the game proceeds to the next period. Both the seller and the buyers are risk-neutral and have a common discount rate $r > 0$. This implies a discount factor per period equal to $\delta = e^{-r\Delta} < 1$. If buyer i wins in period t and has to make a payment π^i, then his payoff is $e^{-rt}(v^i - \pi^i)$, and the seller's payoff is $e^{-rt}\pi^i$.

We assume that the seller has limited commitment power. She can commit to the reserve price that she announces for the current period: if a valid bid is placed, then the object is sold according to the rules of the announced auction and she cannot renege. She cannot commit,

[6] The reservation value can be interpreted as a production cost. Alternatively, if the seller has a constant flow value of using the object, the opportunity cost is the net present value of the seller's stream of flow values. What is important here is that the seller's reservation value is the same as the value of the lowest possible buyer type. In Section 6, we discuss the case that the seller's reservation value is in the interior of the type distribution which introduces uncertainty about the number of potential buyers.

however, to future reserve prices: if the object was not sold in a period, the seller can always run another auction with a new reserve price in the next period. She cannot promise to stop auctioning an unsold object, or commit to a predetermined sequence of reserve prices.

We denote by $h_t = (p_0, p_\Delta, \ldots, p_{t-\Delta})$ the public history at the beginning of $t > 0$ if no bidder has placed a valid bid up to t, and write $h_0 = \emptyset$ for the history at which the seller chooses the first reserve price.[7] Let H_t be the set of such histories. A (behavior) strategy for the seller specifies a Borel-measurable function $p_t : H_t \to P[0,1]$ for each $t = 0, \Delta, 2\Delta, \ldots$, where $P[0,1]$ is the space of Borel probability measures endowed with the weak* topology.[8] A (behavior) strategy for buyer i specifies a function $b_t^i : H_t \times [0,1] \times [0,1] \to P[0,1]$ for each $t = 0, \Delta, 2\Delta, \ldots$, where we assume that $b_t^i(h_t, p_t, v^i)$ is Borel-measurable in v^i, for all $h_t \in H_t$, and all $p_t \in [0,1]$, and that supp $b_t^i(h_t, p_t, v^i) \subset \{0\} \cup [p_t, 1]$, where "0" denotes no bid or an invalid bid.

We consider perfect Bayesian equilibria (PBE), and we will focus on equilibria that are buyer symmetric.[9] We will not distinguish between strategies that coincide with probability one for all histories. In the rest of the paper, "equilibrium" is used to refer to this class of symmetric perfect Bayesian equilibria. Let $\mathcal{E}(\Delta)$ denote the set of equilibria of the game for given Δ.[10] Let $\Pi^\Delta(p,b)$ denote seller's expected revenue in any equilibrium $(p,b) \in \mathcal{E}(\Delta)$. We are interested in the entire set of profits that the seller can achieve in the limit when the period length vanishes. The maximal profit in the limit is

$$\Pi^* := \limsup_{\Delta \to 0} \sup_{(p,b) \in \mathcal{E}(\Delta)} \Pi^\Delta(p,b).$$

The minimal profit in the limit is

$$\Pi_* := \liminf_{\Delta \to 0} \inf_{(p,b) \in \mathcal{E}(\Delta)} \Pi^\Delta(p,b).$$

The analysis of the continuous-time limit allows us to formulate a tractable optimization problem. We will justify our approach by providing approximations through discrete time equilibria. An alternative approach is to set up the model directly in continuous time. This approach, however, has unresolved conceptual issues regarding the definition of strategies and equilibrium concepts in continuous-time games of perfect monitoring, which are beyond

[7]We do not have to consider other histories because the game ends if someone places a valid bid.

[8]We slightly abuse notation by using p_t both for the seller's strategy and the announced reserve price at a given history.

[9]See Fudenberg and Tirole (1991) for the definition of PBE in finite games. The extension to infinite games is straightforward.

[10]We establish equilibrium existence in Proposition 2.(i) (see Appendix A.1).

the scope of this paper.[11]

Remark 1 (**Larger Class of Permissible Auction Formats**)**.** Our exposition and analysis are formulated in terms of second-price auctions. In Appendix E, we establish a revenue equivalence result for our problem, and show that all of our results hold for a larger class of symmetric bidding mechanisms in which only the winner pays. This class includes not only standard first price and second price auctions with reserve prices, but also exotic mechanisms like third price auctions and auctions where the winner's payment may depend on his own bid and his rivals' bids. In these mechanisms, the object is always allocated to the bidder with the highest valid bid. The main substantial restriction is allocative efficiency. This rules out posted prices with a rationing rule (as for example in Hörner and Samuelson, 2011), lotteries, or raffles. Formally, we show that any equilibrium allocation and equilibrium payoff in the game where the seller can choose a (potentially different) mechanism from this larger class of mechanisms in every period can be replicated in the game where the seller is restricted to choose only second-price auctions with reserve prices and vice versa.

Remark 2 (**Interpretation of the Continuous Time Limit**)**.** We take $\Delta \to 0$ in computing the limiting payoff. This need not be interpreted literally as running auctions frequently in real time. As in the dynamic games literature, this formulation is equivalent to taking $\delta \to 1$ in a discrete-time problem. The continuous-time limit, however, is more convenient when we consider limiting price paths.

Remark 3 (**Relation to Milgrom, McAfee-Vincent**)**.** In the terminology of the durable goods monopoly literature, we consider the "no-gap" case as in Milgrom (1987). It is well-known that in the "gap case", where F has a support $[\varepsilon, 1]$ for $\varepsilon > 0$, the game essentially has a finite horizon and all equilibria are stationary (i.e., equilibria where buyers' strategies depend only on their own valuation and the current reserve price), which is the focus of McAfee and Vincent (1997).

Before we proceed, we introduce several mild assumptions on the distribution function F. Most of our analysis only depends on a subset of these assumptions, and we will note explicitly which assumption is used for which result.[12] Examples of distributions that simultaneously satisfy all assumptions include the uniform distribution and more generally all power function distributions $F(v) = v^k$ with support $[0, 1]$ and $k > 0$.

Assumption 1. *$J(v) := v - (1 - F(v))/f(v)$ is strictly increasing on $[0, 1]$.*

[11] See Bergin and MacLeod (1993) and Fuchs and Skrzypacz (2010) for related discussions.

[12] All four assumptions are independent. Details can be found in Appendix F in the Supplemental Material.

Assumption 1 is the standard monotone virtual value. This corresponds to assuming decreasing marginal revenues (see Bulow and Roberts, 1989). The following two assumptions are regularity conditions on the distribution in the neighborhood of 0.

Assumption 2. $\phi := \lim_{v\to 0} (f'(v)v)/f(v)$ *exists and* $-1 < \phi < \infty$.

It is easy to see that $\phi = \lim_{v\to 0} (f(v)v)/F(v) - 1$, so $\phi \geq -1$ if the limit exists. Assumption 2 rules out the knife-edge cases of $\phi = -1$ and $\phi = \infty$.[13] Assumption 2 is satisfied, for example, if the density function f is bounded away from 0 and has a bounded derivative. It is also satisfied for a class of distributions which includes densities with $f(0) = 0$ or $f(0) = \infty$ such as the power function distributions $F(v) = v^k$ with $k > 0$.

Assumption 3. *There exist constants* $0 < M \leq 1 \leq L < \infty$ *and* $\alpha > 0$ *such that* $Mv^\alpha \leq F(v) \leq Lv^\alpha$ *for all* $v \in [0, 1]$.

Assumption 3 is adopted from Ausubel and Deneckere (1989) who use it to prove the uniform Coase conjecture. We use it when we extend this result to the auction setting.

Assumption 4. *The revenue function* $v(1 - F(v))$ *is concave on* $[0, 1]$.

Assumption 4 is equivalent to assuming that $J(v)f(v)$ is increasing. It is also equivalent to $(f'(v)v)/f(v) > -2$. Note that, under Assumption 2, $\phi = \lim_{v\to 0} (f'(v)v)/f(v) > -1$, so $v(1 - F(v))$ is concave for v sufficiently close to 0. This will allow us to dispense with Assumption 4 for all but one of our results.

3 A Heuristic Example

To get an intuitive idea of non-stationary equilibria, let us heuristically construct a particular class of equilibria in continuous time in a simple example. This example will also illustrate why the constructive method is not useful for us to characterize the maximal profit. Consider two bidders ($n = 2$) whose values are uniformly distributed on $[0, 1]$. At any $t \geq 0$, on the equilibrium path, the seller posts a reserve price p_t. Buyers use a cutoff strategy, that is, a buyer bids before time t if and only if his value v is weakly above some cutoff v_t, so that v_t is the highest type remaining at time t. If the seller deviates from the reserve price path p_t, the off-path play stipulates that the seller posts a constant reserve price $p_t \equiv 0$ and buyers place valid bids if and only if $p_t = 0$.[14]

[13] An example for the knife-edge cases, due to Yuliy Sannikov, is the distribution function $F(v) = v^{(\ln(1/v))^k}$ defined on $[0, 1]$. For this distribution function, $\phi = -1$ if $k = -1/2$, and $\phi = \infty$ if $k = 1/2$.

[14] We can ignore continuations after deviations by a buyer because they either remain undetected or lead to a successful sale which ends the game. For this heuristic construction, we also assume that both p_t and

The Buyers' Incentives Consider a buyer whose valuation equals the cutoff type v_t at $t > 0$. This buyer must be indifferent between buying at p_t, and waiting for a period of length dt to accept a lower price p_{t+dt}. The latter exposes him to the risk of losing, if his opponent has a valuation between v_{t+dt} and v_t. Therefore, the indifference condition is

$$v_t - p_t = (1 - rdt)\left(\frac{v_{t+dt}}{v_t}\right)(v_t - p_{t+dt}). \tag{3.1}$$

The left-hand side of equation (3.1) is the marginal bidder's profit from trading immediately at t, conditional on being the bidder with the higher valuation. The right-hand side is the option value from waiting: $(1 - rdt)$ is the discounting, $\frac{v_{t+dt}}{v_t}$ is the probability that the opponent's valuation is below v_{t+dt} conditional on the fact that her valuation is below v_t (this is the probability that v_t wins the object at $t + dt$), and $v_t - p_{t+dt}$ is the payoff the marginal bidder gets from the delayed trade at $t + dt$. Using a first-order approximation, we obtain the following differential equation governing p_t and v_t:

$$\dot{p}_t = \left(\frac{\dot{v}_t}{v_t} - r\right)(v_t - p_t). \tag{3.2}$$

The Seller's Incentive We look for an equilibrium in which the seller is indifferent between following the equilibrium path and deviating at any time $t > 0$. This condition is given by,

$$\int_t^{\infty} e^{-r(s-t)} p_s \frac{2v_s}{(v_t)^2}(-\dot{v}_s)\, ds = \frac{1}{3}v_t. \tag{3.3}$$

The left-hand side is the expected present value of the seller's equilibrium revenue at $t > 0$: Since v_t is continuously differentiable, at each moment $s > t$, only the marginal buyer type v_s buys at the reserve price p_s. The marginal type has a conditional density $2v_s/(v_t)^2$, the density of the higher value of two buyers, and it declines with the rate $-\dot{v}_s$. The right-hand side is the seller's revenue after a deviation: running an efficient second-price auction with an expected revenue of $\Pi^E(v_t) = \frac{1}{3}v_t$.

Combining the Seller's and the Buyers' Incentives Equations (3.2) and (3.3) together give rise to a second-order differential equation in v_t:[15]

$$\ddot{v}_t + r\dot{v}_t = 0. \tag{3.4}$$

v_t are continuously differentiable and decreasing over time.

[15]Since (3.3) holds for all $t > 0$, we can differentiate it twice with respect to t and then combine it with (3.2) to eliminate p_t.

Boundary conditions for this ODE are given by the cutoff after the first instant which we write as v_0^+, and the fact that the seller cannot maintain a positive price forever, which implies $\lim_{t\to\infty} v_t = 0$. Using these boundary conditions, we obtain the following solution for the cutoff path

$$v_t = v_0^+ e^{-rt}. \tag{3.5}$$

Substituting v_t in the indifference condition we obtain the corresponding price sequence

$$p_t = \frac{2}{3} v_0^+ e^{-rt}. \tag{3.6}$$

Equilibrium Profit For every given initial cutoff v_0^+, (3.6) describes an equilibrium selling strategy. We now determine the seller-optimal cutoff within this particular class of trigger strategy equilibria. The equilibrium yields the following expected profit for the seller:

$$2v_0^+ \left(1 - v_0^+\right) p_0 + \left(1 - v_0^+\right)^2 \left(v_0^+ + \frac{1 - v_0^+}{3}\right) + \frac{1}{3} \left(v_0^+\right)^3. \tag{3.7}$$

This expected profit consists of two parts. The first is the expected revenue from the initial auction in which the reserve price is $p_0 = \frac{2}{3} v_0^+$, and buyers with a type higher than v_0^+ participate. The transaction price is p_0 if exactly one buyer has a valuation above v_0^+, which occurs with probability $2v_0^+ \left(1 - v_0^+\right)$; when both valuations are above v_0^+, which occurs with probability $\left(1 - v_0^+\right)^2$, the average transaction price is $v_0^+ + \frac{1-v_0^+}{3}$—that is, the expected value of the lower valuation conditional on both being above v_0^+. The second part is the seller's revenue from the continuation after time $t = 0$, which equals $\frac{1}{3} \left(v_0^+\right)^3$ by (3.3) . The expected profit in (3.7) is maximized by $v_0^+ = \frac{2}{3}$, which implies $p_0 = \frac{4}{9}$.

Comparing with Profit under Full Commitment The profit associated with the equilibrium just constructed can be computed by evaluating (3.7) for $v_0^+ = \frac{2}{3}$. This yields $\frac{31}{81} \approx 0.38$. How does this figure compare with the benchmarks achieved under full commitment and in an efficient auction? The profit is larger than the profit of an efficient auction with zero reserve price, $\Pi^E = \frac{1}{3} \approx 0.33$, but is smaller than the profit of Myerson's optimal auction $\Pi^M = \frac{5}{12} \approx 0.42$.[16] Even though the full commitment profit is not achievable, we have $\frac{0.38-\Pi^E}{\Pi^M-\Pi^E} > 50\%$. Put differently, commitment accounts for less than 50% of the profit increase from running Myerson's optimal auction in an environment with two buyers and

[16] The seller's reserve price in Myerson's optimal auction with full commitment is $\frac{1}{2}$. The optimal reserve price is such that the virtual valuation $v - \frac{1-F(v)}{f(v)}$ equals 0.

uniformly distributed valuations.

Remark In the above construction, we only consider one particular class of trigger strategy equilibria, which is similar in spirit to Ausubel and Deneckere (1989). Is the equilibrium profit $\frac{31}{81}$ the highest profit across all possible equilibria with two bidders? With three or more bidders, we can follow the same heuristics, but it does not lead to an equilibrium because the resulting ODE does not possess a declining solution. In this case, what is the maximal revenue attainable across all equilibria? In particular, are there equilibria that employ strategies more complicated than the simple trigger strategy and attain a revenue strictly higher than the revenue from an efficient auction? None of these questions can be addressed by extending the idea of Ausubel and Deneckere (1989). Our main results, obtained through solving an auxiliary mechanism design problem, show that with uniform distribution $\frac{31}{81}$ is indeed the maximal profit for the two bidder case (Theorem 3), and efficient auctions indeed attain the maximal profit in the case of three or more bidders (Theorem 2).

4 Results

This section presents the results of the paper. Our first theorem formalizes our earlier observation that with limited commitment, the revenue from Myerson's optimal auction is not attainable in any perfect Bayesian equilibrium.[17]

Theorem 1. *Suppose Assumption 1 holds. The maximal profit, Π^*, that the seller can achieve in equilibrium as $\Delta \to 0$, is strictly below the seller's profit in Myerson's optimal auction Π^M.*

Note that in order to attain the Myerson's optimal auction profit Π^M, the seller must maintain a constant reserve price in equilibrium. This is impossible because in all equilibria of our game prices must decline to zero. In fact, we prove that, for any fixed $\Delta > 0$, as well as in the limit as $\Delta \to 0$, the maximal profit the seller can attain is strictly below the full commitment profit Π^M.

The main analysis of the paper concerns the characterization of Π^* as well as the set of perfect Bayesian equilibrium payoffs for the seller in the limit as $\Delta \to 0$. The characterization depends on the type distribution and the number of buyers. To state the dependence

[17]Theorem 1 holds without Assumption 1. We focus on the regular case where this assumption holds. Otherwise, Myerson's optimal auction may involve bunching and is not contained in the class of auction formats that we consider.

formally, we define a distribution-specific cutoff $\overline{N}(F)$ for the number of buyers:[18]

$$\overline{N}(F) := 1 + \frac{\sqrt{2+\phi}}{1+\phi}.$$

The first main result shows that if the number of buyers is above this threshold, the maximal equilibrium profit the seller can achieve in the limit is the efficient auction profit. Since the seller can guarantee this profit in any equilibrium (see Lemma 3 below), the set of achievable payoffs contains just Π^E. This result is reminiscent of the finding in Milgrom (1987) and McAfee and Vincent (1997), but in contrast to theirs and as we explain later, limited commitment, rather than stationarity, is the driving force here.

Theorem 2. *Suppose Assumptions 1 and 2 hold. If $n > \overline{N}(F)$, then the set of equilibrium profits in the limit is a singleton and $\Pi^* = \Pi_* = \Pi^E$. There exists a sequence of equilibria for which the profit converges to Π^E and the reserve prices for all $t > 0$ converge to 0 as $\Delta \to 0$.*

Depending on the type distribution, the cutoff $\overline{N}(F)$ can take any value above one. For example, if valuations are distributed according to $F(v) = v^k$ with support $[0, 1]$ and $k > 0$, we have $\phi = k - 1$ and $\overline{N}(F) = 1 + \sqrt{1+k}/k$. If $k = 1$ (corresponding to the uniform example in Section 3), $\overline{N}(F) \in (2, 3)$. If $k < 1$, the density is unbounded at zero and $\overline{N}(F)$ can be large. In many economic applications, however, we study distribution functions with finite densities. For this common class of distributions, $\overline{N}(F)$ remains small, and Π^* equals Π^E if there are at least two or three buyers, as shown in the following corollaries.

Corollary 1. *Suppose Assumptions 1 and 2 hold. If the density f satisfies $f(0) > 0$ and has a finite derivative at 0, then $\Pi^* = \Pi_* = \Pi^E$ if $n \geq 3$.*

Corollary 2. *Suppose Assumptions 1 and 2 hold. If the density f is twice continuously differentiable at zero, $f(0) = 0$ and $f'(0) \neq 0$, then $\Pi^* = \Pi_* = \Pi^E$ if $n \geq 2$.*

Corollary 1 is applicable to any distribution with a finite and strictly positive density and a bounded derivative. Corollary 2 shows that the cutoff is even lower if the density vanishes at zero.

From Theorem 2 (and the complementary Theorem 3.(i) below), we observe that the optimality of the efficient auction in the limit only depends on the lower tail of the distribution. The intuition is as follows. At any time t, the seller's posterior is a truncation from

[18] Recall that $\phi = \lim_{v\to 0} \frac{f'(v)v}{f(v)}$, which exists and is greater than -1 by Assumption 2.

above of the original distribution. Therefore, the tail of the distribution determines the set of equilibria in subgames which start after sufficiently many periods. Suppose the tail of the distribution allows multiple equilibria in every subgame starting in period $t+\Delta$. Then, there are also multiple equilibria in any subgame starting at t. In contrast, if the tail of the distribution pins down a unique continuation equilibrium for all possible histories after sufficiently many periods, then there is a unique equilibrium in the whole game. Therefore, the degenericity of the equilibrium set hinges on properties of the tail of the distribution.

If $n < \overline{N}(F)$, the efficient auction no longer attains the highest equilibrium revenue. We construct a sequence of equilibria that achieves $\Pi^* > \Pi^E$ and characterize the entire set of limiting profits that the seller can obtain in equilibrium. To do this, we need to introduce some notation. We define a function $g:(0,1] \to \mathbb{R}$ that will be used to characterize the limiting outcome (in terms of v_t) that achieves Π^*:

$$g(x) = \frac{f'(x)}{f(x)} - \frac{\left[x\,(F(x))^{n-1} - 2\int_0^x (F(v))^{n-1}\,dv\right] f(x)}{(n-1)\int_0^x \left[F(x) - F(v)\right](F(v))^{n-2} f(v)\,v dv}.$$

Theorem 3. *Suppose Assumptions 1, 2, and 3 hold, and $n < \overline{N}(F)$.*

(i) $\Pi^* > \Pi_* = \Pi^E$.

If in addition, Assumption 4 holds:

(ii) $\Pi^* > \Pi_* = \Pi^E$ *is achieved by a sequence of equilibria with positive reserve prices in which the buyers' equilibrium cutoff paths converge to a cutoff path that starts with some $v_0^+ > 0$, and it is given by the unique solution of the differential equation*

$$\dot{v}_t = -\int_0^{v_t} r e^{\int_v^{v_t} g(x)dx} dv. \tag{4.1}$$

The corresponding path of reserve prices is given by[19]

$$p_t = v_t + \int_t^\infty e^{-r(s-t)} \left(\frac{F(v_s)}{F(v_t)}\right)^{n-1} \dot{v}_s ds, \quad \forall t > 0. \tag{4.2}$$

(iii) Any $\Pi \in \left[\Pi^E, \Pi^*\right]$ *is a limit of a sequence of equilibrium payoffs as* $\Delta \to 0$.

[19]The initial price at $t=0$ is given by $p_0 = v_0^+ + \int_0^\infty e^{-rs}\left(F(v_s)/F(v_0^+)\right)^{n-1} \dot{v}_s ds$.

Assumption 4 is used in parts (ii) and (iii) of Theorem 3 to show that the seller's incentive constraint must become binding in the limit as $\Delta \to 0$ in order to achieve Π^*.[20] In particular, this implies that Π^* is achieved by an initial auction followed by a continuously declining reserve price that satisfies the ODE (4.1).[21] Without Assumption 4, we cannot rule out that the reserve price jumps down at times $t > 0$, so that a positive measure of types is induced to participate in an auction at the same point in time. It can be verified that, in the example of Section 3, equations (4.1) and (4.2) pin down the limiting equilibrium that attains the maximal profit $\Pi^* = \frac{31}{81}$.

Next we discuss the logic behind the results in Theorems 2 and 3. Theorem 2 and its two corollaries can be interpreted as Coase conjecture results without stationarity restriction. A related Coase conjecture result is obtained in Milgrom (1987) and McAfee and Vincent (1997), but their result is entirely driven by their stationarity restriction. In stationary equilibria, all bidders follow stationary bidding strategies which can be interpreted as a *demand curve* faced by the seller. The seller would like to collect the *surplus* below the demand curve as quickly as possible. As $\Delta \to 0$, she can collect the whole surplus by setting more and more finely spaced reserve prices in shorter and shorter intervals. Prices must therefore decline to zero immediately which implies that the demand curve collapses to zero as well, and the Coase conjecture follows. This logic works independent of the type distribution and the number of buyers but crucially relies on stationarity. In contrast, Theorem 2 imposes no stationarity restriction, and shows that limited commitment alone can obtain the efficient auction as the unique limit equilibrium outcome if the number of bidders is above the distribution-specific cutoff. Therefore, Theorem 2 helps clarify the role of limited commitment in the auction setting.

We give an intuitive explanation why the efficient auction is revenue-maximizing if the number of buyers is above the distribution-specific cutoff, and otherwise we obtain an intermediate result between the folk theorem of Ausubel and Deneckere (1989) for ($n = 1$) and the Coase conjecture. As shown by Lemma 3 below, the seller in the auction setting can guarantee herself a strictly positive profit because she can always run an efficient auction immediately. Hence, at any point in time, to deter the seller from running the efficient auction, the continuation profit from screening must exceed the profit from immediately running the efficient auction. This creates a tension between screening types optimally from the perspec-

[20]For part (i), Assumption 4 is not needed because it suffices to construct a limiting outcome that achieves a profit greater than Π^E but not necessarily equal to Π^*.

[21]We explain in Section A.1 how (4.1) is obtained from the seller's incentive constraint.

tives of any two times t and s.[22] This tension can be resolved more easily and a declining price path can be part of an equilibrium, if the number of buyers is small $\left(n < \overline{N}(F)\right)$ so that running the efficient auction is less attractive relative to continued screening. Otherwise, active screening is strictly dominated by running the efficient auction immediately, so the efficient auction is the unique limit equilibrium outcome.

The comparison between the profit from an efficient auction and the potential benefits from screening can also help understand the gap case, as analyzed by McAfee and Vincent (1997), where the buyers' type distribution has support $[\varepsilon, 1]$. By posting price $p_t = \varepsilon$, the seller can guarantee herself a profit $\varepsilon > 0$, even with one buyer. In contrast to the no-gap auction case where the lower bound on the seller's profit at time t (i.e., the profit from running the efficient auction at time t) goes to zero as $v_t \to \varepsilon$, here the profit bound ε is a constant independent of v_t. In fact, for v_t sufficiently close to ε, the profit attainable by setting $p_t = \varepsilon$ coincides with the full commitment profit. As a result, the game ends in finite time which implies that all equilibria must be stationary.[23] Hence, in the gap case, the Coase conjecture directly follows from stationarity.

5 Methodology and Overview of Proofs

Our strategy to characterize Π^*, the corresponding limit price path, and the set of limit equilibrium profits for the seller is to analyze an auxiliary dynamic mechanism design problem. To formulate the problem, we identify basic properties of equilibria of the discrete time game (Section 5.1). These properties are necessary conditions for equilibrium outcomes. We then formulate the same restrictions in continuous time and use them to define the feasible set of mechanisms in the dynamic mechanism design problem (Section 5.2). Necessity of the constraints implies that the value of the auxiliary problem is an upper bound for Π^*. To establish sufficiency, we show that the optimal value of the auxiliary problem is attained by a sequence of discrete time equilibria as period length goes to zero. Therefore, the optimal value of the auxiliary problem is exactly the maximal profit attainable in any equilibrium in the continuous time limit.

[22] To see this, note that to induce the seller to continue screening at time t, one can choose a price path $(p_\tau)_{\tau > t}$ to induce high types to trade early and low types to delay trades. At a later time $s > t$ after most high types have traded, however, the seller must speed up the trade with some of these low types to generate sufficiently high continuation profit in order for her to resist the temption to run the efficient auction, creating a conflict in inducing her to continue screening at the earlier time t.

[23] In the gap-case where the last period is endogenous, as well as in a game with an exogenous last period, the equilibrium can be found by backward induction. This implies that it is essentially unique. In both cases reputational equilibria are ruled out by uniqueness.

5.1 Equilibrium Properties

In any equilibrium of the discrete time game, all buyers play pure strategies that are characterized by history-dependent cutoffs. This is captured by the following Lemma which establishes the "skimming property," an auction analog of a result by Fudenberg, Levine, and Tirole (1985). Its proof is standard and thus omitted.

Lemma 1 (Skimming Property). *Let $(p, b) \in \mathcal{E}(\Delta)$. Then, for each $t = 0, \Delta, 2\Delta, \ldots$, there exists a function $\beta_t : H_t \times [0,1] \to [0,1]$ such that every bidder with valuation above $\beta_t(h_t, p_t)$ places a valid bid and every bidder with valuation below $\beta_t(h_t, p_t)$ waits if the seller announces reserve price p_t at history h_t.*

The next lemma shows that randomization on the equilibrium path is not necessary to attain the maximal profit.

Lemma 2. *For every equilibrium $(p, b) \in \mathcal{E}(\Delta)$, there exists an equilibrium $(p', b') \in \mathcal{E}(\Delta)$ in which the seller does not randomize on the equilibrium path and achieves a profit $\Pi^\Delta(p', b') \geq \Pi^\Delta(p, b)$.*

Lemma 1 implies that at any history, the posterior of the seller is given by a truncation of the prior. Lemmas 1 and 2 together imply that for the characterization of Π^*, we can restrict attention to equilibrium allocation rules which are deterministic (up to tie-breaking).[24] Symmetric deterministic equilibrium allocation rules can be described in terms of a trading time function $T : [0,1] \to \{0, \Delta, 2\Delta, \ldots\}$ which must be non-increasing because of Lemma 1. Given that buyers bid truthfully in a second-price auction, in any symmetric equilibrium the object will be allocated at time $T(v^{(n)})$, to the bidder with the highest valuation.

The last lemma in this section shows that the seller can ensure a continuation profit no smaller than the profit of an efficient auction, even though running an efficient auction is not a part of an equilibrium.

Lemma 3. *Fix any equilibrium $(p, b) \in \mathcal{E}(\Delta)$ and any history h_t. If the seller announces the reserve price $p_t = 0$ at h_t (this may not be part of an equilibrium strategy), then every bidder bids his true value and the game ends.*

Lemma 3 provides a lower bound for the seller's payoff on and off the equilibrium path which provides a constraint for continuation payoffs in the auxiliary problem introduced below. It also follows from Lemma 3 that $\Pi_* \geq \Pi^E$. See the Supplemental Material for proofs of Lemmas 2 and 3.

[24] The proof of our main results shows that this restriction is also without loss for the *set* of limit profits achievable for the seller.

5.2 The Auxiliary Mechanism Design Problem

In the auction context, limited commitment invalidates the full commitment solution as a target for equilibrium construction, so we have to first find the maximal equilibrium profit in order to characterize the entire set of equilibrium profits for the seller. In this subsection, we set up the auxiliary mechanism design problem with full commitment which we use to chacterize the maximal profit, and briefly explain why solving the auxiliary problem constitutes the crucial step in proving the main results.

5.2.1 Mechanisms

The auxiliary mechanism design problem is formulated in continuous time and assumes the seller has full commitment power. Buyers participate in a direct mechanism and make a single report of their valuations at time zero. The mechanism awards the object to the buyer with the highest reported type (up to tie breaking). It specifies a deterministic and non-increasing trading time function $T : [0, 1] \to [0, \infty]$. If the mechanism awards the object to buyer i, then the allocation takes place at time $T(v^i)$. This is motivated by Lemmas 1 and 2. Moreover, the mechanism specifies a payment for the winning bidder.[25]

The discounted trading probability of a bidder with type v is $e^{-rT(v)}$ if he is the highest bidder and zero otherwise. The (interim) expected discounted winning probability of a buyer is thus $\Pr\{v^i = \max_j v^j\} e^{-rT(v)}$, and this is non-decreasing since T is non-increasing. Therefore, any non-increasing trading time function is implementable, and following standard arguments, individual rationality and incentive compatibility constraints for the buyers can be used to express the seller's profit as

$$\int_0^1 J(v)\, e^{-rT(v)} F^{(n)}(v). \tag{5.1}$$

Let us define cutoff types as

$$v_t := \sup\{v \,|\, T(v) \geq t\}.$$

v_t is the highest type that does not trade before time t. Since all buyers with types $v > v_t$ trade before t, the posterior distribution at t, conditional on the event that the object has not yet been allocated, is given by the truncated distribution $F(v|v \leq v_t)$. Therefore, we call v_t the *posterior at time* t. We denote the posterior distribution functions and the virtual

[25] We restrict attention to mechanisms that only require payments from the winning bidder as is the case for second-price auctions. This can be generalized easily to other mechanisms.

valuation for the posterior at time t by

$$F_t(v) := \frac{F(v)}{F(v_t)}, \qquad \text{and} \qquad F_t^{(n)}(v) := \frac{F^n(v)}{F^n(v_t)},$$

and

$$J_t(v) := v - \frac{F(v_t|v \le v_t) - F(v|v \le v_t)}{f(v|v \le v_t)} = v - \frac{F(v_t) - F(v)}{f(v)}.$$

Generally, v_t is continuous from the left, and since it is non-increasing, the right limit exists everywhere. We will denote the right limit at t by

$$v_t^+ := \lim_{s \searrow t} v_s.$$

For each t, v_t^+ is the highest type in the posterior after time t if the object is not yet sold.

Given the assumption of full commitment, the dynamic mechanism design problem of maximizing (5.1) without further constraints, reduces to the static problem of Myerson (1981). Under Assumption 1, the optimal solution is to allocate to the buyer with the highest valuation if his virtual valuation is non-negative, and otherwise to withhold the object. Formally, in terms of trading times, Myerson's solution is given by

$$T^M(v) := \begin{cases} 0 & \text{if } J(v) \ge 0, \\ \infty & \text{if } J(v) < 0. \end{cases} \tag{5.2}$$

5.2.2 Payoff Floor Constraint

To obtain an auxiliary problem that captures the seller's incentives under limited commitment, we add an additional constraint. Motivated by Lemma 3 we assume that the continuation payoff of the seller must be bounded below by the revenue of an efficient auction for the given posterior at each point in time. To state this "payoff floor constraint" formally, we denote the revenue from an efficient auction for the posterior v_t as

$$\Pi^E(v_t) = \frac{1}{F^{(n)}(v_t)} \int_0^{v_t} J_t(x) dF^{(n)}(x).$$

The seller's continuation payoff from the dynamic mechanism at time t can be formulated as

$$\frac{1}{F^{(n)}(v_t)} \int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x).$$

Therefore, the payoff floor constraint (PF) is given by (where we have dropped the term $1/F^{(n)}(v_t)$ on both sides):

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) \geq \int_0^{v_t} J_t(x) dF^{(n)}(x), \text{ for all } t \geq 0. \tag{5.3}$$

The payoff floor constraint introduces a dynamic element into the auxiliary problem that distinguishes it from a standard static mechanism design problem under full commitment.

5.2.3 Auxiliary Problem

To summarize, we can formulate the auxiliary problem as the following dynamic mechanism design problem:

$$\sup_{T:[0,1]\to[0,\infty]} \int_0^1 e^{-rT(x)} J(x) dF^{(n)}(x) \tag{5.4}$$

$$\text{s.t.} \quad \begin{array}{l} \text{IC: } T \text{ is non-increasing,} \\ \text{PF: } \displaystyle\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) \geq \int_0^{v_t} J_t(x) dF^{(n)}(x), \forall t \geq 0. \end{array}$$

We call any $T : [0, 1] \to [0, \infty]$ that satisfies (IC) and (PF) a *feasible solution* of the auxiliary problem. We denote the value of the auxiliary problem by V and standard techniques can be used to show that an optimal solution exists (see Proposition 1 in Appendix A).

The payoff floor constraint rules out a deviation by the seller to an efficient auction, which is a necessary condition for an equilibrium. Therefore, V is an upper bound for the seller's maximal profit Π^*, which is formally proved in Proposition 3 in Appendix A. We then show that V is achievable by a sequence of discrete-time equilibria as $\Delta \to 0$. If $V = \Pi^E$, this follows directly from the existence of stationary equilibria (Proposition 2.(i) in Appendix A). If $V > \Pi^E$ the construction uses the simple trigger strategy with stationary equilibria as off-path punishment. This is possible because the profit of stationary equilibria converges to the right-hand side of the payoff floor constraint as $\Delta \to \infty$ (see Proposition 2.(ii)). Therefore, the payoff floor constraint exactly captures limited commitment and the optimal value of the auxiliary problem is exactly the maximum revenue attainable in any equilibrium as the seller's commitment ability vanishes.

In order to characterize Π^*, the revenue maximizing cutoffs and reserve prices, and the set of limiting profits achievable for the seller, it is adequate to solve the auxiliary problem. Theorem 1 directly follows from Proposition 3 because Myerson's optimal auction is not a feasible solution to the auxiliary problem. To prove Theorems 2 and 3, we start by constructing a solution candidate to the auxiliary problem by assuming that the payoff floor constraint binds for all $t > 0$. This yields the ODE in (4.1). We show that if $n > \overline{N}(F)$, the solution to this ODE is increasing, and therefore, we cannot obtain a feasible (i.e. decreasing) solution from the binding payoff floor constraint. The crucial step to obtain Theorem 2 is to show that no other feasible solution except the efficient auction exists if the binding payoff floor constraint is infeasible. For Theorem 3, we show that the binding payoff floor constraint yields a decreasing (and thus feasible) solution if $n < \overline{N}(F)$. Moreover, we show that under Assumption 4 the payoff floor must be binding at the optimal solution to the auxiliary problem. The formal development of these steps can be found in Appendix A.

6 Concluding Remarks

In this paper we have studied the role of commitment power in auctions where the seller cannot commit to future reserve prices. Our analysis draws insights from the bargaining literature, and the auction and mechanism design literature. We conclude the paper with a discussion of alternative modeling assumptions and extensions of our framework.

Modeling Limited Commitment. Our way of modeling limited commitment assumes that the seller can commit to the terms of trade within a single period: if $\Delta = \infty$, there is full commitment; as $\Delta \to 0$, the seller's commitment power vanishes. This approach is taken by Milgrom (1987) and McAfee and Vincent (1997).

An alternative modeling approach is to assume that the seller's opportunity of running an additional auction is uncertain. This can be cast into a continuous-time framework as follows. There is a Poisson arrival of auction opportunities, with constant arrival rate λ. An auction can only be held at time $t = 0$ or when there is an arrival. If $\lambda = 0$, there is full commitment; if $\lambda \to \infty$, the commitment power vanishes. This model is similar to ours except that the period length Δ is random, but $\Delta \to 0$ in distribution as $\lambda \to \infty$.

Another way to formulate the problem of limited commitment is to allow long-term contracts and renegotiation (see Hart and Tirole, 1988; Strulovici, 2013, and references therein). In our setup with multiple bidders, however, modeling renegotiation introduces new conceptual issues, such as the protocol of multiple-person bargaining and signaling in the renegotiation phase.

Unknown Number of Bidders. We assume that the seller knows the number of serious bidders, and normalize the seller's commonly known reservation value to be 0. This is a natural assumption because a bidder who knows that his value is below the seller's reservation value will not obtain the object in any case and will not show up in an auction. A natural research question is what happens when the seller is uncertain about the number of serious buyers. With full commitment, the problem of an uncertain number of bidders has first been studied by McAfee and McMillan (1987). Without commitment, a possible modeling approach is to assume that there are n bidders whose values are distributed over $[0, 1]$, but the seller's reservation value c is interior. In this case, the seller is uncertain about the number of bidders whose values are above c; indeed, it is possible that no bidder has a value above c. Over time, the seller will update her belief about the number of serious bidders and their valuations. Eventually, the seller will believe that the number of bidders is small, and hence the seller will slow down the decline of the reserve price, which can be used to support an equilibrium that fares better than an efficient auction.

A Appendix

In this appendix, we sketch the key steps in characterizing the optimal solutions to the auxiliary problem, which will form the basis of our proofs of Theorems 1–3. Except the proof of Proposition 2 (existence of stationary equilibria and uniform Coase conjecture) which is in Section C, all other proofs omitted from this appendix are collected in Section B of the Supplemental Material where one can also find several omitted intermediate steps in solving the auxiliary problem. Section D of the Supplemental Material constructs equilibria that approximate the solution to binding payoff floor constraint and proves Proposition 6 which is used in the proof of Theorem 3.

A.1 Analysis of the Auxiliary Problem

A.1.1 Basic Results

For $n = 1$, the case of a single buyer, the right-hand side of the payoff floor constraint is zero, and the optimal solution is T^M.[26] For $n \geq 2$ this is not the case, as shown in the following lemma.

Lemma 4. *For any T in the feasible set of the auxiliary problem, $T(v) < \infty$ for all $v > 0$ and $\lim_{t\to\infty} v_t = 0$.*

Proof. Suppose by contradiction that T is feasible but $T(v) = \infty$ for some $v > 0$. Since T is non-increasing, there exists $w \in (0, 1)$ such that $T(v) = \infty$ for all $v \in [0, w)$ and $T(v) < \infty$

[26]This also implies the folk-theorem obtained by Ausubel and Deneckere (1989).

for all $v \in (w, 1]$. The left-hand side of the payoff floor constraint can be rewritten as, for all $t < \infty$,

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) = \int_w^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x).$$

Since $T(v) < \infty$ for all $v \in (w, 1]$, we have $v_t \to w$ as $t \to \infty$. Hence, as $t \to \infty$, the limit of the left-hand side is zero:

$$\lim_{t\to\infty} \int_w^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) = 0.$$

The limit of right-hand side of the payoff floor constraint as $t \to \infty$, however, is strictly positive:

$$\lim_{t\to\infty} \int_0^{v_t} J_t(x) dF^{(n)}(x) = \int_0^{w} \left(x - \frac{F(w) - F(x)}{f(x)} \right) dF^{(n)}(x) > 0.$$

Therefore, the payoff floor constraint must be violated for sufficiently large t, which contradicts the feasibility of T. □

The following existence result is standard, and its proof (as well as all other omitted proofs) can be found in the Supplemental Material.

Proposition 1. *An optimal solution to the auxiliary problem exists.*

A.1.2 Optimal Value as Equilibrium Revenue Upper Bound

Based on Ausubel and Deneckere (1989) we start by showing existence of stationary equilibria, i.e., equilibria with stationary buyer-strategies that only depend on the valuation and the current reserve price. We also generalize the uniform Coase conjecture for stationary equilibria to the auction setting.

Proposition 2. *(i) (Existence) A stationary equilibrium exists for every $r > 0$ and $\Delta > 0$.*

(ii) (Uniform Coase Conjecture) Suppose Assumption 3 holds. For every $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta < \Delta_\varepsilon$, all $x \in [0, 1]$, and every symmetric stationary equilibrium (p, b) of the game with period length Δ and a truncated distribution $F(v|v \leq x)$ on $[0, x]$, the seller's profit associated with this equilibrium, $\Pi^\Delta(p, b|x)$, is bounded above by $(1 + \varepsilon) \Pi^E(x)$, where $\Pi^E(x)$ is the seller's profit from the efficient auction under this truncated distribution.

The second part of the proposition shows that the seller's profit in every symmetric stationary equilibrium converges to the profit of the efficient auction.[27] Uniform convergence, in the sense that $\Pi^{\Delta}(p, b|x) / \Pi^{E}(x) \to 1$ uniformly for all $x \in (0, 1]$, will be used in the construction of trigger strategy equilibria for Theorem 3.

Clearly, the lower bound of the seller's profit for all equilibria is achievable by $T^E(v) \equiv 0$. This corresponds to a second-price auction with reserve price $p_t = 0$ at time $t = 0$, and $T^E(v) \equiv 0$ implies $v_t = 0$ for all $t > 0$. Therefore, the payoff floor constraint is trivially satisfied for both $t > 0$ and $t = 0$. The following result shows that the optimal value of the auxiliary problem is an upper bound for all equilibrium revenues in the original game.

Proposition 3. *Let (Δ_m) be a decreasing sequence with $\Delta_m \searrow 0$, and let (p_m, b_m) be a sequence of equilibria in which the seller does not randomize on the equilibrium path. Then* $\limsup_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) \in [\Pi^E, V]$. *In particular* $\Pi^* \leq V$.

Proof. We first define an ε-relaxed continuous-time auxiliary problem. We replace the payoff floor constraint by

$$\int_0^{v_t} e^{-r\,(T(x)-t)} J_t(v) dF_t^{(n)}(v) \geq (1-\varepsilon)\Pi^E(v_t).$$

By the maximum theorem, the value of this problem, which we denote by V_ε, is continuous in ε—that is, $\lim_{\varepsilon\to 0} V_\varepsilon = V$.

Next, we formulate a discrete version of the auxiliary problem. For given Δ, the feasible set of this problem is given by

$$\begin{aligned} &T : [0,1] \to \{0, \Delta, 2\Delta, \ldots\} \text{ non-increasing,} \\ \text{and} \quad &\int_0^{v_{k\Delta}} e^{-r\,(T(x)-k\Delta)} J_{k\Delta}(v) dF_{k\Delta}^{(n)}(v) \geq \Pi^E(v_{k\Delta}) \qquad \forall k \in \mathbb{N}. \end{aligned}$$

We denote the value of this problem by $V(\Delta)$. Let $\mathcal{E}^d(\Delta) \subset \mathcal{E}(\Delta)$ denote the set of equilibria in which the seller does not randomize on the equilibrium path. The first constraint is clearly satisfied for outcomes of any equilibrium $\mathcal{E}^d(\Delta)$. The second constraint requires that in each period, the seller's continuation profit on the equilibrium path exceeds the revenue from an efficient auction given the current posterior. This is a necessary condition for an equilibrium. Therefore, the seller's expected revenue in any equilibrium $(p, b) \in \mathcal{E}^d(\Delta)$ cannot exceed $V(\Delta)$. Moreover, for given ε, the feasible set of the discrete auxiliary problem is contained

[27] Notice that in contrast to the Coase conjecture for one buyer, Proposition 2.(ii) does not show that the initial reserve price p_0 converges to zero. This is in fact not the case in the auction setting as was noted by McAfee and Vincent (1997). However, reserve prices for $t > 0$ converge to zero which is sufficient for the convergence of equilibrium profits to the profit of an efficient auction—the counterpart of the Coase conjecture in the auction setting.

in the feasible set of the ε-relaxed continuous-time auxiliary problem if Δ is sufficiently small. Formally, we have:

Claim: Let $\varepsilon > 0$ and $\Delta_\varepsilon = -\ln(1-\varepsilon)/r$. For all $\Delta < \Delta_\varepsilon$ we have

$$\sup_{(p,b)\in\mathcal{E}^d(\Delta)} \Pi^\Delta(p,b) \le V(\Delta) \le V_\varepsilon.$$

Proof of the claim: The first inequality has been shown in the text above. For the second, let T^Δ be an element of the feasible set of the discrete auxiliary problem for $\Delta \le \Delta_\varepsilon$. Let v_t^Δ be the corresponding cutoff path. Note that for $t \in (k\Delta, (k+1)\Delta]$ we have $v_t^\Delta = v_{(k+1)\Delta}^\Delta$ and hence

$$\begin{aligned}
&\int_0^{v_t^\Delta} e^{-r\,(T^\Delta(v)-t)} J_t(v) n(F(v))^{n-1} f(v)dv \\
=&e^{-r((k+1)\Delta-t)} \int_0^{v_{(k+1)\Delta}^\Delta} e^{-r\,(T^\Delta(v)-(k+1)\Delta)} J_{(k+1)\Delta}(v) n(F(v))^{n-1} f(v)dv \\
\ge&e^{-r\Delta} \int_0^{v_{(k+1)\Delta}^\Delta} e^{-r\,(T^\Delta(v)-(k+1)\Delta)} J_{(k+1)\Delta}(v) n(F(v))^{n-1} f(v)dv \\
\ge&e^{-r\Delta}\Pi^E(v_{(k+1)\Delta}^\Delta) \\
=&e^{-r\Delta}\Pi^E(v_t^\Delta) \\
\ge&(1-\varepsilon)\Pi^E(v_t^\Delta).
\end{aligned}$$

The first inequality holds because $t \ge k\Delta$, the second inequality follows from the payoff floor constraint of the discretized auxiliary problem, and the last inequality holds because $\Delta \le \Delta_\varepsilon$. Therefore, T^Δ is a feasible solution for the ε-relaxed continuous time auxiliary problem, and hence $V(\Delta) \le V_\varepsilon$ if $\Delta < \Delta_\varepsilon$. Thus the claim is proved.

To complete the proof for Proposition 3, it suffices to show $\Pi^* \le V$, which follows directly from Lemma 2 and the claim above:

$$\Pi^* = \limsup_{\Delta\to 0} \sup_{(p,b)\in\mathcal{E}^d(\Delta)} \Pi^\Delta(p,b) \le \lim_{\varepsilon\to 0} V_\varepsilon = V.$$

□

A.1.3 Implementing Decreasing Trading Time through Price Path

Any non-increasing trading time function T (with cutoffs v_t) can be implemented, that is, there exists a sequence of reserve prices p_t such that for all t, all types above v_t^+ strictly prefer to bid before or at time t, all lower types strictly prefer to wait, and type v_t^+ is indifferent between buying immediately at price p_t and waiting.[28] This price sequence can be obtained from the envelope formula for the buyers' payoff.

Lemma 5. *Let $T : [0,1] \to [0,\infty]$ be non-increasing and $(v_t)_{t\in\mathbb{R}}$ the corresponding sequence of cutoffs. Then the following sequence of prices implements $(v_t)_{t\in\mathbb{R}}$:*

$$p_t = v_t^+ - e^{rT\left(v_t^+\right)} \int_0^{v_t^+} e^{-rT(v)} \left(\frac{F(v)}{F\left(v_t^+\right)} \right)^{n-1} dv. \tag{A.1}$$

If v_t is differentiable, we have $v_t^+ = v_t$, and obtain equation (4.2)

$$p_t = v_t + \int_t^{\infty} e^{-r(s-t)} \left(\frac{F(v_s)}{F(v_t)} \right)^{n-1} \dot{v}_s ds.$$

Proof. First consider period t when at least one type trades, that is, $t \in T([0,1])$. If p_t is the price that a buyer who trades at time t has to pay, then we have

$$\begin{aligned} Q^i\left(v_t^+\right) &= F\left(v_t^+\right)^{n-1} e^{-rT\left(v_t^+\right)}, \\ \text{and} \qquad U^i\left(v_t^+\right) &= F\left(v_t^+\right)^{n-1} e^{-rT\left(v_t^+\right)} \left(v_t^+ - p_t\right). \end{aligned}$$

Inserting this into the payoff equivalence formula, we obtain

$$\left(F\left(v_t^+\right)\right)^{n-1} e^{-rT\left(v_t^+\right)} \left(v_t^+ - p_t\right) = \int_0^{v_t^+} e^{-rT(v)} \left(F(v)\right)^{n-1} dv,$$

which can be rearranged to (A.1). Next, for $t \notin T([0,1])$, we can set $p_t = p_{\underline{t}}$ where $\underline{t} = \inf\{s \,|\, (s,t] \cap T = \emptyset\}$ is the latest time s before t for which we have already defined p_s. Since v_t^+ is constant on $[\underline{t}, t]$ this yields (A.1) again. ☐

A.1.4 Characterizing Optimal Solutions

We now prove intermediate results which are useful in characterizing optimal solutions to the auxiliary problem and the set of feasible profits. In Section B.6 in the Supplemental material we provide proofs and in some cases more general statements for these results that are not used in the analysis of the auxiliary problem but are only needed for the equilibrium

[28] Note that v_t^+ is the infimum of all types that trade at time t. Therefore, if the reserve price at time t is p_t, the buyer with valuation v_t^+ will pay price p_t if she makes a truthful bid at time t and this bid wins.

approximation. To understand the main argument that leads to the characterization of V, the following intermediate results are sufficient.

First, we show that the efficient auction (T^E) is optimal if and only if it is the only feasible solution to the auxiliary problem. It is clear that any feasible solution yields a profit that is at least as high as the profit of the efficient auction. Otherwise, the payoff floor constraint would be violated at $t = 0$. The following proposition shows that if positive reserve prices are feasible, that is, if the feasible set includes a solution with delayed trade for low types, then the seller can achieve a strictly higher revenue than in the efficient auction.

Proposition 4. *An efficient auction (T^E) is an optimal solution to the auxiliary problem if and only if it is the only feasible solution.*

To get an intuition for this result, compare the efficient auction in which all types trade at time zero, to an alternative feasible solution in which only the types in $(v_0^+, 1]$ trade at time zero, where $v_0^+ < 1$.[29] There are two effects that determine how the profits of these two solutions are ranked. First, in the alternative, the trade of low types is delayed, which creates an inefficiency. Second, the delay for the low types reduces information rents for higher types. We must argue that the total reduction in information rents exceeds the inefficiency, so that the ex-ante profit is higher under the alternative solution. We first consider the reduction in information rents only for the types in $[0, v_0^+]$. This is what matters for the continuation profit at time 0^+, that is, right after the initial trade. Feasibility implies that the reduction in information rents for the types in $[0, v_0^+]$ must already (weakly) exceed the revenue loss from inefficiency. Otherwise, the continuation profit at 0^+ would be smaller than the profit from an efficient auction given the posterior v_0^+, and thus the payoff floor constraint would be violated. If we now include the types in $(v_0^+, 1]$ in the comparison, we must add the reduction in information rents for these types but there is no additional inefficiency because these types trade at time zero in both solutions. Therefore, the total reduction in information rents is strictly higher than the inefficiency, and the ex-ante profit under the alternative is strictly higher than under the efficient auction.

Before proving Proposition 4, we first establish a lemma. We consider solutions where a strictly positive measure of types trade at the same time t so that $v_t > v_t^+$. In other words, there is an "atom" of types that trade at t. The following lemma shows that if the payoff floor constraint is satisfied right after the atom, then the payoff floor constraint at t (right before the atom) is strictly slack. Moreover, if we reduce the size of the atom by lowering v_t

[29] In the proof of Proposition 4, we show that we can always construct a feasible solution with $0 < v_0^+ < 1$, if there exists any feasible solution that differs from the efficient auction.

to $v \in (v_t^+, v_t)$ so that some types in the atom trade earlier than t, the payoff floor constraint at t remains strictly slack for all choices $v \in (v_t^+, v_t)$. This lemma is more general than needed for the proof of Proposition 4 which will be convenient later. The proof can be found in the Supplemental Material.

Lemma 6. *Let $T : [0,1] \to [0,1]$ be non-increasing (not necessarily feasible) and denote the corresponding cutoff sequence by v_t. Suppose there is an "atom" at $t \geq 0$, that is, $v_t > v_t^+$. If the payoff floor constraint is satisfied at t^+, that is*

$$\int_0^{v_t^+} e^{-r(T(x)-t)} \left(x - \frac{F(v_t^+) - F(x)}{f(x)} \right) dF^{(n)}(x) \geq \int_0^{v_t^+} \left(x - \frac{F(v_t^+) - F(x)}{f(x)} \right) dF^{(n)}(x), \tag{A.2}$$

then we have, for all $v \in (v_t^+, v_t]$,

$$\int_0^{v} e^{-r(T(x)-t)} \left(x - \frac{F(v) - F(x)}{f(x)} \right) dF^{(n)}(x) > \int_0^{v} \left(x - \frac{F(v) - F(x)}{f(x)} \right) dF^{(n)}(x). \tag{A.3}$$

In particular, the payoff floor constraint is satisfied at t.

Proof of Proposition 4. The "if" part is trivial. For the "only if" part, suppose there is another feasible solution $\tilde{T}$ other than the efficient auction $T^E \equiv 0$. Let $\tilde{v}_t$ denote the cutoff path corresponding to $\tilde{T}$. Note first that the range of $\tilde{T}$ cannot be a singleton because this would imply that $\tilde{T}(v) = t$ for all $v \in [0,1]$ for some $t > 0$. Then the expected revenue would be given by

$$e^{-rt} \int_0^1 J(v) dF^{(n)}(v),$$

which is strictly lower than the revenue from an efficient auction at time 0. Therefore, the payoff floor constraint would be violated at $t = 0$, contradicting the feasibility of $\tilde{T}$. Hence, we can assume that there exists some time s with $0 < \tilde{v}_s < 1$ such that $\tilde{T}(v) < s$ for all $v > \tilde{v}_s$, and $\tilde{T}(v) > s$ for all $v < \tilde{v}_s$. Then we can define a new feasible solution

$$\hat{T}(v) := \begin{cases} 0 & \text{if } v > \tilde{v}_s, \\ \tilde{T}(v) - s & \text{if } v \leq \tilde{v}_s, \end{cases}$$

with corresponding cutoff path $\hat{v}_t$. Solution $\hat{T}$ is feasible because $\tilde{T}$ satisfies the payoff floor constraint for all $t \geq s$. Moreover, we have $0 < \hat{v}_0^+ < 1$ because $\hat{v}_0^+ = \tilde{v}_s$. We can invoke

Lemma 6 by setting $t = 0$ and $v = v_0 = 1$ to obtain

$$\int_0^1 e^{-r\hat{T}(x)} J(x) dF^{(n)}(x) > \int_0^1 J(x) dF^{(n)}(x).$$

The left hand side of the above inequality is the revenue from $\hat{T}$, while the right hand side is the revenue from $T^E \equiv 0$. This completes the proof. □

Proposition 4 implies that in order to decide whether the efficient auction is optimal or not, it suffices to determine whether it is the unique feasible solution. This will be particularly useful, if we are able to construct solutions with non-zero trading times. We approach such a construction by considering the binding payoff floor constraint.

Lemma 7. *Let v_t be a sequence of cutoffs for which the payoff floor constraint is binding for all $t \in (a, b)$, where $0 \leq a < b \leq \infty$. Then v_t is twice continuously differentiable on (a, b) and satisfies the differential equation*

$$\frac{\ddot{v}_t}{\dot{v}_t} + g(v_t)\dot{v}_t + r = 0. \tag{A.4}$$

This lemma is a consequence of Lemmas 10 and 11 in Appendix B.6.1. The next lemma studies the solutions to the differential equation (A.4). In particular, we characterize precise conditions under which there exists a non-trivial solution that is decreasing and thus is feasible in the auxiliary problem. It turns out that a non-trivial feasible solution exists if $n < \overline{N}(F)$ and does not exist if $n > \overline{N}(F)$.

Lemma 8. *(i) If $n > \overline{N}(F)$, there exists no decreasing solution to (A.4) that satisfies $v_0^+ > 0$ and $\lim_{t\to\infty} v_t = 0$.*

(ii) If $n < \overline{N}(F)$, there exists a decreasing solution to (A.4) that satisfies $v_0^+ > 0$ and $\lim_{t\to\infty} v_t = 0$. Among all such solutions, the unique solution of by (4.1) uniquely maximizes the seller's revenue for a given boundary value v_0^+.

Note that, when feasible solutions exist, they are not unique for a given boundary value v_0^+, because (A.4) is a second-order differential equation.[30] The second part of Lemma 8 identifies the optimal solution for a given boundary value. Lemma 8 follows from Lemmas 12–14 in Appendix B.6.2.

[30]The ODE (A.4) does not satisfy the Lipschitz condition at $v = 0$ because $g(v)$ may be unbounded. Therefore a boundary condition for $t \to \infty$ does not pin down a unique solution.

The last intermediate result is the following proposition which shows that the payoff floor constraint must be locally binding for an optimal solution if the monopoly profit is locally concave. It is clear from this proposition why the previous two lemmas are crucial for the analysis of optimal solutions to the auxiliary problem.

Proposition 5. *If $v(1 - F(v))$ is locally concave over an interval (a, b), then for every optimal solution, the payoff floor constraint binds for all t such that $v_t \in (a, b)$.*

Proposition 5 is a direct consequence of Lemmas 18 and 19 in Appendix B.6.3. In order to clarify the role of local concavity, we briefly outline the proof of Proposition 5. To show that the payoff floor constraint must bind at the optimal solution, we consider solutions for which the payoff floor constraint is slack for a time interval (a, b) and construct feasible variations. Roughly speaking, the variation we consider spreads out the trades that happen between a and b. For the high types in the interval $(v_b^+, v_a]$, we decrease the trading time, and for the low types we increase the trading time. Such a variation is always possible. If the monopoly profit $v(1 - F(v))$ is concave on the interval of valuations that trade between a and b, then we prove that such a variation is not only feasible but also improves the seller's ex-ante expected profit. If $v(1 - F(v))$ is convex, we have to construct a variation that concentrates the trading times of the types that trade between a and b, rather than spreading them out. Such a variation, however, is only feasible if the trade is not already concentrated on a single point in time. Therefore, with a non-concave monopoly profit, we cannot rule out that the payoff floor constraint is slack on some interval if there is an atom of trade at the end of the interval.[31]

In the proof of Theorem 2, we will use Proposition 5 on intervals of the form $(0, \varepsilon)$. In this case, the requirement of local concavity is satisfied for any distribution function without imposing Assumption 4 (see the discussion in Section 2). For Proposition 5 to have bite in this case, we show that a feasible solution cannot end with a single atom.

Lemma 9. *Let T be a feasible solution. Then for all $t > 0$ such that $v_t > 0$, there exists $w < v_t$ such that $T(v) > t$ for all $v \leq w$.*

Now we are ready to state the proofs of our main results.

[31] So far, we have not been able to rule out this possibility or to construct an example where a solution with this feature is optimal.

A.2 Proof of Theorem 1

Proof. From Proposition 1 we know that an optimal solution to the auxiliary problem exists and hence V is attained by an element in the feasible set. Lemma 4 implies that T^M is not in the feasible set of the auxiliary problem. Moreover, T^M is the only non-increasing trading time function that attains Π^M. Therefore $V < \Pi^M$. Proposition 3 then implies $\Pi^* \leq V < \Pi^M$. □

A.3 Proof of Theorems 2 and 3

A.3.1 Overview

The proofs of Theorems 2 and 3 each has two parts. The first characterizes the solution to the auxiliary problem. The second part shows that the value of the auxiliary problem is Π^* and that its optimal solution can be approximated by discrete time equilibria.

Theorem 2 assumes $n > \overline{N}(F)$. We use an indirect argument to show that in this case, the feasible set of the auxiliary problem only contains the efficient auction. Suppose by contradiction, that there exists another element T in the feasible set (we identify trading time functions that coincide almost everywhere). Proposition 4 implies that this solution yields strictly higher revenue than the efficient auction. T need not be optimal but Proposition 1 implies that an optimal solution to the auxiliary problem exists, which we call $\hat{T}$ with cutoffs denoted by $\hat{v}_t$. Remember that for any distribution, $v(1-F(v))$ is locally concave for v sufficiently small. Therefore, Proposition 5 implies that the payoff floor constraint is locally binding for v sufficiently small. In other words, if t is large enough, so that $\hat{v}_t$ is sufficiently small, $\hat{v}_t$ must satisfy the ODE (A.4) and $\lim_{t\to\infty} \hat{v}_t = 0$.[32] This is a contradiction because Lemma 8 shows that for $n > \overline{N}(F)$ the ODE (A.4) does not admit a solution that satisfies $\lim_{t\to\infty} v_t = 0$. Therefore, we have shown that the feasible set of the auxiliary problem collapses to a singleton—the efficient auction—if $n > \overline{N}(F)$. In other words, $V = \Pi^E$. Note that this proof does not require Assumption 4 because local concavity around zero is a property of any distribution function with support [0,1].

For the second step in the proof of Theorem 2, note that $V = \Pi^E$, together with Proposition 3 and Lemma 3, implies that $\Pi^* = \Pi^E$ if $n > \overline{N}(F)$. Here we implicitly used equilibrium existence (Proposition 2.(i)), but do not require the uniform Coase conjecture in Proposition 2.(ii). Proposition 3 and Lemma 3 alone imply that equilibrium profits converge to Π^E.

[32]For this step, we need to ensure that the sequence $\hat{v}_t$ does not jump over the range of values where local concavity is guaranteed (see Lemma 9 in Appendix B.6.3).

Hence, the proof of Theorem 2 does not rely on Assumption 3.

Theorem 3 assumes $n < \overline{N}(F)$. In this case, Lemma 8 implies that the ODE (A.4) yields a feasible solution for the auxiliary problem. Taking $v_0^+ > 0$ we thus obtain a feasible solution that is different from the efficient auction. By Proposition 4, this solution must yield strictly higher revenue than the efficient auction. This establishes that the value of the auxiliary problem exceeds Π^E if $n < \overline{N}(F)$. For parts (ii) and (iii), Theorem 3 assumes global concavity (Assumptions 4). Under this assumption, Proposition 5 and Lemma 8 imply that the solution to the ODE (4.1) is an optimal solution (for an optimally chosen boundary condition v_0^+). By varying v_0^+ between 0 and the optimal value, we thus obtain a family of feasible solutions of the auxiliary problem that achieve any profit in $[\Pi^E, V]$.

For the second step in the proof of Theorem 3, we show that each solution in this family can be approximated by discrete time equilibria and thus establish sufficiency of the auxiliary problem (see Appendix D in the Supplemental Material). The approximation uses a discrete trading time $T^\Delta : [0,1] \to \{0, \Delta, 2\Delta, ...\}$, where $\Delta > 0$ is an arbitrarily chosen period length. T^Δ is constructed such that the payoff floor constraint is slack for all $t \in \{0, \Delta, 2\Delta, ...\}$. This approximation, together with (A.1), will be used to define the equilibrium price path for a game with given Δ. On the equilibrium path, buyers best respond to this price path. If the seller deviates from the equilibrium price path, the buyers use a continuation strategy given by a stationary equilibrium. Note that buyers can react to a deviation by the seller in the same period. Therefore, the response to a deviation is immediate and the seller cannot obtain profits in excess of the stationary equilibrium profit. The uniform Coase conjecture (Proposition 2.(ii)) thus implies that the profit after a deviation converges to the profit of the efficient auction. The equilibrium path, on the other hand is carefully constructed such that it yields a profit above the profit of stationary equilibria. As $\Delta \to \infty$, T^Δ is constructed such that it converges to the solution to the binding payoff floor constraint, but sufficiently slowly so that stationary equilibria can be used to provide incentives for the seller.

A.3.2 Proof of Theorem 2

Proof. Since $\phi > -1$, there exists a valuation $\bar{v} > 0$ such that for all $v \in [0, \bar{v}]$, $(f'(v)v)/f(v) > -2$ which implies that $v(1 - F(v))$ is concave on this interval. Lemma 9 shows that the optimal solution to the auxiliary problem does not end with an atom. Therefore, Proposition 5 implies that there exists a time $\bar{t}$ with $v_{\bar{t}} \leq \bar{v}$ after which the payoff floor must be binding for all t at the optimal solution. Lemma 8 shows that this is not possible if $n > \overline{N}(F)$. Proposition 4 and the existence of an optimal solution (Proposition 1) therefore imply that the efficient auction is the only element in the feasible set of the auxiliary problem if $n > \overline{N}(F)$. This shows $V = \Pi^E$. Proposition 3 and Lemma 3 then imply that $\Pi^* = V = \Pi^E = \Pi_*$. Proposition 2 shows the existence of stationary equilibria, and since $\Pi^* = \Pi^E$, there must exist a sequence of stationary equilibria for which the seller's profit converges to Π^E. □

A.3.3 Proof of Theorem 3

Proof. (i) Lemma 8 shows that, if $n < \overline{N}(F)$, then there exists a feasible solution to the auxiliary problem that differs from the efficient auction. This result, together with Proposition 4, implies that the efficient auction is not the optimal solution of the auxiliary problem if $n < \overline{N}(F)$. Again by Lemma 8, a profit $\tilde{\Pi} > \Pi^E$ can be achieved by the solution to the binding payoff floor constraint for some $v_0^+ \in (0,1)$. By Proposition 6 in Appendix D in the Supplemental Material, there exists a sequence of equilibria $(p_m, b_m) \in \mathcal{E}(\Delta_m)$, for $\Delta_m \to 0$ as $m \to \infty$, such that $\lim_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) = \tilde{\Pi}$.

(ii) By Proposition 5 and Assumption 4, the payoff floor constraint must be binding at the optimal solution to the auxiliary problem. By Lemma 8 the optimal solution must satisfy (4.1) and is unique. If we choose v_0^+ optimally, we thus obtain the optimal solution to the auxiliary problem which achieves V. As in (i), Proposition 6 in Appendix D in the Supplemental Material implies that there exists a sequence of equilibria $(p_m, b_m) \in \mathcal{E}(\Delta_m)$, for $\Delta_m \to 0$ as $m \to \infty$, such that $\lim_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) = V$.

(iii) Let v_t^x be the sequence of cutoffs obtained from the ODE in (4.1) with boundary condition $v_0^+ = x \in [0,1]$ and denote the value of the objective function of the auxiliary problem evaluated at v_t^x by $\Pi(x)$. The argument used in (ii) imply that for any choice $v_0^+ = x \in [0,1]$, there exists a sequence of equilibria for which the equilibrium profits converge to $\Pi(x)$. This it remains to show that the range of $\Pi(x)$ is $[\Pi^E, \Pi^*]$. It is clear that $x = 0$ leads to $\Pi(x) = \Pi^E$ and from (ii) we know that there exists x^* such that $\Pi(x^*) = \Pi^*$. To complete the proof we show that $\Pi(x)$ is continuous. To see this, denote the trading time function corresponding to v_t^x by T^x. $\Pi(x)$ is obtained by substituting $T(v) = T^x(v)$ in the objective function of the auxiliary problem. Note that

$$T^x(v) = \begin{cases} 0, & \text{if } v \geq x, \\ T^1(v) - T^1(x), & \text{if } v \leq x. \end{cases}$$

Hence $T^x(v)$ is continuous in x for all $v > 0$ and therefore $e^{-rT^x(v)}$ is continuous in x for all $v > 0$. Since $e^{-rT^x(v)}$ is bounded, $\Pi(x)$ is continuous in x, which completes the proof. □

A.3.4 Proof of Corollary 1

Proof. Since the density satisfies $f(0) > 0$ and $f'(0) < \infty$, we have $\phi := \lim_{v\to 0} \frac{vf'(v)}{f(v)} = 0$, and thus $\overline{N}(F) := 1 + \frac{\sqrt{2+\phi}}{1+\phi} = 1 + \sqrt{2} \in (2, 3)$. □

A.3.5 Proof of Corollary 2

Proof. We use a Taylor expansion of $f(v)$ at zero to obtain

$$\phi = \lim_{v\to 0} \frac{f'(v)v}{f(v)} = \lim_{v\to 0} \frac{f'(v)v}{f'(0)v} = 1.$$

This implies $\overline{N}(F) = 1 + \sqrt{3}/2 < 2$. □

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Supplemental Material For Online Publication

B Omitted Proofs

B.1 Proof of Lemma 2

Proof. In the main paper we slightly abuse notation by using p_t both for the seller's (possibly mixed) strategy and the announced reserve price at a given history. This should not lead to confusion in the main part but for this proof we make a formal distinction. We denote the reserve price announced in period t by x_t. A history is therefore given by $h_t = (x_0, \dots, x_{t-\Delta})$. Furthermore we denote by $h_{t^+} = (h_t, x_t) = (x_0, \dots, x_{t-\Delta}, x_t)$ a history in which the reserve prices $x_0, \dots, x_{t-\Delta}$ have been announced in periods $t = 0, \dots, t-\Delta$ but no buyer has bid in these periods, and the seller has announced x_t in period t, but buyers have not yet decided whether they bid or not. For any two histories $h_t = (x_0, x_\Delta, ..., x_{t-\Delta})$ and $h'_s = (x'_0, x'_\Delta, ..., x'_{s-\Delta})$, with $s \le t$, we define a new history

$$h_t \oplus h'_s = (x'_0, x'_\Delta, ..., x'_{s-\Delta}, x_s, ..., x_{t-\Delta}).$$

That is, $h_t \oplus h'_s$ is obtained by replacing the initial period s sub-history in h_t with h'_s. Finally, we can similarly define $h_{t^+} \oplus h'_s$ for $s < t$. With this notation we can state the proof of the lemma.

Consider any equilibrium $(p, b) \in \mathcal{E}(\Delta)$ in which the seller randomizes on the equilibrium path. The idea of the proof is that we can inductively replace randomization on the equilibrium path by a deterministic reserve price and at the same time weakly increase the seller's ex-ante revenue. We first construct an equilibrium $(p^0, b^0) \in \mathcal{E}(\Delta)$ in which the seller earns the same expected profit as in (p, b), but does not randomize at $t = 0$. If the seller uses a pure action at $t = 0$, we can set $(p^0, b^0) = (p, b)$. Otherwise, if the seller randomizes over several prices at $t = 0$, she must be indifferent between all prices in the support of $p_0(h_0)$. Therefore, we can define $p^0_0(h_0)$ as the distribution that puts probability one on a single price $x_0 \in \operatorname{supp} p_0(h_0)$. If we leave the seller's strategy unchanged for all other histories ($p^0_t(h_t) = p_t(h_t)$, for all $t > 0$ and all $h_t \in H_t$) and set $b^0 = b$, we have defined an equilibrium (p^0, b^0) that gives the seller the same payoff as (p, b) and specifies a pure action for the seller at $t = 0$.

Next we proceed inductively. Suppose we have already constructed an equilibrium (p^m, b^m) in which the seller does not randomize on the equilibrium path up to $t = m\Delta$, but uses a mixed action on the equilibrium path at $(m+1)\Delta$. We want to construct an equilibrium (p^{m+1}, b^{m+1}) with a pure action for the seller on the equilibrium path at $(m+1)\Delta$. Suppose that in the equilibrium (p^m, b^m), the highest type in the posterior at $(m+1)\Delta$ is some type $\beta^0_{(m+1)\Delta} > 0$. We select a price in the support of the seller's mixed action at $(m+1)\Delta$, which we denote by $x^0_{(m+1)\Delta}$, such that the expected payoff of $\beta^0_{(m+1)\Delta}$ at $h_{t^+} = (h_t, x^0_{(m+1)\Delta})$ is weakly smaller than the expected payoff at h_t. In other words, we pick a price that is (weakly) bad news for the buyer with type $\beta^0_{(m+1)\Delta}$. This will be the equilibrium price announced in period $t = (m+1)\Delta$ in the equilibrium (p^{m+1}, b^{m+1}). The formal construction of the equilibrium is rather complicated. The rough idea is that, first we posit that after $x^0_{(m+1)\Delta}$ was announced in period $(m+1)\Delta$, (p^{m+1}, b^{m+1}) prescribes the same continuation as (p^m, b^m). Second, on the equilibrium path up to period $m\Delta$, we change the reserve prices such that the same marginal types as before are indifferent between buying immediately and waiting in all periods $t = 0, \ldots, m\Delta$. Since we have chosen $x^0_{(m+1)\Delta}$ to be bad news, this leads to (weakly) higher prices for $t = 0, \ldots, m\Delta$, and therefore we can show that the seller's expected profit increases weakly. Finally, we have to specify what happens after a deviation from the equilibrium path by the seller in periods $t = 0, \ldots, (m+1)\Delta$. Consider the on-equilibrium history h_t in period t for (p^{m+1}, b^{m+1}). We identify a history $\hat{h}_t$ for which the posterior in the original equilibrium (p, b) is the same posterior as at h_t in the new equilibrium. If at h_t, the seller deviates from p^{m+1} by announcing the reserve price $\hat{x}_t$, then we define (p^{m+1}, b^{m+1}) after $h_{t^+} = (h_t, \hat{x}_t)$ using the strategy prescribed by (p, b) for the subgame starting at $\hat{h}_{t^+} = (\hat{h}_{t^+}, \hat{x}_t)$. We will show that with this definition, the seller does not have an incentive to deviate.

Next, we formally construct the sequence of equilibria (p^m, b^m), $m = 1, 2, ...$, and show that this sequence converges to an equilibrium (p^∞, b^∞) in which the seller never randomizes on the equilibrium path and achieves an expected revenue at least as high as the expected revenue in (p, b). We first identify a particular equilibrium path of (p^0, b^0) with a sequence of reserve prices $h^0_\infty = (x^0_0, x^0_\Delta, ...)$ and the corresponding buyer cutoffs $\beta^0 = (\beta^0_0, \beta^0_\Delta, ...)$ that specify the seller's posteriors along the path $h^0_\infty = (x^0_0, x^0_\Delta, ...)$.[33] Then we construct an equilibrium (p^m, b^m) such that the following properties hold: for $t = 0, ..., m\Delta$, the equilibrium prices x^m_t chosen by the seller are weakly higher than x^0_t and the equilibrium cutoffs β^m_t are exactly β^0_t; for $t > m\Delta$, or off the equilibrium path, the strategies coincide with what (p^0, b^0) prescribes at some properly identified histories, so that the two strategy profiles prescribe the same continuation payoffs at their respective histories.

In order to determine $h^0_\infty = (x^0_0, x^0_\Delta, ...)$ and $\beta^0 = (\beta^0_0, \beta^0_\Delta, ...)$ we start at $t = 0$ and define x^0_0 as the seller's pure action in period zero in the equilibrium (p^0, b^0) and set $\beta^0_0 = 1$. Next we proceed inductively. Suppose we have fixed x^0_t and β^0_t for $t = 0, \Delta, \ldots$. To define $x^0_{t+\Delta}$, we select a price in the support of the seller's mixed action at history $h^0_{t+\Delta} = (x^0_0, ..., x^0_t)$ in the equilibrium (p^0, b^0) such that the expected payoff of the cutoff buyer type β^0_t, conditional on $x^0_{t+\Delta}$ is announces, is no larger than this type's expected payoff at the beginning of period $t + \Delta$ before a reserve price is announces.[34] We then pick $\beta^0_{t+\Delta}$ as the cutoff buyer type following history $\left(x^0_0, ..., x^0_t, x^0_{t+\Delta}\right)$.

(p^0, b^0) was already defined. We proceed inductively and construct equilibrium (p^{m+1}, b^{m+1}) for $m = 0, 1, \ldots$ as follows.

(1) On the equilibrium path at $t = (m+1)\Delta$, the seller plays a pure action and announces the reserve price $x^{m+1}_{(m+1)\Delta} := x^0_{(m+1)\Delta}$.

(2) On the equilibrium path at $t = 0, \Delta, ..., m\Delta$, the seller's pure action x^{m+1}_t is chosen such that the buyers' on-path cutoff types in periods $t = \Delta, ..., (m+1)\Delta$ is $\beta^{m+1}_t = \beta^0_t$, where β^0_t was defined above.

(3) On the equilibrium path at the history $h_{t^+} = (x_0, \ldots, x_t)$ for $t = 0, \Delta, (m+1)\Delta$, each buyer bids if and only if $v^i \geq \beta^{m+1}_t = \beta^0_t$.

[33] Note that the cutoffs β^0_t are the equilibrium cutoffs which may be different from the cutoffs that would arise if the seller used pure actions with prices $x^0_0, x^0_\Delta, ...$ on the equilibrium path.

[34] If the seller plays a pure action at $h^0_{t+\Delta}$, then $x^0_{t+\Delta}$ the price prescribed with probability one by the pure action. If the seller randomizes at $h^0_{t+\Delta}$, there must be one realization, which, together with the continuation following it, gives the buyer a payoff weakly smaller than the average.

(4) at $t > (m+1)\Delta$: for any history $h_t = (x_0, ..., x_{t-\Delta})$ in which no deviation has occurred at or before $(m+1)\Delta$, the seller's (mixed) action is $p^{m+1}(h_t) := p^0\left(h_t \oplus \left(x_0^0, ..., x_{(m+1)\Delta}^0\right)\right)$. For any history $h_{t^+} = (x_0, ..., x_{t-\Delta}, x_t)$ in which no deviation has occurred at or before $(m+1)\Delta$, the buyer's strategy is defined by $b^{m+1}(h_{t^+}) := b^0\left(h_{t^+} \oplus \left(x_0^0, ..., x_{(m+1)\Delta}^0\right)\right)$.

(5) For any off-path history $h_t = (x_0, ..., x_{t-\Delta})$ in which the seller's first deviation from the equilibrium path occurs at $s \leq (m+1)\Delta$, the seller's (mixed) action is prescribed by $p^{m+1}(h_t) := p^0\left(h_t \oplus \left(x_0^0, ..., x_{s-\Delta}^0\right)\right)$. For any off-path history $h_{t^+} = (x_0, ..., x_{t-\Delta}, x_t)$ in which the seller's first deviation from the equilibrium path occurs in period $s \leq (m+1)\Delta$, the buyer's strategy is $b^{m+1}(h_{t^+}) := b^0\left(h_{t^+} \oplus \left(x_0^0, ..., x_{s-\Delta}^0\right)\right)$.

In this definition, (1) and (2) define the seller's pure actions on the equilibrium path up to $(m+1)\Delta$. The prices defined in (1) and (2) are chosen such that bidding according to the cutoffs β_t^{m+1} is optimal for the buyers. Part (4) defines the equilibrium strategies for all remaining on-path histories and after deviations that occur in periods after $(m+1)\Delta$, that is, in periods where the seller can still mix on the equilibrium path. The equilibrium proceeds as in (p^0, b^0) at the history where the seller used the prices $x_0^0, ..., x_{(m+1)\Delta}^0$ in the first $m+1$ periods. This ensures that the continuation strategy profile is taken from the continuation of an on-path history of the equilibrium (p^0, b^0), where the seller's posterior in period $(m+1)\Delta$ is the same as in the equilibrium (p^{m+1}, b^{m+1}). Finally, (5) defines the continuation after a deviation by the seller at a period in which we have already defined a pure action. If the seller deviates at a history $h_t = \left(x_0^m, ..., x_{s-\Delta}^m\right)$, then we use the continuation strategy of (p^0, b^0), at the history $\left(x_0^0, ..., x_{s-\Delta}^0\right)$.

We proceed by proving a series of claims showing that we have indeed constructed an equilibrium.

Claim 1. *The expected payoff of the cutoff buyer $\beta_{(m+1)\Delta}^m = \beta_{(m+1)\Delta}^0$ at the on-path history $h_{(m+1)\Delta}^m = (x_0^m, ..., x_{m\Delta}^m)$ in the candidate equilibrium (p^m, b^m) is the same as its payoff at the on-path history $h_{(m+1)\Delta}^0 = (x_0^0, ..., x_{m\Delta}^0)$ in the candidate equilibrium (p^0, b^0).*

Proof. This follows immediately from (1)–(3) above. □

Claim 2. *The expected payoff of the cutoff buyer $\beta_{(m+1)\Delta}^{m+1} = \beta_{(m+1)\Delta}^0$ at the on-path history $h_{((m+1)\Delta)^+}^{m+1} = \left(x_0^{m+1}, ..., x_{m\Delta}^{m+1}, x_{(m+1)\Delta}^{m+1}\right)$ in the candidate equilibrium (x^{m+1}, x^{m+1}) is the same as this cutoff type's expected payoff at the on-path history $h_{((m+1)\Delta)^+}^0 = \left(x_0^0, ..., x_{m\Delta}^0, x_{(m+1)\Delta}^0\right)$ in the candidate equilibrium (p^0, b^0).*

Proof. By construction, $x_{(m+1)\Delta}^{m+1} = x_{(m+1)\Delta}^0$. It follows from part (4) that (p^{m+1}, b^{m+1}) and (p^0, b^0) are identical on the equilibrium path from period $(m+2)\Delta$ onwards. The claim follows. □

Claim 3. *The expected payoff of the cutoff buyer $\beta_{(m+1)\Delta}^{m+1} = \beta_{(m+1)\Delta}^0$ at the on-path history $h_{(m+1)\Delta}^{m+1} = \left(x_0^{m+1}, ..., x_{m\Delta}^{m+1}\right)$ in the candidate equilibrium (p^{m+1}, b^{m+1}) is weakly lower than this cutoff type's expected payoff at the on-path history $h_{(m+1)\Delta}^0 = (x_0^0, ..., x_{m\Delta}^0)$ in the equilibrium (p^0, b^0).*

Proof. In the candidate equilibrium (p^{m+1}, b^{m+1}), the cutoff type's payoffs at histories $h^{m+1}_{(m+1)\Delta}$ and $h^{m+1}_{((m+1)\Delta)^+}$ are the same because the seller plays a pure action in period $(m+1)\Delta$. In the equilibrium (p^0, b^0), the cutoff type's payoff at history $h^{m+1}_{((m+1)\Delta)^+}$ is weakly lower than his payoff at history $h^0_{(m+1)\Delta}$ because of the definition of $x^0_{(m+1)\Delta}$ (which chosen to give the cutoff type a lower expected payoff than the expected payoff at $h^0_{(m+1)\Delta}$). The claim then follows from Claim 2. □

Claim 4. *The expected payoff of the cutoff buyer* $\beta^{m+1}_{(m+1)\Delta} = \beta^0_{(m+1)\Delta}$ *at the on-path history* $h^{m+1}_{(m+1)\Delta} = \left(x^{m+1}_0, ..., x^{m+1}_{m\Delta}\right)$ *in the candidate equilibrium* (p^{m+1}, b^{m+1}) *is weakly lower than this cutoff type's expected payoff at the on-path history* $h^m_{(m+1)\Delta} = (x^m_0, ..., x^m_{m\Delta})$ *in the candidate equilibrium* (p^m, b^m).

Proof. By Claim 1, the cutoff type's expected payoff at the on-path history $h^m_{(m+1)\Delta} = (x^m_0, ..., x^m_{m\Delta})$ in the candidate equilibrium (p^m, b^m) is the same as its payoff at the on-path history $h^0_{(m+1)\Delta} = (x^0_0, ..., x^0_{m\Delta})$ in the candidate equilibrium (p^0, b^0). The claim then follows from Claim 3. □

Claim 5. *For each* $m = 0, 1, ...$ *and* $t = 0, 1, ..., m\Delta$, *we have* $x^{m+1}_t \geq x^m_t$.

Proof. By Claim 4, the cutoff type $\beta^{m+1}_{(m+1)\Delta} = \beta^m_{(m+1)\Delta} = \beta^0_{(m+1)\Delta}$ in period $(m+1)\Delta$ on the equilibrium path in the candidate equilibrium (p^{m+1}, b^{m+1}) has a weakly lower payoff than its expected payoff in the candidate equilibrium (p^m, b^m). To keep this cutoff indifferent in period $m\Delta$ in both candidate equilibria, we must have $x^{m+1}_{m\Delta} \geq x^m_{m\Delta}$. Then to keep the cutoff type $\beta^{m+1}_{m\Delta} = \beta^m_{m\Delta} = \beta^0_{m\Delta}$ indifferent in period $(m-1)\Delta$, we must have $x^{m+1}_{(m-1)\Delta} \geq x^m_{(m-1)\Delta}$. The proof is then completed by induction. □

Claim 6. *The seller's (time* 0*) expected payoff in the candidate equilibrium* (p^{m+1}, b^{m+1}) *is weakly higher than the seller's expected payoff in the equilibrium* (p^0, b^0).

Proof. By parts (1)–(3) of the construction, at $t = 0, ..., m\Delta$, (p^{m+1}, b^{m+1}) and (p^m, b^m) have the same buyer cutoffs on the equilibrium path. At $t = (m+1)\Delta$, the seller in (p^{m+1}, b^{m+1}) chooses $x^{m+1}_{(m+1)\Delta}$ that is in the support of the seller's strategy in (p^m, b^m) in that period (note that even though we haven't show that (p^m, b^m) is an equilibrium, the seller is indeed indifferent in (p^m, b^m) at $(m+1)\Delta$ because the play switch to (p^0, b^0) with identical continuation payoffs by Part (4) of the construction). It then follows from Claim 5 that the seller's (time 0) expected payoff in (p^{m+1}, b^{m+1}) is weakly higher than the seller's (time 0) expected payoff in (p^m, b^m). The claim is proved by repeating this argument. □

Claim 7. *For* $t = \Delta, ..., (m+1)\Delta$, *the seller's expected payoff at the on-path history* $\left(x^{m+1}_0, ..., x^{m+1}_{t-\Delta}\right)$, *in the candidate equilibrium* (p^{m+1}, b^{m+1}) *is weakly higher than the seller's expected at the history* $\left(x^0_0, ..., x^0_{t-\Delta}\right)$ *in equilibrium* (p^0, b^0).

Proof. Denote $m_t = t/\Delta$ so that $t = m_t\Delta$ and consider (p^{m_t}, b^{m_t}). By parts (1)–(3) of the construction, the buyer's cutoff type at $\left(x^{m_t}_0, ..., x^{m_t}_{t-\Delta}\right)$ in this equilibrium is the same as the buyer's cutoff type at $\left(x^0_0, ..., x^0_{t-\Delta}\right)$ in equilibrium (p^0, b^0). By part (4) of the construction, the seller's payoff at history $\left(x^{m_t}_0, ..., x^{m_t}_{t-\Delta}\right)$ in (p^{m_t}, b^{m_t}) coincides with the seller's payoff at history $\left(x_0^0, ..., x_{t-\Delta}^0\right)$ in equilibrium (p^0, b^0). Now consider the candidate equilibrium (p^{m_t+1}, b^{m_t+1}) and the history $\left(x_0^{m_t+1}, ..., x_{t-\Delta}^{m_t+1}\right)$. By claim 5, $\left(x_0^{m_t+1}, ..., x_{t-\Delta}^{m_t+1}\right) \geq \left(x_0^{m_t}, ..., x_{t-\Delta}^{m_t}\right)$. Note that the candidate equilibrium (p^{m_t+1}, b^{m_t+1}) further differs from the equilibrium (p^{m_t}, b^{m_t}) on the equilibrium path in period $t+\Delta$. But $x_t^{m_t+1}$ is in the support of the seller's randomization in (p^{m_t}, b^{m_t}) (which makes the seller indifferent by part (4) of the equilibrium construction — see the proof in Claim 6). Therefore, the seller's payoff at $\left(x_0^{m_t+1}, ..., x_{t-\Delta}^{m_t+1}\right)$ in the equilibrium (p^{m_t+1}, b^{m_t+1}) is weakly greater than at $\left(x_0^{m_t}, ..., x_{t-\Delta}^{m_t}\right)$ in the equilibrium (p^{m_t+1}, b^{m_t+1}). This completes the proof of the claim. □

Claim 8. *For each $m = 0, 1, ...,$ (p^{m+1}, b^{m+1}) such constructed is indeed an equilibrium.*

Proof. The buyer's optimality condition follows immediately from the construction. Now consider the seller. By part (5) of the construction, for any off-path history $h_t = (x_0, ..., x_{t-\Delta})$ in which the seller's first deviation from the equilibrium path occurs at $s \leq (m+1)\Delta$, the continuation strategy profile prescribed by (p^{m+1}, b^{m+1}) is exactly that prescribed by (p^0, b^0) at a corresponding history $h_t \oplus \left(x_0^0, ..., x_{s-\Delta}^0\right)$ with exactly the same expected payoff (the payoff is the same due to the fact that the seller's strategies coincide and the fact that the buyer's cutoff at h_t in (p^{m+1}, b^{m+1}) is the same as that at $h_t \oplus \left(x_0^0, ..., x_{s-\Delta}^0\right)$ in (p^0, b^0)). Hence there is no profitable deviation at h_t in (p^{m+1}, b^{m+1}) just as there is no profitable deviation at $h_t \oplus \left(x_0^0, ..., x_{s-\Delta}^0\right)$ in (p^0, b^0).

By part (4) of the construction, at $t > (m+1)\Delta$, for any history $h_t = (x_0, ..., x_{t-\Delta})$ in which no deviation has occurred at or before $(m+1)\Delta$, the seller's strategy at h_t in (p^{m+1}, b^{m+1}) coincides with the seller's strategy at $h_t \oplus \left(x_0^m, ..., x_{(m+1)\Delta}^m\right)$, with exactly the same continuation payoffs (see the previous paragraph). Hence there is no profitable deviation at h_t in (p^{m+1}, b^{m+1}).

Now consider parts (1)–(3) of the construction, for $t = 0, ..., (m+1)\Delta$. By Claim 6 and 7, staying on the equilibrium path gives the seller a weakly higher payoff than that from the equilibrium (p^0, b^0) at the corresponding history. But deviation from the equilibrium path triggers a switch to (p^0, b^0) at a corresponding history. Since there is no deviation in (p^0, b^0), deviation becomes even less desirable in (p^{m+1}, b^{m+1}). This completes the proof of the claim. □

So far, we have obtained a sequence of equilibria $\{(p^m, b^m)\}_{m=0}^{\infty}$. Denote the limit of this sequence by (p^∞, b^∞). It is easy to check that the limit is well-defined. It remains to show that (p^∞, b^∞) is an equilibrium. It is clear that buyers do not have an incentive to deviate. For the seller, suppose the seller has a profitable deviation at some history $h_{m\Delta}$. By the definition of (p^∞, b^∞) and the construction of the sequence $\{(p^m, b^m)\}_{m=0}^{\infty}$, the continuation play at h_t in the candidate equilibrium (p^∞, b^∞), where h_t is a history with $h_{m\Delta}$ as its sub-history, will coincide with continuation play at h_t prescribed by equilibrium $\left(p^{m'}, b^{m'}\right)$ for any $m' \geq m$, which is in turn described by $p^0\left(h_t \oplus \left(x_0^0, ..., x_{(m-1)\Delta}^0\right)\right)$ and $b^0\left(h_{t^+} \oplus \left(x_0^0, ..., x_{(m-1)\Delta}^0\right)\right)$ by part (5) of the equilibrium construction. Since $\left(p^{m'}, b^{m'}\right)$ is an equilibrium, this particular deviation is not profitable in the equilibrium $\left(p^{m'}, b^{m'}\right)$ for any $m' \geq m$. But the on-path payoff of $\left(p^{m'}, b^{m'}\right)$ converges to that of (p^∞, b^∞), and we have just argued that the payoff after this particular deviation is the same for both $\left(p^{m'}, b^{m'}\right)$ and (p^∞, b^∞). This contradicts the assumption of profitable deviation. □

B.2 Proof of Lemma 3

Proof. Fix a history h_t. Note that if all buyers bid, then by the standard argument, it is optimal for each bidder to bid their true values. Therefore, it is sufficient to show that each buyer will submit a bid. By Lemma 1, we only need to show $\beta_t(h_t, p_t) = 0$. Suppose by contradiction that $\beta_t(h_t, p_t) > 0$. Consider a positive type $\beta_t(h_t, p_t) - \varepsilon$, where $\varepsilon > 0$. By Lemma 1, if this type follows the equilibrium strategy and waits, he wins only if his opponents all have types lower than $\beta_t(h_t, p_t) - \varepsilon$, and he can only win in period $t + \Delta$ or later at a price no smaller than 0. If he deviates and bids his true value in period t, it follows from Lemma 1 that he wins in period t at a price 0 if all of his opponents have types lower than $\beta_t(h_t, p_t)$. Therefore, the deviation is strictly profitable for type $\beta_t(h_t, p_t) - \varepsilon$, contradicting the definition of $\beta_t(h_t, p_t)$. □

B.3 Proof of Proposition 1

Proof. Let $\delta(v) := e^{-rT(v)}$ denote the discount factor for type v who trades at time $T(v)$. We can rewrite the auxiliary problem as a maximization problem with $\delta(v)$ as the choice variable:

$$\begin{aligned} &\sup_{\delta} \int_0^1 \delta(v)\, J(v)\, f^{(n)}(v)\, dv \\ &\text{s.t. } \delta(v) \in [0,1], \text{ and non-decreasing,} \\ \forall v \in [0,1] : &\int_0^v \delta(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds \ge \delta(v^+) \int_0^v J(s|s \le v)\, f^{(n)}(s)\, ds. \end{aligned}$$

Let $\bar{\pi}$ be the supremum of this maximization problem and let (δ_k) be a sequence of feasible solutions of this problem such that

$$\lim_{k\to\infty} \int_0^1 \delta_k(v)\, J(v)\, f^{(n)}(v)\, dv = \bar{\pi}.$$

By Helly's selection theorem, there is a subsequence (δ_{k_ℓ}), and a non-decreasing function $\bar{\delta} : [0,1] \to [0,1]$ such that $\delta_{k_\ell}(v) \to \bar{\delta}(v)$ for all points of continuity of $\bar{\delta}$. Hence (after selecting a subsequence), we can take (δ_k) to be almost everywhere convergent with a.e.-limit $\bar{\delta}$. By Lebesgue's dominated convergence theorem, we also have convergence w.r.t. the L^2-norm and hence weak convergence in L^2. Therefore

$$\int_0^1 \bar{\delta}(v)\, J(v)\, f^{(n)}(v)\, dv = \lim_{k\to\infty} \int_0^1 \delta_k(v)\, J(v)\, f^{(n)}(v)\, dv = \bar{\pi}.$$

It remains to show that $\bar{\delta}$ satisfies the payoff floor constraint. Suppose not. Then there exists $\hat{v} \in [0,1)$ such that

$$\int_0^{\hat{v}} \bar{\delta}(s)\, J(s|s \leq \hat{v})\, f^{(n)}(s)\, ds < \bar{\delta}(\hat{v}^+) \int_0^{\hat{v}} J(s|s \leq \hat{v})\, f^{(n)}(s)\, ds.$$

Then there also exists $v \geq \hat{v}$ such that $\bar{\delta}$ is continuous at v, and

$$\int_0^{v} \bar{\delta}(s)\, J(s|s \leq v)\, f^{(n)}(s)\, ds < \bar{\delta}(v) \int_0^{v} J(s|s \leq v)\, f^{(n)}(s)\, ds.$$

Define

$$S := \bar{\delta}(v) \int_0^{v} J(s|s \leq v)\, f^{(n)}(s)\, ds - \int_0^{v} \bar{\delta}(s)\, J(s|s \leq v)\, f^{(n)}(s)\, ds.$$

Since v is a point of continuity we have $\bar{\delta}(v) = \lim_{k\to\infty} \delta_k(v)$. Therefore, there exists k_v such that for all $k > k_v$,

$$\left| \bar{\delta}(v) \int_0^{v} J(s|s \leq v)\, f^{(n)}(s)\, ds - \delta_k(v) \int_0^{v} J(s|s \leq v)\, f^{(n)}(s)\, ds \right| < \frac{S}{2},$$

and furthermore, since $\delta_k \to \bar{\delta}$ weakly in L^2, we can choose k_v such for all $k > k_v$ also

$$\left| \int_0^{v} \bar{\delta}(s)\, J(s|s \leq v)\, f^{(n)}(s)\, ds - \int_0^{v} \delta_k(s)\, J(s|s \leq v)\, f^{(n)}(s)\, ds \right| < \frac{S}{2}.$$

Together, this implies that for all $k > k_v$,

$$\int_0^{v} \delta_k(s)\, J(s|s \leq v)\, f^{(n)}(s)\, ds < \delta_k(v) \int_0^{v} J(s|s \leq v)\, f^{(n)}(s)\, ds,$$

which contradicts the assumption that δ_k is an feasible solution of the reformulated auxiliary problem defined above. □

B.4 Proof of Lemma 6

Proof. Fix $v \in (v_t^+, v_t]$. We obtain a lower bound for the LHS of (A.3) as follows:

$$\begin{aligned}
&\int_0^{v} e^{-r(T(x)-t)} \left(x - \frac{F(v)-F(x)}{f(x)} \right) dF^{(n)}(x) \\
&= \int_{v_t^+}^{v} \left(x - \frac{F(v)-F(x)}{f(x)} \right) dF^{(n)}(x) + \int_0^{v_t^+} e^{-r(T(x)-t)} \left(x - \frac{F(v_t^+)-F(x)}{f(x)} \right) dF^{(n)}(x) \\
&\quad - \int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v)-F(v_t^+)}{f(x)} \right) dF^{(n)}(x) \\
&\geq \int_{v_t^+}^{v} \left(x - \frac{F(v)-F(x)}{f(x)} \right) dF^{(n)}(x) + \int_0^{v_t^+} \left(x - \frac{F(v_t^+)-F(x)}{f(x)} \right) dF^{(n)}(x)
\end{aligned}$$

$$-\int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x).$$

The equality follows because all types in $(v_t^+, v]$ trade at time t, and the inequality follows from (A.2). We will show that the RHS of (A.3) is smaller than the lower bound. The RHS can be written as

$$\int_{v_t^+}^{v} \left(x - \frac{F(v) - F(x)}{f(x)} \right) dF^{(n)}(x) + \int_0^{v_t^+} \left(x - \frac{F(v_t^+) - F(x)}{f(x)} \right) dF^{(n)}(x) - \int_0^{v_t^+} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x).$$

The condition that the RHS is smaller than the lower bound for the LHS is sufficient for (A.3) to hold. Canceling terms, the sufficient condition simplifies to.

$$-\int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x) > -\int_0^{v_t^+} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x),$$

or equivalently

$$\int_0^{v_t^+} (1 - e^{-r(T(x)-t)}) \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x) > 0.$$

Since $T(x) > t$ for $x < v_t^+$ and $F(v) - F(v_t^+) > 0$ for $v > v_t^+$, the last inequality holds and the proof is complete. □

B.5 Proof of Lemma 9

Proof. Suppose by contradiction that for some t with $v_t > 0$, we have $T(v) = t$ for all $v \in [0, v_t]$. Then for all $\varepsilon > 0$ the payoff floor constraint at $t - \varepsilon$ is

$$\int_0^{v_t} e^{-r\varepsilon} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) + \int_{v_t}^{v_{t-\varepsilon}} e^{-r(T(v)-(t-\varepsilon))} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) \geq \int_0^{v_{t-\varepsilon}} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v).$$

Rearranging this we get

$$\int_{v_t}^{v_{t-\varepsilon}} \left(e^{-r(T(v)-(t-\varepsilon))} - 1 \right) J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) \geq \left(1 - e^{-r\varepsilon} \right) \int_0^{v_t} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v).$$

The RHS is strictly positive for $\varepsilon > 0$ but sufficiently small because, by the left-continuity of v_t and continuity of $J_t(v)$ in t, we have

$$\lim_{\varepsilon \to 0} \int_0^{v_t} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) = \int_0^{v_t} J_t(v) dF_t^{(n)}(v) > 0.$$

On the other hand, since $J_t(v_t) = v_t > 0$, we have $J_{t-\varepsilon}(v) > 0$ for $v \in (v_t, v_{t-\varepsilon})$ with $\varepsilon > 0$ but sufficiently small. Note that

$$T(v) \geq t - \varepsilon \text{ for all } v \in (v_t, v_{t-\varepsilon})$$

Therefore, $e^{-r(T(v)-(t-\varepsilon))} \leq 1$ for all $v \in (v_t, v_{t-\varepsilon})$, and thus the LHS is non-positive. A contradiction. □

B.6 Analysis of the Auxiliary Problem Omitted from Appendix A

B.6.1 Candidate Solution to the Auxiliary Problem

The (binding) payoff floor constraint we study here will be more general than needed to prove Theorems 2 and 3. The extra generality is important for our later analysis in Appendix D in the Supplemental Material, where we use equilibria of discrete time games to approximate the solution to the auxiliary problem. Our discrete approximation requires a strictly slack payoff floor constraint for feasible solutions, that is, for all $t \geq 0$,

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) = K \int_0^{v_t} J_t(x) dF^{(n)}(x), \tag{B.1}$$

where $K \in [1, \Gamma]$ with some $\Gamma > 0$. We will refer to constraint (B.1) as the generalized (binding) payoff floor constraint. Note that our earlier binding payoff floor constraint is a special case with $K = 1$. The following lemma shows that the generalized payoff floor constraint (B.1) can be reduced to an ODE. For $K = 1$, this ODE reduces to (A.4).

We assume for now that the solutions T and v_t for which the generalized payoff floor constraint is binding are continuously differentiable. We will show later in Lemma 11 that this differentiability property holds for every solution for which the payoff floor is binding.

Lemma 10. *Suppose $T(x)$ satisfies (B.1) for all $t \in (a, b)$ and suppose T is continuously differentiable with $-\infty < T'(v) < 0$ for all $v \in (v_b, v_a)$ and v_t is continuously differentiable for all $t \in (a, b)$. Then v_t is twice continuously differentiable on (a, b) and is characterized by*

$$\frac{\ddot{v}_t}{\dot{v}_t} + g(v_t, K)\dot{v}_t + h(v_t, K)\left(\dot{v}_t\right)^2 + r = 0,$$

where

$$g(v_t, K) = \frac{f'(v_t)}{f(v_t)} - \frac{\left\{\left(2 - \frac{1}{K}\right) v_t F^{n-1}(v_t) - 2\int_0^{v_t} F^{n-1}(v)\, dv\right\} f(v_t)}{(n-1)\int_0^{v_t} \left[F(v_t) - F(v)\right] F^{n-2}(v) f(v)\, v dv},$$

and

$$h(v_t, K) = \frac{K-1}{rK} \frac{F^{n-2}(v_t) f^2(v_t) v_t}{\int_0^{v_t} \left[F(v_t) - F(v)\right] F^{n-2}(v) f(v)\, v dv}.$$

Next we show that, if the payoff floor is binding for T and v_t, then they must be continuously differentiable. Therefore, the differentiability assumption in Lemma 10 is not necessary. However, we will formally prove differentiability only for the original payoff floor constraint, because we need differentiability to show that the solution to the ODE is the only solution to the original binding payoff floor constraint, while the uniqueness result for the generalized payoff floor constraint is not needed for our purpose.

Lemma 11. *Let T be a feasible solution for which (5.3) holds with equality for all $t > 0$. Then* *(i) T is strictly decreasing for $v \in [0, v_0^+]$.*

(ii) T is continuously differentiable with $T'(v) < 0$ for all $v \in (0, v_0^+)$.

(iii) v_t is twice continuously differentiable for all $t > 0$ where $v_t > 0$.

B.6.2 Feasibility of the Candidate Solution

If the ODE in (A.4) admits a decreasing solution ($\dot{v}_t \leq 0$, $\forall t$) with $\lim_{t\to\infty} v_t = 0$, then the binding payoff floor constraint yields non-trivial feasible solution to the auxiliary problem. It turns out that the existence of such a solution depends on the behavior of $g(v)v$ for $v \to 0$. We denote this limit by κ. The following lemma gives an explicit expression for this constant. Again we prove a more general result that will be used in the discrete time approximation.

Lemma 12. *If Assumption 2 is satisfied, we have*

$$\kappa := \lim_{v\to 0} g(v)v = \phi - \frac{((n-1)\,\phi + n - 2)\,(n\phi + n + 1)}{(n-1)\,(1+\phi)}, \tag{B.2}$$

$$\lim_{v\to 0} g(v,K)v = \kappa - \frac{K-1}{K}\left(n\phi + n + 2 + \frac{\phi + 2}{(n-1)\,(1+\phi)}\right), \tag{B.3}$$

and

$$\lim_{v\to 0} h(v,K)v^2 = \frac{1}{r}\frac{K-1}{K}(n + \phi n + 1)(n + \phi n - \phi). \tag{B.4}$$

The constant κ is related to the cutoff $\overline{N}(F)$ as follows:

Lemma 13. *If Assumption 2 is satisfied, $\kappa > -1$ is equivalent to $n < \overline{N}(F)$.*

Proof. If $\phi > -1$, the condition $\kappa > -1$ is equivalent to

$$(1+\phi)^2\,(n-1) - ((n-1)\,\phi + n - 2)\,(n\phi + n + 1) > 0.$$

By collecting terms with respect to n, we can change the condition into

$$-\,(\phi+1)^2\,n^2 + 2\,(\phi+1)^2\,n - \left(\phi^2 + \phi - 1\right) > 0,$$

or equivalently

$$n < 1 + \frac{\sqrt{2+\phi}}{1+\phi} = \overline{N}(F).$$

□

With this notation, we can give a sufficient condition for the existence of a feasible solution to the ODE in (A.4), and we can also provide a sufficient condition under which such a feasible solution does not exist. It turns out that these two sufficient conditions are almost mutually exclusive, depending on whether $\kappa = \lim_{v\to 0} g(v)v$ is above or below -1.

Lemma 14. *(i) If $\kappa < -1$, there exists no decreasing solution to (A.4) that satisfies $v_0 > 0$ and $\lim_{t\to\infty} v_t = 0$.*

(ii) If $\kappa > -1$, there exists a decreasing solution to (A.4) that satisfies $v_0 > 0$ and $\lim_{t\to\infty} v_t = 0$.

(iii) Among all such solutions, the unique solution that maximizes the seller's revenue for a given boundary value v_0^+ is given by the unique solution of (4.1) for given v_0^+.

B.6.3 Optimality of the Candidate Solution

In this section we prove that local concavity of the monopoly profit implies that the payoff floor constraint must be locally binding in the optimal solution, as stated in Proposition 5. To prove this key result, it suffices to show that feasible solutions with a strictly slack payoff floor constraint for a time interval (a, b) are never optimal if $v(1 - F(v))$ is concave on the interval of valuations $[v_b, v_a]$ that trade between a and b. Specifically, suppose we have a feasible solution T with corresponding cutoff path v_t for which the payoff floor constraint is strictly slack for all $t \in (a, b)$ where $0 \leq a < b$. We want to construct a new feasible solution $\hat{T}$ with corresponding cutoff path $\hat{v}_t$ that strictly improves the seller's expected profit. Our construction will only change the trading times of the valuations in the interval $(v_b^+ - \varepsilon, v_a)$ where $\varepsilon > 0$ can be arbitrarily small. This implies that the new solution satisfies the payoff floor constraint for all t for which $\hat{v}_t < v_b^+ - \varepsilon$ because the continuation is unchanged for such t. For times t such that $\hat{v}_t \in (v_b^+ - \varepsilon, v_a)$, we exploit that the payoff floor constraint was slack before the modification. This implies that a small variation in trading times will not lead to a violation of the payoff floor constraint by the new solution. Depending on whether types trade in the interior of the slack interval or types trade only at the end of the interval, the constructed variations are different and are covered in Lemmas 18 and 19, respectively. Finally, we exploit the following lemma to show that the payoff floor constraint for $t < a$ remains satisfied.

Lemma 15. *Let T and $\hat{T}$ be non-increasing solutions with corresponding cutoff paths v_t and $\hat{v}_t$ such that $v_t = \hat{v}_t$ for $t \leq a$. Suppose T is feasible and that the slack in the payoff floor constraint at a is the same for T and $\hat{T}$. If the ex-ante revenue of the seller under $\hat{T}$ is greater than or equal to the revenue under T, then $\hat{T}$ satisfies the payoff floor constraint for all $t \leq a$.*

In light of Lemma 15, we construct the new solution in such a way that the payoff floor constraint at a is unchanged and ex-ante revenue is improved. The lemma then shows that the payoff floor constraint is fulfilled for all $t \in [0, a]$ for the new solution.

Before we take this approach, we prove two observations that will be useful in the subsequent proofs. First, concavity of the monopoly profit is equivalent to the monotonicity of $J(v)f(v)$ or the monotonicity of $J(v|v \leq x)f(v)$ for all $x \in [0, 1]$, as shown in the following lemma.

Lemma 16. *Suppose $v(1-F(v))$ is strictly concave for on an interval $[a, b]$ where $a < b \leq x$. Then $J(s|v \leq x)\, f(s)$ is strictly increasing in s on the interval $[a, b]$.*

Second, we show that, whenever the payoff floor constraint is slack for an interval (a, b), the types that trade within the interval must have positive virtual valuation evaluated at any point of the time interval. Otherwise, one can construct alternative feasible trading times that delay the trade for types with negative virtual valuation and increase revenue.

Lemma 17. *Let T be an optimal solution for which the payoff floor constraint is slack for all $t \in (a, b)$. Then $J_t(v) \geq 0$ for all $t \in (a, b]$ and $v \in \left[v_b^+, v_a\right]$. If v_t is continuous at a, $J_a(v) \geq 0$ for all $v \in \left[v_b^+, v_a\right]$.*

Now we construct a feasible variation that improves revenue. We have to consider two scenarios. In the first scenario, there is a time interval $[s, s'] \subset (a, b)$ such that trade occurs with positive probability between s and s'. In this case, there exists a variation of the trading times for those types who trade in the interval $[s, s']$. Roughly speaking, we construct an alternative solution by splitting the types trading in (s, s'), and then clustering them to the endpoints s and s'. In particular, we advance the trading time of high types who previously traded in (s, s') and delay the trading times of low types who previously traded in (s, s'). The variation is constructed such that the payoff floor constraint at s is unchanged. Furthermore, our concavity assumption ensures that the alternative trading time $\hat{T}$ also leads to a higher ex ante revenue than T. It follows from Lemma 15, that the payoff floor constraint is fulfilled for all $t < s$. Formally, we have the following result.

Lemma 18. *Let T be a feasible solution for which the payoff floor constraint is strictly slack for all $t \in (a, b)$. Suppose there is a positive measure of types $v \in [v_b, v_a]$ for which $T(v) \notin \{a, b\}$. If $v(1 - F(v))$ is strictly concave for all $v \in [v_b, v_a]$, then T is not optimal.*

Lemma 18 implies that the probability of trade at times in the interior of the slack interval must be zero. It leaves open the scenario in which the slack interval consists of a single "quiet period" without trade in (a, b) followed by a single "atom" at b, formally, $v_a = v_b > v_b^+$. In this case, we construct an alternative trading scheme by splitting the atom so that the trading times of high types in the atom are advanced, while the trading times of low types in the atom are delayed. The latter requires that we also delay the trading time for types $v \in [v_b^+ - \varepsilon, v_b^+]$ for some $\varepsilon > 0$. Otherwise the new solution would violate monotonicity of the trading times. This modification can be constructed in a way such that the slack in the payoff floor constraint at a remains unchanged and the payoff floor constraint is satisfied on the newly created second quiet period. Again, concavity implies that ex-ante revenue is increased by this variation which implies that the payoff floor constraint at $t \leq a$ is still satisfied after the variation. Formally, we have the following result.

Lemma 19. *Let T be a feasible solution for which the payoff floor constraint is strictly slack for all $t \in (a, b]$ and binding for a and b_+. Suppose $T(v) = b$ for all $v \in (v_b^+, v_a)$. If $v(1 - F(v))$ is strictly concave for all $v \in \left[v_b^+ - \varepsilon, v_b\right]$ for some $\varepsilon > 0$, then T is not optimal.*

B.7 Proofs for Section B.6

B.7.1 Proof of Lemma 10

Proof. We first rewrite (B.1) as

$$\int_0^{v_t} e^{-rT(x)} J_t(x) dF^{(n)}(x) = Ke^{-rt} \int_0^{v_t} J_t(x) dF^{(n)}(x).$$

Since v_t is continuously differentiable on (a,b), we can differentiate (B.1) on both sides to obtain

$$\begin{aligned}
&e^{-rt}v_t f^{(n)}(v_t)\dot{v}_t - \int_0^{v_t} e^{-rT(x)}\frac{f(v_t)\dot{v}_t}{f(x)}dF^{(n)}(x)\\
=&-Kre^{-rt}\int_0^{v_t} J_t(x)dF^{(n)}(x) + Ke^{-rt}v_t f^{(n)}(v_t)\dot{v}_t - Ke^{-rt}\int_0^{v_t}\frac{f(v_t)\dot{v}_t}{f(x)}dF^{(n)}(x),
\end{aligned}$$

where we have used $\frac{\partial J_t(x)}{\partial t} = -\frac{f(v_t)\dot{v}_t}{f(x)}$. This equation can be further simplified

$$\begin{aligned}
&-\int_0^{v_t} e^{-rT(x)}\frac{f(v_t)\dot{v}_t}{f(x)}dF^{(n)}(x)\\
=&-Kre^{-rt}\int_0^{v_t} J_t(x)dF^{(n)}(x) + (K-1)e^{-rt}f^{(n)}(v_t)v_t\dot{v}_t - Ke^{-rt}\int_0^{v_t}\frac{f(v_t)\dot{v}_t}{f(x)}dF^{(n)}(x).
\end{aligned}$$

Since T is continuous and has a bounded derivative, $\dot{v}_t > 0$. By assumption, $f(v_t) > 0$, so we can divide the previous equation by $-f(v_t)\dot{v}_t$ to obtain

$$\begin{aligned}
&\int_0^{v_t} e^{-rT(x)}\frac{1}{f(x)}dF^{(n)}(x)\\
=&K\frac{re^{-rt}}{f(v_t)\dot{v}_t}\int_0^{v_t} J_t(x)dF^{(n)}(x) + Ke^{-rt}\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x) - (K-1)e^{-rt}\frac{f^{(n)}(v_t)}{f(v_t)}v_t.
\end{aligned} \tag{B.5}$$

This equation, together with our assumption that $f(v)$ is continuously differentiable, implies that v_t is twice continuously differentiable. Hence, we may differentiate on both sides.

$$\begin{aligned}
&\frac{d}{dt}\left(re^{-rt}\frac{\int_0^{v_t} J_t(x)\, f^{(n)}(x)\, dx}{f(v_t)\,\dot{v}_t}\right)\\
=\;& -r^2 e^{-rt}\frac{\int_0^{v_t} J_t(x)\, f^{(n)}(x)\, dx}{f(v_t)\,\dot{v}_t}\\
&+re^{-rt}\left(\frac{v_t f^{(n)}(v_t)\dot{v}_t + \dot{v}_t\int_0^{v_t}\left(-\frac{f(v_t)}{f(x)}\right)f^{(n)}(x)dx}{f(v_t)\,\dot{v}_t} - \frac{\left(f'(v_t)(\dot{v}_t)^2 + f(v_t)\ddot{v}_t\right)\int_0^{v_t} J_t(x)f^{(n)}(x)dx}{(f(v_t)\,\dot{v}_t)^2}\right)\\
=\;& re^{-rt}\left(\frac{v_t f^{(n)}(v_t)\dot{v}_t - f(v_t)\int_0^{v_t}\frac{f^{(n)}(x)}{f(x)}dx\,\dot{v}_t}{f(v_t)\,\dot{v}_t} - \frac{\left(\dot{v}_t\frac{f'(v_t)}{f(v_t)} + \frac{\ddot{v}_t}{\dot{v}_t} + r\right)\int_0^{v_t} J_t(x)f^{(n)}(x)dx}{f(v_t)\,\dot{v}_t}\right),
\end{aligned}$$

where we have used $\frac{\partial^2 J_t(x)}{\partial t^2} = -\frac{f'(v_t)(\dot{v}_t)^2 + f(v_t)\ddot{v}_t}{f(x)}$. Next, note that

$$\begin{aligned}
\frac{d}{dt}\left(e^{-rt}\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x)\right) =& -re^{-rt}\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x) + e^{-rt}\frac{f^{(n)}(v_t)}{f(v_t)}\dot{v}_t\\
=& -re^{-rt}\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x) + e^{-rt}nF^{n-1}(v_t)\dot{v}_t,
\end{aligned}$$

and

$$\frac{d}{dt}\left(e^{-rt}\frac{f^{(n)}(v_t)}{f(v_t)}v_t\right) = \frac{d}{dt}\left(e^{-rt}nF^{n-1}(v_t)v_t\right)$$
$$= -re^{-rt}nF^{n-1}(v_t)v_t + e^{-rt}n(n-1)F^{n-2}(v_t)f(v_t)v_t\dot{v}_t + e^{-rt}nF^{n-1}(v_t)\dot{v}_t.$$

Therefore, differentiating (B.5) on both sides yields

$$\begin{aligned}
&e^{-rt}nF^{n-1}(v_t)\dot{v}_t \\
=&Kre^{-rt}\left(\frac{v_t f^{(n)}(v_t)\dot{v}_t - f\left(v_t\right)\int_0^{v_t}\frac{f^{(n)}(x)}{f(x)}dx\,\dot{v}_t}{f\left(v_t\right)\dot{v}_t} - \frac{\left(\dot{v}_t\frac{f'(v_t)}{f(v_t)}+\frac{\ddot{v}_t}{\dot{v}_t}+r\right)\int_0^{v_t}J_t(x)f^{(n)}(x)dx}{f\left(v_t\right)\dot{v}_t}\right) \\
&-Kre^{-rt}\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x) + Ke^{-rt}nF^{n-1}(v_t)\dot{v}_t \\
&+(K-1)re^{-rt}nF^{n-1}(v_t)v_t - (K-1)e^{-rt}n(n-1)F^{n-2}(v_t)f(v_t)v_t\dot{v}_t - (K-1)e^{-rt}nF^{n-1}(v_t)\dot{v}_t.
\end{aligned}$$

This can be simplified into

$$\begin{aligned}
0 =&Kr\left(\frac{v_t f^{(n)}(v_t)\dot{v}_t - f\left(v_t\right)\int_0^{v_t}\frac{f^{(n)}(x)}{f(x)}dv\,\dot{v}_t}{f\left(v_t\right)\dot{v}_t} - \frac{\left(\dot{v}_t\frac{f'(v_t)}{f(v_t)}+\frac{\ddot{v}_t}{\dot{v}_t}+r\right)\int_0^{v_t}J_t(x)f^{(n)}(x)dx}{f\left(v_t\right)\dot{v}_t}\right) \\
&-Kr\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x) + (K-1)rnF^{n-1}(v_t)v_t - (K-1)n(n-1)F^{n-2}(v_t)f(v_t)v_t\dot{v}_t.
\end{aligned}$$

Multiplying both sides by $f\left(v_t\right)\dot{v}_t$, we obtain

$$\begin{aligned}
0 =&Krv_t f^{(n)}(v_t)\dot{v}_t - Kr\int_0^{v_t}\frac{f^{(n)}(x)}{f\left(x\right)}dx\,f\left(v_t\right)\dot{v}_t - Kr\left(\dot{v}_t\frac{f'\left(v_t\right)}{f\left(v_t\right)}+\frac{\ddot{v}_t}{\dot{v}_t}+r\right)\int_0^{v_t}J_t(x)f^{(n)}(x)dx, \\
&-Kr\int_0^{v_t}\frac{1}{f(x)}dF^{(n)}(x)f\left(v_t\right)\dot{v}_t \\
&+(K-1)rnF^{n-1}(v_t)f\left(v_t\right)\dot{v}_t v_t - (K-1)n(n-1)F^{n-2}(v_t)\left(f(v_t)\right)^2 v_t\left(\dot{v}_t\right)^2.
\end{aligned}$$

Collecting terms we obtain

$$\begin{aligned}
Kr\left(\dot{v}_t\frac{f'\left(v_t\right)}{f\left(v_t\right)}+\frac{\ddot{v}_t}{\dot{v}_t}+r\right)\int_0^{v_t}J_t(x)f^{(n)}(x)dx =&(2K-1)rv_t f^{(n)}(v_t)\dot{v}_t - 2Kr\int_0^{v_t}\frac{f^{(n)}(x)}{f\left(x\right)}dx\,f\left(v_t\right)\dot{v}_t \\
&-(K-1)n(n-1)F^{n-2}(v_t)\left(f(v_t)\right)^2 v_t\left(\dot{v}_t\right)^2.
\end{aligned}$$

Hence we have

$$\frac{\ddot{v}_t}{\dot{v}_t} + \underbrace{\left(\frac{f'\left(v_t\right)}{f\left(v_t\right)} - \frac{\frac{(2K-1)}{K}f^{(n)}(v_t)v_t - 2f\left(v_t\right)n\int_0^{v_t}F^{n-1}(x)dx}{\int_0^{v_t}J_t(x)f^{(n)}(x)dx}\right)\dot{v}_t}_{=:g(v_t,K)}$$

$$+\underbrace{\frac{(K-1)}{rK}\frac{n(n-1)F^{n-2}(v_t)\left(f(v_t)\right)^2 v_t}{\int_0^{v_t} J_t(x)f^{(n)}(x)dx}}_{=:h(v_t,K)}\left(\dot{v}_t\right)^2+r=0.$$

Using

$$\begin{aligned}
\int_0^{v_t} J_t(x)f^{(n)}(x)dx &= n\int_0^{v_t}\left(x-\frac{F(v_t)-F(x)}{f(x)}\right)F^{n-1}(x)f(x)dx \\
&= n\int_0^{v_t}\left(xF^{n-1}(x)f(x)-F(v_t)F^{n-1}(x)+F^n(x)\right)dx \\
&= n\int_0^{v_t}F^{n-1}(x)f(x)xdx - nF(v_t)\int_0^{v_t}F^{n-1}(x)dx + n\int_0^{v_t}F^n(x)dx \\
&= n\int_0^{v_t}F^{n-1}(x)f(x)xdx - nF(v_t)F^{n-1}(v_t)v_t + n(n-1)F(v_t)\int_0^{v_t}F^{n-2}(x)f(x)xdx \\
&\quad + nF^n(v_t)v_t - n\int_0^{v_t} nF^{n-1}(x)f(x)xdx \\
&= -(n-1)n\int_0^{v_t}F^{n-1}(x)f(x)xdx + n(n-1)F(v_t)\int_0^{v_t}F^{n-2}(x)f(x)xdx \\
&= (n-1)n\int_0^{v_t}\left(F(v_t)-F(x)\right)F^{n-2}(x)f(x)xdx,
\end{aligned}$$

we have

$$g(v_t,K)=\frac{f'(v_t)}{f(v_t)}-\frac{\left\{\left(2-\frac{1}{K}\right)v_tF^{n-1}(v_t)-2\int_0^{v_t}F^{n-1}(v)\,dv\right\}f(v_t)}{(n-1)\int_0^{v_t}\left[F(v_t)-F(v)\right]F^{n-2}(v)\,f(v)\,vdv},$$

and

$$h(v_t,K)=\frac{K-1}{rK}\frac{F^{n-2}(v_t)\,f^2(v_t)\,v_t}{\int_0^{v_t}\left[F(v_t)-F(v)\right]F^{n-2}(v)\,f(v)\,vdv}.$$

□

B.7.2 Proof of Lemma 11

Proof. Note that part (i) and part (ii) imply that v_t is continuously differentiable for all $t>0$ where $v_t>0$. Part (iii) then follows from Lemma 10.

(i) Suppose by contradiction, that there exists a trading time $s>0$ such that $T^{-1}(s)=(v_s^+,v_s]$ where $v_s^+<v_s$.

We have the following jump on the LHS of the payoff floor constraint at s:

$$\begin{aligned}
A &:= \int_0^{v_s} e^{-r(T(x)-s)}\left(x-\frac{F(v_s)-F(x)}{f(x)}\right)dF^{(n)}(x) - \int_0^{v_s^+} e^{-r(T(x)-s)}\left(x-\frac{F(v_s^+)-F(x)}{f(x)}\right)dF^{(n)}(x) \\
&= \int_{v_s^+}^{v_s} J_s(x)dF^{(n)}(x) + \left(F(v_s^+)-F(v_s)\right)\int_0^{v_s^+} e^{-r(T(x)-s)}\frac{1}{f(x)}dF^{(n)}(x),
\end{aligned}$$

where last equation follows from $T(x) = s$ for $x \in (v_s^+, v_s)$. The jump on the RHS is

$$\begin{aligned} B &:= \int_0^{v_s} \left(x - \frac{F(v_s) - F(x)}{f(x)} \right) dF^{(n)}(x) - \int_0^{v_s^+} \left(x - \frac{F(v_s^+) - F(x)}{f(x)} \right) dF^{(n)}(x) \\ &= \int_{v_s^+}^{v_s} J_s(x) dF^{(n)}(x) + \Big(F(v_s^+) - F(v_s) \Big) \int_0^{v_s^+} \frac{1}{f(x)} dF^{(n)}(x). \end{aligned}$$

Since the payoff floor constraint is binding for all $t' > 0$, taking a right limit $t' \searrow t$ on both sides implies

$$\int_0^{v_t^+} e^{-r(T(x)-t)} \left(x - \frac{F(v_t^+) - F(x)}{f(x)} \right) dF^{(n)}(x) = \int_0^{v_t^+} \left(x - \frac{F(v_t^+) - F(x)}{f(x)} \right) dF^{(n)}(x). \tag{B.6}$$

This implies

$$A - B = \int_0^{v_s^+} e^{-r(T(x)-s)} \frac{1}{f(x)} dF^{(n)}(x) - \int_0^{v_s^+} \frac{1}{f(x)} dF^{(n)}(x) = 0.$$

Since $T(x) \neq s$ for $x < v_s^+$, this expression can only hold if $v_s^+ = 0$. We show in a separate Lemma (Lemma 9) that this contradicts the feasibility of T. This concludes the proof of part (i).

We prove part (ii) in three steps. First, suppose T is not continuous. Then there exists a time interval (b, c) such that v_t is positive and constant on (b, c). Since $c > b$ we have

$$\begin{aligned} e^{-r(c-b)} \int_0^{v_c} J_c(x) dF^{(n)}(x) &< \int_0^{v_c} J_c(x) dF^{(n)}(x) \Leftrightarrow \\ e^{-r(c-b)} \int_0^{v_c} e^{-r(T(x)-c)} J_c(x) dF^{(n)}(x) &< \int_0^{v_c} J_c(x) dF^{(n)}(x) \Leftrightarrow \\ \int_0^{v_b^+} e^{-r(T(x)-b)} J_b(x) dF^{(n)}(x) &< \int_0^{v_b^+} J_b(x) dF^{(n)}(x). \end{aligned}$$

The first equivalence follows from the binding payoff floor constraint at c, and the second follows from the fact that $v_b^+ = v_c$ and $J_b(x) = J_c(x)$. But the assumption that the payoff floor constraint is satisfied at all t implies that (B.6) holds for $t = b$. This contradicts the last inequality. Therefore, T is continuous. This concludes the first step.

Second, we show that T is continuously differentiable on $(0, v_0^+)$. Since T is continuous and strictly decreasing for $v \in (0, v_0^+)$, a binding payoff floor constraint for all $t > 0$ is equivalent to the condition that, for all $v \in (0, v_0^+)$,

$$\int_0^v e^{-rT(x)} \left(x - \frac{F(v) - F(x)}{f(x)} \right) dF^{(n)}(x) = e^{-rT(v)} \int_0^v \left(x - \frac{F(v) - F(x)}{f(x)} \right) dF^{(n)}(x),$$

which can be rearranged into

$$e^{-rT(v)} = \frac{\int_0^v e^{-rT(x)} \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}{\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}.$$

Continuity of T and continuous differentiability of F imply that the right-hand side of this expression is continuously differentiable, and thus T is also continuously differentiable. This concludes the second step.

Finally, we compute the derivative to show that it is strictly negative. We obtain

$$\begin{aligned}
-re^{-rT(v)}T'(v) =& \frac{e^{-rT(v)} f^{(n)}(v)v - \int_0^v e^{-rT(x)} \frac{f(v)}{f(x)} dF^{(n)}(x)}{\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)} \\
&- \frac{\left[f^{(n)}(v)v - \int_0^v \frac{f(v)}{f(x)} dF^{(n)}(x)\right] \int_0^v e^{-rT(x)} \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}{\left(\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)\right)^2} \\
\iff re^{-rT(v)}T'(v) =& \frac{e^{-rT(v)} \left[f^{(n)}(v)v - \int_0^v \frac{f(v)}{f(x)} dF^{(n)}(x)\right] \int_0^v e^{-r(T(x)-T(v))} \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}{\left(\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)\right)^2} \\
&- \frac{e^{-rT(v)} \left[f^{(n)}(v)v - \int_0^v e^{-r(T(x)-T(v))} \frac{f(v)}{f(x)} dF^{(n)}(x)\right]}{\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}.
\end{aligned}$$

Hence

$$\begin{aligned}
T'(v) =& \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v \frac{f(v)}{f(x)} dF^{(n)}(x)\right] \int_0^v e^{-r(T(x)-T(v))} \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}{\left(\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)\right)^2} \\
&- \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v e^{-r(T(x)-T(v))} \frac{f(v)}{f(x)} dF^{(n)}(x)\right] \int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}{\left(\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)\right)^2} \\
=& \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v \frac{f(v)}{f(x)} dF^{(n)}(x)\right] \int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}{\left(\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)\right)^2} \\
&- \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v e^{-r(T(x)-T(v))} \frac{f(v)}{f(x)} dF^{(n)}(x)\right] \int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}{\left(\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)\right)^2} \\
=& \frac{f(v)}{r} \frac{\left[\int_0^v e^{-r(T(x)-T(v))} \frac{1}{f(x)} dF^{(n)}(x) - \int_0^v \frac{1}{f(x)} dF^{(n)}(x)\right] \int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}{\left(\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)\right)^2} \\
=& \frac{f(v)}{r} \frac{\int_0^v \left(e^{-r(T(x)-T(v))} - 1\right) \frac{1}{f(x)} dF^{(n)}(x)}{\int_0^v \left(x - \frac{F(v)-F(x)}{f(x)}\right) dF^{(n)}(x)}.
\end{aligned}$$

where the second equality follows from the binding payoff floor constraint. In the last line, the numerator is strictly negative and the denominator is positive. Therefore $T'(v) < 0$. This concludes the proof of part (ii). □

B.7.3 Proof of Lemma 12

Proof. We define the following functions:

$$X(v) := \frac{F^{n-1}(v)f(v)v}{(n-1)\int_0^v [F(v)-F(s)]F^{n-2}(s)f(s)sds},$$
$$Y(v) := \frac{2f(v)\int_0^v F^{n-1}(s)ds}{(n-1)\int_0^v [F(v)-F(s)]F^{n-2}(s)f(s)sds}.$$

With these definitions we have

$$g(v) = g(v,1) = \frac{f'(v)}{f(v)} - X(v) + Y(v),$$

and

$$g(v,K) = g(v) - \frac{(K-1)}{K}X(v).$$

It is also useful to note that

$$\lim_{v\to 0}\frac{vf(v)}{F(v)} = \lim_{v\to 0}\frac{f'(v)v+f(v)}{f(v)} = 1+\phi \quad \text{and} \quad \lim_{v\to 0}\frac{F(v)}{vf(v)} = \frac{1}{1+\phi},$$

which will be used repeatedly below.

We now show that

$$\lim_{v\to 0} X(v)v = n\phi + n + 2 + \frac{\phi+2}{(n-1)(1+\phi)}$$
$$\lim_{v\to 0} Y(v)v = 2 + \frac{2(\phi+2)}{(n-1)(1+\phi)}.$$

For the first limit, note that

$$\begin{aligned}
&\lim_{v\to 0} X(v)v\\
&= \lim_{v\to 0}\frac{(n-1)F^{n-2}(v)f^2(v)v^2 + F^{n-1}(v)f'(v)v^2 + F^{n-1}(v)f(v)2v}{(n-1)f(v)\int_0^v s\,F^{n-2}(s)f(s)\,ds}\\
&= \lim_{v\to 0}\frac{F^{n-2}(v)f^2(v)v^2}{f(v)\int_0^v s\,F^{n-2}(s)f(s)\,ds} + \lim_{v\to 0}\frac{F^{n-1}(v)f'(v)v^2}{(n-1)f(v)\int_0^v s\,F^{n-2}(s)f(s)\,ds}\\
&\quad + \lim_{v\to 0}\frac{F^{n-1}(v)f(v)2v}{(n-1)f(v)\int_0^v s\,F^{n-2}(s)f(s)\,ds}\\
&= \lim_{v\to 0}\frac{F^{n-2}(v)f(v)v^2}{\int_0^v s\,F^{n-2}(s)f(s)\,ds} + \lim_{v\to 0}\frac{F^{n-1}(v)f'(v)v^2}{(n-1)f(v)\int_0^v s\,F^{n-2}(s)f(s)\,ds} + \lim_{v\to 0}\frac{F^{n-1}(v)2v}{(n-1)\int_0^v s\,F^{n-2}(s)f(s)\,ds},
\end{aligned}$$

where we have used l'Hospital's rule in the first step and then rearranged the expression.

The limit of the first term is

$$\begin{aligned}
\lim_{v\to 0}\frac{F^{n-2}(v)f(v)v^2}{\int_0^v s\,F^{n-2}(s)f(s)\,ds} &= \lim_{v\to 0}\frac{(n-2)\,F^{n-3}(v)f^2(v)v^2+F^{n-2}(v)f'(v)v^2+F^{n-2}(v)f(v)2v}{v\,F^{n-2}(v)f(v)} \\
&= \lim_{v\to 0}\frac{(n-2)\,f^2(v)v+F(v)f'(v)v+F(v)f(v)2}{F(v)f(v)} \\
&= \lim_{v\to 0}\frac{(n-2)\,f(v)v}{F(v)}+\lim_{v\to 0}\frac{f'(v)v}{f(v)}+2 \\
&= (n-2)\,(\phi+1)+\phi+2 \\
&= (n-1)\,\phi+n,
\end{aligned}$$

where we have have used l'Hospital's rule to obtain the first equality. For the second term we have

$$\begin{aligned}
\lim_{v\to 0}\frac{F^{n-1}(v)f'(v)v^2}{(n-1)f\,(v)\int_0^v s\,F^{n-2}(s)f(s)\,ds} &= \lim_{v\to 0}\frac{f'(v)v}{f\,(v)}\frac{F^{n-1}(v)v}{(n-1)\int_0^v s\,F^{n-2}(s)f(s)\,ds} \\
&= \lim_{v\to 0}\frac{f'(v)v}{f\,(v)}\lim_{v\to 0}\frac{F^{n-1}(v)v}{(n-1)\int_0^v s\,F^{n-2}(s)f(s)\,ds} \\
&= \phi\lim_{v\to 0}\frac{(n-1)\,F^{n-2}(v)f\,(v)\,v+F^{n-1}(v)}{(n-1)v\,F^{n-2}(v)f(v)} \\
&= \phi\lim_{v\to 0}\left\{1+\frac{F(v)}{(n-1)vf(v)}\right\} \\
&= \phi+\frac{1}{n-1}\frac{\phi}{1+\phi}.
\end{aligned}$$

The limit for the third term is

$$\begin{aligned}
\lim_{v\to 0}\frac{F^{n-1}(v)2v}{(n-1)\int_0^v s\,F^{n-2}(s)f(s)\,ds} &= \lim_{v\to 0}\frac{(n-1)\,F^{n-2}(v)f\,(v)\,2v+F^{n-1}(v)2}{(n-1)v\,F^{n-2}(v)f(v)} \\
&= \lim_{v\to 0}\frac{F^{n-2}(v)f\,(v)\,2v}{v\,F^{n-2}(v)f(v)}+\lim_{v\to 0}\frac{F^{n-1}(v)2}{(n-1)v\,F^{n-2}(v)f(v)} \\
&= 2+\frac{2}{n-1}\lim_{v\to 0}\frac{F(v)}{vf(v)} \\
&= 2+\frac{2}{n-1}\frac{1}{1+\phi}.
\end{aligned}$$

We can put the three limits together to obtain the desired result.

$$\begin{aligned}
\lim_{v\to 0}X(v)v &= ((n-1)\,\phi+n)+\left(\phi+\frac{1}{n-1}\frac{\phi}{1+\phi}\right)+\left(2+\frac{2}{n-1}\frac{1}{1+\phi}\right) \\
&= n\phi+n+2+\frac{\phi+2}{(n-1)\,(1+\phi)}.
\end{aligned}$$

For the limit of $Y(v)v$ we have

$$\begin{aligned}
\lim_{v\to 0} Y(v)\,v &= \lim_{v\to 0} \frac{2vf(v)\int_0^v F^{n-1}(s)\,ds}{(n-1)\int_0^v s\,[F(v)-F(s)]\,F^{n-2}(s)\,f(s)\,ds} \\
&= 2\lim_{v\to 0} \frac{f(v)\int_0^v F^{n-1}(s)\,ds + vf'(v)\int_0^v F^{n-1}(s)\,ds + vf(v)\,F^{n-1}(v)}{(n-1)\int_0^v s\,F^{n-2}(s)f(s)f(v)\,ds} \\
&= 2\left\{\lim_{v\to 0} \frac{\int_0^v F^{n-1}(s)\,ds}{(n-1)\int_0^v s\,F^{n-2}(s)f(s)ds} + \lim_{v\to 0}\frac{vf'(v)}{f(v)}\frac{\int_0^v F^{n-1}(s)\,ds}{(n-1)\int_0^v s\,F^{n-2}(s)f(s)ds}\right\} \\
&\quad +2\lim_{v\to 0}\frac{vF^{n-1}(v)}{(n-1)\int_0^v s\,F^{n-2}(s)f(s)\,ds} \\
&= 2\left\{\lim_{v\to 0}\frac{F^{n-1}(v)}{(n-1)vF^{n-2}(v)f(v)} + \phi\lim_{v\to 0}\frac{F^{n-1}(v)}{(n-1)v\,F^{n-2}(v)f(v)}\right\} \\
&\quad +2\lim_{v\to 0}\frac{F^{n-1}(v)+(n-1)\,vF^{n-2}(v)\,f(v)}{(n-1)v\,F^{n-2}(v)f(v)} \\
&= 2\left\{\lim_{v\to 0}\frac{F(v)}{(n-1)vf(v)} + \phi\lim_{v\to 0}\frac{F(v)}{(n-1)vf(v)} + \lim_{v\to 0}\frac{F(v)}{(n-1)vf(v)} + 1\right\} \\
&= 2 + 2\frac{2+\phi}{n-1}\lim_{v\to 0}\frac{F(v)}{vf(v)} \\
&= 2 + 2\frac{2+\phi}{n-1}\frac{1}{1+\phi}.
\end{aligned}$$

Adding up terms we have

$$\begin{aligned}
&\lim_{v\to 0}\frac{f'(v)v}{f(v)} - \lim_{v\to 0} X(v)v + \lim_{v\to 0} Y(v) \\
=\;& \phi - \left(n\phi + n + 2 + \frac{\phi+2}{(n-1)(1+\phi)}\right) + \left(2 + 2\frac{2+\phi}{n-1}\frac{1}{1+\phi}\right) \\
=\;& \phi - \frac{((n-1)\,\phi + n - 2)\,(n\phi + n + 1)}{(n-1)(1+\phi)},
\end{aligned}$$

and hence we have (B.2) and (B.3).

To show (B.4), note that

$$\begin{aligned}
&\lim_{v\to 0} rh(v,K)v^2 \\
=&\frac{K-1}{K}\lim_{v\to 0}\frac{F^{n-2}(v)f^2(v)v^3}{\int_0^v s\,F^{n-2}(s)f(s)(F(v)-F(s))\,ds} \\
=&\frac{K-1}{K}\lim_{v\to 0}\frac{(n-2)F^{n-3}(v)f^3(v)v^3 + F^{n-2}(v)2f(v)f'(v)v^3 + F^{n-2}(v)f^2(v)3v^2}{\int_0^v s\,F^{n-2}(s)f(s)f(v)\,ds} \\
=&\frac{K-1}{K}\left\{\lim_{v\to 0}\frac{(n-2)F^{n-3}(v)f^2(v)v^3}{\int_0^v s\,F^{n-2}(s)f(s)\,ds} + \lim_{v\to 0}\frac{f'(v)v}{f(v)}\frac{F^{n-2}(v)2f(v)v^2}{\int_0^v s\,F^{n-2}(s)f(s)\,ds} + \lim_{v\to 0}\frac{F^{n-2}(v)f(v)3v^2}{\int_0^v s\,F^{n-2}(s)f(s)\,ds}\right\}
\end{aligned}$$

$$=\frac{K-1}{K}\lim_{v\to 0}\frac{(n-2)F^{n-3}(v)f^2(v)v^3}{\int_0^v s\,F^{n-2}(s)f(s)\,ds}+\frac{K-1}{K}(3+2\phi)\lim_{v\to 0}\frac{F^{n-2}(v)f(v)v^2}{\int_0^v s\,F^{n-2}(s)f(s)\,ds}.$$

For the first limit we have

$$\begin{aligned}
\lim_{v\to 0}\frac{(n-2)F^{n-3}(v)f^2(v)v^3}{\int_0^v s\,F^{n-2}(s)f(s)\,ds} &= (n-2)\lim_{v\to 0}\frac{(n-3)F^{n-4}(v)f^3(v)v^3+F^{n-3}(v)2f(v)f'^3+F^{n-3}(v)f^2(v)3v^2}{v\,F^{n-2}(v)f(v)}\\
&= (n-2)\lim_{v\to 0}\frac{(n-3)f^2(v)v^2+F(v)2f'^2+F(v)f(v)3v}{F^2(v)}\\
&= (n-2)\lim_{v\to 0}\frac{(n-3)f^2(v)v^2}{F^2(v)}+\frac{2f(v)v}{F(v)}\frac{f'(v)v}{f(v)}+\frac{f(v)3v}{F(v)}\\
&= (n-2)\lim_{v\to 0}\frac{f(v)v}{F(v)}\left((n-3)\frac{f(v)v}{F(v)}+2\frac{f'(v)v}{f(v)}+3\right)\\
&= (n-2)(1+\phi)\left((n-3)(1+\phi)+2\phi+3\right)\\
&= (n+\phi n-2-2\phi)(n+\phi n-\phi).
\end{aligned}$$

For the second limit we have

$$\begin{aligned}
\lim_{v\to 0}\frac{F^{n-2}(v)f(v)v^2}{\int_0^v s\,F^{n-2}(s)f(s)\,ds} &= \lim_{v\to 0}\frac{(n-2)F^{n-3}(v)f^2(v)v^2+F^{n-2}(v)f'^2+F^{n-2}(v)f(v)2v}{v\,F^{n-2}(v)f(v)}\\
&= \lim_{v\to 0}\frac{(n-2)F^{n-3}(v)f^2(v)v^2}{v\,F^{n-2}(v)f(v)}+\lim_{v\to 0}\frac{F^{n-2}(v)f'^2}{v\,F^{n-2}(v)f(v)}+\lim_{v\to 0}\frac{F^{n-2}(v)f(v)2v}{v\,F^{n-2}(v)f(v)}\\
&= (n-2)\lim_{v\to 0}\frac{f(v)v}{F(v)}+\lim_{v\to 0}\frac{f'(v)v}{f(v)}+2\\
&= (n-2)(1+\phi)+\phi+2\\
&= n+n\phi-\phi.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\lim_{v\to 0} rh(v,K)v^2 &= \frac{K-1}{K}(n+\phi n-2-2\phi)(n+\phi n-\phi)+\frac{K-1}{K}(3+2\phi)\,(n+n\phi-\phi)\\
&= \frac{K-1}{K}(n+\phi n+1)(n+\phi n-\phi).
\end{aligned}$$

□

B.7.4 Proof of Lemma 14

Proof. We transform the ODE (A.4) using the change of variables $y=\dot{v}_t$. This yields

$$y'(v)+g(v)y(v)+r=0.$$

The general solution is given by

$$y(v) = e^{-\int_m^v g(x)dx}\left(C - \int_m^v r e^{\int_m^w g(x)dx} dw\right), \tag{B.7}$$

where $m > 0$.[35] Feasibility requires that $y(v) < 0$ for all $v \in (0, v_0^+)$.

(i) Suppose $\kappa < -1$. Since $\kappa = \lim_{v\to 0} g(v)v$, there must exist $\gamma > 0$ such that $g(v) \leq -\frac{1}{v}$ for all $v \in (0, \gamma]$. Then there does not exist a finite C such that the general solution in (B.7) satisfies $y(v) < 0$ for all $v \in (0, v_0^+)$. Suppose by contradiction, that such $C \in \mathbb{R}$ exists. Then for all $v \in (0, v_0^+)$,

$$C < \int_m^v r e^{\int_m^w g(x)dx} dw.$$

Since the right-hand side is increasing in v this implies

$$\lim_{v\to 0} \int_m^v r e^{\int_m^w g(x)dx} dw > -\infty. \tag{B.8}$$

We may assume that $0 < m < \gamma$. In this case, the limit can be computed as follows:

$$\begin{aligned}
\lim_{v\to 0} \int_m^v r e^{\int_m^w g(x)dx} dw &= \lim_{v\to 0} - \int_v^m r e^{-\int_w^m g(x)dx} dw \\
&\leq \lim_{v\to 0} - \int_v^m r e^{\int_w^m \frac{1}{x}dx} dw \\
&= \lim_{v\to 0} - \int_v^m r \frac{m}{w} dw \\
&= -\infty.
\end{aligned}$$

A contradiction. This shows part (i).

To prove part (ii), we first set

$$C = -\int_0^m r e^{\int_m^w g(x)dx} dw, \tag{B.9}$$

and show that the resulting solution

$$y(v) = -e^{-\int_m^v g(x)dx} \int_0^v r e^{\int_m^w g(x)dx} dw = -\int_0^v r e^{-\int_w^v g(x)dx} dw, \tag{B.10}$$

is negative and finite for all v. It is clear that $y(v) < 0$, so it suffices to rule out $y(v) = -\infty$. Since $\kappa = \lim_{v\to 0} g(v)v > -1$, there exist $\hat{\kappa} > -1$ and $\gamma > 0$ such that $g(v) \geq \frac{\hat{\kappa}}{v}$ for all $v \in (0, \gamma]$. Hence the limit in (B.8) can be computed as (where we may again assume that $0 < m < \gamma$):

$$\lim_{v_t\to 0} \int_m^{v_t} r e^{\int_m^v g(x)dx} dv = \lim_{v_t\to 0} - \int_{v_t}^m r e^{-\int_v^m g(x)dx} dv$$

[35] For $m = 0$, the solution candidate is not well defined for all κ because $e^{-\int_m^v g(x)dx} = \infty$.

$$
\begin{aligned}
&\geq \lim_{v_t \to 0} - \int_{v_t}^{m} r e^{-\hat{\kappa} \ln \frac{m}{v}} dv \\
&= \lim_{v_t \to 0} - \int_{v_t}^{m} r \left(\frac{v}{m}\right)^{\hat{\kappa}} dv \\
&= - r m^{-\hat{\kappa}} \frac{1}{\hat{\kappa}+1} \lim_{v_t \to 0} \left(m^{\hat{\kappa}+1} - v_t^{\hat{\kappa}+1}\right) \\
&> - \infty.
\end{aligned}
$$

Therefore, $y(v)$ is finite and $y(v) < 0$ for all v. Next we have to show that (B.10) can be integrated to obtain a feasible solution of the auxiliary problem. It suffices to verify that the following boundary condition from Lemma 4:

$$
\lim_{t \to \infty} v_t = 0, \tag{B.11}
$$

is satisfied. (This condition must hold for any solution as we show in the proof of Theorem 1.) Recall that $\dot{v}_t = y(v_t)$. Therefore, we have

$$
\dot{v}_t = -e^{-\int_m^{v_t} g(v)dv} \left(\int_0^{v_t} r e^{\int_m^v g(x)dx} dv \right).
$$

We first show that, for any $v_0^+ \in [0,1]$, the solution to this differential equation satisfies (B.11). Since the term in the parentheses is strictly positive we have

$$
\frac{e^{\int_m^{v_t} g(v)dv} \dot{v}_t}{\int_0^{v_t} e^{\int_m^v g(x)dx} dv} = -r.
$$

Integrating both sides we get

$$
\ln \int_0^{v_t} e^{\int_m^v g(x)dx} dv - \ln \int_0^{v_0} e^{\int_m^v g(x)dx} dv = -rt.
$$

Now take $t \to \infty$. The RHS diverges to $-\infty$ and the second term on the LHS is constant, so we must have

$$
\lim_{t \to \infty} \ln \int_0^{v_t} e^{\int_m^v g(x)dx} dv = -\infty
$$

which holds if and only if $\lim_{t \to \infty} v_t = 0$. Therefore, we have found a solution that satisfies the boundary condition and is decreasing for all starting values v_0^+. This concludes the proof for part (ii).

To prove part (iii), it suffices to rule out the possibility that other solutions may yield a higher value of the objective function. In light of (B.8), any decreasing solution must satisfy (B.7) with

$$
C = -\hat{C} - \int_0^m r e^{\int_m^w g(x)dx} dw,
$$

where $\hat{C} \geq 0$, because $\hat{C} < 0$ implies $y(v) > 0$ for v sufficiently small. Notice that if $\hat{C} = 0$,

$\dot{v}_t$ is given by (B.10):

$$\dot{v}_t = -e^{-\int_m^{v_t} g(v)dv} \int_0^{v_t} re^{\int_m^v g(x)dx} dv = -\int_0^{v_t} re^{-\int_v^{v_t} g(x)dx} dv.$$

Let y denote the solution for $\hat{C} = 0$ and z denote the solution for some $\hat{C} > 0$. If $\hat{C} > 0$, then we have for all $v \in (0, 1]$:

$$z(v) = y(v) - \hat{C}e^{-\int_m^v g(x)dx} < y(v).$$

Let v_t be the cutoff path for $\hat{C} = 0$ and w_t be the cutoff path for $\hat{C} > 0$. If we fix $v_0 = w_0$, then $z(v) < y(v)$ implies that for all $t > 0$, $w_t < v_t$. To see this, note that whenever $v_t = w_t \neq 0$, we have $\dot{w}_t = z(w_t) < y(v_t) = \dot{v}_t$. Hence, at every point where the two cutoff paths coincide, w_t must cross v_t from above. But since $w_0 = v_0$, this cannot happen (except at $t = 0$). As a result, w_t cannot be part of the optimal solution.[36] Therefore, the optimal solution is given by $\hat{C} = 0$, which means it satisfies (4.1). Uniqueness of the solution to (4.1) follows from the standard Lipschitz condition. □

B.7.5 Proof of Lemma 15

Proof. If the seller's revenue is weakly higher under $\hat{T}$, then

$$\int_0^1 \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right) \left(v - \frac{1 - F(v)}{f(v)}\right) nF(v)^{n-1} f(v)dv \geq 0.$$

Using the assumption that $v_t = \hat{v}_t$ for all $t \leq a$ and hence $T(v) = \hat{T}(v)$ for all $v > v_a$, we can rewrite this expression as

$$\int_0^{v_a} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right) \left(v - \frac{1 - F(v)}{f(v)}\right) nF(v)^{n-1} f(v)dv \geq 0. \tag{B.12}$$

Since both cutoff sequences have the same slack in the payoff floor constraint at a and $v_a = \hat{v}_a$, we have

$$\int_0^{v_a} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right) \left(v - \frac{F(v_a) - F(v)}{f(v)}\right) nF(v)^{n-1} f(v)dv = 0. \tag{B.13}$$

[36] If $J(v_0) < 0$, then the cutoff path v_t leads to later trading times for types with negative virtual valuation, hence the seller's expected profit is higher. Next suppose that $J(v_0) > 0$. Let x be defined by $J(x) = 0$. Let s_v be the time where $v_{s_v} = x$ and s_w be the time where $w_{s_w} = x$. Since $w_t < v_t$ for all t, we must have $s_w < s_v$. Now we construct a new feasible cutoff path that yields a higher expected profit than w. The idea is to take v_t and advance all trading times by $\Delta_s = s_v - s_w$. Formally, we define $\hat{w}_t = v_{t+\Delta_s}$. This implies that and $\dot{\hat{w}}_t = \dot{v}_{t+\Delta_s}$. By construction $\hat{w}_t = w_t$, $\hat{w}_t < w_t$ for $t < s_w$, and $\hat{w}_t > w_t$ for $t > s_w$. Hence, with the new cutoff path $\hat{w}_t$, all types with $J(v) > 0$ trade (weakly) earlier and all types with $J(v) < 0$ trade (strictly) later that with the old cutoff path w_t. Therefore the expected revenue of the seller is strictly higher.

Subtracting equation (B.13) from inequality (B.12) we obtain

$$\int_0^{v_a} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right)\left(\frac{F(v_a)-1}{f(v)}\right) nF(v)^{n-1}f(v)dv \geq 0,$$

which is equivalent to, for all $t < a$,

$$\int_0^{v_a} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right)\left(\frac{F(v_a)-F(v_t)}{f(v)}\right) nF(v)^{n-1}f(v)dv \geq 0.$$

Adding equality (B.13) to the above inequality, we get

$$\int_0^{v_a} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right)\left(v - \frac{F(v_t)-F(v)}{f(v)}\right) nF(v)^{n-1}f(v)dv \geq 0.$$

But this means that the slack in the payoff floor constraint at $t < a$ is greater under $\hat{T}$ than for T. Hence, the payoff floor constraint is fulfilled under $\hat{T}$ for all $t < a$. □

B.7.6 Proof of Lemma 16

Proof. Note that

$$\begin{aligned} J(s|v \leq x)f(s) &= \left(s - \frac{F(x)-F(s)}{f(s)}\right) f(s) \\ &= \left(s - \frac{1-F(s)}{f(s)}\right) f(s) + 1 - F(x) \\ &= J(s)f(s) + (1-F(x)). \end{aligned}$$

Hence, $J(s|v \leq x)f(s)$ is strictly increasing in s if and only if $J(s)f(s)$ is strictly increasing.

$$\frac{d}{ds}\left(J(s)f(s)\right) = \frac{d}{ds}\left(sf(s) - (1-F(s))\right) = -\frac{d^2}{ds^2}\left(s(1-F(s))\right).$$

□

B.7.7 Proof of Lemma 17

Proof. Let us first assume that the payoff floor constraint is strictly slack for all $t \in [a, b]$. Suppose by contradiction that $J_t(\tilde{v}) < 0$ for some $t \in (a, b]$ and some $\tilde{v} \in [v_b^+, v_a]$. Then $J_t(\tilde{v}) < 0$ for all $t \leq a$, since $J_t(\tilde{v})$ is non-decreasing in t. We claim that the following modification is feasible and improves revenue for $\delta > 0$ sufficiently small:

$$\hat{T}(v) = \begin{cases} T(v), & \text{if } v > \tilde{v} \\ T(v) + \delta, & \text{if } v \leq \tilde{v} \end{cases}.$$

With the new trading times, the payoff floor constraint at $t + \delta$ for $t \geq T(\tilde{v})$ is the same as the payoff floor constraint for t at the original trading times. For $t < a$ the RHS of the payoff floor constraint is unchanged and the LHS is increased because we delay trade of types that have a negative virtual valuation at $t \leq a$. For $a < t < T(\tilde{v}) + \delta$, the RHS of the payoff floor constraint for $\hat{T}(v)$ is equal to the RHS at $\min\{t, T(\tilde{v})\}$ for $T(v)$. To show that for $\hat{T}(v)$ the LHS is greater or equal than the RHS, we distinguish two cases. If the type $\tilde{v}$ is the only type that trades at $T(\tilde{v})$ then for δ sufficiently small, the payoff floor constraint is fulfilled because it was strictly slack for $a \leq t < T(\tilde{v})$ before the change and the LHS is continuous in T and hence in δ. If $\tilde{v}$ is part of an atom of types that all trade at the same time, the same argument applies to the payoff floor constraint at $t \in [a, T(\tilde{v})]$. After the modification, however, the posterior at times $t \in (T(\tilde{v}), T(\tilde{v}) + \delta)$ is the prior truncated to $[0, \tilde{v}]$. Before the change, this posterior did not arise on the equilibrium path. By Lemma 6 we have.

$$\int_0^{\tilde{v}} \left(e^{-r(T(v)-T(\tilde{v}))} - 1\right) \left(v - \frac{F(\tilde{v}) - F(v)}{f(v)}\right) f(v)\, n\, (F(v))^{n-1}\, dv > 0.$$

This implies that after the modification, the payoff floor constraint is strictly slack at $T(\tilde{v})+\delta$. For $t \in (T(\tilde{v}), T(\tilde{v}) + \delta)$, the RHS is the same as at $T(\tilde{v}) + \delta$ because there is not trade on that interval in the modified solution. By continuity, for δ sufficiently small it is also fulfilled for all $t \in (T(\tilde{v}), T(\tilde{v}) + \delta)$.

It remains to show the result for the case that the payoff floor constraint is binding at a and b but strictly slack on (a, b). In this case, we know that the result holds true for $t \in [a + \varepsilon, b - \varepsilon]$ for any $\varepsilon > 0$ and $v \in [v^+_{b-\varepsilon}, v_{a+\varepsilon}]$. Since $v - \frac{F(v_t)-F(v)}{f(v)}$ is continuous in v_t and v and there is no atom at a or b, respectively if the payoff floor constraint binds, $J_t(v)$ is continuous in (t, v) at the endpoints of the interval if the payoff floor constraint is binding. By continuity, the result for $[a + \varepsilon, b - \varepsilon]$ extends to the endpoints. □

B.7.8 Proof of Lemma 18

Proof. Suppose by contradiction that T is optimal. We consider a variation of the trading times on small interval $[s, s''] \subset (a, b)$ constructed as follows. First we choose $[s, s'']$ such that $v_{s''} < v_s$ and there is a positive measure of types with trading times in (s, s'') and such that there are no atoms of trade at s or s''. In words, we choose an interval of types that do not trade at the endpoints of $[s, s'']$ (they could all trade in an atom). Next we pick some type w with $s < T(w) < s''$ and define $\hat{T}$ such that all types in $[v_{s''}, w]$ trade at s'' and all types in $(w, v_s]$ trade at s. Formally the modification of the trading time can be written as

$$\hat{T}(v) = \begin{cases} T(v) & \text{if } v \geq v_s, \\ s & \text{if } v \in (w, v_s], \\ s'' & \text{if } v \in [v_{s''}, w], \\ T(v) & \text{if } v \leq v_{s''}. \end{cases}$$

We want so choose w such that the payoff floor constraint at time s remains unchanged, formally:

$$\int_{w}^{v_s} \left(e^{-r\,s} - e^{-r\,T(v)}\right) \left(J_s(v) f(v)\right)\, n(F(v))^{n-1} dv + \int_{v_{s''}}^{w} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) \left(J_s(v) f(v)\right)\, n(F(v))^{n-1} dv = 0. \tag{B.14}$$

Note that if $w = v_s$ the first integral vanishes so that the LHS is negative, and if $w = v_{s''}$, the second integral vanishes and the LHS is positive. Since the LHS is continuous in w, we can choose w such that (B.14) is satisfied. Note also that if we choose s and s'' sufficiently close together, then the payoff-floor constraint remains satisfied for all $t \in [s, s'']$ because it was strictly slack before the variation. Also the payoff-floor constraint for $t > s''$ is not affected by this change. Finally, if we can show that the ex-ante revenue increases, Lemma 15 implies that the payoff floor constraint is also satisfied for $t < a$. The ex-ante revenue is increased if

$$\int_{w}^{v_s} \left(e^{-r\,s} - e^{-r\,T(v)}\right) \left(J(v) f(v)\right)\, n(F(v))^{n-1} dv + \int_{v_{s''}}^{w} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) \left(J(v) f(v)\right)\, n(F(v))^{n-1} dv > 0.$$

By subtracting (B.14) from the above inequality, we get

$$\int_{w}^{v_s} \left(e^{-r\,s} - e^{-r\,T(v)}\right) \left(F(v_s) - 1\right)\, n(F(v))^{n-1} dv + \int_{v_{s''}}^{w} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) \left(F(v_s) - 1\right)\, n(F(v))^{n-1} dv > 0,$$

which is equivalent to

$$\int_{w}^{v_s} \left(e^{-r\,s} - e^{-r\,T(v)}\right) n(F(v))^{n-1} dv + \int_{v_{s''}}^{w} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) n(F(v))^{n-1} dv < 0.$$

Multiplying by $J_s(w) f(w)$ we have (by Lemma 17, $J_s(s'') \geq 0$ and hence $J_s(w) > 0$.)

$$\int_{w}^{v_s} \left(e^{-r\,s} - e^{-r\,T(v)}\right) J_s(w) f(w)\, n(F(v))^{n-1} dv + \int_{v_{s''}}^{w} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) J_s(w) f(w)\, n(F(v))^{n-1} dv < 0.$$

By Lemma 16, we have that $J_s(v) f(v)$ is strictly increasing. This implies

$$\begin{aligned}
&\int_{w}^{v_s} \underbrace{\left(e^{-r\,s} - e^{-r\,T(v)}\right)}_{>0} J_s(w) f(w)\, n(F(v))^{n-1} dv + \int_{v_{s''}}^{w} \underbrace{\left(e^{-r\,s''} - e^{-r\,T(v)}\right)}_{<0} J_s(w) f(w)\, n(F(v))^{n-1} dv \\
&< \int_{w}^{v_s} \left(e^{-r\,s} - e^{-r\,T(v)}\right) J_s(v) f(v)\, n(F(v))^{n-1} dv + \int_{v_{s''}}^{w} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) J_s(v) f(v)\, n(F(v))^{n-1} dv \\
&= 0,
\end{aligned}$$

where the last equality follows from (B.14). □

B.7.9 Proof of Lemma 19

Proof. The logic of the proof is similar to proof of Lemma 18. Again, suppose by contradiction that T is optimal. We construct a variation by splitting the atom at some type $w \in (v_b^+, v_b)$. First, we let types $[w, v_b]$ trade at $s < b$. Second, we want to delay the trading times for types $[v_b^+, w)$ to $s'' > b$, where $v_{s''} \geq v_b^+ - \varepsilon$. In order to maintain monotonicity we also have to delay the trading time of all types $v \in [v_{s''}, v_b^+)$. To summarize we have:

$$\hat{T}(v) = \begin{cases} T(v) & \text{if } v > v_b, \\ s & \text{if } v \in [w, v_b], \\ s'' & \text{if } v \in (v_{s''}, w), \\ T(v) & \text{if } v \leq v_{s''}. \end{cases}$$

We choose w, s, and s'' such that the payoff floor constraint at a is unchanged:

$$\begin{aligned} &\int_w^{v_b} \left(e^{-r\,s} - e^{-r\,b}\right) (J_a(v) f(v))\; n(F(v))^{n-1} dv \\ &+ \int_{v_b^+}^{w} \left(e^{-r\,s''} - e^{-r\,b}\right) (J_a(v) f(v))\; n(F(v))^{n-1} dv \\ &+ \int_{v_{s''}}^{v_b^+} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) (J_a(v) f(v))\; n(F(v))^{n-1} dv = 0. \end{aligned} \tag{B.15}$$

We argue that it is feasible to choose such w, s, and s''. First note that, if we set $s'' = b$, $v_{s''} = v_b^+$, and $s < b$, the left hand side of the equality is strictly positive since $J_a(w) \geq 0$ by Lemma 17. Next, we show that for $s = b$ we can choose $s'' > b$ such that the left hand side of the expression is strictly negative. If $J_a(v_b^+) f(v_b^+) > 0$, we can choose s'' such that $J_a(v_{s''}) f(v_{s''}) \geq 0$. In this case the last two integrals are strictly negative. If $J_a(v_b^+) f(v_b^+) = 0$ ("$<$" is ruled out by Lemma 17) then $J_a(v) f(v) < 0$ for $v < v_b^+$, and we have

$$\begin{aligned} & \int_{v_b^+}^{w} \left(e^{-r\,s''} - e^{-r\,b}\right) (J_a(v) f(v))\; n(F(v))^{n-1} dv \\ & + \int_{v_{s''}}^{v_b^+} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) (J_a(v) f(v))\; n(F(v))^{n-1} dv \\ \leq\; & \int_{v_b^+}^{w} \left(e^{-r\,s''} - e^{-r\,b}\right) (J_a(v) f(v))\; n(F(v))^{n-1} dv \\ & + \int_{v_{s''}}^{v_b^+} \left(e^{-r\,s''} - e^{-r\,b}\right) (J_a(v) f(v))\; n(F(v))^{n-1} dv \\ =\; & \left(e^{-r\,s''} - e^{-r\,b}\right) \left[\int_{v_b^+}^{w} (J_a(v) f(v))\; n(F(v))^{n-1} dv + \int_{v_{s''}}^{v_b^+} (J_a(v) f(v))\; n(F(v))^{n-1} dv \right] \\ \leq\; & \left(e^{-r\,s''} - e^{-r\,b}\right) \left[\int_{v_b^+}^{w} (J_a(v) f(v))\; n(F(v))^{n-1} dv + (J_a(v_{s''}) f(v_{s''})) \int_{v_{s''}}^{v_b^+} n(F(v))^{n-1} dv \right]. \end{aligned}$$

We want to show that for some s''

$$\int_{v_b^+}^{w} (J_a(v)f(v))\; n(F(v))^{n-1}dv + (J_a(v_{s''})f(v_{s''})) \int_{v_{s''}}^{v_b^+} n(F(v))^{n-1}dv > 0.$$

Note that $(J_a(v_{s''})f(v_{s''})) \int_{v_{s''}}^{v_b^+} n(F(v))^{n-1}dv$ is continuous in $v_{s''}$. Hence, there is a $v_{s''} < v_b^+$ such that

$$\int_{v_b^+}^{w} (J_a(v)f(v))\; n(F(v))^{n-1}dv + (J_a(v_{s''})f(v_{s''})) \int_{v_{s''}}^{v_b^+} n(F(v))^{n-1}dv > 0.$$

Moreover, for every $v_{s''} < v_b$ there is an $\hat{s}$ with $v_{\hat{s}} \in (v_{s''}, v_b)$ such that there is no atom at $\hat{s}$. Hence we can take $v_{s''}$ be a type that is not part of an atom. To summarize, we have shown that for some (s, b) the payoff floor constraints at a decreases and for some (b, s'') it increases. We can select s'' such that the last two integrals in (B.15) become arbitrary small. Since the first integral is continuous in s we can find a value for s such that the whole expression is equal to zero. This proves that our construction is possible.

If the payoff-floor constraint binds at a it must be slack for all $t \in (a, s]$ since there is no trade in this interval. Next we argue that the variation does not violate the payoff floor constraint for $t > s$. If we choose both s and s'' sufficiently close to b, then the payoff-floor constraint remains satisfied for all $t \in (s, s'']$ because for every $v_b^+ < w < v_b$ Lemma 6 implies

$$\int_0^w \left(e^{-r(T(v)-b)} - 1\right) \left(v - \frac{F(w) - F(v)}{f(v)}\right) f(v)\, n\, (F(v))^{n-1}\, dv > 0.$$

Also the payoff-floor constraint for $t > s''$ is not affected by this change. Finally, if we can show that the ex-ante revenue increases, Lemma 15 implies that the payoff floor constraint is also satisfied for $t < a$.

Ex-ante revenue increases if

$$\begin{aligned}
&\left(e^{-r\,s} - e^{-r\,b}\right) \int_w^{v_b} \left(v - \frac{1 - F(v)}{f(v)}\right) f(v)\, n(F(v))^{n-1}dv \\
&+ \left(e^{-r\,s''} - e^{-r\,b}\right) \int_{v_b^+}^{w} \left(v - \frac{1 - F(v)}{f(v)}\right) f(v)\, n(F(v))^{n-1}dv \\
&+ \int_{v_{s''}}^{v_b^+} \left(e^{-r\,s''} - e^{-r\,T(v)}\right) \left(v - \frac{1 - F(v)}{f(v)}\right) f(v)\, n(F(v))^{n-1}dv > 0.
\end{aligned}$$

By subtracting the condition (B.15) from the above inequality, we obtain:

$$\begin{aligned}
&\left(e^{-r\,s} - e^{-r\,b}\right) \int_w^{v_b} \left(\frac{F(w) - 1}{f(v)}\right) f(v)\, n(F(v))^{n-1}dv \\
&+ \left(e^{-r\,s''} - e^{-r\,b}\right) \int_{v_b^+}^{w} \left(\frac{F(w) - 1}{f(v)}\right) f(v)\, n(F(v))^{n-1}dv
\end{aligned}$$

$$+\int_{v_{s''}}^{v_b^+}\left(e^{-r\,s''}-e^{-r\,T(v)}\right)\left(\frac{F(w)-1}{f(v)}\right)f(v)\,n(F(v))^{n-1}dv>0.$$

This can be rearranged to

$$\left(e^{-r\,s}-e^{-r\,b}\right)\int_{w}^{v_b} n(F(v))^{n-1}dv+\left(e^{-r\,s''}-e^{-r\,b}\right)\int_{v_b^+}^{w} n(F(v))^{n-1}dv$$
$$+\int_{v_{s''}}^{v_b^+}\left(e^{-r\,s''}-e^{-r\,T(v)}\right)n(F(v))^{n-1}dv<0.$$

If we multiply the LHS by $J_a(w)f(w)$, we get

$$\begin{aligned}
&\left(e^{-r\,s}-e^{-r\,b}\right)\int_{w}^{v_b} J_a(w)f(w)\,n(F(v))^{n-1}dv\\
&+\left(e^{-r\,s''}-e^{-r\,b}\right)\int_{v_b^+}^{w} J_a(w)f(w)\,n(F(v))^{n-1}dv\\
&+\int_{v_{s''}}^{v_b^+}\left(e^{-r\,s''}-e^{-r\,T(v)}\right)J_a(w)f(w)\,n(F(v))^{n-1}dv\\
<\;&\left(e^{-r\,s}-e^{-r\,b}\right)\int_{w}^{v_b} J_a(v)f(v)\,n(F(v))^{n-1}dv\\
&+\left(e^{-r\,s''}-e^{-r\,b}\right)\int_{v_b^+}^{w} J_a(v)f(v)\,n(F(v))^{n-1}dv\\
&+\int_{v_{s''}}^{v_b^+}\left(e^{-r\,s''}-e^{-r\,T(v)}\right)J_a(v)f(v)\,n(F(v))^{n-1}dv=0.
\end{aligned}$$

where the last equality is the condition for the unchanged payoff floor constraint at a. □

C Existence and Uniform Coase Conjecture

In this section, we follow the approach of Ausubel and Deneckere (1989) to prove the existence of stationary (weak-Markov) equilibria and establish the uniform Coase conjecture (Proposition 2). Weak-Markov equilibria are defined as follows:

Definition 1. An equilibrium $(p, b) \in \mathcal{E}(\Delta)$ is a *weak-Markov (or stationary) equilibrium* if the buyers' strategies only depend on the reserve price announced for the current period.

We adopt Ausubel and Deneckere (1989)'s notation and assume that the types of the bidders are i.i.d. draws from $U[0, 1]$. We denote the type of buyer i by q^i. The valuation for each type is given by the function $v(q) := F^{-1}(q)$. Assumption 3 implies that the same condition also holds for $v(q)$ and corresponds to the assumption made in Definition 5.1 in Ausubel and Deneckere (1989). In the following we will use that F is continuous and strictly increasing (as in Ausubel and Deneckere (1989) we could relax this even further to general distribution functions but this is not necessary for the purpose of the present paper).[37] Since the proof of Proposition 2 follows closely the approach of Ausubel and Deneckere (1989), we only state proofs for the parts of the proof of Ausubel and Deneckere (1989) that need to be modified for the case of $n \geq 2$.

C.1 Proof of Proposition 2.(i)

In a weak-Markov equilibrium, the buyers' strategy can be described by a function $P : [0, 1] \to [0, 1]$. A bidder with type q^i places a valid bid if and only if the announced reserve price is smaller than $P(q^i)$. Given that v is strictly increasing, Lemma 1 implies that P is non-decreasing.

Also by Lemma 1, the posterior of the seller at any history is described by the supremum of the support, which we denote by q. If all buyers play according to P, the seller's (unconditional) continuation profit for given q is[38]

$$R(q) := \max_{y \in [0,q]} \int_y^q v(z) d \left[n z^{n-1} - (n-1) z^n \right] + P(y)\, n\, (q - y) y^{n-1} + e^{-r\Delta} R(y) \qquad \text{(C.1)}$$

Let $Y(q)$ be the argmax correspondence and define $y(q) := \sup Y(q)$. Because the objective satisfies a single-crossing property, $Y(q)$ is increasing and hence single-valued almost everywhere. If $Y(q)$ is single-valued at q the seller announces a reserve price $S(q) = P(y(q))$ if the posterior has upper bound q.

The buyers' indifference condition for the case that $Y(q)$ is single-valued so that the seller does not randomize, is given by:

$$v(q) - P(q) = e^{-r\Delta} \left[v(q) - \frac{(y(q))^{n-1}}{q^{n-1}} S(q) - \frac{1}{q^{n-1}} \int_{y(q)}^q v(x) dx^{n-1} \right]. \qquad \text{(C.2)}$$

[37] In Ausubel and Deneckere (1989) the valuation is decreasing in the type. We define v to be increasing so that higher types have higher valuations.

[38] Dividing the RHS by q^n and replacing $R(y)$ by $y^n R(y)$ would yield the conditional continuation profit. The unconditional version is more convenient for the subsequent development.

If the seller randomizes over $Y(q)$ according to some probability measure μ, then

$$v(q) - P(q) = e^{-r\Delta}\left[v(q) - \int_{Y(q)}\left\{\frac{y^{n-1}}{q^{n-1}}P(y) + \frac{1}{q^{n-1}}\int_y^q v(x)dx^{n-1}\right\}d\mu(y)\right], \tag{C.3}$$

which may require that μ depends on $P(q)$.[39]

We will be looking for left-continuous functions R and P such that (C.1) and (C.2) are satisfied.[40] If this is true for all $q \in [0, \bar{q}]$, then we say that (P, R) support a weak-Markov equilibrium on $[0, \bar{q}]$. The goal is to show the existence of a pair (P, R) that supports a weak-Markov equilibrium on $[0, 1]$. As in Ausubel and Deneckere (1989), we can show that the seller's continuation profit is Lipschitz-continuous in q.

Lemma 20 (cf. Lemma A.2 in Ausubel and Deneckere (1989))**.** *If (P, R) supports a weak-Markov equilibrium on $[0, \bar{q}]$, then R is increasing and Lipschitz continuous satisfying*

$$0 < R(q_1) - R(q_2) \leq n(q_1 - q_2)$$

for all $0 \leq q_2 < q_1 \leq \bar{q}$.

Proof. First, we show monotonicity:

$$\begin{aligned}
R(q_1) &= \int_{y(q_1)}^{q_1} v(z)d\left[nz^{n-1} - (n-1)z^n\right] + P(y(q_1))\, n\,(q_1 - y(q_1))(y(q_1))^{n-1} + e^{-r\Delta}R(y(q_1)) \\
&\geq \int_{y(q_2)}^{q_1} v(z)d\left[nz^{n-1} - (n-1)z^n\right] + P(y(q_2))\, n\,(q_1 - y(q_2))(y(q_2))^{n-1} + e^{-r\Delta}R(y(q_2)) \\
&> \int_{y(q_2)}^{q_2} v(z)d\left[nz^{n-1} - (n-1)z^n\right] + P(y(q_2))\, n\,(q_2 - y(q_2))(y(q_2))^{n-1} + e^{-r\Delta}R(y(q_2)) \\
&= R(q_2)
\end{aligned}$$

To show Lipschitz continuity, notice that the revenue from sales to types below q_2 in the continuation starting from q_1 is at most $R(q_2)$ and the revenue from types between q_2 and q_1 is bounded above by $P(q_1)(q_1^n - q_2^n)$.[41] Hence

$$\begin{aligned}
R(q_1) - R(q_2) &\leq P(q_1)(q_1^n - q_2^n) \\
&\leq (q_1^n - q_2^n) \\
&\leq n(q_1 - q_2)
\end{aligned}$$

□

[39] In the following, we give details for the case that the seller does not randomize and refer to Ausubel and Deneckere (1989) for the discussion of randomization by the seller.

[40] Left-continuity will be used in the proof of Proposition 6 in the next section.

[41] Suppose by contradiction that for the posterior $[0, q_1]$, the expected payment that the seller can extract from some type $q \in [q_2, q_1]$ is greater or equal than $P(q_1)$. In order to arrive at a history where the posterior is $[0, q_1]$, the seller must have used reserve price $P(q_1)$ in the previous period. But then all types in $[q, q_1]$ would prefer to bid in the previous period because they expect to make higher payments if they wait. This is a contradiction.

Using this Lemma, we can show that an existence result for $[0,\bar{q}]$ can be extended to the whole interval $[0,1]$.

Lemma 21 (cf. Lemma A.3 in Ausubel and Deneckere (1989))**.** *Suppose* $(P_{\bar{q}}, R_{\bar{q}})$ *supports a weak-Markov equilibrium on* $[0,\bar{q}]$*, then there exists* (P,R) *which supports a weak-Markov equilibrium on* $[0,1]$*.*

Proof. We extend $(R_{\bar{q}}, P_{\bar{q}})$ to some $[0,\bar{q}']$. Define

$$R_{\bar{q}'}(q) = \max_{0\le y\le \min\{\bar{q},q\}} \int_y^q v(z)d\left[nz^{n-1}-(n-1)z^n\right] + P_{\bar{q}}(y)\,n\,(q-y)y^{n-1} + e^{-r\Delta}R_{\bar{q}}(y)$$

with $y_{\bar{q}'}(q)$ as the supremum of the argmax correspondence. Moreover, we define $P_{\bar{q}'}(q)$ by

$$v(q) - P_{\bar{q}'}(q) = e^{-r\Delta}\left[v(q) - \frac{(y_{\bar{q}'}(q))^{n-1}}{q^{n-1}}P_{\bar{q}}(y_{\bar{q}'}(q)) - \frac{1}{q^{n-1}}\int_{y_{\bar{q}'}(q)}^{q} v(x)dx^{n-1}\right].$$

For $\bar{q}' = \min\left\{1, \sqrt[n]{\bar{q}^n + (1-e^{-r\Delta})R_{\bar{q}}(\bar{q})}\right\}$, the constraint in the maximization in the definition of $R_{\bar{q}'}(q)$ is not binding and moreover

$$R_{\bar{q}'}(q) = \max_{0\le y\le q} \int_y^q v(z)d\left[nz^{n-1}-(n-1)z^n\right] + P_{\bar{q}'}(q)\,n\,(q-y)y^{n-1} + e^{-r\Delta}R_{\bar{q}'}(y)$$

For $y\in[\bar{q},q]$ we have

$$\begin{aligned}
&\int_y^q v(z)d\left[nz^{n-1}-(n-1)z^n\right] + P_{\bar{q}'}(q)\,n\,(q-y)y^{n-1} + e^{-r\Delta}R_{\bar{q}'}(y)\\
\le& q^n - y^n + e^{-r\Delta}R_{\bar{q}'}(q)\\
\le&(1-e^{-r\Delta})R_{\bar{q}}(\bar{q}) + e^{-r\Delta}R_{\bar{q}'}(q)\\
\le&(1-e^{-r\Delta})R_{\bar{q}'}(q) + e^{-r\Delta}R_{\bar{q}'}(q)\\
\le& R_{\bar{q}'}(q).
\end{aligned}$$

In the first step, we have used that the payments $v(z)$ and $P_{\bar{q}'}(q)$ are less than or equal to one. In the second step, we have used that $\bar{q}' = \min\left\{1, \sqrt[n]{\bar{q}^n + (1-e^{-r\Delta})R_{\bar{q}}(\bar{q})}\right\}$; since $\bar{q}\le y\le q\le \bar{q}'$, this implies $q^n - y^n \le (1-e^{-r\Delta})R_{\bar{q}}(\bar{q})$. The third step uses $R_{\bar{q}}(\bar{q}) = R_{\bar{q}'}(\bar{q})$ and that $R_{\bar{q}'}$ is increasing. Thus $(P_{\bar{q}'}, R_{\bar{q}'})$ supports a weak-Markov equilibrium on $[0,\bar{q}']$. Since $R_{\bar{q}}(\bar{q}) > 0$, a finite number of repetitions suffices to extend $(P_{\bar{q}}, R_{\bar{q}})$ to the entire interval $[0,1]$. □

To complete the proof, we follow Ausubel and Deneckere (1989) by replacing the lower tail distribution on the interval $[0,\bar{q}]$ by a uniform distribution. For the uniform distribution, a weak-Markov equilibrium can be constructed explicitly. In the auction case, this has been shown by McAfee and Vincent (1997). Therefore, Lemma 21 implies that for the modified distribution with a uniform part at the lower end, a weak-Markov equilibrium exists. The final step is to show that the functions (P,R) that support the equilibrium for the modified distribution converge to functions that support a weak-Markov equilibrium for the original distribution as $\bar{q} \to 1$.

Proof of Proposition 2.(i). As in Ausubel and Deneckere (1989), we consider a sequence of valuation functions

$$v_\eta(q) = \begin{cases} v(q), & \text{if } q \geq \frac{1}{\eta} \\ v\left(\frac{1}{\eta}\right) \eta q, & \text{otherwise.} \end{cases}$$

This corresponds to the original distribution except that on the interval $[0, 1/\eta]$, we have made the distribution uniform. McAfee and Vincent (1997) show that there exist $(\tilde{P}_{1/\eta}, \tilde{R}_{1/\eta})$ that support a weak-Markov equilibrium on $[0, 1/\eta]$. Hence, by Lemma 21, for each $\eta = 1, 2, \ldots$, there exists a pair (P_η, R_η) that supports a weak-Markov equilibrium on $[0, 1]$. As in Ausubel and Deneckere (1989), we can assume that P_η converges point-wise for all rationals to some function $\Phi(s)$, $s \in \mathbb{Q} \cap [0, 1]$ and taking left limits we can extend this limit to a non-decreasing, left-continuous function $P : [0, 1] \to [0, 1]$. Also, by Lemma 20, after taking a sub-sequence, we may assume that (R_n) converges uniformly to a continuous function R. We have to show that (P, R) supports a weak-Markov equilibrium for v. But given Lemma 20 and 21, only minor modifications are needed to apply the proof of Theorem 4.2 from Ausubel and Deneckere (1989). □

C.2 Proof of Proposition 2.(ii)

Before we begin with the proof, we note that in contrast to the case of one buyer analyzed by Ausubel and Deneckere (1989), the first reserve price in a continuation game where the seller's posterior is v_t need not converge to zero as $\Delta \to 0$.[42] Nevertheless, we obtain the Coase conjecture because prices fall arbitrarily quickly as $\Delta \to 0$. On the buyer side, the strategy is described by a cutoff for the reserve price. A buyer places a bid if and only if the current reserve price is below the cutoff. The Markov property of the buyer's strategy implies that the cutoff only depends on the buyer's type, it is independent of time and of the history of previous reserve prices. As $\Delta \to 0$, the equilibrium cutoff of a buyer with type v converges to the payment that this type would make in a second-price auction without reserve price. Also reserve prices decline arbitrarily quickly so that the delay of the allocation vanishes for all buyers as $\Delta \to 0$. Therefore, the seller's profit converges to the profit of an efficient auction.

We want to show that the profit of the seller in any weak-Markov equilibrium of a subgame that starts with the posterior $[0, q]$, converges (uniformly over q) to $\Pi^E(q)$ as $\Delta \to 0$. The proof consists of two main steps. The first step shows that for any type $\xi \in [0, 1]$, any $\Delta > 0$, and any weak-Markov equilibrium supported by some pair (P, R), the expected payment that the seller can extract from type ξ is bounded by $\xi^{n-1} P(\xi)$. We prove this by showing that the expected payment conditional on winning is bounded by $P(\xi)$.

Lemma 22. *Let (P, R) support a weak-Markov equilibrium in the game for $\Delta > 0$. Suppose that in this equilibrium, type $\xi \in [0, 1]$ trades in period t, let the posterior in period t be*

[42] For the uniform distribution, this was already noted by McAfee and Vincent (1997).

$q_t \geq \xi$, and denote the marginal type in period t by $q_t^+ \leq \xi$. Then we have

$$P(\xi) \geq \int_{q_t^+}^{\xi} v(x) \frac{dx^{n-1}}{\xi^{n-1}} + \frac{\left(q_t^+\right)^{n-1}}{\xi^{n-1}} P(q_t^+), \qquad \forall \xi \in [0,1],$$

and hence

$$R(q) \leq \int_0^q P(x)\, dx^n, \qquad \forall q \in (0,1].$$

Proof. For $q_t^+ = \xi$ the RHS of the first inequality becomes $P(q_t^+) = P(\xi)$. Hence it suffices to show that

$$\int_q^{\xi} v(x) dx^{n-1} + q^{n-1} P(q)$$

is increasing in q. For $q > \hat{q}$ we have

$$\begin{aligned}
&\int_q^{\xi} v(x) dx^{n-1} + q^{n-1} P(q) - \int_{\hat{q}}^{\xi} v(x) dx^{n-1} - \hat{q}^{n-1} P(\hat{q}) \\
=& q^{n-1} P(q) - \hat{q}^{n-1} P(\hat{q}) - \int_{\hat{q}}^{q} v(x) dx^{n-1}
\end{aligned}$$

Using (C.2), we have

$$\begin{aligned}
& q^{n-1} P(q) - \hat{q}^{n-1} P(\hat{q}) \\
=& \left(1 - e^{-r\Delta}\right) q^{n-1} v(q) + e^{-r\Delta} \int_{y(q)}^{q} v(x) dx^{n-1} + e^{-r\Delta} \left(y(q)\right)^{n-1} P(y(q)) \\
& - \left(1 - e^{-r\Delta}\right) \hat{q}^{n-1} v(\hat{q}) - e^{-r\Delta} \int_{y(\hat{q})}^{\hat{q}} v(x) dx^{n-1} - e^{-r\Delta} \left(y(\hat{q})\right)^{n-1} P(y(\hat{q})) \\
=& \left(1 - e^{-r\Delta}\right) \left(q^{n-1} v(q) - \hat{q}^{n-1} v(\hat{q})\right) + e^{-r\Delta} \left(\left(y(q)\right)^{n-1} P(y(q)) - \left(y(\hat{q})\right)^{n-1} P(y(\hat{q}))\right) \\
& + e^{-r\Delta} \int_{\hat{q}}^{q} v(x) dx^{n-1} - e^{-r\Delta} \int_{y(\hat{q})}^{y(q)} v(x) dx^{n-1}
\end{aligned}$$

and hence

$$\begin{aligned}
& q^{n-1} P(q) - \hat{q}^{n-1} P(\hat{q}) - \int_{\hat{q}}^{q} v(x) dx^{n-1} \\
=& \left(1 - e^{-r\Delta}\right) \left(q^{n-1} v(q) - \hat{q}^{n-1} v(\hat{q})\right) + e^{-r\Delta} \left(\left(y(q)\right)^{n-1} P(y(q)) - \left(y(\hat{q})\right)^{n-1} P(y(\hat{q}))\right) \\
& - \left(1 - e^{-r\Delta}\right) \int_{\hat{q}}^{q} v(x) dx^{n-1} - e^{-r\Delta} \int_{y(\hat{q})}^{y(q)} v(x) dx^{n-1} \\
=& e^{-r\Delta} \left(\left(y(q)\right)^{n-1} P(y(q)) - \left(y(\hat{q})\right)^{n-1} P(y(\hat{q})) - \int_{y(\hat{q})}^{y(q)} v(x) dx^{n-1} \right) \\
& + \left(1 - e^{-r\Delta}\right) \int_{\hat{q}}^{q} v'(x) x^{n-1} dx
\end{aligned}$$

Proceeding inductively, we get

$$q^{n-1}P(q) - \hat{q}^{n-1}P(\hat{q}) - \int_{\hat{q}}^{q} v(x)dx^{n-1} = \sum_{k=0}^{\infty} e^{-k\Delta}\left(1 - e^{-r\Delta}\right)\int_{y^k(\hat{q})}^{y^k(q)} v'(x)x^{n-1}dx > 0,$$

where $y^k(\cdot)$ denotes the function obtained by applying $y(\cdot)$ k times. This shows the first inequality.

For the second inequality, notice that the RHS of the first inequality is the payment that the seller can extract from type ξ if ξ wins the auction. This is bounded by $P(\xi)$ as the first inequality shows. The seller's profit if the posterior at time t is q, therefore satisfies

$$R(q) \leq \int_0^q e^{-r(T(x)-t)}P(x)dx^n,$$

where $T(x)$ denotes the trading time of type x in the weak-Markov equilibrium. This implies the second inequality. □

For the second step, fix the distribution and the corresponding function v and define $v_x : [0,1] \to [0,1]$ for all $x \in (0,1]$.

$$v_x(q) := \frac{v(qx)}{v(x)}.$$

Using Helly's selection theorem, we can extend this definition to $x = 0$, by taking the a.e.-limit of a subsequence of functions v_x. Denote by $\mathcal{E}^{wM}(\Delta, x)$ the weak-Markov equilibria of the game with discount factor Δ and distribution given by v_x where $x \to 0$. Slightly abusing notation we write $(P,R) \in \mathcal{E}^{wM}(\Delta, x)$ for a weak-Markov equilibrium that is supported by functions (P,R). We show that there is an upper bound for $P(1)$ that converges to the expected payment in a second price auction without reserve price as $\Delta \to 0$, and the convergence is uniform over x.

Lemma 23. *Fix $v(\cdot)$. For all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in [0,1]$, and all $(P,R) \in \mathcal{E}^{wM}(\Delta, x)$,*

$$P(1) \leq \int_0^1 v_x(s)\, ds^{n-1} + \varepsilon.$$

Proof. Suppose not. Then there exist sequences $\Delta_m \to 0$ and $x_m \to \bar{x}$ such that for all $m \in \mathbb{N}$, there exist equilibria $(P_m, R_m) \in \mathcal{E}^{wM}(\Delta_m, x_m)$ such that for all m,

$$P_m(1) > \int_0^1 v_{x_m}(s)\, ds^{n-1} + \varepsilon.$$

By a similar argument as in the proof of Theorem 4.2 of Ausubel and Deneckere (1989), we can construct a limiting pair $(\overline{P}, \overline{R})$, where $\overline{P}$ is left-continuous and non-decreasing, P_m converges point-wise to $\overline{P}$ for all rationals, and R_m converges uniformly to $\overline{R}$. Obviously, we have

$$\overline{P}(1) \geq \int_0^1 v_{\bar{x}}(s)\, ds^{n-1} + \varepsilon.$$

Left-continuity implies that there exists $\bar{q} < 1$ such that

$$\overline{P}(\bar{q}) \geq \int_0^1 v_{\bar{x}}(s)\, ds^{n-1} + \frac{\varepsilon}{2}. \tag{C.4}$$

Using an argument from the proof of Theorem 5.4 in Ausubel and Deneckere (1989), we can show that

$$\overline{R}(1) \geq \int_{\bar{q}}^1 \overline{P}(s)\, ds^n + \Pi^E(\bar{q}) \geq \Pi^E(1) + (1-\bar{q})\frac{\varepsilon}{2},$$

where we have used (C.4) to show the second inequality. Hence, we have

$$R_m(1) \to \overline{R}(1) \geq \Pi^E(1) + (1-\bar{q})\frac{\varepsilon}{2}. \tag{C.5}$$

But this implies that there must exist a type $\hat{q} > 0$, a time $t > 0$, and $\bar{m}$ such that for all $m > \bar{m}$,

$$T_m(\hat{q}) \geq t.$$

where $T_m(\cdot)$ is the trading time function in the weak-Markov equilibrium supported by (P_m, R_m). To see this, note that delay for low types is needed to increase the seller's revenue beyond the revenue from an efficient auction.

With this observation, we can conclude the proof using a similar argument as in Case I of the proof of Theorem 5.4 in Ausubel and Deneckere (1989). From Lemma 22 we know that the maximal expected payment conditional on winning that a buyer of type q has to make in equilibrium is given by $P_m(q)$. This implies that

$$R_m(1) \leq \int_{\hat{q}}^1 P_m(z)dz^n + e^{-rt}R_m(\hat{q}).$$

In the limit we have

$$\overline{R}(1) \leq \int_{\hat{q}}^1 \overline{P}(z)dz^n + e^{-rt}\overline{R}(\hat{q}). \tag{C.6}$$

On the other hand, the same argument that we used to obtain (C.5) yields

$$\overline{R}(1) \geq \int_0^1 \overline{P}(z)dz^n. \tag{C.7}$$

Combining (C.6) and (C.7) we get

$$\int_0^{\hat{q}} \overline{P}(z)dz^n \leq e^{-rt}\overline{R}(\hat{q}),$$

which implies

$$\overline{R}(\hat{q}) > \int_0^{\hat{q}} \overline{P}(z)dz^n,$$

since $t > 0$. But Lemma 22 implies the opposite inequality which is a contradiction. □

Using this Lemma, we can show that for a given $v(\cdot)$, the difference between the continuation profit at $[0, q]$ and $\Pi^E(q)$, divided by $v(q)$ converges uniformly to zero.

Lemma 24. *Fix $v(\cdot)$. For all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in [0, 1]$, and all $(P, R) \in \mathcal{E}^{wM}(\Delta, 1)$,*

$$\frac{R(x)}{x^n} - \Pi^E(v(x)) \leq \varepsilon v(x).$$

Proof. The statement of the Lemma is equivalent to the statement that for all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in [0, 1]$, and all $(P, R) \in \mathcal{E}^{wM}(\Delta, x)$,

$$R(1|v_x) - \Pi^E(1|v_x) \leq \varepsilon. \tag{C.8}$$

This equivalence holds because truncating and rescaling the function $v(\cdot)$ leads to the following transformations:

$$\frac{R(x|v)}{x^n} = v(x)R(1|v_x),$$
$$\Pi^E(v(x)) = v(x)\Pi^E(1|v_x).$$

To show (C.8), we combine Lemmas 22 and 23, and use that $P(z|v_x) = v_x(z)P(1|v_{z\cdot x})$ to get for all $x \in [0, 1]$,

$$\begin{aligned}
R(1) &\leq \int_0^1 P(z|v_x)dz^n \\
&= \int_0^1 v_x(z)P(1|v_{x\cdot z})dz^n \\
&\leq \int_0^1 v_x(z)\left(\int_0^1 v_{x\cdot z}(s)ds^{n-1} + \varepsilon\right)dz^n \\
&= \int_0^1 \left(\int_0^1 v_x(sz)ds^{n-1}\right)dz^n + \varepsilon\int_0^1 v_x(z)dz^n \\
&\leq \int_0^1 \left(\int_0^z v_x(s)\frac{ds^{n-1}}{z^{n-1}}\right)dz^n + \varepsilon \\
&= \Pi^E(1|v_x) + \varepsilon
\end{aligned}$$

□

This allows us to complete the proof of Proposition 2.(ii).

Proof of Proposition 2.(ii). Translated into the notation of the main paper, Lemma 24 implies that for a given distribution function F, for all $\tilde{\varepsilon} > 0$, there exists $\Delta_{\tilde{\varepsilon}} > 0$ such that for all $\Delta \leq \Delta_{\tilde{\varepsilon}}$, all $v \in [0,1]$, and all weak-Markov equilibria $(p,b) \in \mathcal{E}^{wM}(\Delta)$, we have

$$\Pi^{\Delta}(p,b|v) \leq \Pi^{E}(v) + \tilde{\varepsilon}v.$$

As in the proof of Lemma 26, we can show that under Assumption 3, there exits a constant $B > 0$ such that $\Pi^{E}(v) \geq Bv$ for all $v \in [0,1]$. If we chose $\tilde{\varepsilon}$ sufficiently small we have

$$\begin{aligned} & \tilde{\varepsilon} \leq B\varepsilon, \\ \iff \quad & \tilde{\varepsilon}v \leq B\varepsilon v, \\ \iff \quad & \tilde{\varepsilon}v \leq \varepsilon\Pi^{E}(v), \\ \iff \quad & \Pi^{E}(v) + \tilde{\varepsilon}v \leq (1+\varepsilon)\Pi^{E}(v). \end{aligned}$$

This implies that

$$\Pi^{\Delta}(p,b|v) \leq (1+\varepsilon)\Pi^{E}(v)$$

for all $\Delta \leq \Delta_{\varepsilon} := \Delta_{\tilde{\varepsilon}}$ for $\tilde{\varepsilon}$ sufficiently small. □

D Equilibrium Approximation of the Solution to the Binding Payoff Floor Constraint

D.1 Equilibrium Approximation (Proposition 6)

In this section we construct equilibria that approximate the solution to the binding payoff floor constraint. We proceed in three steps. First, we show that if the binding payoff floor constraint has a decreasing solution, then there exists a nearby solution for which the payoff floor constraint is strictly slack. In particular, we show that for each $K > 1$ sufficiently small, there exists a solution with a decreasing cutoff path to the following *generalized* payoff floor constraint:

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF_t^{(n)}(x) = K\,\Pi^E(v_t). \tag{D.1}$$

For $K = 1$, (D.1) reduces to the original payoff floor constraint in (5.3) (divided by $F_t(v_t)$). Therefore, a decreasing solution that satisfies (D.1) for $K > 1$ is a feasible solution to the auxiliary problem. Moreover, the slack in the original payoff floor constraint is proportional to $\Pi^E(v_t)$.

Lemma 25. *Suppose $n < \overline{N}(F)$. Then there exists $\Gamma > 1$ such that for all $K \in [1, \Gamma]$, there exists a feasible solution T^K to the auxiliary problem that satisfies (D.1). For $K \searrow 1$, $T^K(v)$ converges to $T(v)$ for all $v \in [0, 1]$, and the seller's expected revenue converges to the value of the auxiliary problem.*

In the second step, we discretize the solution obtained in the first step so that all trades take place at times $t = 0, \Delta, 2\Delta, \ldots$. For given K and Δ, we define the discrete approximation $T^{K,\Delta}$ of T^K by delaying all trades in the time interval $(k\Delta, (k+1)\Delta]$ to $(k+1)\Delta$:

$$T^{K,\Delta}(v) := \Delta \min\left\{k \in \mathbb{N} \,\middle|\, k\Delta \geq T^K(v)\right\}. \tag{D.2}$$

In other words, we round up all trading times to the next integer multiple of Δ. Clearly, for all $v \in [0, 1]$ we have,

$$\lim_{K\to 1} \lim_{\Delta\to 0} T^{K,\Delta}(v) = \lim_{\Delta\to 0} \lim_{K\to 1} T^{K,\Delta}(v) = T(v),$$

and the seller's expected revenue also converges. Therefore, if we show that the functions T^{K_m,Δ_m} for some sequence (K_m, Δ_m) describe equilibrium outcomes for a sequence of equilibria $(p^m, b^m) \in \mathcal{E}(\Delta_m)$, we have obtained the desired approximation result.

The discretization changes the continuation revenue, but we can show that the approximation loss vanishes as Δ becomes small. In particular, if Δ is sufficiently small, then the approximation loss is less than half of the slack in the payoff floor constraint at the solution T^K. More precisely, we have the following lemma.

Lemma 26. *Suppose $n < \overline{N}(F)$. For each $K \in [1, \Gamma]$, where Γ satisfies the condition of Lemma 25, there exists $\bar{\Delta}_K^1 > 0$ such that for all $\Delta < \bar{\Delta}_K^1$, and all $t = 0, \Delta, 2\Delta, \ldots$,*

$$\int_0^{v_t^{K,\Delta}} e^{-r\left(T^{K,\Delta}(x)-t\right)} J_t(x) dF_t^{(n)}(x) \geq \frac{K+1}{2} \Pi^E(v_t^{K,\Delta}).$$

This lemma shows that if Δ is sufficiently small, at each point in time $t = 0, \Delta, 2\Delta, \ldots$, the continuation payoff of the discretized solution is at least as high as $1 + (K-1)/2$ times the profit of the efficient auction.

In the final step, we show that the discretized solution $T^{K,\Delta}$ can be implemented in an equilibrium of the discrete time game. To do this, we use weak-Markov equilibria as a threat to deter any deviation from the equilibrium path by the seller. The threat is effective because the uniform Coase conjecture (Proposition 2.(ii)) implies that the profit of a weak-Markov equilibrium is close to the profit of an efficient auction for any posterior along the equilibrium path. More precisely, let $\Pi^{\Delta}(p, b|v)$ be the continuation profit at posterior v for a given equilibrium $(p, b) \in \mathcal{E}(\Delta)$ as before.[43] Then Proposition 2.(ii) implies that for all $K \in [1, \Gamma]$, where Γ satisfies the condition of Lemma 25, there exists $\bar{\Delta}^2_K > 0$ such that, for all $\Delta < \bar{\Delta}^2_K$, there exists an equilibrium $(p, b) \in \mathcal{E}(\Delta)$ such that, for all $v \in [0, 1]$,

$$\Pi^{\Delta}(p, b|v) \leq \frac{K+1}{2}\Pi^E(v). \tag{D.3}$$

Now suppose we have a sequence $K_m \searrow 1$, where $K_m \in [1, \Gamma]$ as in Lemma 25. Define $\bar{\Delta}_K := \min\left\{\bar{\Delta}^1_K, \bar{\Delta}^2_K\right\}$. We can construct a decreasing sequence $\Delta_m \searrow 0$ such that for all m, $\Delta_m < \bar{\Delta}_{K_m}$. By Lemma 26 and (D.3), there exists a sequence of (punishment) equilibria $(\hat{p}^m, \hat{b}^m) \in \mathcal{E}(\Delta_m)$ such that for all m and all $t = 0, \Delta_m, 2\Delta_m, \ldots$

$$\int_0^{v_t^{K_m,\Delta_m}} e^{-r\left(T^{K_m,\Delta_m}(x)-t\right)} J_t(x) dF^{(n)}(x) \geq \frac{K_m+1}{2}\Pi^E(v_t^{K_m,\Delta_m}) \geq \Pi(\hat{p}^m, \hat{b}^m|v_t^{K_m,\Delta_m}). \tag{D.4}$$

The left term is the continuation profit at time t on the candidate equilibrium path given by T^{K_m,Δ_m}. This is greater or equal than the second expression by Lemma 26. The term on the right is the continuation profit at time t if we switch to the punishment equilibrium. This continuation profit is smaller than the middle term by Proposition 2.(ii). Therefore, for each m, $(\hat{p}^m, \hat{b}^m)$ can be used to support T^{K_m,Δ_m} as an equilibrium outcome of the game indexed by Δ_m. Denote the equilibrium that supports T^{K_m,Δ_m} by $(p^m, b^m) \in \mathcal{E}(\Delta_m)$. It is defined as follows: On the equilibrium path, the seller posts reserve prices given by T^{K_m,Δ_m} and (A.1). A buyer with type v bids at time $T^{K_m,\Delta_m}(v)$ as long as the seller does not deviate. By Lemma 5, this is a best response to the seller's on-path behavior. After a deviation by the seller, she is punished by switching to the equilibrium $(\hat{p}^m, \hat{b}^m)$. Since the seller anticipates the switch to $(\hat{p}^m, \hat{b}^m)$ after a deviation, her deviation profit is bounded above by $\Pi(\hat{p}^m, \hat{b}^m|v_t^{K_m,\Delta_m})$. Therefore, (D.4) implies that the seller does not have a profitable deviation. To summarize, we have an approximation of the solution to the binding payoff floor constraint by discrete time equilibrium outcomes.

Proposition 6. *Suppose Assumption 3 is satisfied and $n < \overline{N}(F)$. Then there exists a decreasing sequence $\Delta_m \searrow 0$ and a sequence of equilibria $(p^m, b^m) \in \mathcal{E}(\Delta_m)$ such that the sequence of trading functions T^m implemented by (p^m, b^m) and the seller's ex-ante revenue $\Pi^{\Delta}(p^m, b^m)$ converge to the profit achieved by the solution given by (4.1) for any v_0^+.*

[43]If the profit differs for different histories that lead to the same posterior, we could take the supremum, but this complication does not arise with weak-Markov equilibria.

Proof. The result follows directly from Lemmas 25 and 26. □

For the case that Assumption 4 is satisfied, Proposition 6 shows that the optimal solution to the auxiliary problem is the limit of a sequence of discrete time equilibria for $\Delta \to 0$. For the case that Assumption 4 is not satisfied, we did not obtain an optimal solution to the auxiliary problem from the binding payoff floor constraint. In this case, Proposition 6 shows that a feasible solution to the auxiliary problem exists, which involves strictly positive reserve prices and yields a higher profit than the efficient auction, and which can be obtained as the limit of a sequence of discrete time equilibria for $\Delta \to 0$.

D.2 Proof of Lemma 25

The key step of the approximation is to discretize the solution to the binding payoff floor constraint. In order to do that, we first need to find a feasible solution such that the payoff floor constraint is strictly slack. We use the change of variables $y = \dot{v}_t$ to rewrite the ODE obtained in Lemma 10 as

$$y'(v) = -r - g(v,K)y(v) - h(v,K)\left(y(v)\right)^2. \tag{D.5}$$

Any solution to the above ODE with $K > 1$ would lead to a strictly slack payoff floor constraint. Our goal is to show that the solution to the ODE exists for any K sufficiently close to zero and converges to the solution given by (4.1) as $K \searrow 1$. We will verify below that (4.1) satisfies the boundary condition $\lim_{v\to 0} y(v) = 0$. Given this observation, we want to show the existence of a solution $y_K(v) < 0$ of (D.5) that satisfies the same boundary condition. If the RHS is locally Lipschitz continuous in y for all $v \geq 0$ the Picard-Lindelof Theorem would imply existence and uniqueness and moreover, Lipschitz continuity would imply that the $y_K(v)$ is continuous in K. Unfortunately, although the RHS is locally Lipschitz continuous for all $v > 0$, its Lipschitz continuity may fail at $v = 0$. Therefore, for v strictly away from 0, the standard argument applies given Lipschitz continuity, but for neighborhood around 0, we need a different argument. In what follows, we will center our analysis on the neighborhood of $v = 0$.

We start by rewriting (D.5) by changing variables again, $z(v) = y(v)v^m$:

$$z'(v) = -rv^m - (g(v,K)v - m)\frac{z(v)}{v} - h(v,K)\frac{z(v)^2}{v^m}. \tag{D.6}$$

First, we show that the operator

$$L_K(z)(v) = \int_0^v -rs^m - (g(s,K)s - m)\frac{z(s)}{s} - h(s,K)\frac{z(s)^2}{s^m}ds. \tag{D.7}$$

is a contraction mapping on a Banach space of solutions that includes (4.1). This extends the Picard-Lindelof Theorem to our setting and thus implies existence and uniqueness. Next, we show that the fixed point of L_K converges uniformly to the fixed point of L_1 as $K \searrow 1$. Finally, we show that we can obtain a sequence of solutions T^K that converge (pointwise) to the solution of the binding payoff floor constraint (with $K = 1$) and show that the revenue of these solutions also converges to the value of the auxiliary problem.

Before we introduce the Banach space on which the contraction mapping is defined, we first derive bounds for the RHS of (D.6).

Lemma 27. *For any $\kappa > -1$, there exist $\overline{K} > 1$, an integer $m \geq 0$, and strictly positive real numbers α, η, ξ such that the following holds.*

(a) $m < |\kappa| + \eta$,

(d) $\frac{(|\kappa|+\eta-m)\alpha+\eta\alpha^2+r}{m+1} \in [0, \alpha]$,

(c) $\frac{|\kappa|+\eta(1+2\alpha)-m}{m+1} \in (0,1)$,

(d) $\frac{\kappa+\eta(1+\alpha)-m}{m+1}, \frac{\kappa-\eta(1+\alpha)-m}{m+1} \begin{cases} \in (0,1) & \textit{if } \kappa > m \\ \in (-\frac{1}{2}, \frac{1}{2}) & \textit{if } \kappa = m \\ \in (-1, 0) & \textit{if } \kappa < m \end{cases}$.

(e) $|h(v,K)v^2| < \eta$ *for any* $v < \xi$ *and* $K \in [1, \overline{K}]$,

(f) $|g(v,K)v - \kappa| < \eta$ *for any* $v < \xi$ *and* $K \in [1, \overline{K}]$,

Proof. First we choose m. If $\kappa \geq 1$, let $m = \lfloor \kappa \rfloor$; if $\kappa \in (-1,1)$, let $m = 0$. Thus $0 \leq m \leq |\kappa|$ and (a) is satisfied for any $\eta > 0$. In addition, $0 \leq \frac{|\kappa|-m}{m+1} < 1$ and $0 \leq |\kappa| < m+1$. Note that by the choice of m, $\kappa < m$ if and only if $\kappa < 0$; $\kappa = m$ if and only if $\kappa = 0, 1, ...$; $\kappa > m$ if and only if $\kappa > 0$ and κ is not an integer.

Next we choose α. Consider (b) . By the choice of m, the expression in (b) is non-negative for any $\eta, \alpha > 0$. Given this, Part (b) is equivalent to

$$\eta\alpha^2 - (2m + 1 - |\kappa| - \eta)\alpha + r \leq 0.$$

Hence, $\frac{(2m+1-|\kappa|-\eta)-\left[(2m+1-|\kappa|-\eta)^2-4r\eta\right]^{\frac{1}{2}}}{2\eta} \leq \alpha \leq \frac{(2m+1-|\kappa|-\eta)+\left[(2m+1-|\kappa|-\eta)^2-4r\eta\right]^{\frac{1}{2}}}{2\eta}$. Since $2m + 1 - |\kappa| > 0$, as $\eta \to 0$, the upper bound of α goes to $+\infty$ while the lower bound converge to $\frac{r}{2m+1-|\kappa|}$ by L'Hospital rule. We choose $\alpha = \frac{2r}{2m+1-|\kappa|}$. Then there exists $\eta_0 > 0$ such that Part (b) holds for any $\eta \in (0, \eta_0)$.

For m, α,and η_0 chosen above, since $0 \leq \frac{|\kappa|-m}{m+1} < 1$, there exists $\eta_1 \in (0, \eta_0)$ such that Part (c) holds for any $\eta \in (0, \eta_1)$.

For Part (d), consider the limit

$$\lim_{\eta \to 0} \frac{\kappa \pm \eta(1+\alpha) - m}{m+1} = \frac{\kappa - m}{m+1} \begin{cases} \in (0,1) & \text{if } \kappa > m \\ = 0 & \text{if } \kappa = m \\ \in (-1,0) & \text{if } \kappa < m \end{cases}$$

By continuity in both cases there exists $\eta \in (0, \eta_1)$ such that Part (f) holds.

Finally, given η chosen for Part (f) , it follows from Lemma 12 that we choose ξ and $\overline{K}$ jointly such that (e) and (f) hold. The proof of Lemma 12 shows that ξ can be chosen independently of K if $K < \overline{K}$. ☐

Note that $(\overline{K}, m, \alpha, \eta, \xi)$ in Lemma 27 only depend on the number of bidders n and the distribution function F. Since Lemma 25 is a statement for a fixed distribution and fixed n, we treat $(\overline{K}, m, \alpha, \eta, \xi)$ as fixed constants for the rest of this section. In the following, we slightly abuse notation by using n as an index for sequences. The number of bidders does not show up in the notation in the remainder of this section except in the final proof of Lemma 25.

We define a space of real-valued functions

$$\mathcal{Z}_0 = \left\{ z : [0,\xi] \to \mathbb{R} \,\middle|\, \sup_v |\frac{z(v)}{v^{m+1}}| \in \mathbb{R} \right\},$$

and equip it with the norm

$$||z||_m = \sup_v \left| \frac{z(v)}{v^{m+1}} \right|.$$

Define a subset of $\mathcal{Z}_0$ by

$$\mathcal{Z} = \{ z : [0,\xi] \to \mathbb{R} \,|\, ||z||_m \leq \alpha \}.$$

Note that these definitions are independent of $K < \overline{K}$.

Lemma 28. *$\mathcal{Z}_0$ is a Banach space with norm $||\cdot||_m$ and $\mathcal{Z}$ is a complete subset of $\mathcal{Z}_0$.*

Proof. For any $\gamma_1, \gamma_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathcal{Z}_0$ and $v \in [0,\xi]$, we have

$$\begin{aligned} \left| \frac{\gamma_1 z_1(v) + \gamma_2 z_2(v)}{v^{m+1}} \right| &\leq |\gamma_1| \left| \frac{z_1(v)}{v^{m+1}} \right| + |\gamma_2| \left| \frac{z_2(v)}{v^{m+1}} \right| \\ &\leq |\gamma_1| ||z_1||_m + |\gamma_2| ||z_2||_m \\ &< \infty. \end{aligned}$$

Therefore $\mathcal{Z}_0$ is a linear space. It's straight forward to see that $||\cdot||_m$ is a norm on $\mathcal{Z}_0$. We now show $\mathcal{Z}_0$ is complete. Consider a Cauchy sequence $\{z_n\} \subset \mathcal{Z}_0$: for any $\varepsilon > 0$, there exists N_ε such that $||z_{n'} - z_n||_m < \varepsilon$ for any $n', n \geq N_\varepsilon$.

First, notice that for any $n > 0$, $||z_n||_m \leq \beta := \max_{n' \leq N_\varepsilon} \{||z_{n'}||_m\} + \varepsilon < \infty$. Next we claim that z_n converges pointwise. To see this, note that $\sup_v |\frac{z_{n'}(v) - z_n(v)}{v^{m+1}}| < \varepsilon$ implies that $|\frac{z_{n'}(v) - z_n(v)}{v^{m+1}}| = |\frac{z_{n'}(v)}{v^{m+1}} - \frac{z_n(v)}{v^{m+1}}| < \varepsilon$ for any v. Since $|\frac{z_n(v)}{v^{m+1}}| \leq \beta$, completeness of real interval with the regular norm implies that there exists $x(\cdot)$ such that $\frac{z_n(v)}{v^{m+1}} \to x(v)$ pointwise and $|x(v)| \leq \beta$. Now define $z(v) = x(v) v^{m+1}$. It's straightforward that $z_n(v) \to z(v)$ pointwise.

Finally, we show that z_n converges under $||\cdot||_m$. To see this notice that $||z_n - z|| = \sup_v |\frac{z_n(v)}{v^{m+1}} - x(v)| \leq \varepsilon$ for any $n > N_\varepsilon$. In addition, since $|x(v)| \leq \beta$, $||z||_m \leq \beta$. This proves that $\mathcal{Z}$ is complete. The same argument shows that $\mathcal{Z}$ is complete, by replacing the bound β by α. □

To study the ODE (D.6) for each $K \in [1, \overline{K}]$,we define an operator L_K on $\mathcal{Z}$ as in (D.7).

Lemma 29. *The operator L_K is a contraction mapping on $\mathcal{Z}$ with a common contraction parameter $\rho < 1$ for all $K \in [1, \overline{K}]$.*

Proof. First we show that $L_K \mathcal{Z} \in \mathcal{Z}$. For any $z \in \mathcal{Z}$ and $v \in [0,\xi]$,

$$|L_K(z)(v)| = \left| \int_0^v -r s^m - (g(s,K)s - m)\frac{z(s)}{s} - h(s,K)s^2 \frac{z(s)^2}{s^{m+2}} ds \right|$$

$$
\begin{aligned}
\leq & \frac{rv^{m+1}}{m+1} + \left| \int_0^v (g(s,K)s - m)\frac{z(s)}{s} ds \right| + \eta \left(||z||_m\right)^2 \int_0^v s^{2m+2-m-2} ds \\
\leq & \frac{rv^{m+1}}{m+1} + \sup_{s\in[0,\xi]} |g(s,K)s - m| \, ||z||_m \int_0^v \frac{s^{m+1}}{s} ds + \eta\alpha^2 \frac{v^{m+1}}{m+1} \\
\leq & \frac{rv^{m+1}}{m+1} + (|\kappa| + \eta - m)\alpha \frac{v^{m+1}}{m+1} + \eta\alpha^2 \frac{v^{m+1}}{m+1} \\
= & \frac{(|\kappa| + \eta - m)\alpha + \eta\alpha^2 + r}{m+1} v^{m+1} \\
\leq & \alpha v^{m+1}.
\end{aligned}
$$

The first inequality follows from the triangle inequality of real numbers, Part (e) of Lemma 27 and $|z(s)| \leq ||z||s^{m+1}$. The second inequality follows from $|z(s)| \leq ||z||s^{m+1}$ and $||z|| \leq \alpha$. The third inequality follows from Lemma 27: for any $s \in [0, \xi]$ and $K \in [1, \overline{K}]$:

$$
\begin{aligned}
|g(s,K)s - m| &\leq |g(s,K)s - \kappa| + |\kappa - m| \\
&\leq \begin{cases} \eta + \kappa - m \text{ if } \kappa \geq 1 \\ \eta + |\kappa| \text{ if } \kappa \in (-1,1) \end{cases} \\
&= |\kappa| + \eta - m.
\end{aligned}
$$

We now show $L_K : \mathcal{Z} \to \mathcal{Z}$ is a contraction mapping. For any $z_1, z_2 \in \mathcal{Z}$ and $v \in [0, \xi]$,

$$
\begin{aligned}
|L_K(z_1)(v) - L_K(z_2)(v)| =& \left| \int_0^v -(g(s,K)s - m)\frac{z_1(s) - z_2(s)}{s} - h(s,K)s^2 \frac{z_1(s)^2 - z_2(s)^2}{s^{m+2}} ds \right| \\
\leq & \int_0^v \sup_{s\in[0,\xi]} |g(s,K)s - m| \frac{|z_1(s) - z_2(s)|}{s} \\
& + \sup_{s\in[0,\xi]} |h(s,K)s^2| \frac{|z_1(s) + z_2(s)||z_1(s) - z_2(s)|}{s^{m+2}} ds \\
\leq & (|\kappa| + \eta - m) \int_0^v ||z_1 - z_2||_m \frac{s^{m+1}}{s} ds \\
& + \int_0^v \eta(||z_1||_m + ||z_2||_m)||z_1 - z_2||_m \frac{s^{2m+2}}{s^{m+2}} ds \\
\leq & (|\kappa| + \eta - m)\frac{v^{m+1}}{m+1}||z_1 - z_2||_m + \eta 2\alpha \frac{v^{m+1}}{m+1}||z_1 - z_2||_m \\
= & v^{m+1} \frac{|\kappa| + \eta - m + \eta 2\alpha}{m+1} ||z_1 - z_2||_m
\end{aligned}
$$

The first inequality follows from the triangle inequality for real numbers. The second inequality follows from $\sup |g(s,K)s - m| < |\kappa| + \eta - m$ which was shown above, $|z_1(s) - z_2(s)| \leq ||z_1 - z_2||_m s^{m+1}$, $\sup |h(s,K)s^2| < \eta$, and $|z_1(s) + z_2(s)| \leq |z_1(s)| + |z_2(s)| \leq (||z_1||_m + ||z_2||_m)s^{m+1}$. The third inequality follows from $||z||_m \leq \alpha$.

It follows immediately that $||L_K(z_1) - L_K(z_2)||_m \leq \frac{|\kappa| + \eta - m + \eta 2\alpha}{m+1}||z_1 - z_2||_m$. By Part (c) of Lemma 27, $\rho := \frac{|\kappa| + \eta - m + \eta 2\alpha}{m+1} \in (0,1)$, which is independent of $K \in \overline{K}$. Hence L_K is contraction mapping on $\mathcal{Z}$, with a common contraction parameter for all $K \in [1, \overline{K}]$. $\square$

Since $L_K : \mathcal{Z} \to \mathcal{Z}$ is a contraction mapping, the Banach fixed point theorem implies that there exists a unique fixed point of L_K in $\mathcal{Z}$. For any $K \in \left[1, \overline{K}\right]$, we denote the fixed point by z_K, i.e., $z_K = L_K(z_K) \in \mathcal{Z}$. By the Banach fixed point theorem we have $z_K = \lim_{n\to\infty} L_K^n(0)$.

Lemma 30. *The fixed point of L_K on $\mathcal{Z}$, and hence the solution to the ODE (D.6) must be strictly negative for $v > 0$.*

Proof. Let $\rho_1 = \frac{\kappa+\eta-m+\eta\alpha}{m+1}$, $\rho_2 = \frac{\kappa-\eta-m-\eta\alpha}{m+1}$. We claim that there exists M_1, M_2 such that

$$M_1 \le \frac{L_K^n(0)\,(v)}{v^{m+1}} \le M_2 < 0 \tag{D.8}$$

for any $n \ge 1$.

For any $n > 1$,

$$\begin{aligned} L_K^n(0)(v) &= -\frac{r}{m+1}v^{m+1} - \int_0^v (g(s,K)s - m)\,\frac{L_K^{n-1}(0)\,(s)}{s} + h(s,K)s^2\frac{\left(L_K^{n-1}(0)\,(s)\right)^2}{s^{m+2}}ds \\ &= -\frac{r}{m+1}v^{m+1} + \int_0^v \left((g(s,K)s-m)\frac{1}{s} + h(s,K)s^2\frac{L_K^{n-1}(0)\,(s)}{s^{m+2}}\right)\left(-L_K^{n-1}(0)\,(s)\right)ds \end{aligned} \tag{D.9}$$

We prove separate the three cases $\kappa > m$, $\kappa = m$, $\kappa < m$ (which is equivalent to $\kappa < 0$) separately.

Case 1: $\kappa > m$. In this case, $\rho_1, \rho_2 > 0$ by Lemma 27. Let $M_1 = -\frac{r}{m+1}$ and $M_2 = -\frac{r}{m+1}(1-\rho_1)$. By part (d) of Lemma 27: $M_1 \le \frac{-r}{m+1} \le M_2 < 0$. Therefore we have $L_K^1(0)\,(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (D.8). We prove the desired result by induction. For $n > 1$, consider (D.9):

$$\begin{aligned} L_K^n(0)(v) &\le -\frac{r}{m+1}v^{m+1} + \int_0^v \left(\frac{\kappa-m+\eta}{s} + \frac{\eta\alpha s^{m+1}}{s^{m+2}}\right)\left(-L_K^{n-1}(0)\right)ds \\ &\le -\frac{r}{m+1}v^{m+1} + (\kappa - m + \eta(1+\alpha))\int_0^v \left(-M_1\frac{s^{m+1}}{s}\right)ds \\ &= \left(-\frac{r}{m+1} - \rho_1 M_1\right)v^{m+1} \\ &= M_2 v^{m+1} \end{aligned}$$

The first inequality follows from $-L_K^{n-1}(0) > 0$ and replacing the the coefficient of $-L_K^{n-1}(0)$ by its upper bound. The second inequality follows from $\kappa - m + \eta(1+\alpha) > 0$ and replacing $-L_K^{n-1}(0)$ with its upper bound $-M_1 s^{m+1}$ (by the induction hypothesis). In addition,

$$L_K^n(0)(v) \ge -\frac{r}{m+1}v^{m+1} + \int_0^v \left(\frac{\kappa-m-\eta}{s} - \frac{\eta\alpha s^{m+1}}{s^{m+2}}\right)\left(-L_K^{n-1}(0)\,(s)\right)ds$$

$$
\begin{aligned}
&\geq -\frac{r}{m+1}v^{m+1} + (\kappa - m - \eta(1+\alpha)) \int_0^v \left(-M_2 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_2 M_2\right) v^{m+1} \\
&\geq M_1 v^{m+1}
\end{aligned}
$$

The first inequality follows from $-L_K^{n-1}(0)\,(s) > 0$ and replacing the coefficient of $\left(-L_K^{n-1}(0)\,(s)\right)$ by its lower bound. The second inequality follows from $\kappa - m - \eta(1+\alpha) > 0$ and replacing $-L_K^{n-1}(0)$ with its upper bound $-M_2 s^{m+1}$ (by the induction hypothesis). The last inequality follows from $-\rho_2 M_2 > 0$ and the choice of M_1.

Case 2: $\kappa < m$. In this case, $\rho_1, \rho_2 \in (-1, 0)$ by part (d) of Lemma 27. Let $M_1 = -\frac{r}{m+1}\frac{1}{1+\rho_2}$ and $M_2 = -\frac{r}{m+1}$. $\rho_2 < 0$ implies $M_1 \leq -\frac{r}{m+1} \leq M_2 < 0$. Therefore we have $L_K^1(0)\,(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (D.8). For $n > 1$, consider $(D.9)$:

$$
\begin{aligned}
L_K^n(0)(v) &\leq -\frac{r}{m+1}v^{m+1} + (\kappa - m + \eta(1+\alpha)) \int_0^v \left(-M_2 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_1 M_2\right) v^{m+1} \\
&\leq -\frac{r}{m+1}v^{m+1} \\
&= M_2 v^{m+1}
\end{aligned}
$$

The first inequality follows from a similar derivation as in the case $\kappa > m$. However here $\kappa - m + \eta(1+\alpha) < 0$, therefore $-L_K^{n-1}(0)$ is replaced by its lower bound $-M_2 s^{m+1}$. The second inequality follows because $\rho_1 M_2 > 0$. In addition,

$$
\begin{aligned}
L_K^n(0)(v) &\geq -\frac{r}{m+1}v^{m+1} + (\kappa - m - \eta(1+\alpha)) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_2 M_1\right) v^{m+1} \\
&= M_1 v^{m+1}.
\end{aligned}
$$

Case 3: $\kappa = m$. Then $\rho_1 = -\rho_2 = \frac{\eta(1+\alpha)}{m+1} \in (-1/2, 1/2)$ by part (d) of Lemma 27. Let $M_1 = -\frac{r}{m+1}\frac{1}{1-\rho_1}$ and $M_2 = -\frac{r}{m+1}\frac{1-2\rho_1}{1-\rho_1}$. Since $m \geq 0$ we have $\rho_1 \in (0, 1/2)$. This implies $M_1 \leq -\frac{r}{m+1} \leq M_2 < 0$. Therefore we have $L_K^1(0)\,(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (D.8). For $n > 1$, consider $(D.9)$:

$$
\begin{aligned}
L_K^n(0)(v) &\leq -\frac{r}{m+1}v^{m+1} + \eta(1+\alpha) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \frac{\eta(1+\alpha)}{m+1}M_1\right) v^{m+1} \\
&= M_2 v^{m+1}
\end{aligned}
$$

To obtain the first inequality, we replace $-L_K^{n-1}(0)$ by its upper bound $-M_1 s^{m+1}$ since

$\eta(1+\alpha) > 0$. In addition,

$$\begin{aligned} L_K^n(0)(v) &\geq -\frac{r}{m+1}v^{m+1} - \eta(1+\alpha)\int_0^v \left(-M_1\frac{s^{m+1}}{s}\right)ds \\ &= \left(-\frac{r}{m+1} + \frac{\eta(1+\alpha)}{m+1}M_1\right)v^{m+1} \\ &= M_1 v^{m+1} \end{aligned}$$

To obtain the first inequality, we replace $-L_K^{n-1}(0)\,(v)$ by its upper bound $-M_2 s^{m+1}$ since $-\eta(1+\alpha) < 0$. □

Lemma 31. *$\sup_{v\in[0,\xi]}\left|\frac{z_K(v)}{v^m} - \frac{z_1(v)}{v^m}\right| \to 0$ as $K \to 1$.*

Proof. First note that for any $\varepsilon > 0$, it follows from Lemma 12 that $g(v,K)v$ and $h(v,K)v^2$ are bounded over $v \in [0,\xi]$ and $K \in [1,\overline{K}]$. Hence there exists $\Gamma \in \left(1,\overline{K}\right)$ such that

$$\begin{aligned} \sup_{v\in[0,\xi],K\in[1,\Gamma]} |g(v,K)v - g(v,1)v| < \varepsilon, \\ \sup_{v\in[0,\xi],K\in[1,\Gamma]} |h(v,K)v^2 - h(v,1)v^2| < \varepsilon. \end{aligned}$$

Since $\sup_{v\in[0,\xi]}\left|\frac{z_K(v)}{v^m} - \frac{z_1(v)}{v^m}\right| \leq \sup_v ||z_K - z_1||_m \frac{v^{m+1}}{v^m} \leq \xi||z_K - z_1||_m$, it's sufficient to show that $\lim_{K\to1}||z_K - z_1||_m = 0$. The proof follows from Lee and Liu (2013, Lemma 13(b)). Let $\rho = \frac{|\kappa|+\eta-m+\eta2\alpha}{m+1} < 1$ be the contraction parameter, which is independent of K. For all $z \in \mathcal{Z}$ and $K \in [1,\Gamma]$,

$$\begin{aligned} |L_K(z)(v) - L_1(z)(v)| =& \left|\int_0^v (g(s,K)s - g(s,1)s)\frac{z(s)}{s} + (h(s,K)s^2 - h(s,1)s^2)\frac{z(s)^2}{s^{m+2}}ds\right| \\ \leq& \varepsilon\int_0^v \frac{z(s)}{s}ds + \varepsilon\int_0^v \frac{z(s)^2}{s^{m+2}}ds \\ \leq& \varepsilon\left(||z||_m\frac{v^{m+1}}{m+1} + ||z||_m^2\frac{v^{m+1}}{m+1}\right) \\ \leq& \varepsilon\frac{\alpha+\alpha^2}{m+1}v^{m+1} \end{aligned}$$

Therefore, $||L_K(z) - L_1(z)||_m \leq \varepsilon\frac{\alpha+\alpha^2}{m+1}$.

For any $n > 1$,

$$\begin{aligned} ||L_K^n(z) - L_1^n(z)||_m =& ||L_K\left(L_K^{n-1}(z)\right) - L_1\left(L_K^{n-1}(z)\right) + L_1\left(L_K^{n-1}(z)\right) - L_1\left(L_1^{n-1}(z)\right)||_m \\ \leq& ||L_K\left(L_K^{n-1}(z)\right) - L_1\left(L_K^{n-1}(z)\right)|| + ||L_1\left(L_K^{n-1}(z)\right) - L_1\left(L_1^{n-1}(z)\right)||_m \\ \leq& \varepsilon\frac{\alpha+\alpha^2}{m+1} + \rho||L_K^{n-1}(z) - L_1^{n-1}(z)||_m \\ \leq& \varepsilon\frac{\alpha+\alpha^2}{m+1}\sum_{k=0}^{n-1}\rho^k \end{aligned}$$

$$\leq \varepsilon \frac{\alpha+\alpha^2}{m+1}\frac{1}{1-\rho}$$

Given $z_K = \lim_{n\to\infty} L_K^n(0)$, there exists N_ε s.t. $\forall n \geq N_\varepsilon$, $||z_K - L_K^n(0)|| \leq \varepsilon$:

$$\begin{aligned}||z_K - z_1||_m &\leq ||z_K - L_K^n(0)||_m + ||z_1 - L_1^n(0)||_m + ||L_K^n(0) - L_1^n(0)||_m \\ &\leq 2\varepsilon + \varepsilon\frac{\alpha+\alpha^2}{m+1}\frac{1}{1-\rho} \\ &= \left(2 + \frac{\alpha+\alpha^2}{m+1}\frac{1}{1-\rho}\right)\varepsilon\end{aligned}$$

Therefore $\lim_{K\to 1} ||z_K - z_1||_m = 0$. □

Given definition $z(v) = y(v)v^m$, let $y_K(v) = \frac{z_K(v)}{v^m}$, where z_K is the fixed point of L_K. It follows from the previous two lemmas that $y_K(v)$ is negative and $\lim_{K\to 1} ||y_K - y_1|| = 0$ under standard sup norm. Now we have all the ingredients necessary to prove Lemma 25.

Proof of Lemma 25. The uniform convergence of y_K implies that the cutoff sequence v_t^K given by $v(t) = v(0) + \int_0^t y_K(v(s))\,ds$ converges pointwise to the cutoff sequence $v_t = v_t^1$ associated with the trading time function $T(v) = T^1(v)$. Since v_t is continuous and strictly decreasing (by Lemma 11), this implies that the trading time function

$$T^K(v) = \sup\left\{t : v_t^K \geq v\right\}$$

converges pointwise to $T(v)$. To see this, note that $\sup\{t : v_t \geq v\} = \sup\{t : v_t > v\}$, since v_t is continuous and strictly decreasing. Now, for all t such that $v_t > v$, there exists K^t such that $v_t^K > v$ for all $K < K^t$. Hence,

$$\lim_{K\searrow 1} \sup\left\{t : v_t^K \geq v\right\} \geq \sup\{t : v_t > v\}.$$

Similarly, for all t such that $v_t < v$, there exists K^t such that $v_t^K < v$ for $K < K^t$. Hence,

$$\lim_{K\searrow 1} \sup\left\{t : v_t^K \geq v\right\} \leq \sup\{t : v_t \geq v\}.$$

Therefore, for all v, we have

$$\lim_{K\searrow 1} \sup\left\{t : v_t^K \geq v\right\} = \sup\{t : v_t \geq v\},$$

or equivalently,

$$\lim_{K\searrow 1} T^K(v) = T(v).$$

It remains to show that the seller's ex ante revenue converges. Notice that the sequence $e^{-rT^K(v)}$ is uniformly bounded by 1. Therefore, the dominated convergence theorem implies that

$$\lim_{K\searrow 1} \int_0^1 e^{-rT^K(x)} J(x) dF^{(n)}(x) = \int_0^1 e^{-rT(x)} J(x) dF^{(n)}(x).$$

□

D.3 Proof of Lemma 26

Proof. For $t \in \{0, \Delta, 2\Delta, \ldots\}$, define

$$\tilde{v}_t^{K,\Delta} = \inf \left\{ v \middle| J(v|v \leq v_t^{K,\Delta}) \geq 0 \right\}.$$

Consider the LHS of the payoff floor constraint at $t = k\Delta$, $k \in \mathbb{N}_0$. Notice that, for $k > 0$, the new posterior at this point in time is equal to the old posterior at $((k-1)\Delta)^+$. Therefore, we can approximate the LHS of the payoff floor at $t = k\Delta$ as:

$$\begin{aligned}
&\int_0^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K,\Delta}(v)-k\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&= \int_0^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} e^{-r\,(T^{K,\Delta}(v)-T^{K}(v)-\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&= \int_{\tilde{v}_{k\Delta}^{K,\Delta}}^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} e^{-r\,(T^{K,\Delta}(v)-T^{K}(v)-\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad + \int_0^{\tilde{v}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} e^{-r\,(T^{K,\Delta}(v)-T^{K}(v)-\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\geq \int_{\tilde{v}_{k\Delta}^{K,\Delta}}^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad + \int_0^{\tilde{v}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} e^{r\Delta} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\geq \int_{\tilde{v}_{k\Delta}^{K,\Delta}}^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad + \int_0^{\tilde{v}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad - \int_0^{\tilde{v}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} \left(1 - e^{r\Delta}\right) J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&= \int_0^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad - \int_0^{\tilde{v}_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K}(v)-(k-1)\Delta)} \left(1 - e^{r\Delta}\right) J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv.
\end{aligned}$$

The first term in the last expression is equal to the LHS of the payoff floor constraints at $((k-1)\Delta)^+$ for the original solution v^K. Hence it is equal to $K\Pi^E(v_{k\Delta}^{K,\Delta})$. Therefore, we have

$$
\begin{aligned}
&\int_0^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K,\Delta}(v)-k\Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
=& K\Pi^E(v_{k\Delta}^{K,\Delta}) + \left(e^{r\Delta}-1\right) \int_0^{\tilde{v}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
\ge& K\Pi^E(v_{k\Delta}^{K,\Delta}) + \left(e^{r\Delta}-1\right) \int_0^{\tilde{v}_{k\Delta}^{K,\Delta}} J(v|v \le v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \le v_{k\Delta}^{K,\Delta}) dv \\
=& K\Pi^E(v_{k\Delta}^{K,\Delta}) - \left(e^{r\Delta}-1\right) \left[\Pi^M(v_{k\Delta}^{K,\Delta}) - \Pi^E(v_{k\Delta}^{K,\Delta})\right] \\
=& K\Pi^E(v_{k\Delta}^{K,\Delta}) - \left(e^{r\Delta}-1\right) \left[\frac{\Pi^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1\right] \Pi^E(v_{k\Delta}^{K,\Delta}).
\end{aligned}
$$

Next we show that $\frac{\Pi^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1$ is uniformly bounded. Recall that by Assumption 4, there exist $0 < M \le 1 \le L < \infty$ and $\alpha > 0$ such that $Mv^\alpha \le F(v) \le Lv^\alpha$ for all $v \in [0,1]$. This implies that the rescaled truncated distribution

$$\tilde{F}_x(v) := \frac{F(vx)}{F(x)},$$

for all $v \in [0,1]$ is dominated by a function that is independent of x:

$$\tilde{F}_x(v) \le \frac{Lv^\alpha x^\alpha}{Mx^\alpha} = \frac{L}{M} v^\alpha.$$

Next, we observe that the revenue of the efficient auction can be written in terms of the rescaled expected value of the second-highest order statistic of the rescaled distribution:

$$\Pi^E(v) = \int_0^1 vs\tilde{F}_v^{(n-1:n)}(s)ds.$$

If we define $\hat{F}(v) := \min\left\{1, \frac{L}{M}v^\alpha\right\}$ and $B := \int_0^1 s\hat{F}^{(n-1:n)}(s)ds$, then given $\tilde{F}_x(v) \le \frac{L}{M}v^\alpha$ we can apply Theorem 4.4.1 in David and Nagaraja (2003) to obtain $\Pi^E(v) \ge Bv > 0$. Since $\Pi^M(v) \le v$, we have

$$\frac{\Pi^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1 \le \frac{1}{B} - 1.$$

Therefore, LHS of the payoff floor at $t = k\Delta$ is bounded below by

$$\left[K - \left(e^{r\Delta}-1\right)\left(\frac{1}{B}-1\right)\right] \Pi^E(v_{k\Delta}^{K,\Delta}).$$

Clearly, for Δ sufficiently small, the term in the square bracket is no less than $(K+1)/2$. $\square$

E More General Mechanisms

We focus on a class of bidding mechanisms which are symmetric and accept one-dimensional bids. We can thus denote the message space by $M = \{\emptyset\} \cup [b_0, \infty)$, where $\emptyset$ indicates non-participation and b_0 is the minimal bid. We further impose three additional restrictions on the feasible mechanisms.

First, the mechanism always chooses the bidder with the highest valid bid as the winner (ties are resolved randomly). Hence, the allocation rule of the mechanism is given by

$$q^i(b^i, b^{-i}) = q(b^i, b^{-i}) = \begin{cases} \frac{1}{\#\{k: b^k = \max_j b^j\}} & \text{if } b^i \geq \max\{b_0, \max_{j \neq i} b^j\} \\ 0 & \text{otherwise (including } b^i = \emptyset) \end{cases}.$$

Second, we restrict attention to the class of *winner-pay-only mechanisms* where all bidders other than the winner do not make or receive any payments. More precisely, we allow for any mechanism that belongs to one of the following two sub-classes. The first sub-class, the winner's payment does not depend on his own bid if he is the only bidder who places a valid bid, that is, the payment rule satisfies the following property (where we write $p^i(b^i, b^{-i}) = p(b^i, b^{-i})$ by the symmetry assumption):

$$p(b^i, \emptyset, \ldots, \emptyset) = p(\tilde{b}^i, \emptyset, \ldots, \emptyset) \quad \forall b^i = \tilde{b}^i.$$

Clearly, second-price auctions with arbitrary reserve price b_0 belong to this sub-class, but it also includes more exotic formats like third-price auctions. In the second sub-class, the winner's payment is strictly increasing in his own bid if he is the only bidder who places a valid bid. This subclass includes first-price auctions with arbitrary reserve price b_0, and also mechanisms in which the winner's payment may depend on his own bid as well as bids placed by other bidders.

Finally, we assume that, regardless of the continuation payoff that bidders can get from abstaining in the current period, each mechanism has a unique symmetric equilibrium, which has the following properties: There exists a cutoff valuation such that all buyers with valuations below a cutoff do not place a valid bid, and all buyers with valuations above the cutoff submit valid bids that are strictly increasing in their valuations. This restriction, together with the first one, implies that the mechanism allocates efficiently if the object is allocated: the winner is always the bidder with the highest valuation.

Let $\overline{\mathcal{M}}$ be the set of all mechanisms (M, q, p) that satisfy the above three restrictions. Let $\mathcal{M} \subset \overline{\mathcal{M}}$ be a subset that contains second-price auctions with arbitrary reserve prices $b_0 \in [0, 1]$. We consider the dynamic game $\Gamma^{\mathcal{M}}$ in which the seller can randomize over mechanisms $m_t \in \mathcal{M}$ at all non-terminal histories h_t. Let Γ^{SPA} denote the game considered in the main text where the seller is restricted to use second price auctions.

We use $\Pi^*_{\mathcal{M}}$ to denote the maximal profit the seller can achieve in the game $\Gamma^{\mathcal{M}}$ in the continuous time limit, and use Π^* as defined in the main text. The purpose of this appendix is to show that $\Pi^*_{\mathcal{M}} = \Pi^*$, for all choices of $\mathcal{M}$. *This implies that the restriction to second-price auctions is without loss of generality, because our results would remain valid even if we allowed the seller to choose among mechanisms in $\mathcal{M}$.*

To see this, consider a symmetric perfect Bayesian equilibrium of $\Gamma^{\mathcal{M}}$. First, we note that, given our restrictions on $\mathcal{M}$, the object must be allocated to the buyer with the highest valuation. This implies that the equilibrium outcome is given by a non-increasing cutoff path. We use Lemma 5 to obtain a sequence of reserve prices that the seller can use to implement the same allocation with a sequence of second-price auctions. The payoff equivalence theorem then implies that we can replicate the seller's equilibrium profit and the buyers' equilibrium utilities in the equilibrium of $\Gamma^{\mathcal{M}}$ by using only second price auctions. Therefore, on the equilibrium path, it is sufficient to consider equilibria in which the seller only uses second-price auctions.

Next, we observe that the necessary condition for an equilibrium given by the payoff floor constraint remains valid in the game $\Gamma^{\mathcal{M}}$, because, by assumption, $\mathcal{M}$ contains the efficient auction, so that the seller can guarantee the profit of an efficient auction at any point in time. Therefore, we can consider the same auxiliary problem as in the case of Γ^{SPA}.

It remains to show that we can extend the construction of equilibria that approximate the optimal solution to the auxiliary problem to the game $\Gamma^{\mathcal{M}}$. The main step is to show the existence of equilibria of $\Gamma^{\mathcal{M}}$ for arbitrary $\Delta > 0$ that satisfy the uniform Coase conjecture. We will use weak-Markov equilibria of Γ^{SPA} to construct corresponding equilibria of $\Gamma^{\mathcal{M}}$ that yield the same expected revenue for the seller. More precisely, we will construct a weak-Markov equilibrium for $\Gamma^{\mathcal{M}}$ in which the seller always uses second-price auctions—on and off the equilibrium path. Let us fix $\Delta > 0$ and suppose (p^{SPA}, b^{SPA}) is a weak-Markov equilibrium of Γ^{SPA}. Given our assumptions, any equilibrium of $\Gamma^{\mathcal{M}}$ must satisfy the skimming property, so that the buyers' strategy defines a cutoff function $\beta_t(h_t, m_t)$ where $(h_t, m_t) \in \mathcal{M}^t$. This does not fully describe the buyers' strategy. The function $\beta_t(h_t, m_t)$ only describes the types that place a valid bid at any history: A buyer bids if $v > \beta_t(h_t, m_t)$ and waits if his valuation is below $\beta_t(h_t, m_t)$.

We need to consider two types of histories. Consider first a history h_t where the seller has never deviated to a mechanism different from a second price auction. Note that h_t can be off the equilibrium path if the seller has deviated to off-equilibrium reserve prices but still used second-price auctions throughout. Such a history is also a history of Γ^{SPA}, and we use (p^{SPA}, b^{SPA}) to define the equilibrium behavior of the buyers and the seller at any such history.

Next, consider a history h_t where the seller has never deviated, and suppose that at h_t, she deviates to a mechanism $m_t = (M_t, q_t, p_t) \in \mathcal{M}$ with $M_t = [b_{0t}, \infty)$, which is not a second-price auction. If some bidder places a valid bid, the game ends. If nobody bids at (h_t, m_t), the seller continues as if m_t was a second price auction with reserve price $p_t(b_{0t}, \emptyset, \ldots, \emptyset)$. To define the buyer's equilibrium behavior at (h_t, m_t) we choose the cutoff $v_{t+1} = \beta_t(h_t, m_t) := \beta_t^{SPA}(h_t, p_t(b_{0t}, \emptyset, \ldots, \emptyset)))$. All buyers with valuations greater than or equal to v_{t+1} place a bid and use a strictly increasing bidding function which we leave unspecified for the moment. All buyers below v_{t+1} do not bid. If everybody follows this strategy, the payoff of a buyer with valuation v_{t+1} is given by

$$\left(\frac{F(v_{t+1})}{F(v_t)}\right)^{n-1} \left[v_{t+1} - p_t(b_{0t}, \emptyset, \ldots, \emptyset))\right]. \tag{E.1}$$

To see this, first suppose m_t belongs to the first sub-class of mechanisms considered above. Since buyers use a strictly increasing bidding strategy, the marginal type only wins if no other buyer places a valid bid and in this case his payment is independent of his own bid and equal to $p_t(b_{0t}, \emptyset, \dots, \emptyset))$. Next suppose m_t belongs to the second sub-class. Again, in equilibrium, the marginal type can only win if all other buyers do not bid. His payment is thus given by $p_t(b, \emptyset, \dots, \emptyset))$ where b is his bid. Since the payment is strictly increasing in b, we must have $b = b_{0t}$ in equilibrium. Therefore, the payoff of the marginal type v_{t+1} is given by (E.1). Note that this payoff is also equal to the payoff of v_{t+1} at history $(h_t, p_t(b_{0t}, \emptyset, \dots, \emptyset))$ in Γ^{SPA}. Since v_{t+1} is the marginal type at that history, and the continuation payoff at $(h_t, p_t(b_{0t}, \emptyset, \dots, \emptyset))$ in Γ^{SPA} is the same as the continuation payoff at (h_t, m_t) in our constructed equilibrium of $\Gamma^{\mathcal{M}}$, v_{t+1} is indifferent between bidding and waiting at (h_t, m_t). Therefore, by replacing mechanism m_t by a second-price auction with reserve price $p_t(b_{0t}, \emptyset, \dots, \emptyset)$, we can replicate the incentives of the marginal type v_{t+1}. To complete the definition of the buyers' equilibrium behavior at (h_t, m_t), we use the unique symmetric equilibrium for m_t, given the outside option implied by the continuation payoff obtained from the buyers continuation strategy defined above. (Existence and uniqueness follows from our assumptions on $\mathcal{M}$.)

For history h_t that involves multiple deviations to mechanisms different from second price auctions, we can similarly define the equilibrium strategy by simply replacing every mechanism m_τ along the history by a second-price auction with a reserve price $p_t(b_{0\tau}, \emptyset, \dots, \emptyset)$.

This construction implies that after a deviation to any mechanism, the seller obtains a profit that she could also obtain by deviating to a second-price auction. Since such deviations are ruled out by the assumption that (p^{SPA}, b^{SPA}) is an equilibrium of Γ^{SPA}, the seller has no incentive to deviate and we have shown that our construction is indeed an equilibrium.

F Independence of the Assumptions

In the following, we present four examples of distributions that violate exactly one of Assumptions 1 to 4 and satisfy all others.

- Example that satisfies A2-A4 but not A1: The Beta distribution parameterized by $k = 1/2$ and $\beta = 1/2$, with density

$$f(v) = \frac{v^{k-1}(1-v)^{\beta-1}}{\int_0^1 x^{k-1}(1-x)^{\beta-1}\,dx}.$$

- Example that satisfies A1-A3 but not A4: The Beta distribution with $k > 1$ and $\beta > 1$.
- Example that satisfies A1, A2, and A4 but not A3. Consider

$$F(v) = v^k(1 - C\ln v)$$

  where $k > 1$ and $0 < C < \min\left\{k, \frac{k^2-k}{2k-1}, \frac{k^2+k}{2k+1}\right\}$.
  We show that F is a well defined CDF that satisfies A1, A2, A4 and fails A3.

  - It's straightforward to see that $F(0) = 0$ and $F(1) = 1$. Note that

$$f(v) = v^{k-1}(k - C(k\ln v + 1)) \geq v^{k-1}(k - C) > 0,$$

    where the first inequality follows because $\ln v \leq 0$ for $v \in [0,1]$, and the second one follows because $C < k$. Therefore, F is a well defined CDF. For later reference, we note that

$$f'(v) = v^{k-2}\left(k^2 - k - C\left((k^2-k)\ln v + 2k - 1\right)\right)$$

  - A1:

$$\begin{aligned} J(v) =& v - \frac{1-F(v)}{f(v)} \\ \implies J'(v) =& 2 + \frac{f'(v)}{f(v)^2}(1-F(v)) \\ =& 2 + \frac{v^{k-2}\left(k^2-k-C\left((k^2-k)\ln v + 2k-1\right)\right)}{v^{2k-2}\left(k - C(k\ln v+1)\right)^2}(1-F(v)) \\ =& 2 + \frac{1}{v^k}\frac{(k^2-k) - C\left((k^2-k)\ln v + 2k - 1\right)}{\left(k - C(k\ln v + 1)\right)^2}(1-F(v)) \\ \geq& 2 \end{aligned}$$

    The inequality follows from $F(v) \leq 1$, $(k - C(k\ln v + 1))^2 > 0$, and $k^2 - k > C(2k-1)$. Therefore J is strictly increasing on $[0,1]$.

– A2:

$$
\begin{aligned}
\phi =& \lim_{v \to 0} \frac{f(v)v}{F(v)} - 1 \\
=& \lim_{v \to 0} \frac{v^k(k - C(k\ln v + 1))}{v^k(1 - C\ln v)} - 1 \\
=& \lim_{v \to 0} \frac{k - C(k\ln v + 1)}{1 - C\ln v} - 1 \\
=& \lim_{v \to 0} \frac{-\frac{Ck}{v}}{-\frac{C}{v}} - 1 \\
=& k - 1 > 0
\end{aligned}
$$

The fourth equality is due to the L'Hospital rule.

– A4:

$$
\begin{aligned}
(v(1 - F(v)))'' =& -vf'(v) - 2f(v) \\
=& -v^{k-1}\left(k^2 - k - C\left((k^2 - k)\ln v + 2k - 1\right)\right) - 2v^{k-1}\left(k - C(k\ln v + 1)\right) \\
=& -v^{k-1}\left(k^2 + k - C\left((k^2 + k)\ln v + 2k + 1\right)\right) \\
\leq& -v^{k-1}\left(k^2 + k - C\left(2k + 1\right)\right) \\
<& 0
\end{aligned}
$$

Therefore $v(1 - F(v))$ is concave on $[0, 1]$.

– A3:

$$
\begin{aligned}
\frac{F(v)}{v^\alpha} =& \frac{1 - C\ln v}{v^{\alpha - k}} \\
\implies \lim_{v \to 0} \frac{F(v)}{v^\alpha} =& \begin{cases} +\infty & \text{if } \alpha - k \geq 0 \\ \lim_{v \to 0} \frac{-C}{(\alpha - k)v^{\alpha - k}} = 0 & \text{if } \alpha - k < 0 \end{cases}
\end{aligned}
$$

Therefore, it is impossible to find $0 < M \leq L < \infty$ such that $M < \frac{F(v)}{v^\alpha} < L$ for some $\alpha > 0$.

- Example that satisfies A1, A3, A4 but not A2:

$$
F(v) = v^k\left(1 + C\sin\left(\ln v\right)\right)
$$

where $k > 1$ and $0 < C < \min\left\{\frac{k}{k+1}, \frac{k^2 - k}{k^2 + k - 2}, \frac{k+1}{k+3}\right\}$.
We show that F is a well defined CDF that satisfies A1, A3, A4 and fails A2:

– It is easy to verify that $F(0) = 0$ and $F(1) = 1$. Furthermore,

$$
f(v) = v^{k-1}\left(k + kC\sin(\ln v)\right) + v^k\left(C\cos(\ln v)\frac{1}{v}\right)
$$

$$
\begin{aligned}
&= v^{k-1}\left(k + C\left(k\sin(\ln v) + \cos(\ln v)\right)\right) \\
&\geq v^{k-1}(k + C(k+1)) > 0
\end{aligned}
$$

Therefore F is a well defined CDF. Note that, $\forall v \in (0,1]$,

$$
\begin{aligned}
f'(v) = & \, v^{k-2}\left(k^2 - k + C\left((k^2-k)\sin(\ln v) + (k-1)\cos(\ln v)\right)\right) \\
& + v^{k-1}\left(C\left(k\cos(\ln v) - \sin(\ln v)\right)\frac{1}{v}\right) \\
& = v^{k-2}\left(k^2 - k + C\left((k^2-k-1)\sin(\ln v) + (2k-1)\cos(\ln v)\right)\right) \\
& \geq v^{k-2}\left(k^2 - k - C(k^2 - k - 1 + 2k - 1)\right) > 0
\end{aligned}
$$

– A1:

$$
J(v) = v - \frac{1-F(v)}{f(v)} \implies J'(v) = 2 + \frac{f'(v)}{f(v)}(1 - F(v))
$$

Since $\forall v > 0,\ f'(v) > 0,\ J$ is strictly increasing.

– A3:

$$
\frac{F(v)}{v^k} = 1 + C\sin(\ln v)
$$

Our assumption that $c < \frac{k}{k+1}$ implies $|C\sin(\ln v)| < \frac{k}{k+1}$. Therefore:

$$
\frac{F(v)}{v^k} \in \left[\frac{1}{k+1}, \frac{2k+1}{k+1}\right]
$$

A3 is satisfied because we can set $\alpha = k, M = \frac{1}{k+1}$, and $L = \frac{2k+1}{k+1}$.

– A4:

$$
\begin{aligned}
(v(1-F(v)))'' = & - vf'(v) - 2f(v) \\
= & - v^{k-1}\left(k^2 - k + C\left((k^2-k-1)\sin(\ln v) + (2k-1)\cos(\ln v)\right)\right) \\
& - 2v^{k-1}\left(k + C\left(k\sin(\ln v) + \cos(\ln v)\right)\right) \\
= & - v^{k-1}\left(k^2 + k + C\left(k^2 + k - 1\right)\sin(\ln v) + (2k+1)\cos(\ln v)\right) \\
\leq & - v^{k-1}\left(k^2 + k - C(k^2 + 3k)\right) < 0
\end{aligned}
$$

– A2:

$$
\frac{f(v)v}{F(v)} = \frac{k + C(k\sin(\ln v) + \cos(\ln v))}{1 + C\sin(\ln v)}
$$

If we take $v_\ell = \exp(-2\ell\pi)$, then $\lim_{\ell\to\infty} \frac{f(v_\ell)v_\ell}{F(v_\ell)} = k + C$. If we take $v_\ell = \exp\left(-2\ell\pi + \frac{\pi}{2}\right)$, then $\lim_{\ell\to\infty} \frac{f(v_\ell)v_\ell}{F(v_\ell)} = \frac{k+Ck}{1+C}$. Therefore, the limit in A2 doesn't exist.