

Supplemental Material for “Optimal Sequential Decision with Limited Attention”

(For Online Publication)

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B Omitted Proofs

B.1 Explicit Expressions for the Value Function

The expressions for the baseline model are obtained by setting $\bar{\lambda} = \lambda$ and $\underline{\lambda} = 0$. If $r \neq 0$ or $\underline{\lambda} \neq 0$ we have

$$\begin{aligned} \underline{V}_{ct}(p) &= \frac{(\underline{\lambda}u_a^B - c)(1-p)}{r + \underline{\lambda}} + \frac{(\bar{\lambda}u_a^A - c)p}{r + \bar{\lambda}} \\ &\quad + \delta \frac{c - u_a^B \underline{\lambda} + u_b^B(r + \underline{\lambda})}{(r + \bar{\lambda})(r + \underline{\lambda})} \left(\frac{1-p}{p}\right)^{\frac{r+\underline{\lambda}}{\delta}} \left(\frac{p^*}{1-p^*}\right)^{\frac{r+\underline{\lambda}}{\delta}} (1-p), \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} \bar{V}_{ct}(p) &= \frac{(\underline{\lambda}u_b^A - c)p}{r + \underline{\lambda}} + \frac{(\bar{\lambda}u_b^B - c)(1-p)}{r + \bar{\lambda}} \\ &\quad + \delta \frac{c - u_b^A \underline{\lambda} + u_a^A(r + \underline{\lambda})}{(r + \bar{\lambda})(r + \underline{\lambda})} \left(\frac{p}{1-p}\right)^{\frac{r+\underline{\lambda}}{\delta}} \left(\frac{1-\bar{p}^*}{\bar{p}^*}\right)^{\frac{r+\underline{\lambda}}{\delta}} p, \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} \underline{V}_{cf}(p) &= \frac{U_b(p)\bar{\lambda} + pu_b^A r - (1-p)c}{r + \bar{\lambda}} - \frac{p(c + ru_b^A)}{r + \underline{\lambda}} \\ &\quad + \delta \frac{\delta c + r(u_b^B \bar{\lambda} - u_a^B \underline{\lambda}) + (u_b^B - u_a^B)\underline{\lambda}\bar{\lambda}}{(r + \bar{\lambda})(r + \underline{\lambda})(2r + \bar{\lambda} + \underline{\lambda})} \left(\frac{p}{1-p}\right)^{\frac{r+\bar{\lambda}}{\delta}} \left(\frac{1-p^*}{p^*}\right)^{\frac{r+\bar{\lambda}}{\delta}} (1-p), \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \bar{V}_{cf}(p) &= \frac{U_a(p)\bar{\lambda} + (1-p)u_a^B r - pc}{r + \bar{\lambda}} - \frac{(1-p)(c + ru_a^B)}{r + \underline{\lambda}} \\ &\quad + \delta \frac{\delta c + r(u_a^A \bar{\lambda} - u_b^A \underline{\lambda}) + (u_a^A - u_b^A)\bar{\lambda}\underline{\lambda}}{(r + \bar{\lambda})(r + \underline{\lambda})(2r + \bar{\lambda} + \underline{\lambda})} \left(\frac{1-p}{p}\right)^{\frac{r+\bar{\lambda}}{\delta}} \left(\frac{p^*}{1-p^*}\right)^{\frac{r+\bar{\lambda}}{\delta}} p, \end{aligned} \quad (\text{B.4})$$

where

$$\begin{aligned} \frac{p^*}{1-p^*} &= \frac{c - u_a^B \underline{\lambda} + u_b^B(r + \underline{\lambda})}{u_a^A \bar{\lambda} - u_b^A(r + \bar{\lambda}) - c}, \\ \frac{1-\bar{p}^*}{\bar{p}^*} &= \frac{c - u_b^A \underline{\lambda} + u_a^A(r + \underline{\lambda})}{u_b^B \bar{\lambda} - u_a^B(r + \bar{\lambda}) - c}, \\ \frac{1-p^*}{p^*} &= \frac{\delta c + u_a^A(r + \underline{\lambda})\bar{\lambda} - u_b^A(r + \bar{\lambda})\underline{\lambda}}{\delta c + u_b^B(r + \underline{\lambda})\bar{\lambda} - u_a^B(r + \bar{\lambda})\underline{\lambda}}. \end{aligned}$$

If $r = \underline{\lambda} = 0$ we have

$$\begin{aligned} \underline{V}_{ct}(p) &= pu_a^A + (1-p)u_b^B + \frac{c}{\lambda} \left(\log \left(\frac{1-p}{p} \frac{p^*}{1-p^*} \right) (1-p) - 1 \right), \\ \bar{V}_{ct}(p) &= pu_a^A + (1-p)u_b^B + \frac{c}{\lambda} \left(\log \left(\frac{p}{1-p} \frac{1-\bar{p}^*}{\bar{p}^*} \right) p - 1 \right), \\ \underline{V}_{cf}(p) &= pu_a^A + (1-p)u_b^B + \frac{c}{\lambda} \left(\log \left(\frac{p}{1-p} \right) p - 2p - 1 \right), \end{aligned}$$

$$\bar{V}_{cf}(p) = pu_a^A + (1-p)u_b^B + \frac{c}{\lambda} \left(\log \left(\frac{1-p}{p} \right) (1-p) - 2(1-p) - 1 \right).$$

At p^* we have the following expressions for the contradictory branches:

$$\begin{aligned} \underline{V}_{ct}(p^*) &= \frac{1}{r(r+\lambda)} \left(\frac{(c+u_b^B r)(c+u_a^A r)}{(c+u_b^B r) + (c+u_a^A r)} \left(\frac{c+ru_a^A}{\lambda u_a^A - (r+\lambda)u_b^A - c} \right)^{\frac{r}{\lambda}} + \frac{u_a^A u_b^B r^2 - c^2}{(c+u_b^B r) + (c+u_a^A r)} \lambda - cr \right) \\ \bar{V}_{ct}(p^*) &= \frac{1}{r(r+\lambda)} \left(\frac{(c+u_b^B r)(c+u_a^A r)}{(c+u_b^B r) + (c+u_a^A r)} \left(\frac{c+ru_b^B}{\lambda u_b^B - (r+\lambda)u_a^B - c} \right)^{\frac{r}{\lambda}} + \frac{u_a^A u_b^B r^2 - c^2}{(c+u_b^B r) + (c+u_a^A r)} \lambda - cr \right) \end{aligned}$$

B.2 Proof of Proposition 1

Proof of Proposition 1. (a) By (A.14), \underline{p}^* and \bar{p}^* are given by the intersections of $U(p)$ and $\widehat{U}(p)$. Since $U(p)$ is independent of r and c , and $\widehat{U}(p)$ is strictly decreasing in both parameters, the experimentation region expands as r or c fall. As $(r, c) \rightarrow (0, 0)$, we have $\widehat{U}(p) \rightarrow U(p)$ for $p \in \{0, 1\}$ (since $\underline{\lambda} = 0$ in the baseline model), hence the experimentation region converges to $(0, 1)$.

(b) The dependence of \underline{p}^* and \bar{p}^* on u_b^A and u_a^B is straightforward from the expressions for the cutoffs in (A.10) and (A.11), where we set $\underline{\lambda} = 0$ for the baseline model (note that $\bar{p}^* < 1$ implies that the numerator of \bar{p}^* in (A.11) is positive).

(c) By (A.14), \underline{p}^* is the intersection between $U_b(p)$ and $\widehat{U}(p)$. The former is independent of u_a^A and the latter is increasing in u_a^A . Hence $\partial \underline{p}^* / \partial u_a^A < 0$. Also by (A.14), \bar{p}^* is the intersection between $U_a(p)$ and $\widehat{U}(p)$. In the baseline model we have

$$\frac{\partial U_a(p)}{\partial u_a^A} = p > \frac{\lambda}{r+\lambda} p = \frac{\partial \widehat{U}(p)}{\partial u_a^A}.$$

This implies that $\partial \bar{p}^* / \partial u_a^A < 0$. The comparative statics with respect to u_b^B is derived similarly.

(d) We prove $\partial \bar{p} / \partial u_b^A > 0$, the other case follows from a symmetric argument. Consider $\bar{V}_{ct}(p)$. Since the right branch of the contradictory value function is obtained from a strategy that takes action b only if a signal has been received, its value is independent of u_b^A . (This can also be seen from (B.2) if we set $\underline{\lambda} = 0$ for the baseline model). On the other hand we have $\partial \underline{V}_{ct}(p) / \partial u_b^A > 0$ from (B.1) (again we set $\underline{\lambda} = 0$ for the baseline model). Therefore the point of intersection of \underline{V}_{ct} and \bar{V}_{ct} is increasing in u_b^A .

(e) It is clear from (A.20) that \underline{c} is decreasing in u_b^A and u_a^B . Therefore, it suffices to consider the case that $c < \underline{c}$. We prove that $\underline{p} \rightarrow 0$ monotonically as $u_b^A \rightarrow -\infty$. If a confirmatory region exists, $\underline{p} \in (\underline{p}^*, p^*)$ is defined as the unique intersection between $\underline{V}_{cf}(p)$ and $\underline{V}_{ct}(p)$. Note that in the baseline model, $\underline{V}_{cf}(p)$ is independent of u_b^A since the confirmatory strategy never leads to a mistake. As in (d) we have $\partial \underline{V}_{ct}(p) / \partial u_b^A > 0$. Moreover, Lemma 5 shows that $\underline{V}_{ct}(p)$ crosses $\underline{V}_{cf}(p)$ from above at \underline{p} . Since \underline{V}_{cf} is

independent of u_b^A this implies that of \underline{p} is monotonically increasing in u_b^A .

Since \underline{p} is bounded from below, there exists $q = \lim_{u_b^A \rightarrow -\infty} \underline{p} < p^*$. Suppose by contradiction that $q > 0$. Notice that for $p \in [q, p^*]$, $\underline{V}_{ct}(p) \rightarrow \underline{V}_{ct}^{\text{lim}}(p)$ as $u_b^A \rightarrow -\infty$, where using $\underline{p}^*/(1 - \underline{p}^*) \rightarrow 0$ we get

$$\underline{V}_{ct}^{\text{lim}}(p) = \frac{\lambda u_a^A p r - \lambda c(1 - p) - cr}{(r + \lambda)r}.$$

Note that the convergence is uniform on $[q, p^*]$ since $q > 0$. Simple algebra yields

$$\begin{aligned} \underline{V}_{ct}^{\text{lim}}(p^*) &\leq \bar{U}(p^*), \\ \text{and } \underline{V}_{ct}^{\text{lim}'}(p) &> \bar{U}'(p). \end{aligned}$$

Since $\underline{V}_{ct}^{\text{lim}}(p)$ is linear in p , this implies that $\underline{V}_{cf}(q) \geq \bar{U}(q) > \underline{V}_{ct}^{\text{lim}}(q)$ which is a contradiction and we must have $q = 0$. The proof for \bar{p} is essentially the same. \square

B.3 Proof of Proposition 2

Proof of Proposition 2. (a) We have

$$\frac{\partial \bar{c}}{\partial r} = -\frac{u_a^A u_b^B - u_b^A u_a^B}{(u_a^A + u_b^B) - (u_b^A + u_a^B)},$$

and hence $\text{sgn}(\partial \bar{c} / \partial r) = \text{sgn}(u_b^A u_a^B - u_a^A u_b^B)$. It is straightforward to verify that $U(\hat{p}) > 0$ if and only if $u_a^A u_b^B - u_b^A u_a^B > 0$.

(b) Denoting $Z(\rho) := (\rho + 1) / \left(1 + (2\rho + 1)^{\frac{1}{\rho}}\right)$, we have

$$\frac{\partial \underline{c}}{\partial r} = \begin{cases} Z'(r/\lambda) (u_a^A - u_b^A) - r u_a^A & \text{if } (\lambda Z(r/\lambda) - r) (u_a^A - u_b^B) - \lambda Z(r/\lambda) (u_b^A - u_a^B) < 0, \\ Z'(r/\lambda) (u_b^B - u_a^B) - r u_b^B & \text{if } (\lambda Z(r/\lambda) - r) (u_a^A - u_b^B) - \lambda Z(r/\lambda) (u_b^A - u_a^B) > 0. \end{cases}$$

Consider the first case. Since $Z'(\rho) \in \left[(1 + 3e^2) / (1 + e^2)^2, 1/2\right]$,

$$Z'(r/\lambda) (u_a^A - u_b^A) - u_a^A < \frac{1}{2} (u_a^A - u_b^A) - u_a^A = -\frac{1}{2} (u_a^A + u_b^A),$$

which is negative if $u_a^A > |u_b^A|$. Conversely, if u_b^A is sufficiently negative $Z'(r/\lambda) (u_a^A - u_b^A) - u_a^A > 0$. The argument for the second case is similar. \square

B.4 Proofs for the Single Experimentation Property

In this Section we prove that (N-SEP) is sufficient for the SEP. To this end, consider the problem of a decision maker who is *not* forced to take an action after the first signal

that she receives. We call this the *multi-experimentation problem*. We have the following result.

Proposition 8. *If (N-SEP) holds, then $V^*(p)$ in (A.1) coincides with the value function of the multi-experimentation problem.*

Proof of Proposition 8. The HJB equation for the multi-experimentation problem is the following variational inequality:

$$\max \left\{ -c - rV(p) + \max_{\alpha \in [0,1]} \left\{ \begin{array}{l} \alpha \lambda^A(p) (V(q^A(p)) - V(p)) \\ +(1 - \alpha) \lambda^B(p) (V(q^B(p)) - V(p)) \\ - (2\alpha_t - 1) \delta p(1 - p) V'(p) \end{array} \right\}, U(p) - V(p) \right\} = 0. \quad (\text{B.5})$$

In the following, we write $V(p) = V_{Env}(p)$ to simplify the notation. We first show that $V(p)$ is a viscosity solution of (B.5). If (EXP) is violated, $V(p) = U(p)$ for all $p \in [0, 1]$ and hence (B.5) is equivalent to (A.3). We have shown in the proof of Proposition 7 that $V(p) = U(p)$ is a viscosity solution of (A.3) if (EXP) is violated. If (EXP) is satisfied and (N-SEP) holds, then (B.5) coincides with (A.3) for all $p \in [\underline{p}^*, \bar{p}^*]$. Therefore, given the arguments in the proof of Proposition 7 there is nothing left to be shown for $p \in [\underline{p}^*, \bar{p}^*]$. For $p \notin [\underline{p}^*, \bar{p}^*]$ on the other hand, the optimal strategy for (A.1) yields $V(p) = U(p)$. We need therefore need to show

$$c + rU(p) \geq \max_{\alpha \in \{0,1\}} \left\{ \begin{array}{l} \alpha \lambda^A(p) (V(q^A(p)) - U(p)) + (1 - \alpha) \lambda^B(p) (V(q^B(p)) - U(p)) \\ - (2\alpha_t - 1) \delta p(1 - p) U'(p) \end{array} \right\},$$

for all $p \notin [\underline{p}^*, \bar{p}^*]$, where we have used the linearity of the objective in α to restrict attention to $\alpha \in \{0, 1\}$. For $\alpha = 1$, we consider several cases. Case A: $q^A(p) \geq \bar{p}^*$. In this case we have

$$c + rU(p) \geq \lambda^A(p) (U_a(q^A(p)) - U(p)) - \delta p(1 - p) U'(p).$$

This is satisfied since V is a viscosity solution of (A.3).

Case B: $q^A(p) \leq \underline{p}^*$. In this case we have $c + rU(p) = c + rU_b(p) \geq \lambda^A(p) (U_b(q^A(p)) - U_b(p)) - \delta p(1 - p) U'_b(p) = 0$, where the inequality follows from Lemma 7 since $q^A(p) \leq \underline{p}^*$ implies $p < \underline{p}^*$.

Case C: It remains to consider p such that $q^A(p) \in (\underline{p}^*, \bar{p}^*)$ where we have

$$\begin{aligned} c + rU_b(p) &\geq \lambda^A(p) (V(q^A(p)) - U_b(p)) - \delta p(1 - p) U'_b(p) \\ \iff c + rU_b(p) &\geq \lambda^A(p) \left(\widehat{U}(q^A(p)) - U_b(p) \right) + \lambda^B(p) (U_b(q^B(p)) - U_b(p)). \end{aligned} \quad (\text{B.6})$$

It is easy to verify that both sides of (B.6) are linear in p . Hence it suffices to check that it holds for $p = q^B(\underline{p}^*)$ and $p = \underline{p}^* > q^B(\bar{p}^*)$. At $p = q^B(\underline{p}^*)$ we have $\widehat{U}(q^A(p)) = U_b(\underline{p}^*)$ and

some algebra shows that the RHS of (B.6) is zero. For $p = \underline{p}^*$ we have $U_b(\underline{p}^*) = \widehat{U}(\underline{p}^*)$. Inserting this into (B.6) and some algebra shows that (B.6) at $p = \underline{p}^*$ is equivalent to (A.22) and therefore satisfied if (N-SEP) holds. Symmetric arguments apply for $\alpha = 0$. We have therefore shown that $V(p)$ is a viscosity solution of (B.5).

The value function must necessarily be a viscosity solution of (B.5) (see, e.g., Theorem 10.8 in Oksendal and Sulem, 2009). While we are not aware of a statement of sufficiency that covers precisely our model, the arguments in Soner (1986) can be easily extended to show uniqueness of the viscosity solution to (B.5). This proves that $V(p)$ is the value function of the multi-experimentation problem. \square

For completeness we consider a relaxed problem where the DM observes all signals, i.e. $\alpha = \beta = 1$, and only has to decide when to stop. She may decide to stop at any point based on the history of signals that have arrived so far. In particular, she is not forced to stop at the first signal. We call this the *full attention stopping problem*. Clearly the value function is an upper bound for the value function of both the single-experimentation version and the multiple-experimentation version of the DM's problem. We do not use the following Lemma in the proofs of Theorems 1 and 2 but it is useful since it demonstrates that at \underline{p}^* and \bar{p}^* , the DM with limited attention achieves the same value as a DM with unlimited attention.

Lemma 8. *If (N-SEP) and (EXP) are satisfied, the value function of the full attention stopping problem is $\widehat{V}(p) = \max \{U(p), \widehat{U}(p)\}$.*

Proof. The HJB equation for this problem is

$$\max \left\{ \left(\lambda^A(p) \widehat{V}(q^A(p)) + \lambda^B(p) \widehat{V}(q^B(p)) \right) - \left(c + (r + \bar{\lambda} + \underline{\lambda}) \widehat{V}(p) \right), U(p) - \widehat{V}(p) \right\} = 0.$$

If (N-SEP) holds, we have $\widehat{V}(q^A(p)) = U_a(q^A(p))$ and $\widehat{V}(q^B(p)) = U_b(q^B(p))$ for all $p \in [\underline{p}^*, \bar{p}^*]$. Therefore the HJB equation is satisfied for $\widehat{V}(p) = \widehat{U}(p)$.

It remains to show that

$$c + (r + \bar{\lambda} + \underline{\lambda}) \widehat{V}(p) \geq \lambda^A(p) \widehat{V}(q^A(p)) + \lambda^B(p) \widehat{V}(q^B(p)) \quad (\text{B.7})$$

for $p \notin [\underline{p}^*, \bar{p}^*]$. We first consider $p \in [q^B(\bar{p}^*), \underline{p}^*]$. In this case (B.7) is equivalent to

$$c + (r + \bar{\lambda} + \underline{\lambda}) U_b(p) \geq \lambda^A(p) U_a(q^A(p)) + \lambda^B(p) U_b(q^B(p)).$$

Some algebra shows that this inequality is equivalent to $p \leq \underline{p}^*$.

Next consider $p \leq q^B(\underline{p}^*)$. In this case, we have $\widehat{V}(q^A(p)) = U_b(q^A(p))$ and the RHS of (B.7) is

$$\lambda^A(p) U_b(q^A(p)) + \lambda^B(p) U_b(q^B(p)) = (\bar{\lambda} + \underline{\lambda}) U_b(p) \leq c + (r + \bar{\lambda} + \underline{\lambda}) U_b(p),$$

where the inequality follows from Lemma 7. Hence (B.7) is satisfied for $p \leq q^B(\underline{p}^*)$.

Finally consider $p \in [q^B(\underline{p}^*), q^B(\bar{p}^*)]$. In this case (B.7) is equivalent to

$$\begin{aligned} c + (r + \bar{\lambda} + \underline{\lambda})U_b(p) &\geq \lambda^A(p)\widehat{U}(q^A(p)) + \lambda^B(p)U_b(q^B(p)) \\ \iff c + rU_b(p) &\geq \lambda^A(p)\left(\widehat{U}(q^A(p)) - U_b(p)\right) + \lambda^B(p)\left(U_b(q^B(p)) - U_b(p)\right). \end{aligned}$$

This holds by exactly the same argument as for (B.6) in the proof of Proposition 8. This proves that $\widehat{V}(p) = \max\{U(p), \widehat{U}(p)\}$. \square

B.5 Proofs for Section 5.2

B.5.1 Proof of Proposition 3

Proof of Proposition 3. Without loss, fix any $p \in (\underline{p}, \bar{p})$, and consider a history that results in the DM choosing a . The DM's belief at the time of choosing a must satisfy $q \geq q^A(p^*)$. The same action can be chosen starting from $p' \in (\underline{p}^*, \underline{p}) \cup (\bar{p}, \bar{p}^*)$ in two different cases: either (i) the initial belief is $p' \in (\underline{p}^*, \underline{p})$ and an A -signal is obtained; or (ii) the initial belief is $p' \in (\bar{p}, \bar{p}^*)$ and no signal is observed until the belief drifts to \bar{p}^* . In case (i), $p' < p$ and hence $q^A(p') < q^A(p)$, so the result follows. In case (ii), the result holds since $\bar{p}^* < q^A(p^*)$, by SEP. \square

B.5.2 Proof of Proposition 4

Proof of Proposition 4. At p^* the DM uses $\alpha = 1/2$. Hence the arrival rate of a signal is $(\underline{\lambda} + \bar{\lambda})/2$ and the expected delay until the DM takes an action is given by the expectation of the exponential distribution:

$$\tau(p^*) = \frac{2}{\underline{\lambda} + \bar{\lambda}}.$$

For $p_0 \in (p^*, \bar{p})$

$$\tau(p_0) = \int_0^t s(p_0 \bar{\lambda} e^{-\bar{\lambda}s} + (1 - p_0) \underline{\lambda} e^{-\underline{\lambda}s}) ds + (p_0 e^{-\bar{\lambda}t} + (1 - p_0) e^{-\underline{\lambda}t})(\tau(p_t) + t).$$

Differentiating both sides by t yields

$$0 = (p_0 e^{-\bar{\lambda}t} + (1 - p_0) e^{-\underline{\lambda}t})(\tau'(p_t) \dot{p}_t + 1) - (\bar{\lambda} p e^{-\bar{\lambda}t} + \underline{\lambda} (1 - p) e^{-\underline{\lambda}t}) \tau(p_t),$$

and evaluating at $t = 0$ gives:

$$\tau'(p_0) = \frac{1 - (\bar{\lambda} p + \underline{\lambda} (1 - p)) \tau(p)}{p(1 - p) \delta}.$$

Solving this differential equation with boundary condition $\tau(p^*) = 2/(\underline{\lambda} + \bar{\lambda})$ and some algebra yields

$$\tau''(p) < 0 \iff p^* < \frac{\bar{\lambda}}{\underline{\lambda} + \bar{\lambda}}.$$

Moreover the right derivative of τ at p^* is given by

$$\tau'(p_+^*) = \frac{1 - \frac{2(\bar{\lambda}p^* + \underline{\lambda}(1-p^*))}{\underline{\lambda} + \bar{\lambda}}}{p^*(1-p^*)\delta} = \frac{(1-2p^*)}{p^*(1-p^*)(\underline{\lambda} + \bar{\lambda})}.$$

Using similar steps for $p_0 \in (p, p^*)$ we have

$$\tau'(p_t) = \frac{(\underline{\lambda}p + \bar{\lambda}(1-p))\tau(p_t) - 1}{p(1-p)\delta},$$

and

$$\tau''(p) < 0 \iff p^* > \frac{\underline{\lambda}}{\underline{\lambda} + \bar{\lambda}}.$$

The left derivative of τ at p^* is given by

$$\tau'(p_-^*) = \frac{\frac{2(\underline{\lambda}p^* + \bar{\lambda}(1-p^*))}{\underline{\lambda} + \bar{\lambda}} - 1}{p^*(1-p^*)\delta} = \frac{(1-2p^*)}{p^*(1-p^*)(\underline{\lambda} + \bar{\lambda})}.$$

Since τ is concave on (p, p^*) and (p^*, \bar{p}) and $\tau'(p_-^*) = \tau'(p_+^*)$, τ is concave on (p, \bar{p}) .

To show that τ is quasi-concave, it remains to show that τ is decreasing on $[\bar{p}, \bar{p}^*]$ and increasing on $[p^*, p]$. Since the argument is essentially the same for both cases, we consider $[\bar{p}, \bar{p}^*]$. The expected delay implied by the contradictory strategy is

$$\tau(p) = \int_0^{\bar{T}^*(p)} s(p\underline{\lambda}e^{-\underline{\lambda}s} + (1-p)\bar{\lambda}e^{-\bar{\lambda}s})ds + (pe^{-\underline{\lambda}\bar{T}^*(p)} + (1-p)e^{-\bar{\lambda}\bar{T}^*(p)})\bar{T}^*(p).$$

where $\bar{T}^*(p)$ is the time it takes for the belief to reach \bar{p}^* in the absence of a signal if the DM follows the contradictory strategy (i.e., seeks B -signals). Since $\bar{T}^*(p)$ is decreasing in p we have

$$\tau'(p) = (pe^{-\underline{\lambda}\bar{T}^*} + (1-p)e^{-\bar{\lambda}\bar{T}^*})\bar{T}^{*'}(p) < 0.$$

Therefore it remains to show that $\tau(\bar{p}_-) \geq \tau(\bar{p}_+)$.

If at \bar{p} , the DM follows the contradictory strategy, she enjoys the payoff of

$$e^{-r\bar{T}^*(\bar{p})}(1-H(\bar{T}^*(\bar{p})))U_a(\bar{p}^*) + \int_0^{\bar{T}^*(\bar{p})} e^{-rt}(U_b(q_b(t))h(t) - c(1-H(t)))dt,$$

where H is the distribution of time at which the DM makes a decision, and $q_b(t)$ is the

posterior she has at t after observing B -signal.

Let $x := \frac{e^{-r\bar{T}^*(\bar{p})}(1-H(T))}{e^{-r\bar{T}^*(\bar{p})}(1-H(T)) + \int_0^{\bar{T}^*(\bar{p})} e^{-rt}h(t)dt}$, and $\bar{q}_b := \frac{\int_0^{\bar{T}^*(\bar{p})} e^{-rt}h(t)q_b(t)dt}{\int_0^{\bar{T}^*(\bar{p})} e^{-rt}h(t)dt}$. We can rewrite the payoff as

$$\begin{aligned} & (e^{-r\bar{T}^*(\bar{p})}(1-H(T)) + \int_0^{\bar{T}^*(\bar{p})} e^{-rt}h(t)dt)[xU_a(\bar{p}^*) + (1-x)U_b(\bar{q}_b)] - c \int_0^{\bar{T}^*(\bar{p})} e^{-rt}(1-H(t))dt \\ = & (1-r \int_0^{\bar{T}^*(\bar{p})} e^{-rt}(1-H(t))dt)[xU_a(\bar{p}^*) + (1-x)U_b(\bar{q}_b)] - c \int_0^{\bar{T}^*(\bar{p})} e^{-rt}(1-H(t))dt, \end{aligned}$$

where the second equality is obtained by integration by parts. Since b decisions are made sooner, the discounting overweights \bar{q}_b relative to \bar{p}^* , so in general $\bar{p} > x\bar{p}^* + (1-x)\bar{q}_b$. If $r = 0$, then the martingale property will mean that $\bar{p} = x\bar{p}^* + (1-x)\bar{q}_b$. The expression for the payoff reduces to:

$$[xU_a(\bar{p}^*) + (1-x)U_b(\bar{q}_b)] - c \int_0^{\bar{T}^*(\bar{p})} (1-H(t))dt. \quad (\text{B.8})$$

Suppose instead that the DM follows the confirmatory strategy. In this case her expected payoff (for $r = 0$) is given by

$$\int_0^\infty (g(t)[z_t U_a(q'_a(t)) + (1-z_t)U_b(q'_b(t))] - c(1-G(t))) dt,$$

where G is the distribution of time at which the DM makes a decision, z_t is the probability that the decision is a , $q'_a(t)$ and $q'_b(t)$ are the posteriors she has when choosing a and b , respectively, at t .

Let $y := \int_0^\infty z_t g(t)dt$, $\bar{q}'_a := \frac{\int_0^\infty z_t q'_a(t)g(t)dt}{\int_0^\infty z_t g(t)dt}$, and $\bar{q}'_b := \frac{\int_0^\infty (1-z_t)q'_b(t)g(t)dt}{\int_0^\infty z_t g(t)dt}$. Using the linearity of U_x , we can rewrite the confirmatory payoff as

$$[yU_a(\bar{q}'_a) + (1-y)U_b(\bar{q}'_b)] - c \int_0^\infty (1-G(t))dt. \quad (\text{B.9})$$

By the martingale property, we have $y\bar{q}'_a + (1-y)\bar{q}'_b = p_0$.

Proposition 3 implies that $\bar{q}'_a > \bar{p}^*$ and $\bar{q}'_b < \bar{q}_b$, so the distribution of posteriors under the confirmatory strategy is a mean-preserving spread of the distribution under the contradictory strategy. This means that the first term in (B.9) exceeds the first term in (B.8). Since at \bar{p} , the DM is indifferent between both strategies we must have

$$\int_0^{\bar{T}^*(\bar{p})} (1-H(t))dt < \int_0^\infty (1-G(t))dt,$$

i.e., the DM will take a longer time for decision if she chooses a confirmatory strategy instead. This proves case 2 of the Proposition for $r = 0$. By continuity the result extends

to r in a neighborhood of zero. For case 1, it suffices to invoke the argument used for (\bar{p}, \bar{p}^*) for the whole interval (\check{p}, \bar{p}^*) . \square

B.5.3 Proof of Proposition 5

Proof of Proposition 5. Take a DM with a prior $p_0 \in [\underline{p}^*, \underline{p}] \cup [\bar{p}^*, \bar{p}]$. Since the belief drifts down in the event of no decision (i.e., no discovery), conditional on choosing a , a later decision is associated with a lower posterior, hence a lower probability that a is the right decision. A symmetric argument proves the second statement. \square

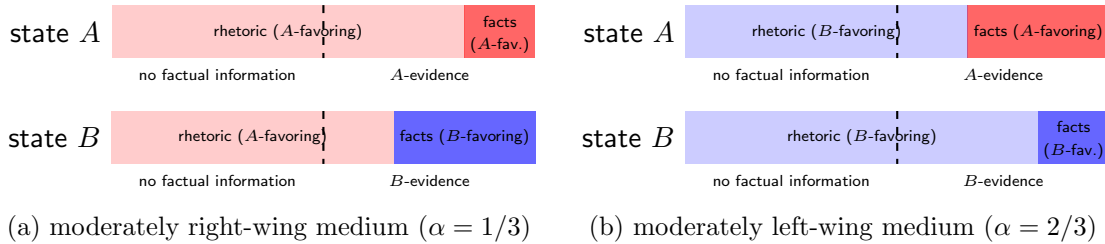


Figure 1: Content of news media with different biases. Labels below the bars describe the information the medium has learned. Colors and labels inside the bars describe informational content (the message) published in each case.

C Micro foundation of Media

In this section we provide a more detailed description of media of various types. Remember from Section 4.2 that each medium publishes content that is interpreted by the voter as one of three possible messages: (i) factual information in favor of A , (ii) factual information in favor of B , and (iii) partisan rhetoric. In Section 4.2, we have described the two most extreme media: the right-wing medium $\alpha = 0$ and the left-wing medium $\alpha = 1$. Now we also describe the messages receive by a subscriber of a medium with intermediate bias $\alpha \in (0, 1)$.

Figure 1 depicts the case of a moderately right-wing medium (Panel (a)) and a moderately left-wing medium (Panel (b)). In contrast to the extreme media, these outlets publish content that sends all three messages, but the rhetoric is still in favor of their bias and voters will find more factual information in favor, say, the left-wing candidate if they subscribe the moderately right-wing medium.

To provide a foundation of the bias of a news medium and the content it sends, we assume that each medium is composed of N journalists and we interpret α as the fraction of *left-wing journalists* employed by the medium. For example the purely-right medium $\alpha = 0$ employs no left-wing journalists. An individual journalist i has access to factual information confirming *each state* at rate λ/N , where N is the number of journalists employed by a medium. The journalist will always write about factual information when he learns about it. In the absence of factual information, however, he will produce a partisan rhetoric in favor of her own preferred candidate, and voters cannot distinguish between rhetoric and factual information. Hence, a voter who reads an article written by a right-leaning journalist effectively receives one of two messages: (i) a B -message which is factual information in favor of B , or (ii) an A -message which could be either rhetoric or factual information in favor of A . A left-leaning journalist is modeled analogously. Figure 2 depicts the messages sent by right- and left-leaning journalists in the two states.

A medium that only employs right-leaning journalists ($\alpha = 0$) therefore sends B -messages when there is conclusive evidence in favor of the left-wing candidate. Otherwise, it sends A -messages which causes a voter to gradually revise her belief in favor of the right-

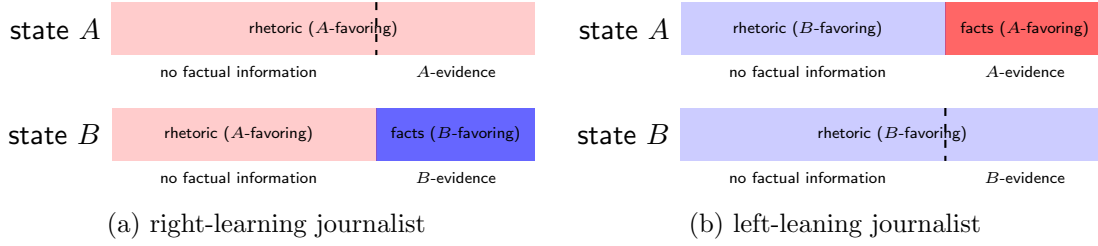


Figure 2: Bias of journalists

wing candidate. *A*-messages correspond to “no discovery” in the baseline model. Hence, for a DM with high p , subscribing to the news channel $\alpha = 0$ corresponds to contradictory learning.

A more moderate news-paper $\alpha \in (0, 1)$ employs both right- and left-leaning journalists and thus produces both *A*-evidence and *B*-evidence. We assume the voter knows the bias of each journalist within a medium α . She therefore interprets articles of right- (left-)leaning journalists in the medium as sending strong *B*-(*A*-)messages. In the absence of strong messages, the mix of weak messages the medium publishes will increase (decrease) the voter’s confidence in candidate *A* in a more right-leaning (left-leaning) medium. Updating is precisely as in the case of no discovery in the baseline model. A voter who reads a right-wing medium ($\alpha < 1/2$) is more likely to receive convincing factual information in favor of the left-wing candidate, but they are more likely to hear the rhetoric favoring *A* from right-leaning journalists (who outnumber the left-leaning journalists).

D Third Action

In this Appendix, we extend the model in Section 5 to include a third action $x = c$ which yields utilities $u_c^A \in (u_b^A, u_a^A)$ and $u_c^B \in (u_a^B, u_b^B)$. To make this extension interesting we assume that for some $p \in (0, 1)$, $U_c(p) > \max\{U_a(p), U_b(p)\}$. Otherwise the DM would never want to take the third action and the problem would be identical to the baseline model. To simplify the analysis we also assume that $U_c(p^*) > \bar{U}(p^*)$, $U_c(\bar{p}^*) < \hat{U}(\bar{p}^*)$, and $U_c(\underline{p}^*) < \hat{U}(\underline{p}^*)$ where $\bar{U}(p)$, $\hat{U}(p)$, \underline{p}^* , \bar{p}^* , and p^* are defined as before.

In this case, the confirmatory strategy as we constructed it in the baseline model is no longer optimal. Indeed, at p^* it is dominated by taking action c immediately. We therefore modify the confirmatory strategy to the following structure:

$$\left| \begin{array}{c} \xrightarrow{\alpha=0} \xrightarrow{\alpha=0} \xrightarrow{\alpha=0} \xrightarrow{\alpha=0} \tilde{p}_* \xrightarrow{\text{immediate action } c} \tilde{p}^* \xleftarrow{\alpha=1} \xleftarrow{\alpha=1} \xleftarrow{\alpha=1} \xleftarrow{\alpha=1} \end{array} \right|_1$$

Note that the absorbing belief p^* is now replaced by an immediate action region for action c with boundaries \tilde{p}_* and \tilde{p}^* . These are obtained by value-matching and smooth-pasting conditions that resemble those used to define \underline{p}^* and \bar{p}^* . For this purpose let us define two beliefs \hat{p}_1 and \hat{p}_2 at which the DM is indifferent between a and c and b and c , respectively.

$$U_a(\hat{p}_1) = U_c(\hat{p}_1) \quad \text{and} \quad U_b(\hat{p}_2) = U_c(\hat{p}_2).$$

Given our assumption we have $\hat{p}_1 < \hat{p}_2$. We define \tilde{p}^* by imposing smooth pasting and value matching with $U_c(p)$ in (A.7)

$$c + rU_c(\tilde{p}^*) = \lambda^A(\tilde{p}^*) (U_a(q^A(\tilde{p}^*)) - U_c(\tilde{p}^*)) - \delta\tilde{p}^*(1 - \tilde{p}^*)U_c'(\tilde{p}^*). \quad (\text{D.1})$$

Similarly we define \tilde{p}_* by imposing smooth pasting and value matching with $U_c(p)$ in (A.6)

$$c + rU_c(\tilde{p}_*) = \lambda^B(\tilde{p}_*) (U_b(q^B(\tilde{p}_*)) - U_c(\tilde{p}_*)) + \delta\tilde{p}_*(1 - \tilde{p}_*)U_c'(\tilde{p}_*). \quad (\text{D.2})$$

Note that if action b in the baseline model is replaced by c , we can apply (A.14) to characterize \tilde{p}^* as the intersection of $U_c(p)$ and

$$\hat{U}^{ac}(p) = \frac{\lambda^A(p)U_a(q^A(p)) + \lambda^B(p)U_c(q^B(p)) - c}{r + \bar{\lambda} + \underline{\lambda}}.$$

Similarly, \tilde{p}_* is characterized as the intersection of $U_c(p)$ and

$$\hat{U}^{bc}(p) = \frac{\lambda^A(p)U_c(q^A(p)) + \lambda^B(p)U_b(q^B(p)) - c}{r + \bar{\lambda} + \underline{\lambda}}.$$

The cutoffs are given by

$$\begin{aligned}\tilde{p}^* &= \frac{(u_c^B - u_a^B) \underline{\lambda} + u_c^B r + c}{r(u_c^B - u_c^A) + (u_a^A - u_c^A) \bar{\lambda} + (u_c^B - u_a^B) \underline{\lambda}}, \\ \tilde{p}_* &= \frac{(u_b^B - u_c^B) \bar{\lambda} - u_c^B r - c}{r(u_c^A - u_c^B) + (u_b^B - u_c^B) \bar{\lambda} + (u_c^A - u_b^A) \underline{\lambda}}.\end{aligned}$$

To simplify the analysis further, we focus on the case that $\tilde{p}_* \leq \tilde{p}^*$. In this case, the value of the modified confirmatory strategy is given by:

$$\tilde{V}_{cf}(p) := \begin{cases} V_0(p; \tilde{p}_*, U_c(\tilde{p}_*)), & \text{for } p \leq \tilde{p}_*, \\ U_c(p), & \text{for } p \in (\tilde{p}_*, \tilde{p}^*), \\ V_1(p; \tilde{p}^*, U_c(\tilde{p}^*)), & \text{for } p \geq \tilde{p}^*. \end{cases}$$

The Lemmas leading to the upper envelope characterization of the value function in Theorem 2 depend on the properties of branches defined by particular solutions to (A.6) and (A.7). Therefore the same steps can be applied in this extension and we obtain that the value function of the extended problem is given by:

$$V(p) = \max \left\{ V_{ct}(p), \tilde{V}_{cf}(p) \right\}.$$

E Signals with Varying Arrival Rates

As an extension to the baseline model in Section 2, we show that the general structure of the solution is preserved in a model with a richer set of news sources which allows us to capture the trade-off between skewness and informativeness discussed in Section 4.2.

A news source is now characterized by two parameters (λ^A, λ^B) . If an amount of attention α is directed to a news-source given by (λ^A, λ^B) , the DM will receive a signal that confirms state A with Poisson arrival rate $\lambda^A\alpha$ if the state is indeed A and she will receive a signal that confirms state B with Poisson arrival rate $\lambda^B\alpha$ if the state is B . Hence, when allocating her attention over two news sources with parameters (λ^A, λ^B) and $(\lambda^{A'}, \lambda^{B'})$ with attention levels α and $\alpha' = 1 - \alpha$, the DM will receive a signal that confirms A with Poisson rate $\bar{\lambda}^A = \alpha\lambda^A + (1 - \alpha)\lambda^{A'}$, and a signal that confirms B with Poisson rate $\bar{\lambda}^B = \alpha\lambda^B + (1 - \alpha)\lambda^{B'}$. The set of feasible arrival rates (λ^A, λ^B) is thus a weakly convex subset of \mathbb{R}_+ that is given by the weakly concave upper bound $\Gamma(\lambda^A)$:

$$\{(\lambda^A, \lambda^B) \in \mathbb{R}_+ \mid \lambda^B \leq \Gamma(\lambda^A)\}.$$

Note that in the baseline model studied before we had $\Gamma(\lambda^A) = \lambda - \lambda^A$, which is the linear boundary that is spanned by the two primitive news sources given by $(\lambda, 0)$ and $(0, \lambda)$.

Clearly, the DM will only chose arrival rates $(\lambda^A, \Gamma(\lambda^A))$ so we can describe her choice by λ^A . To simplify the notation we omit the superscript and write λ instead of λ^A . Moreover we assume that $\lambda \in [0, 1]$.¹ We maintain the following assumptions about the function Γ .

Assumption 1. $\Gamma : [0, 1] \rightarrow [0, 1]$ is twice continuously differentiable, strictly decreasing, strictly convex, and satisfies $\Gamma(0) = 1$, $\Gamma(1) = 0$ and $\Gamma'(\gamma) = -1$, where γ is the unique fixed point of Γ .

Example 1. A parametric example that we will use for numerical calculations is obtained by solving $\rho(\lambda + \Gamma) + (1 - \rho)(\lambda^2 + \Gamma^2)^{1/2} = 1$, where the parameter $\rho \in [0, 1]$ determines the curvature of Γ . For $\rho = 1$, we obtain the baseline model and for $\rho = 0$, the graph of Γ is a segment of a circle.

E.1 The Decision Maker's Problem

The DM's posterior evolves according to

$$\dot{p}_t = -p_t(1 - p_t)(\lambda_t - \Gamma(\lambda_t)), \tag{E.1}$$

¹The normalization of the upper bound is without loss of generality since only the ratios r/λ and c/λ matter.

The objective is given by

$$J((\lambda_t)_{t \geq 0}, T; p_0) := \left\{ \begin{aligned} & \int_0^T e^{-rt} P_t(p_0, (\lambda_\tau)) (p_t \lambda_t u_a^A + (1 - p_t) \Gamma(\lambda_t) u_b^B) dt \\ & + e^{-rT} P_T(p_0, (\lambda_\tau)) U(p_T) \end{aligned} \right\},$$

$$\text{where } P_t(p_0, (\lambda_\tau)) := p_0 e^{-\int_0^t \lambda_s ds} + (1 - p_0) e^{-\int_0^t \Gamma(\lambda_s) ds}.$$

The DM solves the problem (\mathcal{P}^Γ) given by:

$$V(p_0) := \sup_{((\lambda_t)_{t \geq 0}, T)} J((\lambda_t)_{t \geq 0}, T; p_0) \quad \text{s.t. (E.1), and } \lambda_t \in [0, 1]. \quad (\mathcal{P}^\Gamma)$$

We define

$$H(p, V(p), V'(p), \lambda) := \left\{ \begin{aligned} & \lambda p (u_a^A - V(p)) + \Gamma(\lambda)(1 - p) (u_b^B - V(p)) \\ & - p(1 - p)(\lambda - \Gamma(\lambda))V'(p) \end{aligned} \right\}.$$

The HJB equation for (\mathcal{P}^Γ) is

$$c + rV(p) = \max_{\lambda \in [0, 1]} H(p, V(p), V'(p), \lambda) \quad (\text{E.2})$$

The first-order condition is given by

$$\frac{\partial H(p, V(p), V'(p), \lambda)}{\partial \lambda} = \left\{ \begin{aligned} & p (u_a^A - V(p)) + \Gamma'(\lambda)(1 - p) (u_b^B - V(p)) \\ & - p(1 - p)(1 - \Gamma'(\lambda))V'(p) \end{aligned} \right\} = 0. \quad (\text{E.3})$$

For a given policy $\lambda(p)$, we obtain the differential equation

$$\begin{aligned} c + rV(p) &= H(p, V(p), V'(p), \lambda(p)) \\ \iff c + rV(p) &= \left\{ \begin{aligned} & \lambda(p)p (u_a^A - V(p)) + \Gamma(\lambda(p))(1 - p) (u_b^B - V(p)) \\ & - p(1 - p)(\lambda(p) - \Gamma(\lambda(p)))V'(p) \end{aligned} \right\}. \end{aligned} \quad (\text{E.4})$$

As in the baseline model, we will define two candidate value functions. For this purpose, we state the HJB equation for problems in which the DM is either restricted to choose $\lambda \geq \gamma$,

$$c + rV_+(p) = \max_{\lambda \in [\gamma, 1]} H(p, V_+(p), V'_+(p), \lambda), \quad (\text{E.5})$$

or $\lambda \leq \gamma$:

$$c + rV_-(p) = \max_{\lambda \in [0, \gamma]} H(p, V_-(p), V'_-(p), \lambda). \quad (\text{E.6})$$

we denote policies corresponding to solution to (E.5) and (E.6) by $\lambda_+(p)$ and $\lambda_-(p)$,

respectively.

E.2 Preliminary results

We first revisit some definitions made for the baseline model and the generalization in Section 5. The stationary strategy is now given by choosing $\lambda = \gamma$ until a signal arrives and then taking an optimal action according to the signal. The value of this strategy is now given by

$$\bar{U}(p) = \frac{\gamma}{r + \gamma} U^*(p) - \frac{c}{r + \gamma},$$

where

$$U^*(p) = pu_a^A + (1 - p)u_b^B$$

is the first best value that is achieved if the DM can learn the state without any delay.

As in the baseline model, we obtain a crossing condition for functions that satisfy (E.5) and (E.6) and a condition under which solutions to (E.5) and (E.6) satisfy (E.2).

Lemma 9 (Crossing Lemma). *Suppose $V_+(p)$ is \mathcal{C}^1 at p and satisfies (E.5) and $V_-(p)$ is \mathcal{C}^1 at p and satisfies (E.6). If $V_+(p) = V_-(p) \geq \bar{U}(p)$, then $V'_+(p) \leq V'_-(p)$.*

Proof of Lemma 9. Suppose $V(p) := V_+(p) = V_-(p) \geq \bar{U}(p)$ at p and denote the maximizers in (E.5) and (E.6) by $\lambda_+(p)$ and $\lambda_-(p)$ respectively.

From (E.5) and (E.6), we obtain

$$\begin{aligned} & p(1 - p)(\Gamma(\lambda_-(p)) - \lambda_-(p))(\lambda_+(p) - \Gamma(\lambda_+(p)))(V'_-(p) - V'_+(p)) \\ &= (\delta(p)r + \Delta(p)) \left[V(p) - \frac{\frac{\Delta(p)}{\delta(p)}}{\frac{\Delta(p)}{\delta(p)} + r} \bar{u} + \frac{1}{\frac{\Delta(p)}{\delta(p)} + r} c \right] \\ &\geq (\delta(p)r + \Delta(p)) \left[V(p) - \frac{\gamma}{\gamma + r} \bar{u} + \frac{1}{\gamma + r} c \right] \\ &= (\delta(p)r + \Delta(p)) [V(p) - \bar{U}(p)], \end{aligned}$$

where

$$\begin{aligned} \delta(p) &:= \Gamma(\lambda_-(p)) - \lambda_-(p) + \lambda_+(p) - \Gamma(\lambda_+(p)) > 0, \\ \Delta(p) &:= \lambda_+(p)\Gamma(\lambda_-(p)) - \lambda_-(p)\Gamma(\lambda_+(p)) > 0, \end{aligned}$$

since $\lambda_+(p) > \gamma > \lambda_-(p)$. The inequality can be seen as follows. First, one can verify that $(\Delta(p)/\delta(p), \Delta(p)/\delta(p))$ is the point of intersection between the forty-five degrees line and the line segment between two points, $(\lambda_-(p), \Gamma(\lambda_-(p)))$ and $(\lambda_+(p), \Gamma(\lambda_+(p)))$. Since Γ is concave, we must have $\Delta(p)/\delta(p) < \gamma$. Since $\delta(p), \Delta(p) \geq 0$, and since $V(p) \geq \frac{1}{1+r/\gamma} \bar{U}^*(p)$, the last expression is non-negative, as was to be shown. \square

Lemma 10 (Unimprovability). (a) Suppose $V_+(p)$ is \mathcal{C}^1 at p and satisfies (E.5). If $V_+(p) \geq \max\{\bar{U}(p), U(p)\}$, then $V_+(p)$ satisfies (E.2) at p .
(b) Suppose $V_-(p)$ is \mathcal{C}^1 at p and satisfies (E.6). If $V_-(p) \geq \max\{\bar{U}(p), U(p)\}$, then $V_-(p)$ satisfies (E.2) at p .

Proof of Lemma 10. We prove the first statement; the second follows symmetrically. Suppose the optimal policy satisfies $\lambda_+(p) > \gamma$. By the condition, it is not improvable by an immediate action or by any $\lambda \geq \gamma$. Hence, it suffices to show that it is not improvable by any $\lambda < \gamma$.

Substituting $V'_+(p)$ from (E.5) and rearranging we get

$$\begin{aligned} & H(p, V_+(p), V'_+(p), \lambda_+) - H(p, V_+(p), V'_+(p), \lambda_-) \\ &= \frac{\hat{\delta}(p)r + \hat{\Delta}(p)}{\lambda_+(p) - \Gamma(\lambda_+(p))} \left[V_+(p) - \frac{\frac{\hat{\Delta}(p)}{\hat{\delta}(p)}}{\frac{\hat{\Delta}(p)}{\hat{\delta}(p)} + r} \bar{u} + \frac{1}{\frac{\hat{\Delta}(p)}{\hat{\delta}(p)} + r} c \right] \\ &\geq \frac{\hat{\delta}(p)r + \hat{\Delta}(p)}{\lambda_+(p) - \Gamma(\lambda_+(p))} [V_+(p) - \bar{U}(p)], \end{aligned}$$

where

$$\hat{\delta}(p) := \Gamma(\lambda) - \lambda + \lambda_+(p) - \Gamma(\lambda_+(p)) \text{ and } \hat{\Delta}(p) := \lambda_+(p)\Gamma(\lambda) - \lambda\Gamma(\lambda_+(p)).$$

The inequality follows from the same observation as in the proof of Lemma (9). \square

Before constructing the value function for (\mathcal{P}^Γ) , we make one general observation about the boundaries of the experimentation region and the value of obtaining confirmatory evidence at the boundaries.

For this purpose we consider a model in which the DM has full attention. In this case we have $\lambda^A = 1 = \lambda^B$ and the DM only chooses when to stop. Note that Assumption 1 precludes the DM from choosing $\lambda^A = 1 = \lambda^B$ so the full attention model only serves as a hypothetical benchmark.

We have established in Lemma 8 that the value of this problem is given by

$$\hat{V}(p) := \max \left\{ U(p), \hat{U}(p) \right\},$$

where

$$\hat{U}(p) = \frac{1}{r+1} U^*(p) - \frac{c}{r+1}.$$

Lemma 8 requires that (N-SEP) is satisfied but since we consider a model with conclusive signals in this section, this condition is vacuous. Moreover, we note that by Assumption 1, $(\lambda, \Gamma(\lambda)) \leq (1, 1)$ for all $\lambda \in (0, 1)$. Therefore, $\hat{V}(p)$ is an upper bound for the value function of the problem (\mathcal{P}^Γ) .

Remember that in the baseline model, the boundaries of the experimentation region are given by the points of intersection between $\widehat{U}(p)$ and $U(p)$:

$$\widehat{U}(\bar{p}^*) = U_a(\bar{p}^*). \quad (\text{E.7})$$

$$\widehat{U}(\underline{p}^*) = U_b(\underline{p}^*). \quad (\text{E.8})$$

If (EXP) is satisfied, we have $\underline{p}^* < \bar{p}^*$. We now show that the value of (\mathcal{P}^Γ) is equal to \widehat{V} at these boundaries. This immediately shows that \underline{p}^* and \bar{p}^* are the boundaries of the experimentation region in (\mathcal{P}^Γ) . Moreover, we show that under Assumption 1, at these boundaries, the DM does not benefit from interior choices $\lambda \in (0, 1)$.

Proposition 9. *Suppose (EXP) is satisfied. Then \underline{p}^* and \bar{p}^* given by (E.7) and (E.8) are the boundaries of the experimentation region for the optimal solution to (\mathcal{P}^Γ) . At \underline{p}^* and \bar{p}^* , the value of (\mathcal{P}^Γ) coincides with the value of the baseline model and equals $\widehat{V}(p)$. The loss of restricting the DM to choose $\lambda \in \{0, 1\}$ vanishes as $p \downarrow \underline{p}^*$ and $p \uparrow \bar{p}^*$.*

Proof. If the DM is restricted to choose $\lambda \in \{0, 1\}$, her optimal strategy coincides with the optimal strategy in the baseline model. The value of the baseline model is a lower bound for the value of (\mathcal{P}^Γ) . Since at \underline{p}^* and \bar{p}^* the value of the baseline model coincides with the upper bound $\widehat{V}(p)$, it must also coincide with the value of (\mathcal{P}^Γ) . \square

Note that while Assumption 1 requires $\Gamma(\lambda) < 1$ for $\lambda > 0$, it does not rule out an Inada condition like $\lim_{\lambda \rightarrow 0} \Gamma(\lambda) = 0$. This shows that at the boundaries of the experimentation region, the value of a confirmatory signal is zero even if it is cost-less to obtain. We will see below when we characterize the value function that without an Inada condition, there exist neighborhoods of \underline{p}^* and \bar{p}^* such that the DM does not suffer any loss if in these neighborhoods she uses $\lambda = 1$ and $\lambda = 0$, respectively.

E.3 Construction of Solutions to the HJB equation

For the remainder of this section, we will focus on the cases that the payoffs are symmetric. This simplifies the derivations and is sufficient to understand the main features of the optimal solution in the extension. Formally we impose:

Assumption 2. *There exist $\bar{u} > \underline{u} > 0$ such that $u_a^A = u_b^B = \bar{u}$ and $u_b^A = u_a^B = \underline{u}$.*

In contrast to the baseline model, it may now be optimal to choose $\lambda \in (0, 1)$ for beliefs $p \in (\underline{p}^*, \bar{p}^*)$, i.e., in the interior of the experimentation region. For an interval where this is the case, we will obtain a differential equation for $\lambda(p)$ and furthermore an equation that expresses $V(p)$ as a function of $\lambda(p)$. We begin with the latter. To state the result in concise form we define

$$A(\lambda) := \frac{\Gamma(\lambda) - \Gamma'(\lambda)\lambda}{\Gamma(\lambda) - \Gamma'(\lambda)\lambda + r(1 - \Gamma'(\lambda))}, \quad \text{and} \quad B(\lambda) := \frac{1 - \Gamma'(\lambda)}{\Gamma(\lambda) - \Gamma'(\lambda)\lambda + r(1 - \Gamma'(\lambda))}.$$

A basic observation that we will use at several points is that these two functions are (inverse) U-shaped with (maximum) minimum at $\lambda = \gamma$.

Lemma 11. *If Assumption 1 is satisfied,*

$$A'(\lambda) > (<)0 \iff B'(\lambda) < (>)0 \iff \lambda > (<)\gamma.$$

Proof. The Lemma follows from straightforward algebra which we omit here. \square

Lemma 12. *Suppose Assumptions 1 and 2 are satisfied. If $p \in (0, 1)$, $V(p)$ is continuously differentiable at p and satisfies (E.2) with maximizer $\lambda(p) \neq \gamma$, then*

$$V(p) \geq A(\lambda(p))\bar{u} - B(\lambda(p))c \geq \bar{U}(p) \quad (\text{E.9})$$

If λ satisfies (E.3) at p , then the first inequality binds. The statement continues to hold if we replace V , λ , and (E.2), by V_+ , λ_+ and (E.5), or V_- , λ_- and (E.6).

Proof of Lemma 12. We define the LHS of (E.3) as

$$X := (p + (1 - p)\Gamma'(\lambda))(\bar{u} - V(p)) - p(1 - p)(1 - \Gamma'(\lambda))V'(p). \quad (\text{E.10})$$

Eliminating $V'(p)$ from (E.4) and (E.10) we obtain an expression for $V(p)$ in terms of $\lambda(p)$ and X :

$$V(p) = A(\lambda(p))\bar{u} - B(\lambda(p))c + \frac{X(\lambda - \Gamma(\lambda(p)))}{\Gamma(\lambda) - \Gamma'(\lambda)\lambda + r(1 - \Gamma'(\lambda))}.$$

If $\lambda(p)$ is a maximizer in (E.2), we must have

$$X \begin{cases} \geq 0 & \text{if } \lambda = 1, \\ = 0 & \text{if } \lambda \in (0, 1), \\ \leq 0 & \text{if } \lambda = 0. \end{cases}$$

Since $\lambda = 1$ implies $\lambda - \Gamma(\lambda(p)) > 0$ and $\lambda = 0$ implies $\lambda - \Gamma(\lambda(p)) < 0$ we have

$$V(p) \geq A(\lambda(p))\bar{u} - B(\lambda(p))c,$$

and the inequality holds with equality if $X = 0$ which is equivalent to λ satisfying (E.3). This proves the first inequality and the first statement.

The second inequality follows from Lemma 11 and $A(\gamma)\bar{u} - B(\gamma)c = \bar{U}(p)$, which is obtained from straightforward algebra. It is straightforward to adapt the proofs to V_+ and V_- . \square

Using Lemma 12 we can obtain an ODE for λ that holds whenever the optimal policy is interior, i.e., it satisfies (E.3).

Lemma 13. *Suppose Assumptions 1 and 2 are satisfied. If $p \in (0, 1)$, V is continuously differentiable at p and satisfies (E.4) and the maximizer is $\lambda(p) \neq \gamma$ and satisfies (E.3) at p , then*

$$\lambda'(p) = \frac{[p + (1-p)\Gamma'(\lambda(p))] [\Gamma(\lambda(p)) - \Gamma'(\lambda(p))\lambda(p) + r(1 - \Gamma'(\lambda(p)))]}{p(1-p)(\Gamma(\lambda(p)) - \lambda(p))\Gamma''(\lambda(p))}. \quad (\text{E.11})$$

The statement continues to hold if we replace V and λ , by V_+ and λ_+ , or V_- and λ_- .

Proof of Lemma 13. If $\lambda(p) \neq \gamma$ satisfies (E.3), then by Lemma 12

$$\begin{aligned} V(p) &= A(\lambda(p))\bar{u} - B(\lambda(p))c, \\ \text{and } V'(p) &= A'(\lambda(p))\lambda'(p)\bar{u} - B'(\lambda(p))\lambda'(p)c. \end{aligned}$$

Inserting these two equations in (E.4) and solving for $\lambda'(p)$ we get (E.11) \square

Next, we state a Lemma that identifies conditions under which the solution to (E.11) remains bounded away from $\lambda = 0$ or $\lambda = 1$.

Lemma 14. *Suppose Assumptions 1 and 2 are satisfied. Then there exists values $p^1 > 1/2$ and $p^0 < 1/2$ such that*

$$\begin{aligned} \lambda(p) = 1 &\Rightarrow \{\lambda'(p) < 0 \iff p < p^1\}, \\ \lambda(p) = 0 &\Rightarrow \{\lambda'(p) > 0 \iff p > p^0\}. \end{aligned}$$

Proof. Inserting $\lambda(p) = 1$ in (E.11) yields

$$\lambda'(p) = -\frac{(p + (1-p)\Gamma'(1))(r - (1-r)\Gamma'(1))}{p(1-p)\Gamma''(1)}.$$

Hence $\lambda'(p) < 0$ is equivalent to

$$p < p^1 = \frac{|\Gamma'(1)|}{1 + |\Gamma'(1)|}.$$

Since $|\Gamma'(1)| > 1$ $p^1 > 1/2$. The proof for $\lambda(p) = 0$ is similar. \square

Next, we show the following property that relates sufficiency of the FOC (E.3) to convexity of the value function.

Lemma 15. *Suppose Assumptions 1 and 2 are satisfied.*

(a) Let $W : [0, 1] \rightarrow \mathbb{R}$ be weakly convex and satisfy $W(p) = U(p)$ in neighborhoods of 0 and 1. Then $H(p, W(p), W'(p), \lambda)$ is weakly concave in λ for all p and strictly concave whenever $W(p) > U(p)$.

(b) Let $\lambda(p)$ be a solution to (E.11) such that $\lambda(p) \in (0, 1)$ at some p . Let

$$\pi(\ell) = \frac{(r + \ell) \Gamma'(\ell)}{(r + \ell) \Gamma'(\ell) - (r + \Gamma(\ell))}.$$

Then

$$\frac{\partial^2 [A(\lambda(p))\bar{u} - B(\lambda(p))c]}{\partial p^2} \geq 0 \text{ if } \begin{cases} \lambda(p) > \gamma \text{ and } p \leq \pi(\lambda(p)), \\ \text{or } \lambda(p) < \gamma \text{ and } p \leq \pi(\lambda(p)). \end{cases}$$

$\pi(\ell) > 1/2$ if $\ell > \gamma$, and $\pi(\ell) < 1/2$ if $\ell < \gamma$.

Proof. (a) Some algebra yields

$$\frac{\partial^2 H(p, W(p), W'(p), \lambda)}{\partial \lambda^2} \leq 0 \iff W(p) - pW'(p) \leq \bar{u}.$$

The latter inequality is satisfied under the assumptions on W and both are strict if $W(p) > U(p)$.

(b) Differentiating $A(\lambda(p))\bar{u} - B(\lambda(p))c$ with respect to p , substituting $\lambda'(p)$ from (E.11) and differentiating again yields (after some algebra):

$$\begin{aligned} & \frac{\partial^2 [A(\lambda(p))\bar{u} - B(\lambda(p))c]}{\partial p^2} < 0 \\ \iff & -\frac{(p^2 - (1-p)^2\Gamma'(\lambda(p)))(r + \Gamma(\lambda(p)) - (r + \lambda(p))\Gamma'(\lambda(p)))}{p(1-p)(r + p\lambda(p) + (1-p)\Gamma(\lambda(p)))\Gamma''(\lambda(p))} > \lambda'(p). \end{aligned}$$

Substituting $\lambda'(p)$ from (E.11) in the last line and rearranging we get

$$(\lambda(p) - \Gamma(\lambda(p)))(p[r + \Gamma(\lambda(p))] + (1-p)[r + \lambda(p)]\Gamma'(\lambda(p))) < 0.$$

Solving for p this yields an upper bound if $\lambda(p) > \gamma$ so that the first term is positive and a lower bound if $\lambda(p) < \gamma$. The bound is $\pi(\lambda(p))$ in both cases. If $\ell > \gamma > \Gamma(\ell)$ we have

$$\begin{aligned} \pi(\ell) &= \frac{(r + \ell) |\Gamma'(\ell)|}{(r + \ell) |\Gamma'(\ell)| + (r + \Gamma(\ell))} \\ &> \frac{(r + \ell) |\Gamma'(\ell)|}{(r + \ell) |\Gamma'(\ell)| + (r + \ell)} \\ &= \frac{|\Gamma'(\ell)|}{|\Gamma'(\ell)| + 1} \\ &> 1/2. \end{aligned}$$

where the last step follows because Assumption 1 implies that $|\Gamma'(\ell)| > 1$ if $\ell > \gamma$. Similarly we obtain $\pi(\ell) < 1/2$ if $\ell < \gamma$. \square

E.4 Solution Candidates

E.4.1 Contradictory Evidence

The first candidate is obtained by assuming that the DM uses a “contradictory” attention strategy. This involves seeking evidence contrary to the current belief. In contrast to the baseline model, where we choose $\lambda \in \{0, 1\}$, we will now also use interior values for λ . What remains true is that for the contradictory attention strategy, the arrival rate of signal that would contradict the current belief is higher. For instance, for low posterior beliefs p , the contradictory strategy involves $\lambda > \gamma$. We have already identified the boundaries of the experimentation region.

Lemma 16. *Suppose (EXP) is satisfied. Then \underline{p}^* and \bar{p}^* satisfy*

$$p^* := \inf \left\{ p \in [0, \hat{p}] \mid c + rU_b(p) \leq \max_{\lambda \in [\gamma, 1]} \left\{ \begin{array}{l} (\lambda p + \Gamma(\lambda)(1-p))(\bar{u} - U_b(p)) \\ -p(1-p)(\lambda - \Gamma(\lambda))U'_b(p) \end{array} \right\} \right\}, \quad (\text{E.12})$$

$$p^* := \sup \left\{ p \in (\hat{p}, 1] \mid c + rU_a(p) \leq \max_{\lambda \in [0, \gamma]} \left\{ \begin{array}{l} (\lambda p + \Gamma(\lambda)(1-p))(\bar{u} - U_a(p)) \\ -p(1-p)(\lambda - \Gamma(\lambda))U'_a(p) \end{array} \right\} \right\}, \quad (\text{E.13})$$

and the maximizers on the right-hand side are given by $\lambda = 1$ and $\lambda = 0$, respectively. Moreover,

$$U_b(\underline{p}^*) \geq A(1)\bar{u} - B(1)c,$$

and $U_a(\bar{p}^*) \geq A(1)\bar{u} - B(1)c.$

The first inequality is strict if and only if $\Gamma'(1)$ is finite. The second is strict if and only if $\Gamma'(0) < 0$.

Proof of Lemma 16. We only give the proof for \underline{p}^* , the other case is symmetric. Consider the maximization problem in (E.12). The derivative of the objective function simplifies to $p(\bar{u} - \underline{u})$. Therefore we can set $\lambda = 1$ and (E.12) reduces to the definition via smooth pasting and value matching in the baseline model.

The first inequality is equivalent to

$$\frac{1}{(1+r)\Gamma'(1) - r} \leq 0,$$

which holds under Assumption 1. The inequality is strict iff $\Gamma'(1)$ is finite. The second inequality is equivalent to

$$\frac{\Gamma'(0)}{1+r - r\Gamma'(0)} \leq 0,$$

which is strict if and only if $\Gamma'(0) < 0$. □

We are now ready to define the contradictory strategy. Given that we impose Assumption 1, we only describe the construction for the left branch which is used for $p \leq 1/2$. There are up to four intervals where the contradictory strategy takes a different form. First, for $p \leq \underline{p}^*$, the DM takes immediate action. Then there is an interval $(\underline{p}^*, \underline{q}^b]$ there the DM uses the contradictory strategy from the baseline model. \underline{q}^b is given by

$$\frac{\partial H(\underline{q}^b, \underline{V}_{cf}(\underline{q}^b), \underline{V}'_{cf}(\underline{q}^b), 1)}{\partial \lambda} = 0.$$

Rearranging this we get

$$\frac{(1+r)\Gamma'(\underline{q}^b)}{r - (1+r)\Gamma'(\underline{q}^b)} + \underline{q}^b + (1 - \underline{q}^b) \left(\frac{1 - \underline{q}^b}{\underline{q}^b} \frac{\underline{p}^*}{1 - \underline{p}^*} \right)^r = 0,$$

which is equivalent to

$$\underline{V}_{cf}(\underline{q}^b) = A(1)U^*(\underline{q}^b) - B(1)c.$$

By Lemma 16, $\underline{q}^b = \underline{p}^*$ if $\Gamma'(1)$ is infinite and otherwise $\underline{q}^b > \underline{p}^*$. If $\underline{q}^b \geq 1/2$ we define the contradictory strategy as in the baseline model. If $\underline{q}^b < 1/2$, Lemma 14 implies that $\lambda'(\underline{q}^b) < 0$ if we impose the boundary condition $\lambda(\underline{q}^b) = 1$. Denote the unique solution for $p \geq \underline{q}^b$ to (E.11) with $\lambda(\underline{q}^b) = 1$ by $\lambda(p; \underline{q}^b, 1)$. Since by Lemma 14, $\lambda'(p; p, 1) < 0$ for all $p \leq 1/2$, we have $\lambda(p; \underline{q}^b, 1) < 1$ for $p \in (\underline{q}^b, 1/2)$. Finally we need to take care of the possibility that there exists $\underline{q}^s \in (\underline{q}^b, 1/2]$ such that $\lambda(p; \underline{q}^b, 1) = \gamma$. If no such \underline{q}^s exists we set $\underline{q}^s = 1/2$. If Assumption 2 is satisfied, a symmetric construction can be used for the right branch with cutoffs $\bar{q}^b = 1 - \underline{q}^b$ and $\bar{q}^s = 1 - \underline{q}^s$.

We thus define the contradictory strategy as follows. For $p \notin (\underline{p}^*, \bar{p}^*)$: take the optimal immediate action. For $p \in (\underline{p}^*, \bar{p}^*)$, experiment according to the following attention strategy:

$$\lambda_{ct}^\Gamma(p) = \begin{cases} 1, & \text{if } p \in (\underline{p}^*, \underline{q}^b], \\ \lambda(p; \underline{q}^b, 1), & \text{if } p \in (\underline{q}^b, \underline{q}^s], \\ \gamma, & \text{if } p \in (\underline{q}^s, \bar{q}^s), \\ \lambda(p; \bar{q}^b, 0), & \text{if } p \in [\bar{q}^s, \bar{q}^b), \\ 1, & \text{if } p \in [\bar{q}^b, \bar{p}^*), \end{cases}$$

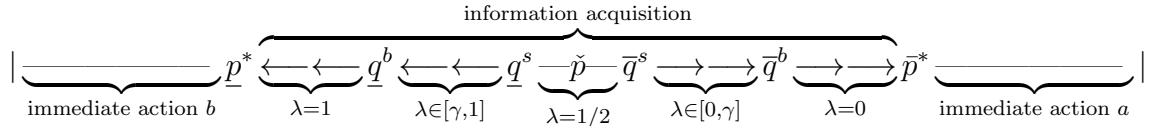
and take an action corresponding to the signal if one is received. The value of this strategy

is given by

$$V_{ct}^\Gamma(p) = \begin{cases} V_{ct}(p), & \text{if } p \leq \underline{q}^b, \\ A(\lambda(p; \underline{q}^b, 1)) U^*(p) - B(\lambda(p; \underline{q}^b, 1)) c, & \text{if } p \in (\underline{q}^b, \underline{q}^s], \\ \bar{U}(p), & \text{if } p \in (\underline{q}^s, \bar{q}^s), \\ A(\lambda(p; \bar{q}^b, 0)) U^*(p) - B(\lambda(p; \bar{q}^b, 0)) c, & \text{if } p \in [\bar{q}^s, \bar{q}^b), \\ V_{ct}(p), & \text{if } p \geq \bar{q}^b, \end{cases}$$

where $V_{ct}(p)$ denotes the value of the contradictory strategy from the baseline model. Note that since we focus attention on the symmetric case (Assumption 2), the belief that separates the “left branch” and the “right branch” of the contradictory solution is given by \check{p} . Note also, that in contrast to the baseline model, we defined the contradictory strategy in a way that it is always weakly greater than $\bar{U}(p)$.

The implied dynamics of the posterior as well as the attention strategy are summarized by the following diagram:



Lemma 17. *Suppose Assumptions 1 and 2 are satisfied. Then V_{ct}^Γ is continuously differentiable and convex on $[0, \underline{q}^s)$ and on $(\bar{q}^s, 1]$, respectively, and satisfies (E.2) on $[\underline{p}^*, \underline{q}^s)$ and on $(\bar{q}^s, \bar{p}^*]$, respectively.*

Proof. We show the Lemma for $p \leq 1/2$. The remaining results follow from a symmetric argument.

We need to show that V_{ct}^Γ is continuously differentiable at \underline{q}^b . For $r > 0$, some algebra yields for $p \leq 1/2$ ²

$$\begin{aligned} V_{ct}^\Gamma &= A(1)\bar{u} - B(1)c \\ \iff \left(\frac{\underline{p}^*}{1 - \underline{p}^*} \frac{1 - p}{p} \right)^r &= 1 - \frac{r}{(1 - p)(r - (1 + r)\Gamma[1])}. \end{aligned}$$

Substituting this expression in $V_{ct}^{\Gamma'}(p)$ yields

$$\begin{aligned} V_{ct}^{\Gamma'}(p) \Big|_{V_{ct}(p)=A(1)U^*(p)-B(1)c} &= \frac{(c + r\bar{u})(p + (1 - p)\Gamma[1])}{(1 - p)p(r - (1 + r)\Gamma[1])} \\ &= \frac{\partial [A(\lambda(p))\bar{u} - B(\lambda(p))c]}{\partial \lambda} \Big|_{\lambda(p)=1}. \end{aligned}$$

²The derivation for $r = 0$ is similar.

Convexity on $[\underline{p}^*, \underline{q}^s]$ follows from strict convexity of V_{ct} (Lemma 3) and strict convexity of $A(\lambda(p))U^*(p) - B(\lambda(p))c$ (Lemma 15.(b)) and continuous differentiability.

Note that by Lemma 10, it suffices to show that V_{ct}^Γ satisfies (E.5) for all $[\underline{p}^*, \underline{q}^s]$ since $V_{ct}^\Gamma(p) > \bar{U}(p)$ for $p < \underline{q}^s$. We have derived V_{ct}^Γ from the first order-condition (E.3) and the respective Kuhn-Tucker condition of $p < \underline{q}^b$. Therefore it suffices to show that the maximization problem in the HJB equation is concave. By Lemma 15.(a), this is the case since we have shown that V_{ct}^Γ is weakly convex. \square

E.4.2 Confirmatory Evidence

The second candidate for the value function is obtained by assuming that the DM uses a “confirmatory” attention strategy. This involves seeking evidence consistent with the current belief. Specifically, we define a “reference belief” p^* such that the DM chooses $\lambda < \gamma$ for lower beliefs $p < p^*$ and $\lambda > \gamma$ for higher beliefs $p > p^*$. The implied dynamics of the posterior as well as the attention strategy are summarized by the following diagram:

$$\left| \underbrace{\rightarrow \rightarrow \rightarrow \rightarrow}_{\lambda \in [0, \gamma)} p^* \underbrace{\leftarrow \leftarrow \leftarrow \leftarrow}_{\lambda \in (\gamma, 1]} \right|$$

The reference belief is absorbing and we assume that once p^* is reached, the DM adopts the stationary attention strategy $\lambda = \gamma$. Under Assumption 2, we have $p^* = 1/2$. This can also be derived from value matching

$$V(p^*) = \bar{U}(p^*) (= \bar{u}), \quad (\text{E.14})$$

and the tangency condition

$$V'(p^*) = \bar{U}'(p^*) (= 0). \quad (\text{E.15})$$

Substituting these two conditions together with $\lambda = \gamma$ in (E.3) yields $p^* = 1/2$.³

We would now like to construct the confirmatory strategy in a similar fashion as the contradictory solution, that is, we will identify two types of regions. If $\lambda \in \{0, 1\}$, we will use solutions to (A.6) or (A.7) (with $\bar{\lambda} = 1$, $\underline{\lambda} = 1$ and α replace by λ .) On the other hand, if $\lambda \in (0, 1)$ we will use solutions to (E.11) with a suitable boundary condition. A problem arises since we want to impose the boundary condition $\lambda(p^*) = \gamma$. Note that this implies $\lambda'(p^*) = 0/0$. We therefore begin by identifying a solution to (E.11) that satisfies $\lambda(p^*) = \gamma$ as well as $\lambda'(p^*) > 0$.

Lemma 18. *Suppose Assumptions 1 and 2 are satisfied. Then there exists a unique continuously differentiable function $\hat{\lambda}_{cf}(p)$ which satisfies (E.11) for all p in a neighborhood*

³Note that in contrast to the linear model, we cannot use the HJB equation because for $\lambda = \gamma$, $V'(p)$ vanishes so that substituting (E.15) has no bite.

of $p^* = 1/2$, such that $\lambda(p^*) = \gamma$ and $\lambda'(p^*) > 0$. The derivative at p^* is given by

$$\hat{\lambda}'_{cf}(p^*) = -(r + \gamma) + \sqrt{(r + \gamma)^2 - \frac{8(r + \gamma)}{\Gamma''(\gamma)}}.$$

Proof of Lemma 18. The ODE (E.11) can be written as

$$\lambda'(p) = \frac{P(p, \lambda(p))}{Q(p, \lambda(p))},$$

where

$$\begin{aligned} P(p, \lambda) &= [\Gamma(\lambda) - \Gamma'(\lambda)\lambda + r(1 - \Gamma'(\lambda))] \times [p + \Gamma'(\lambda)(1 - p)], \\ Q(p, \lambda) &= p(1 - p)\Gamma''(\lambda)[\Gamma(\lambda) - \lambda]. \end{aligned}$$

Since P and Q are both continuous and have continuous partial derivatives, the behavior of solutions that go through points in a neighborhood of (p^*, γ) is, under some conditions (see below), the same as for⁴

$$\lambda'(p) = \frac{a(p - p^*) + b(\lambda(p) - \gamma)}{c(p - p^*) + d(\lambda(p) - \gamma)}, \quad (\text{E.16})$$

where

$$\begin{aligned} a &= \partial_p P(p^*, \gamma) = 4(r + \gamma) > 0, \\ b &= \partial_\lambda P(p^*, \gamma) = (r + \gamma)\Gamma''(\gamma) < 0, \\ c &= \partial_p Q(p^*, \gamma) = 0, \\ d &= \partial_\lambda Q(p^*, \gamma) = -\frac{1}{2}\Gamma''(\gamma) > 0. \end{aligned}$$

The characteristic equation is

$$x^2 - bx - ad = 0.$$

Since $ad > 0$, the characteristic equation has two real roots of opposite sign. This implies that (p^*, γ) is a saddle point and there are two continuously differentiable solutions $\lambda(p)$ that pass through (p^*, γ) . In the case of a saddle point, the behavior of the solutions of (E.16) in a neighborhood of (p^*, γ) corresponds to the behavior of the solutions to (E.11). Hence there exist two continuously differentiable solutions $\lambda(p)$ that satisfy the boundary condition $\lambda(p^*) = \gamma$.

Next we want to obtain $\lambda'(p^*)$ for these solutions, and show that only one of them has

⁴See e.g. Bronshtein, Semendyayev, Musiol, and Muehlig (2007).

a positive derivative. We have

$$\begin{aligned}\lambda'(p^*) &= \lim_{p \rightarrow p^*} \lambda'(p) = \lim_{p \rightarrow p^*} \frac{P(p, \lambda(p))}{Q(p, \lambda(p))} \\ &= \lim_{p \rightarrow p^*} \frac{\partial_p P(p, \lambda(p)) + \partial_\lambda P(p, \lambda(p))\lambda'(p)}{\partial_p Q(p, \lambda(p)) + \partial_\lambda Q(p, \lambda(p))\lambda'(p)} \\ &= \frac{a + b\lambda'(p^*)}{d\lambda'(p^*)}.\end{aligned}$$

Hence $\lambda'(p^*)$ solves

$$\begin{aligned}x^2 - \frac{b}{d}x - \frac{a}{d} &= 0, \\ \lambda'(p^*) &= \frac{b}{2d} \pm \sqrt{\left(\frac{b}{2d}\right)^2 + \frac{a}{d}}.\end{aligned}$$

Since $a/d > 0$, there is one positive and one negative solution. For the confirmatory solution, we are interested in a solution that satisfies $\lambda'(p^*) > 0$. Hence we have

$$\begin{aligned}\lambda'(p^*) &= \frac{b}{2d} + \sqrt{\left(\frac{b}{2d}\right)^2 + \frac{4(r+\gamma)}{d}} \\ &= -(r+\gamma) + \sqrt{(r+\gamma)^2 - \frac{8(r+\gamma)}{\Gamma''(\gamma)}}.\end{aligned}$$

□

Lemma 18 provides the solution $\hat{\lambda}_{cf}$ which together with $V(p) = A(\hat{\lambda}_{cf}(p))U^*(p)$ defines V_{cf} in a neighborhood of p^* . To extend this definition to $(0, 1)$ we first extend $\hat{\lambda}_{cf}$ to the maximal interval (\underline{q}, \bar{q}) where $\hat{\lambda}_{cf}(p) \in (0, 1) \setminus \{\gamma\}$ unless $p = p^*$.

Lemma 19. *Suppose Assumptions 1 and 2 are satisfied. There exist two points $0 \leq \underline{q} < p^* < \bar{q} \leq 1$ such that*

- (a) $\hat{\lambda}_{cf}(p)$ is well defined as the unique \mathcal{C}^1 -solution to (E.11) that satisfies the properties in Lemma 18
- (b) $\hat{\lambda}_{cf}(p) > \gamma$ if $p > p^*$ and $\hat{\lambda}_{cf}(p) < \gamma$ if $p < p^*$.
- (c) Either $\underline{q} = 0$ or $\hat{\lambda}_{cf}(\underline{q}) = 0$.
- (d) Either $\bar{q} = 1$ or $\hat{\lambda}_{cf}(\bar{q}) = 1$.

Note that Properties (c) and (d) mean that the interval (\underline{q}, \bar{q}) is the maximal interval where $\hat{\lambda}_{cf}(p) \in (0, 1)$.

Proof of Lemma 19. Consider the interval (\underline{q}, p^*) . $\hat{\lambda}_{cf}(p) \in (0, \gamma)$ in a neighborhood of p^* . Moreover, (E.11) satisfies local Lipschitz continuity if $p \in (0, p^*)$ and $\lambda \neq \gamma$. Hence, if there exists a \mathcal{C}^1 solution to (E.11) with initial condition $\hat{\lambda}_{cf}(p^* - \varepsilon) \in (0, \gamma)$ that satisfies

$\hat{\lambda}_{cf}(p) \in (0, \gamma)$ for all $p \in (\underline{q}, p^*)$, then it is the unique such solution. We first show that by extending the interval from a neighborhood of p^* to (\underline{q}, p^*) , we do not violate $\hat{\lambda}_{cf}(p) < \gamma$. Suppose by contradiction that there exists $p' < p^*$ such that $\lim_{p \searrow p'} \hat{\lambda}_{cf}(p) \nearrow \gamma$. Note that

$$p' + \Gamma'(\gamma)(1 - p') < p^* + \Gamma'(\gamma)(1 - p^*) = 0.$$

Hence, since $\Gamma'' < 0$, $\lim_{p \searrow p'} \hat{\lambda}'_{cf}(p) \rightarrow \infty$ which contradicts $\lim_{p \searrow p'} \hat{\lambda}_{cf}(p) \nearrow \gamma$. Therefore we can extend the domain of $\hat{\lambda}_{cf}(p)$ to the left until either $p = 0$ or $\hat{\lambda}_{cf}(p) = 0$. This completes the proof for $p < p^*$ and the argument for $p > p^*$ is similar. \square

If $\underline{q} > 0$ and $\bar{q} < 1$, respectively, then we further extend $\lambda_{cf}(p)$ to $(0, 1)$ by setting $\lambda = 0$ for $p < \underline{q}$ and $\lambda = 1$ for $p > \bar{q}$. We define

$$\lambda_{cf}^\Gamma(p) := \begin{cases} 0, & \text{if } p \leq \underline{q}, \\ \hat{\lambda}_{cf}(p), & \text{if } p \in (\underline{q}, \bar{q}), \\ 1, & \text{if } p \geq \bar{q}. \end{cases}$$

The value of this strategy is given by

$$V_{cf}^\Gamma(p) := \begin{cases} V_0(p; \underline{q}, A(0)U^*(\underline{q}) - B(0)c) & \text{if } p \leq \underline{q}, \\ A(\lambda_{cf}^\Gamma(p))U^*(q) & \text{if } p \in (\underline{q}, \bar{q}), \\ V_1(p; \bar{q}, A(1)U^*(\bar{q}) - B(1)c) & \text{if } p \geq \bar{q}. \end{cases}$$

Lemma 20. *Suppose Assumptions 1 and 2 are satisfied. Then $V_{cf}^\Gamma(p)$ is a C^1 solution to (E.2).*

Proof. The proof has several steps. We give arguments for $p \geq 1/2$. The Lemma then follows by symmetry (Assumption 2) and the fact that $V_{cf}^\Gamma(p)$ is constructed in a way that is continuously differentiable at p^* (see (E.15)). Suppose in the following that $p > 1/2$.

First we note that $V_{cf}^\Gamma(p)$ is continuously differentiable. This holds by construction for $p \neq \bar{q}$ and at \bar{q} it follows by the same argument as in the proof of Lemma 17.

Second, we show that $V_{cf}^\Gamma(p)$ is strictly convex. For $p > 1/2$, $\lambda_{cf}^\Gamma(p) > \gamma$. Therefore, by Lemma 15, strict convexity on (p^*, \bar{q}) follows if $p < \pi(\lambda_{cf}^\Gamma(p))$ for all $p \in (p^*, \bar{q})$. Note that $\pi(\lambda_{cf}^\Gamma(p^*)) = \pi(\gamma) = 1/2$. We show that whenever $p = \pi(\lambda_{cf}^\Gamma(p))$, then $\pi'(\lambda_{cf}^\Gamma(p))\lambda_{cf}^{\Gamma'}(p) > 1$. This implies that $p < \pi(\lambda_{cf}^\Gamma(p))$ for all $p \in (p^*, \bar{q})$. We have

$$\begin{aligned} & \pi'(\lambda_{cf}^\Gamma(p^*))\lambda_{cf}^{\Gamma'}(p^*) > 1 \\ \iff & \frac{2 - (r + \gamma)\Gamma''(\gamma)}{4(r + \gamma)} \left(\sqrt{(r + \gamma)^2 - \frac{8(r + \gamma)}{\Gamma''(\gamma)}} - (r + \gamma) \right) > 1 \\ & \iff \Gamma''(\gamma) < 0. \end{aligned}$$

for $p > p^*$, we substitute $p = \pi(\lambda_{cf}^\Gamma(p))$ in (18) which yields (after some algebra)

$$\pi'(\lambda_{cf}^\Gamma(p^*))\lambda_{cf}^{\Gamma'}(p^*) = 1 + \frac{\Gamma'(\lambda_{cf}^\Gamma(p)) (r + \Gamma(\lambda_{cf}^\Gamma(p)) - (r + \lambda_{cf}^\Gamma(p)) \Gamma'(\lambda_{cf}^\Gamma(p)))}{(r + \lambda_{cf}^\Gamma(p)) (r + \Gamma(\lambda_{cf}^\Gamma(p))) \Gamma''(\gamma)} > 1.$$

This completes the proof of convexity on (p^*, \bar{q}) . For $p > \bar{q}$, convexity has been shown in Lemma 4. Since $V_{cf}^\Gamma(p)$ is continuously differentiable at $p = \bar{q}$, $V_{cf}^\Gamma(p)$ is strictly convex on $[0, 1]$.

Third, by Lemma 15.(a), convexity implies that the maximization problem in (E.5) is concave so that the first-order condition is sufficient. Therefore, $V_{cf}^\Gamma(p)$ satisfies (E.5) or for $p > p^*$.

Finally, convexity, together with (E.14) and (E.15) implies that $V_{cf}^\Gamma(p) \geq \bar{U}(p)$ for $p \geq p^*$. Lemma 10 then implies that $V_{cf}^\Gamma(p)$ satisfies (E.2). \square

E.5 Optimal Solution

As in the baseline model we show that the value function V^Γ is the upper envelope of the two solution candidates.

Theorem 1. *Suppose Assumptions 1 and 2 are satisfied.*

- (a) *If (EXP) is violated then $V^\Gamma(p) = U(p)$ for all $p \in [0, 1]$.*
- (b) *If (EXP) is satisfied and $V_{ct}^\Gamma(p) > \bar{U}(p)$ for all $p \neq 1/2$, then $V^\Gamma(p) = V_{ct}^\Gamma(p)$ for all $p \in [0, 1]$.*
- (c) *If (EXP) is satisfied and $V_{ct}^\Gamma(p) = \bar{U}(p)$ for some $p \neq 1/2$, then $V^\Gamma(p) = \max \{V_{ct}^\Gamma(p), V_{cf}^\Gamma(p)\}$.*

Proof of Theorem 1. Follows from the same arguments as the proof of Theorem 2. \square

F Repeated Experimentation

In this appendix, we describe the construction of candidate value functions for several cases of the DMs problem if SEP is violated. We use this construction to compute value functions numerically and verify (numerically) that the HJB equation is satisfied. A complete characterization of the value function without SEP is beyond the scope of this paper and left for future research. We restrict attention to the case of symmetric payoffs (Assumption 2).

The HJB equation without SEP is

$$c + rV(p) = \max_{\alpha} \left\{ \begin{array}{l} \alpha \lambda^A(p) [\max \{V(q^A(t)), U(q^A(t))\} - V(p)] \\ +(1 - \alpha) \lambda^B(p) [\max \{V(q^B(t)), U(q^B(t))\} - V(p)] \\ -p(1 - p)\delta(2\alpha - 1)V'(p) \end{array} \right\}. \quad (\text{F.1})$$

For future reference we note that

$$q^A(p) = 1 - q^B(1 - p) \quad \text{and} \quad q^A(q^B(p)) = p \quad \text{and} \quad q^B(q^A(p)) = p. \quad (\text{F.2})$$

We start by describing the construction of the contradictory strategy, assuming that there is no confirmatory region. Given Assumption 2, we posit that the contradictory strategy is given by

$$\alpha(p) = \begin{cases} 1 & \text{if } p \in [\underline{p}^*, \frac{1}{2}], \\ 0 & \text{if } p \in [\frac{1}{2}, 1 - \underline{p}^*]. \end{cases} \quad (\text{F.3})$$

Suppose first that \underline{p}^* satisfies $q^A(\underline{p}^*) > 1 - \underline{p}^*$. In this case, the contradictory strategy does not involve repeated experimentation and SEP is satisfied. Therefore let us assume that SEP is violated, that is

$$q^A(\underline{p}^*) < 1 - \underline{p}^* \quad \text{or equivalently} \quad \underline{p}^* < 1 - q^A(\underline{p}^*). \quad (\text{F.4})$$

In this case, the boundaries \underline{p}^* and $\bar{p}^* = 1 - \underline{p}^*$ are different from those we obtained in Section 5 when SEP holds. Let us also assume that

$$q^A(\underline{p}^*) > \frac{1}{2}. \quad (\text{F.5})$$

Condition (F.5) implies that receiving a signal under the contradictory policy (F.3) leads the DM to either take an immediate action or to switch from seeking A -evidence to B -evidence (or vice versa). Condition (F.5) also implies that the confirmatory strategy can be constructed by extending the construction in Appendix A. We describe the construction of a solutions under the assumption that (F.5) holds and will verify that this is indeed the case in our numerical examples.

Substituting (F.3) in (F.1) yields for $p < 1/2$:

$$c + rV(p) = \lambda^A(p) [\max \{V(q^A(p)), U(q^A(p))\} - V(p)] - p(1-p)\delta V'(p), \quad (\text{F.6})$$

and for $p > 1/2$

$$c + rV(p) = \lambda^B(p) [\max \{V(q^B(p)), U(q^B(p))\} - V(p)] + p(1-p)\delta V'(p). \quad (\text{F.7})$$

Now suppose we fix some \underline{p}^* and $\bar{p}^* = 1 - \underline{p}^*$. Then by (F.2), the continuation value in (F.6) is

$$\max \{V(q^A(p)), U(q^A(p))\} = \begin{cases} V(q^A(p)) & \text{if } p \in [\underline{p}^*, q^B(\bar{p}^*)], \\ U(q^A(p)) & \text{if } p \in [q^B(\bar{p}^*), \frac{1}{2}], \end{cases}$$

and in (F.7) it is

$$\max \{V(q^B(p)), U(q^B(p))\} = \begin{cases} V(q^B(p)) & \text{if } p \in [q^A(\underline{p}^*), \bar{p}^*], \\ U(q^B(p)) & \text{if } p \in [\frac{1}{2}, q^A(\underline{p}^*)]. \end{cases}$$

Note, that $q^A(p) \in [q^A(\underline{p}^*), \bar{p}^*]$ if $p \in [\underline{p}^*, q^B(\bar{p}^*)]$ and $q^B(p) \in [\underline{p}^*, q^B(\bar{p}^*)]$ if $p \in [q^A(\underline{p}^*), \bar{p}^*]$. Therefore, for a prior belief in the repeated experimentation region $\mathcal{R} := [\underline{p}^*, q^B(\bar{p}^*)] \cup [q^A(\underline{p}^*), \bar{p}^*]$, the posterior never leaves this region unless it reaches one of the absorbing points \underline{p}^* and \bar{p}^* . \mathcal{R} is absorbing and we therefore begin by determining \underline{p}^* and \bar{p}^* together with $V(p)$ for $p \in \mathcal{R}$.

The main difficulty is that value of the contradictory solution is jointly determined by the functional differential equations

$$c + rV(p) = \lambda^A(p) [V(q^A(p)) - V(p)] - p(1-p)\delta V'(p), \quad (\text{F.8})$$

for $p < 1/2$ and

$$c + rV(p) = \lambda^B(p) [V(q^B(p)) - V(p)] + p(1-p)\delta V'(p), \quad (\text{F.9})$$

for $p > 1/2$. We transform these into a pair of ODEs. To do so, we first transform the belief p into a two dimensional state variable $(d, \rho) \in \{-1, 1\} \times [\underline{p}^*, q^B(\bar{p}^*)]$ as follows

$$p \mapsto \rho = \begin{cases} (-1, p) & \text{if } p \in [\underline{p}^*, q^B(\bar{p}^*)], \\ (1, q^B(p)) & \text{if } p \in [q^A(\underline{p}^*), \bar{p}^*]. \end{cases}$$

Note that for $d = -1$ we have

$$\dot{\rho}_t = \dot{p}_t = -p_t(1-p_t)\delta = -\rho_t(1-\rho_t)\delta,$$

and for $d = 1$ we have

$$\begin{aligned} \rho_t &= q^B(p_t) \\ \implies \dot{\rho}_t &= q^{B'}(p_t)\dot{p}_t \\ \iff \dot{\rho}_t &= -\frac{\lambda(\lambda p_t + \bar{\lambda}(1-p_t)) + \lambda p_t(\bar{\lambda} - \lambda)}{(\lambda p_t + \bar{\lambda}(1-p_t))^2} \delta p_t(1-p_t) \\ \iff \dot{\rho}_t &= -\frac{\lambda p_t \bar{\lambda}(1-p_t)}{(\lambda p_t + \bar{\lambda}(1-p_t))^2} \delta \\ \iff \dot{\rho}_t &= -q^B(p_t)(1-q^B(p_t))\delta \\ \iff \dot{\rho}_t &= -\rho_t(1-\rho_t)\delta, \end{aligned}$$

and

$$\begin{aligned} q^A(\rho_t) &= p_t \\ \implies q^{A'}(\rho)\dot{\rho}_t &= \dot{p}_t. \end{aligned}$$

Moreover we define $W(-1, \rho) = V(\rho)$ and $W(1, \rho) = V(q^A(\rho))$ which yields

$$\begin{aligned} W_\rho(-1, \rho) &:= \frac{\partial W(-1, \rho)}{\partial \rho} = V'(\rho), \\ W_\rho(1, \rho) &:= \frac{\partial W(1, \rho)}{\partial \rho} = V'(q^A(\rho))q^{A'}(\rho), \\ \iff \dot{\rho}_t W_\rho(1, \rho) &= V'(q^A(\rho))\dot{p}_t. \end{aligned}$$

Substituting this in (F.8) and (F.9) and using $V(p) = V(q^A(\rho))$ in (F.9), we obtain

$$c + rW(-1, \rho) = \lambda^A(\rho) [W(1, \rho) - W(-1, \rho)] - \rho(1-\rho)\delta W_\rho(-1, \rho), \quad (\text{F.10})$$

$$c + rW(1, \rho) = \lambda^B(q^A(\rho)) [W(-1, \rho) - W(1, \rho)] + \rho(1-\rho)\delta W_\rho(1, \rho). \quad (\text{F.11})$$

(F.10)–(F.11) is a system of ODEs in the two function $W(1, \cdot)$ and $W(-1, \cdot)$ for $\rho \in [\underline{p}^*, q^B(\bar{p}^*)]$. We have boundary conditions given by $W(-1, \underline{p}^*) = V(\underline{p}^*) = U_b(\underline{p}^*)$ and $W(1, q^B(\bar{p}^*)) = V(\bar{p}^*) = U_a(\bar{p}^*)$. For given \underline{p}^* and $\bar{p}^* = 1 - \underline{p}^*$, we can solve (F.10)–(F.11) with these boundary conditions and denote the solution by $W(d, \rho; \underline{p}^*)$. To pin down \underline{p}^* , we impose a smooth pasting condition on $V(p)$ at \underline{p}^* :

$$rU_b(\underline{p}^*) = \lambda^A(\underline{p}^*) [V(q^B(\bar{p}^*)) - U_b(\underline{p}^*)] - \underline{p}^*(1-\underline{p}^*)\delta^A U'_b(\underline{p}^*).$$

For given \underline{p}^* this imposes a restriction on the solutions of the pair of ODEs:

$$\begin{aligned} W(-1, q^B(\bar{p}^*)) &= V(q^B(\bar{p}^*)) = V(q^A(\underline{p}^*)) = W(1, \underline{p}^*), \\ &= U_b(\underline{p}^*) + \frac{rU_b(\underline{p}^*) + \underline{p}^*(1 - \underline{p}^*)\delta^A U'_b(\underline{p}^*)}{\lambda^A(\underline{p}^*)}. \end{aligned}$$

In order to determine \underline{p}^* we solve for $W(-1, q^B(\bar{p}^*); \underline{p}^*) = W(-1, 1 - q^A(\underline{p}^*); \underline{p}^*)$ as a function of \underline{p}^* , and then search numerically for a value \underline{p}^* for which the smooth pasting condition is satisfied. In this way we determine the repeated experimentation region \mathcal{R} and the value function in this region.

To complete the construction of the value function we solve (A.5) with $\alpha = 1$ and boundary condition $V(1 - q^A(\underline{p}^*)) = W(-1, 1 - q^A(\underline{p}^*); \underline{p}^*)$ for $p \in [q^B(\bar{p}^*), 1/2]$ and (A.5) with $\alpha = 0$ and boundary condition $V(q^A(\underline{p}^*)) = W(1, \underline{p}^*; \underline{p}^*)$ for $p \in [1/2, q^A(\underline{p}^*)]$. Combining this with the value function on \mathcal{R} , and using $V(p) = U(p)$ for $p \notin [\underline{p}^*, \bar{p}^*]$, we obtain the value function for the contradictory solution which we denote by $V_{ct}^{rep}(p)$.

If $V_{ct}^{rep}(p) \geq V_{cf}(p)$ for all $p \in (0, 1)$, where $V_{cf}(p)$ is the value of the confirmatory solution defined as in Section 5, we conjecture that V_{ct}^{rep} is the value function of the DM's problem with repeated experimentation and the optimal policy is of the form described in Panel (a) of Figure 3. We verify numerically that the conjectured value function satisfies the HJB equation. In Panel (a) of Figure 4, we give an example of this case.

If $V_{ct}^{rep}(p) < V_{cf}(p)$ for some $p \in (0, 1)$ but $V_{ct}^{rep}(p) \geq V_{cf}(p)$ for all $p \in \mathcal{R}$, we conjecture that the value function is given by $\max\{V_{ct}^{rep}(p), V_{cf}(p)\}$. Again we verify numerically, that this candidate satisfies the HJB equation. In Panel (b) of Figure 4, we give an example of this case.

Two further cases arise if condition (F.5) is satisfied. First, we consider the case depicted in Panels (c) of Figures 3 and 4. This coincides with the example discussed in Section 6. In this case, the value of the confirmatory strategy can still be computed as in the baseline model. The repeated experimentation region is identical to the contradictory region. For more extreme beliefs $p \in (\underline{p}^*, q^B(\bar{p}^*)) \cup (q^A(\underline{p}^*), \bar{p}^*)$, receiving a signal leads to a posterior in the confirmatory region. The continuation value is thus known from the confirmatory strategy, and the value of the contradictory strategy can be obtained by solving the following two differential equations.

$$c + rV_{ct}^{rep}(p) = \lambda^A(p) [V_{cf}(q^A(p)) - V_{ct}^{rep}(p)] - p(1 - p)\delta V_{ct}^{rep'}(p),$$

with boundary condition $V(\underline{p}^*) = U_b(\underline{p}^*)$ for $p < 1/2$, and

$$c + rV_{ct}^{rep}(p) = \lambda^B(p) [V_{cf}(q^B(p)) - V_{ct}^{rep}(p)] + p(1 - p)\delta V_{ct}^{rep'}(p),$$

with boundary condition $V(\bar{p}^*) = U_a(\bar{p}^*)$ for $p > 1/2$. We call these solutions $\underline{V}_{ct}^{rep1}$

$$\left| \underbrace{p^*}_{p=0} \xleftarrow{\text{action } b} \underbrace{p_1 \xleftarrow{q^B(\bar{p}^*)} p_0 \xleftarrow{\check{p}}}_{\alpha=1} \xrightarrow{q^A(p^*)} \underbrace{q^A(p_1)}_{\alpha=0} \xrightarrow{\bar{p}^*} \underbrace{q^A(p_0)}_{\text{action } a} \right|_1$$

(a) contradictory strategy with repeated experimentation

$$\left| \underbrace{p^*}_{p=0} \xleftarrow{\text{action } b} \underbrace{q^B(\bar{p}^*)}_{\alpha=1} \xleftarrow{p} \underbrace{\xrightarrow{\xrightarrow{p^*}}}_{\alpha=0} \xleftarrow{\bar{p}} \underbrace{\xrightarrow{q^A(p^*)}}_{\alpha=0} \xrightarrow{\bar{p}^*} \underbrace{\text{action } a}_1 \right|_1$$

confirmatory region

(b) contradictory and confirmatory strategy with repeated experimentation (no jumps into confirmatory region)

$$\left| \underbrace{p^*}_{p=0} \xleftarrow{\text{action } b} \underbrace{q^B(\bar{p})}_{\alpha=1} \xleftarrow{p} \underbrace{\xrightarrow{\xrightarrow{p^*}}}_{\alpha=0} \xleftarrow{\bar{p}} \underbrace{\xrightarrow{q^A(p)}}_{\alpha=0} \xrightarrow{\bar{p}^*} \underbrace{\text{action } a}_1 \right|_1$$

(c) contradictory and confirmatory strategy with repeated experimentation (jumps into both confirmatory and contradictory region)

$$\left| \underbrace{q^B(\bar{p})}_{p=0} \xleftarrow{\text{action } b} \underbrace{p}_{\alpha=1} \xrightarrow{\xrightarrow{p^*}} \underbrace{\bar{p}}_{\alpha=1} \xrightarrow{\xrightarrow{\bar{p}^*}} \underbrace{q^A(p)}_{\text{action } a} \right|_1$$

(d) contradictory and confirmatory strategy with repeated experimentation (no jumps into contradictory region)

Figure 3: Repeated Experimentation

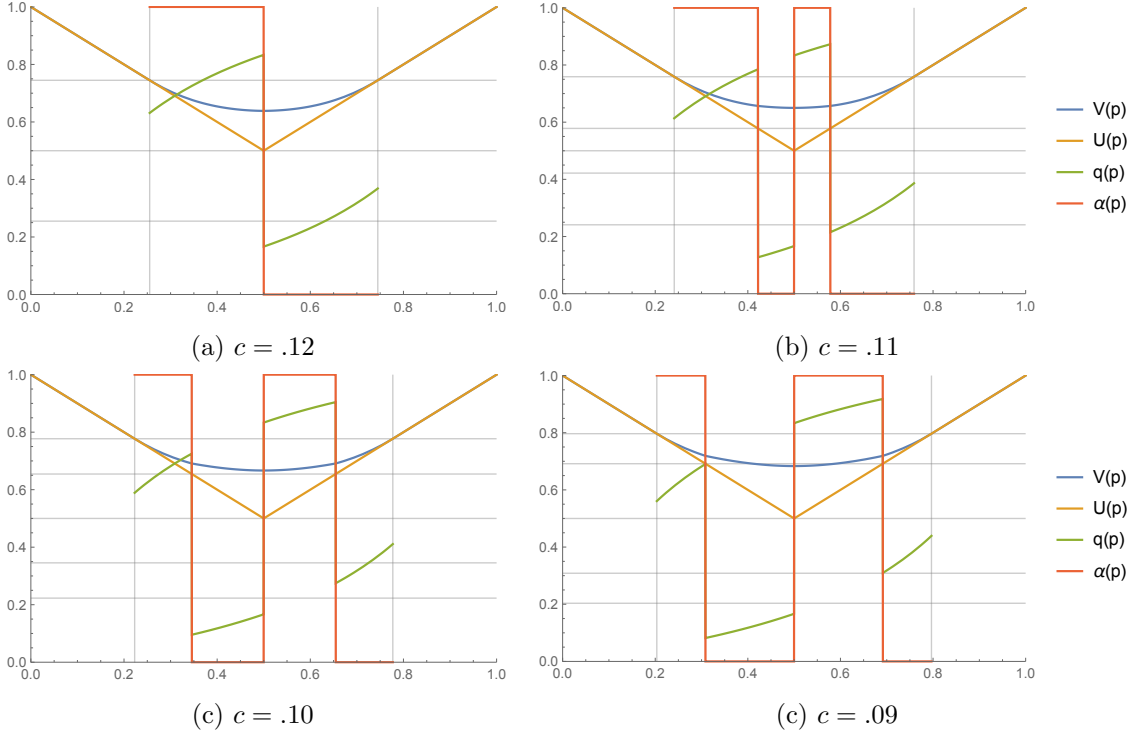


Figure 4: Value functions and optimal policies with repeated experimentation. $q(p)$ is the posterior belief after a signal is observed at p if the DM uses $\alpha(p)$. Common parameters: $\bar{\lambda} = 1$, $\underline{\lambda} = .2$, $r = 0$, $\bar{u} = 1$, $\underline{u} = 0$. Grid lines at: \underline{p}^* , \underline{p} , $1/2$, \bar{p} , \bar{p}^* (left to right and bottom to top).

($p < 1/2$) and \bar{V}_{ct}^{rep1} ($p > 1/2$), respectively.

To determine \underline{p}^* and \bar{p}^* , we use smooth pasting and value matching as in the case where SEP holds, but replace $U_a(q^A(\underline{p}^*))$ by $V_{cf}(q^A(\underline{p}^*))$ and $U_b(q^B(\bar{p}))$ by $V_{cf}(q^B(\bar{p}))$ in (A.6) and (A.7).

For $p \in (q^B(\bar{p}), \underline{p}) \cup (\bar{p}, q^A(\underline{p}))$ receiving a signal leads to a jump of the belief back into $(q^B(\bar{p}), \underline{p}) \cup (\bar{p}, q^A(\underline{p}))$. Therefore we obtain the value function in this domain from a solution to (F.10)–(F.11). We use a similar construction as in cases (a) and (b) to determine the boundary points $\underline{a}^* = q^B(\bar{p})$ and $\bar{a}^* = 1 - \underline{a}^* = q^A(\underline{p})$ (this also determines \underline{p} , \bar{p}). Instead of a smooth pasting condition we now want to find \underline{a}^* such that the solution to (F.10)–(F.11) with boundary conditions $W(-1, \underline{a}^*) = \underline{V}_{ct}^{rep1}(\underline{a}^*)$ and $W(1, q^B(\bar{a}^*)) = \bar{V}_{ct}^{rep1}(q^B(\bar{a}^*))$ satisfies a value matching condition:

$$V_{cf}(q^A(\underline{a}^*)) = W(1, \underline{a}^*),$$

and similarly

$$V_{cf}(q^B(\bar{a}^*)) = W(-1, q^B(\bar{a}^*)).$$

These conditions allow us to pin down \underline{a}^* . We then have

$$V_{ct}^{rep}(p) = \begin{cases} U_b(p) & \text{if } p \leq \underline{p}^*, \\ \underline{V}_{ct}^{rep1}(p) & \text{if } p \in (\underline{p}^*, \underline{a}^*], \\ W(-1, p) & \text{if } p \in (\underline{a}^*, \underline{p}], \\ V_{cf}(p) & \text{if } p \in (\underline{p}, \bar{p}], \\ W(1, q^B(p)) & \text{if } p \in (\bar{p}, \bar{a}^*], \\ \bar{V}_{ct}^{rep1}(p) & \text{if } p \in (\bar{a}^*, \bar{p}^*], \\ U_A(p) & \text{if } p \in > \bar{p}^*. \end{cases}$$

Again we verify numerically, that this candidate satisfies the HJB equation. In Panel (c) of Figure 4, we give an example of this case.

The last case is depicted in Panels (d) of Figures 3 and 4. This case is simpler because the intervals $(\underline{a}^*, \underline{p}]$ and $(\bar{p}, \bar{a}^*]$ are empty and any signal in the contradictory region leads to a jump into the confirmatory region. Therefore this case does not require a solution of the system (F.10)–(F.11).

The examples we have provided here are relatively tractable because the value of the confirmatory strategy can be computed as in the case where SEP holds. Moreover, we have imposed symmetry which (a) simplifies the transformation of the functional equations that arise from the HJB for the contradictory solution to a system of ODEs and (b) allows us to search for a single boundary point at a time (\underline{p}^* or \underline{a}^* , depending on the case). We leave a full characterization of the solution in under these assumptions, as well as an extension of the construction to the case where the confirmatory region overlaps with the repeated experimentation region for future research.

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