Errata for: "Identifying Network Ties from Panel Data: Theory and an Application to Tax Competition"

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Lemma 2

A) de Paula et al. [2025] use the the following paths for ρ and β (Online Appendix, p.7):

$$\rho(t) = t\rho^* + (1-t)\rho$$
$$\beta(t) = (t\rho^*\beta^* + (1-t)\rho\beta)/(t\rho^* + (1-t)\rho)$$

This construction leaves $\beta(t)$ undefined if $\rho(t) = 0$ for some t. That is the case when $\rho \geq 0$ and $\rho^* \leq 0$, or – conversely – when $\rho \leq 0$ and $\rho^* \geq 0$. Any path between ρ and ρ^* then goes through zero at some $\bar{t} \in (0,1)$.

B) de Paula et al. [2025] invoke $\sum_{j=1}^N |W_{ij}(t)| \leq t\sum_{j=1}^N |(W_*)_{ij}| + (1-t)\sum_{j=1}^N |W_{ij}| \leq 1$, where W_* is an intermediate matrix used to construct a path $W(t)$ connecting W and W^* (Online Appendix, p.6), and $|\rho(t)| < 1$ (Online Appendix, p.7) to verify assumption (A2). (This was a legacy from previous unpublished versions of the article that used a different assumption (A2).) Assumption (A2) must nonetheless be verified directly on $\rho(t)$ and the path $W(t)$ connecting W and W^*.

C) When $\left(W^{*2}\right)_{11} \neq \left(W^{*2}\right)_{22}$, de Paula et al. [2025] propose constructing a path between W and W^* via W_*, corresponding to the network of directed connections $\{(1,2),(2,1)\}$ and $\{(3,4),(4,5),\ldots,(N-1,N),(N,3)\}$. If instead $\left(W^{*2}\right)_{11} = \left(W^{*2}\right)_{22}$ propose a path between W and W^* via W_{**}, where W_{**} represents the network of directed connections $\{(1,3),(3,1)\}$ and $\{(2,4),(4,5),\ldots,(N-1,N),(N,2)\}$. Whereas the construction works when $N > 4$, it fails if $N = 3$ as W_* violates assumption (A1) and W_{**} violates assumption (A4').

To address these issues, we present a different proof for Lemma 2, which replaces the proof available in the original Online Appendix. Here we connect any two parameter vectors θ and $\theta^* \in \Theta_+$ via an intermediate parameter vector θ_* with $\rho_* = \beta_* = 0, \gamma_* > 0$ and W_* as previously defined for $N \geq 4$ and a modified W_* for $N = 3$. It thus uses different paths for ρ, β and γ than those presented in the original proof to address point A. The new path for ρ and the path W (through the intermediate parameters ρ_* and W_*) can be seen to address point B. To address point C, we offer a different intermediate matrix W_* as indicated above and apply row and column permutations to W for $N = 3$. The proof for Lemma 2 then becomes the following.

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Proof. As $\Pi(\cdot)$ is a continuous map, it suffices to prove path-connectedness of Θ_+. To do so, take the parameter vector $\theta_* = ((W_*)_{12}, ..., (W_*)_{N,N-1}, \gamma_*, \rho_*, \beta_*)'$ with $\rho_* = \beta_* = 0$, $\gamma_* > 0$. If $N \geq 4$, define W_* such that $(W_*)_{ij} = 1$ for all $(i,j) \in \{(1,2),(2,1),(N,3)\} \cup \{(3,4),(4,5),...,(N-1,N)\}$ and $(W_*)_{ij} = 0$ for all other (i,j). When $N = 3$, let instead $(W_*)_{ij} = 1$ for all $(i,j) \in \{(1,2),(2,1),(3,2)\}$ and $(W_*)_{ij} = 0$ for all other (i,j). It can be shown that $\theta_* \in \Theta_+$. We can then obtain path-connectedness of Θ_+ by proving that every $\theta = ((W)_{12}, ..., (W)_{N,N-1}, \gamma, \rho, \beta)' \in \Theta_+$ is path-connected to θ_*, which in turn makes any two $\theta, \theta^* \in \Theta_+$ path-connected via θ_*.

Take $\theta \in \Theta_+$ and define the paths from θ to θ_* for ρ, β and γ as follows:

$$\begin{aligned}\rho(t) &= t\rho_* + (1-t)\rho = (1-t)\rho,\\ \beta(t) &= t\beta_* + (1-t)\beta = (1-t)\beta,\\ \gamma(t) &= t\gamma_* + (1-t)^2\gamma,\end{aligned}$$

with $t \in [0,1]$. As $\theta \in \Theta_+$, we have $\rho\beta + \gamma > 0$. Since $\gamma_* > 0$, this gives us:

$$\rho(t)\beta(t) + \gamma(t) = (1-t)^2(\rho\beta + \gamma) + t\gamma_* > 0,$$

for all $t \in [0,1]$. As $|\rho| < 1$ and $\rho_* = 0$, we have $|\rho(t)| < 1$ for all $t \in [0,1]$.

Now, consider the subvector $((W_*)_{12}, ..., (W_*)_{N,N-1})$ of θ_* and the subvector $((W)_{12}, ..., (W)_{N,N-1})$ of θ. Define the path

$$W(t) = tW_* + (1-t)W, \quad t \in [0,1]$$

between W and W_*. As $(W)_{ii} = 0$ and $(W_*)_{ii} = 0$ for all $k \in \{1, ..., N\}$, we get $(W(t))_{ii} = 0$ for all $t \in [0,1]$ and $i \in \{1, ..., N\}$. Assumption (A4) implies $\sum_{j=1}^N (W)_{ij} = 1$ for some row i. As we also have $\sum_{j=1}^N (W_*)_{ij} = 1$, we get that for all $t \in [0,1]$:

$$\sum_{j=1}^N (W(t))_{ij} = t\sum_{j=1}^N (W_*)_{ij} + (1-t)\sum_{j=1}^N (W)_{ij} = 1.$$

Next, we prove that $\sum_{j=1}^N |\rho(t)(W(t))_{ij}| < 1$. As $\sum_{j=1}^N |(W_*)_{ij}| = 1$ for all $i \in \{1, ..., N\}$, we get

$$\begin{aligned}\sum_{j=1}^N |\rho(t)(W(t))_{ij}| &\leq (1-t)t|\rho| \cdot \sum_{j=1}^N |(W_*)_{ij}| + (1-t)^2 \sum_{j=1}^N |\rho(W)_{ij}|\\ &= (1-t)t \underbrace{|\rho|}_{<1} + (1-t)^2 \underbrace{\sum_{j=1}^N |\rho(W)_{ij}|}_{<1}\\ &< (1-t)t + (1-t)^2 \leq 1.\end{aligned}$$

for all $t \in [0,1)$. For $t = 1$, we have $\sum_{j=1}^N |\rho(t)(W(t))_{ij}| = 0 < 1$. Taken together, the foregoing proves assumptions (A1)-(A4) for the proposed path and that $\rho(t)\beta(t) + \gamma(t) > 0$.

The last assumption to prove is (A5). As $\theta \in \Theta_+$, we have $(W^2)_{11} \neq (W^2)_{hh}$ for some $h \in \{2, ..., N\}$. Consider

$N \geq 4$. Here, we have two cases. First, $(W^2)_{11} \neq (W^2)_{22}$. Then, we have

$$(W(t)^2)_{11} = t^2 + (1-t)tW_{12} + (1-t)tW_{21} + (1-t)^2(W^2)_{11},$$
$$(W(t)^2)_{22} = t^2 + (1-t)tW_{21} + (1-t)tW_{12} + (1-t)^2(W^2)_{22}.$$

That gives us

$$(W(t)^2)_{11} - (W(t)^2)_{22} = (1-t)^2 \left((W^2)_{11} - (W^2)_{22}\right) \neq 0,$$

for all $t \in [0,1]$, so it satisfies (A5). The second case is $(W^2)_{11} = (W^2)_{22}$. We can assume without loss of generality that $(W^2)_{11} \neq (W^2)_{33}$. Define θ_{**} as follows: $\rho_{**} = \beta_{**} = 0$, $\gamma_{**} = \gamma_* > 0$, $(W_{**})_{ij} = 1$ for $(i,j) \in \{(1,3),(3,1),(2,4)\} \cup \{(4,5),(5,6),\dots,(N-1,N),(N,2)\}$ and $(W_{**})_{ij} = 0$, otherwise. Note that $\theta_{**} \in \Theta$. Now, we can show that there is a path from θ to θ_* via θ_{**}. As $(\rho_{**}, \beta_{**}, \gamma_{**}) = (0,0,\gamma_*) = (\rho_*, \beta_*, \gamma_*)$, we can use the paths for ρ, β, γ defined earlier between θ and θ_{**}. By using similar logic as above, it can be proven that assumption (A5) holds on the path $\widetilde{W}(t) = tW + (1-t)W_*$ from θ to θ_{**} and the path $\widehat{W}(t) = tW_{**} + (1-t)W_*$ from θ_* to θ_{**}.

Now consider $N = 3$. Take two vectors $\theta, \theta^* \in \Theta_+$. Path-connectedness for W and W^* is proven component by component. So, we can permute rows without loss of generality, as long as we apply the same permutation to W and W^*. By symmetry, all possible cases reduce to three:

- $(W^2)_{11} \neq (W^2)_{22}$ and $(W^{*2})_{11} \neq (W^{*2})_{22}$. This is the base case.
- $(W^2)_{11} \neq (W^2)_{22}$ and $(W^{*2})_{11} = (W^{*2})_{22}$. As $\theta_2 \in \Theta_+$, this implies $(W^{*2})_{11} \neq (W^{*2})_{33}$. We have two subcases. First, $(W^2)_{22} = (W^2)_{33}$. Now, swap row 2 and 3 and then swap column 2 and 3 of W and W^*. Second, $(W^2)_{22} \neq (W^2)_{33}$. Here, swap row 1 and 3 and column 1 and 3 in W and W^*.
- $(W^2)_{11} = (W^2)_{22}$ and $(W^{*2})_{11} = (W^{*2})_{22}$. Swap row 2 and 3 and column 2 and 3 in W and W^*.

The resulting matrices W and W^* all satisfy $(W^2)_{11} \neq (W^2)_{22}$ and $(W^{*2})_{11} \neq (W^{*2})_{22}$. We can again path-connect them via the parameter vector θ_*, but now with a slightly different W_* as defined above for $N = 3$. Take the paths

$$\widetilde{W}(t) = tW_* + (1-t)W \qquad \text{and} \qquad \widehat{W}(t) = tW_* + (1-t)W^*$$

between θ and θ_* and between θ^* and θ_*. For these paths, we have

$$(\widetilde{W}(t)^2)_{11} - (\widetilde{W}(t)^2)_{22} = (1-t)^2((W^2)_{11} - (W^2)_{22}) \neq 0$$

and

$$(\widehat{W}(t)^2)_{11} - (\widehat{W}(t)^2)_{22} = (1-t)^2((W^{*2})_{11} - (W^{*2})_{22}) \neq 0$$

for all $t \in [0,1]$. Thus, there is a path in Θ_+ between all pairs $\theta_1, \theta_2 \in \Theta_+$ via θ_*. □

Theorem 2

The proof for Theorem 2 declares that: *"The mapping $\Pi(\theta)$ is continuous and proper (by Corollary 1), with a connected image (Lemma 2), and non-singular Jacobian at any point (as per the proof for Theorem 1), which guarantees local invertibility"* (Online Appendix, p.7). Theorem 1 does not guarantee that the Jacobian of the unrestricted mapping $\Pi(\theta)$ has full rank. One needs to account for the restriction in assumption (A4) to ensure a full-rank Jacobian. The precise statement should read instead: *"The mapping* $\Pi(\theta)$ *is continuous and proper (by Corollary 1), with a connected image (Lemma 2), and it is locally invertible at any point of the domain constrained by assumption (A4) (as per the proof for Theorem 1)."*

References

Aureo de Paula, Imran Rasul, and Pedro Souza. Identifying network ties from panel data: Theory and an application to tax competition. *Review of Economic Studies*, 92(4):2691–2729, July 2025.