MSc Macroeconomics G022

Problem set # 8

Due week of December 6-December 10 : hand in at beginning of tutorial

Problem 1
Pricing by arbitrage

We consider an economy with two dates, $t = 0, 1$. At date 1, there are two states of nature $e =$ high and $e =$ low. There are two securities, $k = 0, 1$. A risk-free bond serves as the numeraire in date 0, where a market is opened where one can trade the risk free bond against a risky stock whose price is denoted S, $S > 0$. The risk free bond serves an interest rate r, $r > -1$. In state *high*, the price of the stock will be $(1 + h)S$, while in state *low*, it is $(1 + \ell)S$, where h and ℓ, $h \geq \ell$, are scalars larger than -1.

1. Write the (2×2) matrix of contingent payoffs of the securities at date 1, with running element a: the first row concerns asset 0, the second asset 1; the first column is associated with state *low* the second with state *high*. Are markets complete? Write the line vector of payoffs of the generic portfolio
$$z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$
In the rest of the problem, the parameters are assumed to be such that markets are complete.

2. Define a call option on the stock at date 1 with strike price K. Compute the portfolio $z(K)$ that replicates the option when $(1 + h)S > K > (1 + \ell)S$. Infer from this computation the value $C(K)$ of the option.

3. Using the definition of an arbitrage opportunity, show that a necessary and sufficient condition for the absence of arbitrage opportunities is
$$\ell < r < h.$$
In the absence of arbitrage opportunities, show that the state prices are
$$q_h = \frac{r - \ell}{h - \ell} \frac{1}{1 + r} \quad q_\ell = \frac{h - r}{h - \ell} \frac{1}{1 + r}.$$
Conmpute the risk adjusted probability. Check that the pricing formula yields a value for the call option consistent with that of the previous question.

Problem 2
The mean-variance criterion

Consider a stock market with two risky securities ($k = 1{,}2$). Their returns (per dollar invested) are denoted by $\tilde{R}_k$, $k = 1{,}2$, with mathematical expectation and standard deviation μ_k and σ_k, respectively. Let ρ be the correlation coefficient between $\tilde{R}_1$ and $\tilde{R}_2$, i.e. $\rho = \mathrm{cov}(\tilde{R}_1, \tilde{R}_2)/(\sigma_1\sigma_2)$. There is also a riskless security ($k = 0$) that yields $R_0 = (1 + r)$ per dollar invested.

Let $\tilde{R}$ be the return of a portfolio of composition x, i.e.:

$$\tilde{R} = x_0 R_0 + x_1 \tilde{R}_1 + x_2 \tilde{R}_2,$$

in which x_k represents the fraction invested in k ($k = 0, 1, 2$):

$$x_0 + x_1 + x_2 = 1.$$

There is no condition on the sign of the x_k, implying that short selling of any security is allowed. Finally, denote the mathematical expectation and the standard deviation of $\tilde{R}$ by μ and σ, respectively.

1. Show that all portfolios that consist exclusively of risky securities define a curve in the plane (σ, μ) of equation

   $$\begin{aligned}(\mu_1 - \mu_2)^2\sigma^2 &= [\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2]\mu^2 \\ &\quad -2[\mu_1\sigma_1^2 + \mu_2\sigma_2^2 - 2\rho\sigma_1\sigma_2(\mu_1 + \mu_2)]\mu + [\mu_1^2\sigma_1^2 + \mu_2^2\sigma_2^2 - 2\rho\sigma_1\sigma_2\mu_1\mu_2].\end{aligned}$$

   Represent this curve by assuming $\mu_1 = 2, \mu_2 = 1, \sigma_1 = 2, \sigma_2 = 1$, in the cases $\rho = 1$, $\rho = 1/2$, $\rho = 0$ and $\rho = -1$. Identify the part of the curve that can be reached when short sales are not allowed. Comment.

2. Assume that $\rho = 1/2$, so that the hyperbola of the previous question has equation

   $$\sigma^2 = 3(\mu - 1)^2 + 1.$$

   Suppose that the return of the risk free asset is μ_0. Write and solve the program that defines the efficient portfolios. When $\mu_0 = 1/2$, show that the equation of the efficiency frontier is

   $$\mu = \frac{1}{2}\left[1 + \sqrt{\frac{7}{3}}\sigma\right].$$

   What is the composition of the mutual fund of risky securities chosen by any investor? Comment.

3. Characterize the demand of securities of an investor with preferences represented by a utility function $U(\mu, \sigma) = \mu - \alpha\sigma^2, \alpha > 0$.

Problem 3
A model of the spot rate curve

Consider an economy with a representative consumer, whose preferences are

$$\frac{1}{1-\alpha} E \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} (c_t)^{1-\alpha},$$

where α is a (strictly) positive scalar, which is different from 1.

We assume that resources ω (i.e. GDP, equal here to aggregate consumption) have a lognormal distribution (more precisely the joint distribution of the $\log(\omega_t)$ is normal).

We shall use the property that, if the real variable z has a lognormal distribution, one has :

$$\log Ez = E \log z + \frac{1}{2} \mathrm{var}(\log z).$$

1. Show that the term τ interest rate, $r(\tau)$, satisfies the following equation

$$\log[1 + r(\tau)] = \log(1+\rho) + \frac{\alpha}{\tau} \left\{ E_0 \left[\log(\tilde{\omega}_\tau / \omega_0) \right] - \frac{\alpha}{2} \mathrm{var}_0(\log \tilde{\omega}_\tau) \right\},$$

   What is the impact of the risk aversion of the representative agent on the interest rate?

2. Use the preceding result to get explicit expressions when resources evolve as an AR(1)

$$\log \omega_t = g + a \log \omega_{t-1} + \varepsilon_t,$$

   where g is a scalar (an approximation of the growth rate in the deterministic case), a is in $(-1, +1)$ and the ε_t's are i.i.d gaussian of mean zero and variance σ^2. Comment!