Expectations and learning

A consistent intertemporal setup, to describe the future states of the economy.

The definition of *rational* or *perfect foresight* expectations is conditional on the **model** of the economy.
Given a model choice, it is meaningful to talk of *expectation errors* and of learning by trials and errors.

Plan

1. The representative agent model
2. The overlapping generations model
3. Learning

The representative agent model

The neoclassical growth model: consumption

Discrete time, infinite horizon, all purpose commodity, no nominal asset.

Inelastic labour supply L_t at date t.

The representative consumer maximizes

$$\sum_{t=1}^{\infty} \beta^t U(C_t)$$

subject to the sequence of budget constraints for $t = 1, \ldots$:

$$p_t C_t + p_t K_t = w_t L_t + \Pi_t + \rho_t K_{t-1} + p_t K_{t-1},$$

given initial capital stock K_0.

Arbitrary numéraire at each date to measure the price p_t and wage w_t, i.e. a good, storable without cost.

The neoclassical growth model: production

The manager of the (aggregate) firm maximizes his short run profit:

$$\Pi_t = p_t Q_t - w_t N_t - \rho_t K_{t-1}$$

subject to the technical constraint (production function) :

$$Q_t = F(K_{t-1}, N_t)$$

Profits are immediately distributed to the owners of the capital stock.

Assumption: The production function is twice continuously differentiable, concave and exhibits constant returns to scale. Also

$$F(0, L) = 0 \quad \lim_{K \to 0} F'_K(K, N) = \infty \quad \lim_{N \to 0} F'_N(K, N) = \infty$$

Net production function (capital stock maintained in original state).

The neoclassical growth model: equilibrium

Definition: *An* equilibrium *is a sequence of prices and quantities* $(w_t/p_t, \rho_t/p_t, Q_t, C_t, N_t, L_t, K_t)$ *such that, every agent maximizing his/her objective subject to his/her constraints given prices, the corresponding allocations satisfy the scarcity constraints:*

$$C_t + K_t = Q_t + K_{t-1},$$

$$N_t = L_t.$$

First order necessary conditions

Profit maximization at each date:

$$F'_K = \frac{\rho}{p} \qquad F'_N = \frac{w}{p}.$$

Intertemporal consumer optimization between t and $t+1$:

$$\beta \frac{U'(C_{t+1})}{U'(C_t)} = \frac{p_{t+1}}{p_{t+1} + \rho_{t+1}}.$$

Eliminating prices yields

$$\beta \frac{U'(C_{t+1})}{U'(C_t)} = \frac{1}{1 + F'_{K,t+1}}.$$

Stationary equilibria

If the stock of capital per labor unit stays constant, a property observed along any equilibrium of the type just described in the absence of technical progress, F'_K is constant, which implies that

$$\beta \frac{U'(C_{t+1})}{U'(C_t)} = \frac{1}{1+F'_K}$$

is also constant. If consumption stays constant, at any equilibrium one has

$$\beta = \frac{1}{1+F'_k}.$$

The marginal productivity of capital is equal to the psychological discount rate of the consumers: this is the simplest version of the *golden rule*. This is only possible when labor supply is constant.

Balanced growth paths

If labor supply L exogenously grows at rate n, consumption and the capital stock must grow at the same rate, if one looks for a constant growth equilibrium (again stressing the absence of technical progress in the model). In general, one cannot expect the first order conditions to hold unless U is a *homogenous function* of C. If utility is logarithmic, one again gets the golden rule :

$$\beta \frac{1}{1+n} = \frac{1}{1+F_k'}.$$

These conditions on the marginal rates of substitution determine the ratio K/N, and as a consequence the initial capital stock that allows to remain from then on on the constant growth path. Consumption can then be computed from the scarcity constraint.

Role of nominal assets: theory

With a finite number of agents and an infinite number of goods, any competitive equilibrium is Pareto optimal.

Checking the first welfare theorem

The optimum allocation maximizes $\sum \beta^t U(C_t)$ subject to

$$C_t + K_t = F(K_{t-1}, L_t) + K_{t-1}.$$

First order conditions: let $\beta^t \lambda_t$ be the multiplier associated with the constraint of period t.
Derivative with respect to C_t:

$$U'(C_t) = \lambda_t$$

Derivative with respect to K_t:

$$\beta(1 + F'_{K_{t+1}})\lambda_{t+1} = \lambda_t.$$

Eliminating λ's yields

$$\beta \frac{U'(C_{t+1})}{U'(C_t)} = \frac{1}{1 + F'_{K,t+1}},$$

identical to the equilibrium condition.

Adding a nominal asset

Budget constraints:

$$p_t C_t + p_t K_t + B_t = w_t L_t + \Pi_t + \rho_t K_{t-1} + p_t K_{t-1} + B_{t-1}.$$

An equilibrium with (positive price) of the nominal asset (and no central bank intervention) should have a demand B_t equal to B_0 for all t.

This is inconsistent with utility maximization as soon as $B_0 > 0$: the plan which consumes B_0 at the first date and keeps a zero money balance from then on dominates the reference plan.

Assets and bubbles

Fundamental value of an asset

Money, nominal assets, physical assets

Arbitrage conditions

$$p_\tau = p_{\tau+1} + \rho_{\tau+1}.$$

If $\sum_{\tau=t}^{\infty} \rho_\tau$ is infinite, the price of the good in terms of numeraire is infinite: the price of the numeraire in terms of good is zero, and the nominal asset has zero real value at the equilibrium.

The overlapping generations model

A simple dynamic framework: overlapping generations

Discrete time, from $t = 1$ onwards.

Generations of identical agents with *finite* lives: here for simplicity life lasts two periods.

Consumers with utility $U(C^y, C^o)$, where C^y (resp. C^o) is the consumption of physical good when young (resp. old).

Initial endowments Y^y and Y^o.

The physical good is not storable.

Nominal asset B storable without cost, which serves as numéraire at all dates. Borrowing is allowed. Nominal interest rate is zero.

Program of typical consumer born at t

$$
\begin{cases}
\max \; U(C^y, C^o) \\
p_t C_t^y + B_t = p_t Y^y \\
p_{t+1} C_{t+1}^o = p_{t+1} Y^o + B_t
\end{cases}
$$

At date t, the old consumer has a quantity B_{t-1} of nominal asset, and consumes all her wealth, i.e. $Y^o + B_{t-1}/p_t$.

Intertemporal equilibrium

Definition 1 *: A* **perfect foresight intertemporal equilibrium with nominal asset** *is a sequence* (p_t, C_t^y, C_t^o)*,* $t = 1, \ldots,$ *with (strictly) positive prices, which satisfies:*

1. $$C_t^y + C_t^o = Y^y + Y^o,$$
2. *For* $t \geq 1$*,* (C_t^y, C_{t+1}^o) *maximizes the program of the consumer born at date* t*, given* (p_t, p_{t+1})*.* C_t^o *is equal to* $B_{t-1}/p_t + Y^o$*.*

Walras' law, nominal assets.

Finding the equilibria

Real interest rate at date t

$$\rho_t = \frac{p_t}{p_{t+1}} - 1$$

$$\begin{cases} \max \;\; U(C^y, C^o) \\ C^y + \dfrac{p_{t+1}}{p_t} C^o = Y^y + \dfrac{p_{t+1}}{p_t} Y^o \end{cases}$$

Let z^y and z^o be the excess demand functions coming out of the program:

$$z^y(\rho_t) = C_t^y - Y^y = -\frac{B}{p_t}$$

$$z^o(\rho_t) = C_{t+1}^o - Y^o = -(1+\rho_t) z^y(\rho_t).$$

Mind indices: $z^o(\rho_t)$ is a transaction of date $t+1$.

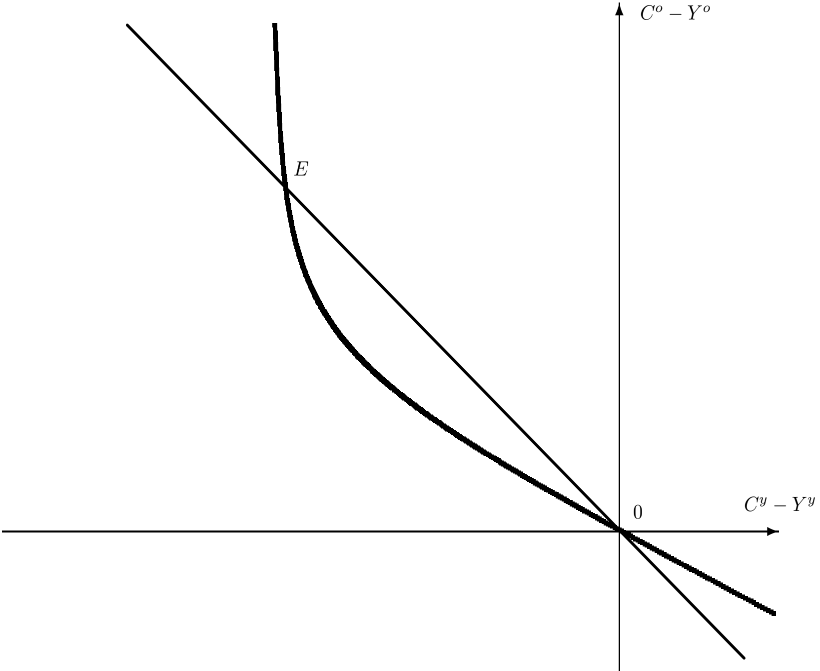
Supply (or offer) curve

The range of values of (z^y, z^o), when the price ratio $p_t/p_{t+1} = 1 + \rho_t$ varies, is the *supply curve* of the consumer.

z^y goes to infinity when p_t/p_{t+1} tends to 0 (a finite z^y would imply $C^o = Y^o$ and C^y finite, something incompatible with the equality of the marginal rate of substitution with the price ratio). Similarly, z^o tends to infinity when p_t/p_{t+1} tends to infinity.

Samuelson case:

$$\frac{U_o'(Y^y, Y^o)}{U_y'(Y^y, Y^o)} = \frac{1}{1+\underline{\rho}} > 1.$$



$C^o - Y^o$
E
0
$C^y - Y^y$

Finding the equilibria: continued

In the plan (C^y, C^o), $-(1+\rho)$ is the slope of the budget line.

The equality between supply and demand at date t is

$$z^y(\rho_t) + z^o(\rho_{t-1}) = 0.$$

Finite difference equation with initial condition $z^o(\rho_0) = B_0/p_1$ (of same sign as B_{t-1}).

A priori, there exists a continuum of equilibria, depending on the initial value ρ_0.

Stationary equilibria

Fixed points of the difference equation

$$z^y(\rho^*) + z^o(\rho^*) = 0,$$

at the intersection of the supply curve with the second bissector.

Since $z^y(\rho^*) + z^o(\rho^*) = -\rho^* z^y(\rho^*)$,

1. Autarky of every generation : $z^y = z^o = 0$. The equilibrium gross interest rate is $\underline{\rho}$. The real aggregate quantity of nominal assets is null: $B = 0$.
2. Transfers between generations, $B \neq 0$, interest rate $\rho^* = 0$, golden rule.

Non stationary equilibria

In the Samuelson case, continuum of indeterminate equilibria converging to the autarkic equilibria.

Isolated (determinate) golden rule equilibrium.

'Classical' case.

The results of the single-type two-period-life case apply to more general environments.

Fundamental and bubbles

Definition: *The* fundamental value *of an asset is the discounted sum of the dividends brought by a unit of this asset.*

The fundamental value of the nominal asset is zero. The competitive equilibrium(a) along which money is priced at its fundamental value are typically inefficient.

The first welfare theorem does not hold in economies with an infinity of goods and an infinity of agents.

Different agents, all with a two-period life

Agents are indexed with i, $i = 1, \ldots, I$.

They reproduce identically, so that at each date the economy is made of I young and I old agents.

Under perfect foresight, the young agent of type i born at t solves

$$\begin{cases} \max \; U^i(C^{yi}, C^{oi}) \\ C^{yi} + \dfrac{p_{t+1}}{p_t} C^{oi} = Y^{iy} + \dfrac{p_{t+1}}{p_t} Y^{oi} \end{cases}$$

The excess demand functions satisfy

$$(1 + \rho_t) z^{yi}(\rho_t) + z^{oi}(\rho_t) \equiv 0.$$

Different agents: continued

Aggregate excess demand: $Z^y(\rho) = \sum_{i=1}^{I} z^{yi}(\rho)$,
$Z^o(\rho) = \sum_{i=1}^{I} z^{oi}(\rho)$.

There exists $\bar{\rho}$ such that $Z^y(\bar{\rho}) = Z^o(\bar{\rho}) = 0$.

Along an equilibrium path

$$Z^y(\rho^t) + Z^o(\rho_{t-1}) = 0.$$

Walras' law: autarkic and efficient stationary equilibria.

Long lives

Assume that agents live for A periods, and that a new agent is born every period (the time period might be 30 years, if one wants to mimic the replacement of generations, and then A would be 3: but here we shall consider an unrestricted, possibly larger A).

$$\begin{cases} \max \ U(C^1, C^2, \ldots, C^A) \\ \displaystyle\sum_{a=1}^{A} \frac{p_{t+a-1}}{p_t}(C^a - Y^a) = 0 \end{cases}$$

The t generation faces a sequence $\rho_t, \ldots, \rho_{t+A-1}$ of one period real interest rates.

$$z^a(\rho_{t-a+1}, \ldots, \rho_{t+A-a})$$

Long lives: continued

Equilibria:

$$\sum_{a=1}^{A} z^a(\rho_{t-a+1}, \ldots, \rho_{t+A-a}) = 0.$$

There is an efficient stationary equilibrium associated with a zero interest rate. Aggregate savings are equal to

$$\sum_{a=1}^{A-1}(z^1 + \ldots + z^a) = \sum_{a=1}^{A-1}(A-a)z^a.$$

There is also an autarkic stationary equilibrium with constant interest rate, solution to

$$\sum_{a=1}^{A} z^a(\rho, \ldots, \rho) = 0.$$

Demographic growth

Suppose that consumers are as before in the basic two-period-life model, but that population grows at the constant rate n.

The (individual) supply curve is unchanged.

The feasibility constraints become

$$(1+n)z^y(\rho_t) + z^o(\rho_t - 1) = 0,$$

$$(1+n)B_t = B_{t-1}.$$

The diagram works as before with the 45^0 line $z^o = -z^y$ replaced with $z^o = -(1+n)z^y$.

The overlapping generations model is useful in a number of circumstances: interaction of demographics and economics, land and housing, capital accumulation.

We shall use it later in these lectures to discuss the role of public debt, and study pensions.

Temporary equilibria vs perfect foresight: learning

- Perfect foresight does not pin down expectations: indeterminateness.
- Stationarity is too strong an assumption.
- Putting things together: learning.

Learning

Learning

I take the overlapping generations model as the reference, because of its rationalization of valued nominal assets in the long run, in spite of the lack of realism of the *two* period-life setup.

The temporary equilibrium price at date t is a solution to the equation

$$z^y \left(\frac{p_t}{p^e_{t+1}} - 1 \right) + \frac{B}{p_t} = 0,$$

which can be rewritten as

$$A(p_t, p^e_{t+1}) = 0.$$

The expected price p^e only can depend on past and current observations.

$$p^e_{t+1} = \psi(p_t, \ldots, p_{t-L}).$$

Local linearization at a stationary point: structural equation

One differentiates and looks for solutions of the form

$$dp_t = \lambda^t.$$

Perfect foresight dynamics:

$$A(p_t, p_{t+1}) = 0.$$

$$a_0 dp_t + a_1 dp_{t+1} = 0$$

$$Q_F(\lambda) = a_0 + a_1 \lambda$$

The perfect foresight dynamics is locally stable if

$$\left| \frac{a_0}{a_1} \right| < 1$$

Local linearization: expectations

Expectation formation near a stationary point:

$$p^e_{t+1} = \psi(p_t, \ldots, p_{t-L}).$$

$$dp^e_{t+1} = \sum_{i=0}^{L} \psi_i dp_{t-i}$$

$$Q_\psi(\lambda) = \lambda^{L+1} - \sum_{i=0}^{L} \psi_i \lambda^{L-i}$$

No forecast errors on stationary trajectories:
$Q_\psi(1) = 1 - \sum_{i=0}^{L} \psi_i = 0.$

No forecast errors on trajectories with rate of growth r:
$Q_\psi(1+r) = (1+r)^{L+1} - \sum_{i=0}^{L} \psi_i (1+r)^{L-i} = 0.$

Trends: real roots. Fluctuations : complex roots (except for period 2 cycles)

Temporary equilibrium dynamics

$$a_0 dp_t + a_1 \sum_{i=0}^{L} \psi_i dp_{t-i} = 0$$

$$Q_T(\lambda) = a_0 \lambda^L + a_1 \sum_{i=0}^{L} \psi_i \lambda^{L-i}$$

This can be rewritten as

$$Q_T(\lambda) = a_0 \lambda^L + a_1 \lambda^{L+1} + a_1(-\lambda^{L+1} + \sum_{i=0}^{L} \psi_i \lambda^{L-i})$$

or

$$Q_T(\lambda) = \lambda^L Q_F(\lambda) - a_1 Q_\psi(\lambda).$$

Note: Q_T has degree L, difference of two polynomials of degree $L+1$.

Learning reverses time

A natural (too simple) example is

$$\psi(p_t, \ldots, p_{t-L}) = p_{t-1}$$

$$Q_\psi(\lambda) = \lambda^2 - 1$$

$$Q_T(\lambda) = \lambda(a_0 + a_1\lambda) - a_1(\lambda^2 - 1) = \lambda a_0 + a_1$$

Then *the learning temporary equilibrium dynamics converges ($|a_1/a_0| < 1$) if and only if the perfect foresight dynamics diverges.*

In general interaction between the structure (perfect foresight dynamics) and the sophistication of learning.

Sophisticated learning creates instability (intuition)

Suppose that the two extreme trends with which the expectation function is consistent are $\lambda_1 \leq -1$ and $1 < \lambda_2$.

Suppose also that the perfect foresight structural characteristic root(s) belong to (λ_1, λ_2).

Then the learning dynamics has an unstable root r outside the interval $[\lambda_1, \lambda_2]$.