# MSC G022 Macroeconomics
# Preliminary notes
# on
# Macroeconomics and money[1]

January 4, 2011

[1]These rough notes have been written by the professor, as a reminder for himself of the content of the lectures. Please report all errors, misprints, etc... to Guy Laroque. Do not circulate.

After Morten Ravn's description of the real economy, I am going to discuss several themes on macroeconomics and finance. I shall not present an integrated full blown theory, but rather shed light on a few important questions: expectations and the determination of the general price level, finance and the allocation of risks, the role of money and finance, monetary policy and public debt, pensions.

# Chapter 1

# Temporary equilibrium and finance

- Important prerequisite for the study of the determination of interest rates, the role of monetary policy,...
- Concept of *temporary equilibrium*: history, expectations, description of resource allocation.
- Status of the *competitive* equilibrium: speed of search (models of unemployment with matching function on the labor market), price or wage flexibility, size of agents with respect to the overall market...
- Agents' horizons and subjectivity (vs. rationality) of expectations.
- Dynamics. Going from one period to the next: adjustment costs, (Bayesian?) revision of expectations, accumulation of assets.

## 1.1 How to describe the typical consumer's behavior

General program of a consumer with a finite horizon, in discrete time.

Definition of an asset: physical vs. financial asset (which may represent (or not) a physical asset: money (numéraire, $r$=0); bonds ($r$ is measured in numéraire, nominal); shares yielding dividends ($r$ is directly linked to the price of real or physical assets, some *fundamental*); derivatives, etc...). To define a financial asset, one needs to make precise who is the debtor, and rules in case of non solvency.

Budget constraints, debt constraints.

$$p^e_\tau C_\tau + q^e_\tau B_\tau = p^e_\tau Y^e_\tau + (q^e_\tau + r^e_\tau) B_{\tau-1}$$

End of game:

$$B_T \geq 0$$

Physical or feasibility constraints:

$$C_\tau \in \mathrm{I\!R}_+^\ell$$

Possibly liquidity constraints:

$$B_\tau \in \mathrm{I\!R}_+^m$$

Expectations on income, prices, returns. Random environment. All expectations can be represented with a vector $(Y_\tau^e, p_\tau^e,\ q_\tau^e,\ r_\tau^e,\ \tau = 2, \ldots,\ T)$, which bears on the whole horizon, and depends on the available information at the decision date, date 1. News sequentially arrive, and lead to revise expectations, following a well defined a priori rule (for instance Bayes rule).

The utility function, defined on random consumptions, is often taken to be separable over time (separability is simplifying in an infinite horizon environment). Von Neumann-Morgenstern assumption: example $E_0 \sum_{t=0}^{T} \beta^t U(C_t)$.

The solution typically uses dynamic programming techniques, with specific tricks to deal with bankruptcy.

Where do the expectations come from? In the spirit of temporary equilibrium, they are exogenously given and revised according to historical observations. Typically this involves prediction errors. When agents learn from their past errors, one *may* converge towards a perfect foresight (in the absence of uncertainty) or rational expectations (if there are shocks) equilibrium trajectory, where expectations are endogenously determined in the long run.

## 1.2 The case of a single asset and the treatment of bankruptcies

### 1.2.1 The model

Aggregation and study of a two periods model, with one asset ('money': numèraire, no dividends), one consumption good. Certain expectations.

Typical agent (superindex $i$ to designate the agent, omitted when there is no risk of confusion):

$$\begin{cases} \max\ U(C_t,\ C_{t+1}) \\ p_t C_t + B_t = p_t Y_t + B_{t-1} \\ p_{t+1}^e C_{t+1} + B_{t+1} = p_{t+1}^e Y_{t+1}^e + B_t \\ C_t,\ C_{t+1},\ B_{t+1}\ \geq\ 0 \end{cases}$$

The good is non durable, non storable. Consumption at date $t$ is denoted $C_t$, production $Y_t$. Its price in terms of the numeraire asset is the general price level

$p_t$. Agents enter period $t$ with a financial wealth $B_{t-1}$ (in case of debt, $B_{t-1}$ is negative). There are no credit constraints: $B_t$ is unconstrained and can take any sign. At the end of the game, the agents cannot leave debts to their successors, so that $B_{t+1}$ is restricted to be nonnegative.

**Assumption**: *$U(C_t,\ C_{t+1})$ is strictly quasi-concave, strictly increasing in $C_t$ and $C_{t+1}$, and continuously differentiable on $\mathbb{R}^2_+$. $Y_t$ and $Y_{t+1}$ are (strictly) positive. For any positive couple $C_1$ and $C_{t+1}$:*

$$\lim_{c\to 0} U(c, C_{t+1}) < U(C_t, C_{t+1}),$$

$$\lim_{c\to 0} U(C_t, c) < U(C_t, C_{t+1}).$$

(the indifference curves are asymptote to the axis).

At date $t$, the consumer program simplifies into

$$\begin{cases} \max\limits_{C_t, C_{t+1}} U(C_t,\ C_{t+1}) \\ p_t C_t + p^e_{t+1} C_{t+1} = p_t Y_t + p^e_{t+1} Y^e_{t+1} + B_{t-1}, \end{cases}$$

with the demand for asset $B_t$ derived from the first period budget constraint

$$p_t C_t + B_t = p_t Y_t + B_{t-1}.$$

The *subjective* real interest rate between dates $t$ and $t+1$ is defined by

$$1 + \rho^e_t = \frac{p_t}{p^e_{t+1}}.$$

Indeed selling 1 unit of good at date $t$ yields $p_t$ units of money, with which the consumer can contemplate buying $(1 + \rho^e_t)$ units of good at date $t + 1$. The consumer's budget constraint can be rewritten as

$$C_t - Y_t + \frac{1}{1+\rho^e_t}(C_{t+1} - Y^e_{t+1}) = \frac{B_{t-1}}{p_t}.$$

The first order condition for the optimum consumption profile is

$$\frac{U'_1(C_t,\ C_{t+1})}{U'_2(C_t,\ C_{t+1})} = \frac{p_t}{p^e_{t+1}} = 1 + \rho^e_t.$$

Under the assumptions bearing on the utility function, the program has a unique maximum $(C_t,\ C_{t+1})$, which is obtained by resolving the system made of the first order condition coupled with the budget constraint. The (continuous) consumption function $C_t$ can be written as

$$C_t = \gamma\left(\frac{p_{t+1}}{p_t},\ Y_t + \frac{B_{t-1}}{p_t},\ Y_{t+1}\right),$$

with asset demand $B_t$ following through

$$B_t = p_t\left[Y_t + \frac{B_{t-1}}{p_t} - \gamma\left(\frac{p_{t+1}}{p_t},\ Y_t + \frac{B_{t-1}}{p_t},\ Y_{t+1}\right)\right] \equiv p_t\beta\left(\frac{p_{t+1}}{p_t},\ Y_t + \frac{B_{t-1}}{p_t},\ Y_{t+1}\right).$$

Definition of a temporary competitive *equilibrium at date t*: value of price $p_t$ such that supply equals demand, both for the consumption good and for the asset:

$$\sum_i C_t^i = \sum_i Y_t^i,$$

$$\sum_i B_t^i = \sum_i B_{t-1}^i.$$

By Walras' law (the budget constraints of date $t$ are verified identically by the consumption functions and asset demands of all agents), one equality automatically implies the other: we in fact have one equation (supply-demand either of good or of assets, as desired) for one unknown to be determined, the general price level:

$$\sum_i \gamma^i\left(\frac{p_{t+1}^{ei}}{p_t},\ Y_t^i + \frac{B_{t-1}^i}{p_t},\ Y_{t+1}^{ei}\right) - \sum_i Y_t^i = 0,$$

$$\sum_i p_t\beta^i\left(\frac{p_{t+1}^{ei}}{p_t},\ Y_t^i + \frac{B_{t-1}^i}{p_t},\ Y_{t+1}^{ei}\right) - \sum_i B_{t-1}^i = 0.$$

The remainder of this section discusses the existence of a temporary equilibrium, i.e. of a (positive finite)price level that clears the markets. The elements of interest are the (exogenous) expectations and whether they may prevent market clearing when they are poorly behaved.

How should we specify expectations? A priori an agent's expectations depend on her whole microeconomic and macroeconomic history. But history at date $t$ is given and fixed: to avoid unnecessary notations, it is omitted from the arguments. The only argument of the expectations that needs to be made explicit when looking for an equilibrium is the current endogenous variable $p_t$. We only consider the case where $Y_{t+1}^{ei}$ is a constant, independent from the endogenous variable, but discuss a general continuous possibly nonlinear form for price expectations

$$p_{t+1}^{ei} = \psi^i(p_t).$$

There are two basic polar cases where an equilibrium could fail to exist:

1. Keynesian unemployment. There is a permanent excess supply of good, whatever the price level (or equivalently excess demand for assets or money). Reducing prices which should stimulate demand and/or reduce supply does not clear the markets.

2. Repressed inflation. There is a permanent excess demand for good, whatever the price level (or equivalently excess supply of assets or money). Increasing prices which should dampen demand and/or stimulate supply does not clear the markets.

Under our standard assumptions on preferences and continuity of the expectations, aggregate excess demand (either of goods or of assets) is a continuous function of the general price level. The existence of a temporary equilibrium, i.e. of a zero of aggregate excess demand, then typically is ensured by conditions that implies that it has a different sign when $p_t$ goes either to zero or to infinity. We first look at the behavior of individual demand, for which we distinguish two effects of a change in the price level, a real balance Pigou effect and a substitution effect.

**Real balance effects**

We define the *real balance effect* as the change of demand caused by variations of the real value of asset holdings $B^i_{t-1}/p_t$, everything else equal. This means that we observe the real balance effects when expectations have a unit elastic shape:

$$p^{ei}_{t+1} = \psi^i(p_t) = \frac{p_t}{1+\bar{\rho}^{ei}},$$

for some constant expected real rate of interest $\bar{\rho}^{ei}$. Then the demand for good is

$$\gamma\left(\frac{1}{1+\bar{\rho}^{ei}},\ Y_t + \frac{B^i_{t-1}}{p_t},\ Y_{t+1}\right)$$

and is obtained by maximizing utility $U^i(C_t, C_{t+1})$ on the budget constraint

$$C_t + \frac{1}{1+\rho^{ei}_t}C_{t+1} = Y^i_t + \frac{1}{1+\rho^{ei}_t}Y^{ei}_{t+1} + \frac{B^i_{t-1}}{p_t} = W^i.$$

When $p_t$ varies, the optimal consumption profile $(C_t, C_{t+1})$, when it exists, moves on a fraction of consumer $i$'s Engel curve, associated with the relative price $1/(1+\rho^{ei}_t)$ and the intertemporal income $W^i_t$.

1. For indebted consumers, $B^i_{t-1} < 0$, price decreases augment the debt burden and reduce intertemporal income. When

$$\frac{B^i_{t-1}}{p_t} + \left[Y^i_t + \frac{1}{1+\rho^{ei}_t}Y^{ei}_{t+1}\right] < 0,$$

intertemporal income is negative, the consumer is bankrupt, the budget set is empty, and there is no solution to the consumer problem! For the time being (see a fuller discussion of bankruptcies below), we assume that a central bank then steps in, seizes all the consumer's properties and leaves her/him with the $(0,0)$ consumption profile.[1]

When $p_t$ goes to infinity, intertemporal income $W^i$ increases and tends to

$$Y_t^i + \frac{1}{1+\rho_t^{ei}} Y_{t+1}^{ei},$$

and $C_t^i$ has a positive upper limit, say $\underline{C}^i$.

2. For creditors, with positive cash balances, price decreases augment the real value of money holdings, which goes to infinity when $p_t$ goes to zero. If consumption today is a *normal* good, i.e. the Engel curve at the current expected real rate of interest has a uniformly positive slope and $C_t^i$ is a decreasing function of the price level $p_t$. At one extremity of its range, $C_t^i$ goes to infinity when $p_t$ goes to zero. This is Pigou *real balance effect.* At the other extremity, when $p_t$ goes to infinity, intertemporal income $W^i$ decreases to

$$Y_t^i + \frac{1}{1+\rho_t^{ei}} Y_{t+1}^{ei},$$

and $C_t^i$ has a positive lower limit, say $\underline{C}^i$.

To summarize, from a theoretical point of view, the real balance effect in itself if empirically strong enough may remedy a situation of keynesian unemployment: the creditors' demands are stimulated by deflation (a decrease of the current price level), so that aggregate demand would increase without limit. An important caveat is that, in the process, all debtors become bankrupt! On the other hand, looking at repressed inflation, increases in the price level may not be enough to reduce demand below the supply of good. In particular, the limit of aggregate demand $\sum_i \underline{C}^i$ may stay higher than aggregate supply $\sum_i \underline{Y}^i$, preventing market clearing.

### Substitution effects

Define *substitution effects* as changes in demand induced by changes in the price level, in the absence of real balance effects, i.e. when we put initial cash balances $B_t^i - 1$ at zero. All these effects go through the expected real interest rate

$$1 + \rho_t^{ei} = \frac{p_t}{\psi^i(p_t)}.$$

Of course, substitution effects are absent when price expectations are unit elastic: indeed they stem from the difference from 1 of the elasticity of price expectations.

[1] Note that it makes the associated consumption function $\gamma$ continuous, since at the limit price, when $W^i = 0$, the only point left in the budget set is precisely $(0,0)$.

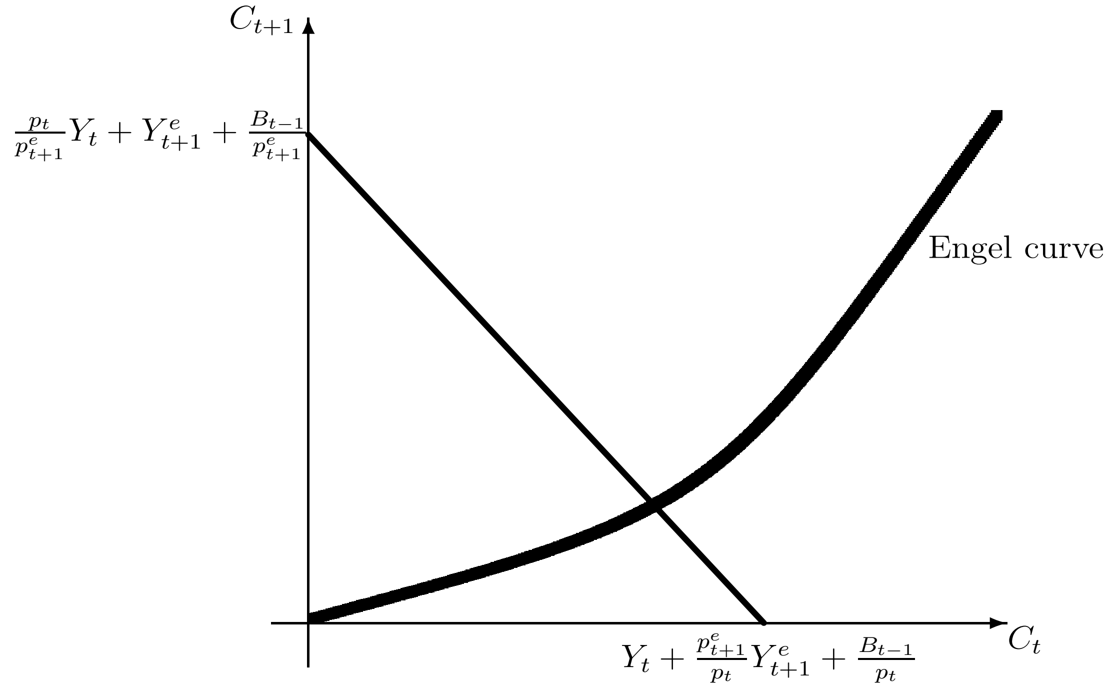


Figure 1.1: Demand and expectations

The easiest way to represent them is on the standard two-good consumer diagram. When the general price level changes, the budget line turns around the initial endowment point $(Y_t^i, Y_{t+1}^i)$, and the optimal consumption plan is the point located on the budget line such that the ratio of the marginal rates of substitution is equal to the ratio of prices

$$\frac{U_t'(C_t, C_{t+1})}{U_{t+1}'(C_t, C_{t+1})} = \frac{p_t}{p_{t+1}^{ei}} = 1 + \rho_t^{ei}.$$

When the general price level varies, so does the real interest rate $\rho$, and therefore demand. We are interested in the limit behavior of demand, in order to deal with the two polar cases of Keynesian unemployment and repressed inflation: this means respectively $C_t$ or $C_{t+1}$ becoming very large, which obtains when the budget line is horizontal or vertical, that is when $\rho$ goes to $-1$ or $+\infty$. Under our assumption on preferences, it is easy to show that

when $\rho$ goes to -1, $C_{t+1}$ tends to $Y_{t+1}^{ei}$ and $C_t$ goes to $+\infty$;

when $\rho$ goes to $+\infty$, $C_t$ tends to $Y_t^i$ and $C_{t+1}$ goes to $+\infty$.

The above remarks guide us towards sufficient conditions on the shape of expectations which guarantee the existence of a temporary equilibrium. Indeed:

**Proposition 1 (curing unemployment by deflation)**: *Assume $\psi(p)/p$ goes to infinity when p goes to zero. Then demand $\gamma(\psi(p_t)/p_t,\ Y_t,\ Y_{t+1})$ tends to infinity when $p_t$ goes to 0.*

The control of the demand for assets is slightly more complicated:

**Proposition 2**: *Assume that $\psi(p)/p$ tends to 0 when $p$ tends to $\infty$. Then the demand for assets become non negative for $p_t$ large enough.*

*Proof :* Otherwise, $C_{t+1} = B_t/\psi(p_t) + Y_{t+1}$ stays smaller than $Y_{t+1}$. $C_t$ tends to $Y_t$. This gives a contradiction with the first order condition. ∎

**Proposition 3 (curing repressed inflation, continued)**: *Suppose $\psi(p)/p$ tends to 0 when $p$ tends to $\infty$, while $\psi(p)$ stays larger than a strictly positive number, say $\underline{p}$. Then the demand of financial assets $p_t\beta\left(\frac{\psi(p_t)}{p_t},\ Y_t,\ Y_{t+1}\right)$ goes to infinity with $p_t$.*

*Proof :* Otherwise, $B_t$ stays bounded. Then $C_t$ tends to $Y_t$, and $C_{t+1} = B_t/\psi(p_t)+Y_{t+1} \leq B_t/\underline{p}+Y_{t+1}$ also is bounded. The marginal rate of substitution converges towards a finite value, and cannot stay equal to (or larger than) $p_t/\psi(p_t)$ which goes to infinity. ∎

Proposition 3 is linked to the debate on the value of money (or of nominal assets). When the price $p_t$ goes to infinity, by definition the value of money in terms of good goes to zero. Why do the economic agents want to keep any money, since it has no intrinsic function in the model? The answer here is that they believe that it will have value later ($\psi(p)$ does not go to infinity as fast as $p$). The asset has value because one expects that it will keep being accepted as a means of payment by the future generations!

### Temporary competitive equilibrium

Sufficient conditions for the existence of a temporary competitive equilibrium can be derived from the above remarks, putting together the real balance effect and the substitution effects.

**Theorem**: *Assume that, for at least* **one** *agent, $B^i_{t-1} \geq 0$, and that $\psi^i(p) \geq \underline{p} > 0$ for all $p$. Furthermore, assume that for* **all** *agents $\psi^j(p)/p$ tends to 0 when $p$ tends to $\infty$. Then there exists a temporary competitive equilibrium.*

*Sketch of a proof :* By Walras' law, one only needs to check that there is a price such that the demand for good is equal to supply. The difference between demand and supply is a continuous function of price. It tends to infinity when $p$ goes to zero from Proposition 1. It is equal to $-p\sum_i(B^i_t - B^i_{t-1})$, from the sum of the budget constraints (Walras' law), and therefore becomes negative when $p$ goes to infinity from Propositions 2 and 3. ∎

The above analysis stresses the importance of substitution effects and expectation formation to determine an equilibrium. Real balance effects, which empirically seem weak anyway, have a very subsidiary role (in the statement of the Theorem, one just needs that one agent has a non negative money balance so that s/he does not go bankrupt).

In practical exercises, one usually computes the demand curves for good or assets as a function of the current price, given some shape of the expectation function. One then looks for a solution of the demand equal supply equation, to obtain an equilibrium.

**Remark**: a similar analysis can be carried out for an economy without credit, where the holdings of nominal assets are restricted to be non negative. The conditions on expectations to ensure the existence of an equilibrium are less restrictive: one does not need that expectations make *all* agents voluntarily have a positive demand for assets, as above, since this is now imposed by the institutional setup. It is enough that *one* agent is such that $\psi(p)/p$ goes to zero when $p$ tends to infinity.

### 1.2.2 Debts without a lender of last resort: Chain of bankruptcies

Initial wealth distribution and chains of bankruptcies (links between debtors). Let $B_t^{ij}$ be the (non negative) sum that agent $j$ owes to agent $i$ at the beginning of period $t+1$, for $i \neq j$, with the notational convention $B^{ii} = 0$. To illustrate, the budget constraints of agent $i$ can be written as:

$$\begin{array}{l} p_t C_t + \sum_j q^j B_t^{ji} - q^i \sum_{j \neq i} B_t^{ij} = p_t Y_t + \sum_j B_{t-1}^{ij}(1 - r_t^j) - \sum_j B_{t-1}^{ji} \\ p_{t+1} C_{t+1} = p_{t+1} Y_{t+1} + \sum_j B_t^{ij}(1 - r_{t+1}^j) - \sum_j B_t^{ji} \end{array}$$

where $q^j$ is today's price of a claim on agent $j$, and $r^j$ stands for the fraction of that claim which will not be honored. We look for an endogenous determination of today's bankruptcies. Assume that the bankruptcy rule stipulates that, when the debtor cannot meet his liabilities, for instance because he cannot get further credits, all his holdings are confiscated, and are shared proportionately to their claims among the creditors. The corresponding algebraic formula looks like

$$1 - r^i = \min\left(1\ ,\ \frac{p_t Y_t + \sum_j B_{t-1}^{ij}(1 - r^j)}{\sum_{j \neq i} B_{t-1}^{ji}}\right),$$

when it is assumed that all previous commitments must be met before proceeding to new borrowing and lending. Such a rule has obvious drawbacks, since it rules out long run debt: one possibly can add to current resources the cash $q^i \sum_j B_t^{ij}$

raised by $i$. But this makes bankruptcy dependent on the debt behavior of agent $i$. To avoid manipulations, other rules can be thought of, with other downsides: in case of bankruptcy, the judge can put a claim on the future income of the agents (if feasible), and send this claim on today's market to add the present value of these incomes to the right hand side $q^i \sum_j B_t^{ij}$. The expectation of such a bankruptcy rule in the future typically creates non convexities or discontinuity in the behavior of today, source of possible non existence of an equilibrium. To the best of my knowledge, there does not exist a systematic comparative study of the theoretical properties of the various rules that can be thought of.

The literature on credit and collateral, together with the attached interest rate, is closely linked to the topic: credit channel of monetary policy (the market value of the assets of a firm, which act as a collateral, determine the amount of new debt,...). The dynamic properties of economies exhibiting this type of credit multiplier has been studied actively in the 90s (Kyotaki Moore, Bernanke Gertler).

# Chapter 2

# The overlapping generations model

There are generations of identical agents who live two periods. The model therefore has a double infinity of (dated) goods and agents. Economically, the savings needs for life cycle considerations, linked to the labor supply profile are *a priori* likely to be different from the stock of capital required for the efficiency of production. This structure makes room for a financial system to create a bridge between the two.

We start with the simplest model, introducing complications progressively.

## 2.1 An exchange economy

A single physical good.

Consumers with utility $U(C^y, C^o)$, where $C^y$ (resp. $C^o$) is the consumption of physical good when young (resp. old).

Initial endowments $Y^y$ and $Y^o$.

**The physical good is not storable.**

Nominal asset $B$ storable without cost. This asset serves as numéraire at all dates. One can borrow or lend using $B$ as support.

The program of the typical consumer born in period $t$ is:

$$\begin{cases} \max \ U(C^y, C^o) \\ p_t C_t^y + B_t = p_t Y^y \\ p_{t+1} C_{t+1}^o = p_{t+1} Y^o + B_t \end{cases}$$

At the initial date 1, the old consumer has a quantity $B_0$ of nominal asset, and maximizes his current consumption.

**Definition 1** *: A* **perfect foresight intertemporal equilibrium with nominal asset** *is a sequence $(p_t, C_t^y, C_t^o)$, $t = 1, \ldots,$ with (strictly) positive prices, which satisfies:*

1. $$C_t^y + C_t^o = Y^y + Y^o,$$

2. *For* $t \geq 1$*,* $(C_t^y, C_{t+1}^o)$ *maximizes the program of the consumer born at date* $t$*, given* $(p_t, p_{t+1})$*.* $C_1^v$ *is equal to* $B_0/p_1 + Y^o$*.*

From Walras' law, the scarcity constraint in physical good implies the equality between supply and demand in nominal assets. The quantity of nominal asset in the economy stays constant along the path.

To determine the set of equilibria, one starts by studying the consumer's program, after eliminating the nominal asset, ignoring for the time being the sign constraint bearing on asset holdings:

$$\begin{cases} \max\ U(C^y, C^o) \\ C^y + \dfrac{p_{t+1}}{p_t} C^o = Y^y + \dfrac{p_{t+1}}{p_t} Y^o \end{cases}$$

Let $z^y$ and $z^o$ be the excess demand functions coming out of the program:

$$z^y\left(\frac{p_t}{p_{t+1}} - 1\right) = C_t^y - Y^y = -\frac{B}{p_t}$$

$$z^o\left(\frac{p_t}{p_{t+1}} - 1\right) = C_{t+1}^o - Y^o = -\frac{p_t}{p_{t+1}} z^y\left(\frac{p_t}{p_{t+1}} - 1\right).$$

By construction, these functions satisfy the budget identity

$$(1+\rho)z^y(\rho) + z^o(\rho) \equiv 0,$$

for all $\rho$. The range of values of $(z^y, z^o)$, when the price ratio $p_t/p_{t+1} = 1 + \rho_t$ varies, is the *supply curve* of the consumer. Figure 2.1 shows the (Samuelson) case where the marginal rate of substitution between future and current consumptions at the initial endowment point is larger than 1:

$$\frac{U_o'(Y^y, Y^o)}{U_y'(Y^y, Y^o)} = \frac{1}{1+\underline{\rho}} > 1.$$

If prices were constant over time, the agents at their initial endowment point would want to save for their old days. It is easy to check that $z^y$ goes to infinity when $p_t/p_{t+1}$ tends to 0 (a finite $z^y$ would imply $C^o = Y^o$ by the budget constraint, and this would be incompatible with the equality of the marginal rate of substitution to the price ratio). Similarly, $z^o$ tends to infinity when $p_t/p_{t+1}$ tends to infinity. Finally the Figure is drawn in the case where $z^y$ (resp. $z^o$) is a decreasing (resp. increasing) function of the price ratio $p_t/p_{t+1}$ : this property holds whenever the assumption of gross substituability of aggregate demand is satisfied, but it is easy to build examples where savings (-$z^y$) is a decreasing function of the interest rate.

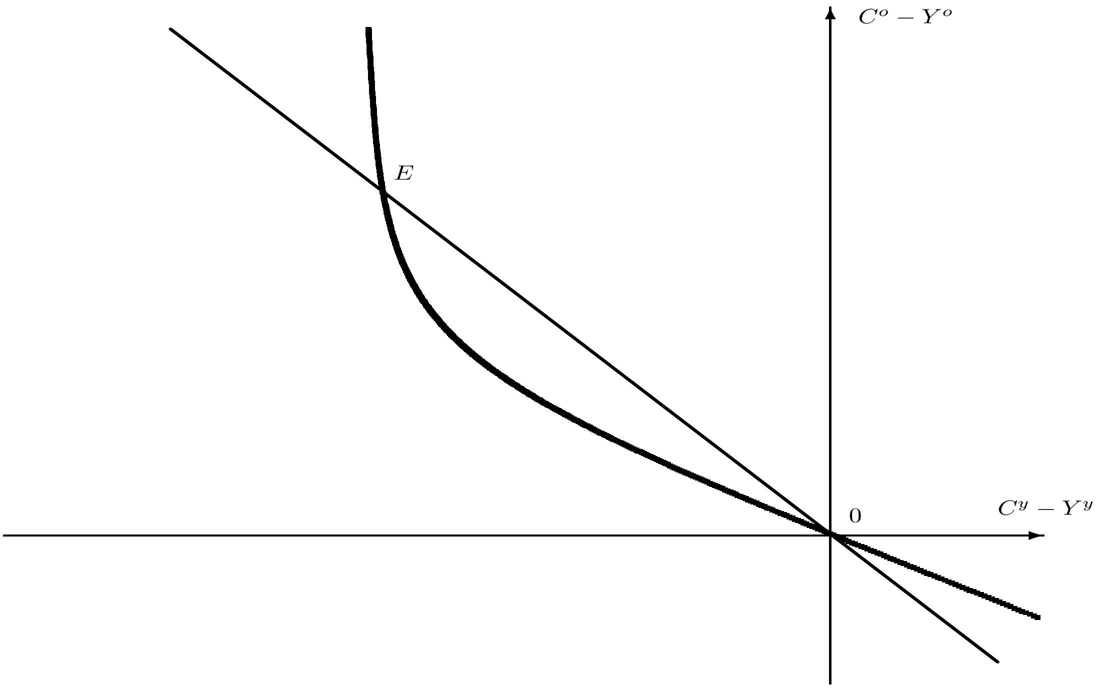


Figure 2.1: Equilibria in the overlapping generations model

Let $1 + \rho_t = p_t/p_{t+1}$. This ratio is the gross real interest rate between dates $t$ and $t+1$, i.e. the quantity of consumption good one can obtain at date $t+1$ in exchange of a unit of good at date $t$. In the plan $(C^y, C^v)$, $-(1+\rho)$ is the slope of the budget line. The equilibria then appear through the study of a finite difference equation, with initial condition $z^o(\rho_0) = B_0/p_1$ of same sign as $B_0$ :

$$z^y(\rho_t) + z^o(\rho_{t-1}) = 0.$$

*A priori*, there may exist a continuum of equilibria, depending on the initial value of $\rho_0$.

The stationary equilibria are located at the intersection of the supply curve with the second bissector. They correspond to fixed points of the difference equation, the values $\rho^*$ such that $z^y(\rho^*) + z^o(\rho^*) = 0$. Using the budget identity, $(1+\rho)z^y(\rho) + z^o(\rho) = 0$, this equation can be rewritten as

$$\rho^* z^y(\rho^*) = 0.$$

As a consequence, there are two sorts of fixed points:

1. those that correspond to the autarky of every generation (in the case of a generation made of a diverse population, there may be lending and borrowing inside a generation, but the balance is zero): $z^y = z^o = 0$. Here the equilibrium gross interest rate is $\underline{\rho}$. The real aggregate quantity of nominal assets is null: $B = 0$.

2. equilibria with transfers between generations, $B \neq 0$. By construction of the supply curve, they are associated with a gross interest rate $\rho^* = 0$. They are golden rule equilibria: the interest rate is equal to the population growth rate.

One can use the graph to find a continuum of (non stationnary) equilibria: starting from any $\rho_0$ between $\underline{\rho}$ and $\rho$, one can draw an infinite sequence which corresponds to a non stationary equilibrium with perfect foresight.

For a positive $\rho_0$, or for a $\rho(0)$ smaller than $\underline{\rho}$, the sequence diverges. The *"difference equation"* is not a difference equation in the precise mathematical meaning of the term: it is only defined on a subset of $[0, \infty]$. Note that, if time is reversed, for a given $p_{t+1}$, under the existence conditions of temporary equilibria, there is always (at least) one associated $p_t$. The *backward dynamics* is well defined. It is furthermore closely linked to a *learning* dynamics, where the forecast of agents at date $t$ would be $\rho^a_{t+1} = \rho_{t-1}$ (along a trajectory, expectations errors imply that the consumptions do not belong to the supply curve: one follows the consumptions when young on the graph, but not consumptions when old).

One can also note that all the non stationary equilibria converge towards the inefficient autarkic stationary equilibrium with the forward dynamics (on the other hand, if expectations are supposed equal to the past price, the real time dynamics is the backward dynamics, and the trajectory, along which there are always expectations errors, converges towards $E$. $E$ is the only isolated equilibrium (in the product topology), it is the only *determinate* equilibrium.

Remark : with either an infinity of agents, or an infinity of goods, the competitive equilibrium is Pareto optimal under standard assumptions. This property does not hold any more when there is a double infinity. In a sense the multiplicity of equilibria partially disappoints the hopes that the assumption of rational expectations might get rid of the lack of determination associated with the choice of expectations in temporary equilibrium theory.

## 2.2 Economies with land

The nominal asset, when its equilibrium value is not zero (price of the good in units of asset different from infinity, price of the asset in units of good different from zero) is a bubble: its price differs from the discounted sum of the values of the dividends which it produces. How robust is this bubble?

Suppose that together with the nominal asset, which in the previous section was the only mean of transferring wealth between periods, there is a physical asset, land. The owner of a piece of land receives the crop, a quantity of non durable good, at each date. Formally, we assume that there are no randomness in the crops, the crops are constant over time, and we choose units so that a unit of land brings a unit of good at each date. The available land area is constant and equal to $T$. The price of land measured in numéraire is $q_t$. The consumer's program becomes:

$$\begin{cases} \max\ U(C^y, C^o) \\ p_t C_t^y + q_t T_t + B_t = p_t Y^y \\ p_{t+1} C_{t+1}^o = p_{t+1} Y^o + (q_{t+1} + p_{t+1}) T_t + B_t \\ T_t \geq 0 \end{cases}$$

A perfect foresight intertemporal equilibrium is a sequence of prices and quantities such that, at each date, consumptions and asset holdings are solutions of the consumers' programs, given prices, and the scarcity constraints are satisfied:

$$C_t^y + C_t^o = Y^y + Y^o + T,$$

$$B_t = B_0,$$

$$T_t = T.$$

**Property 1** : *there is no equilibrium with a (strictly) positive quantity $B_0$ of nominal asset.*

*Proof* : The absence of arbitrage opportunity between the two assets for the consumer gives:

$$q_t \geq q_{t+1} + p_{t+1},$$

with equality if $T_t$ is strictly positive. Therefore, along an equilibrium path:

$$q_t = \sum_{\tau=t+1}^{t'} p_\tau + q_{t'} \geq \lim \sum_{\tau=t+1}^{\infty} p_\tau.$$

In order to have a finite price of land, $p_t$ has to go to zero when $t$ goes to infinity. But this is inconsistent with the budget constraint of the young consumer at date $t$, which becomes in the limit $\lim q_t T + B_0 = 0$. ∎

It therefore is of interest to study equilibria with land, without nominal assets. We take the non durable good as numéraire, instead of the nominal asset, while keeping the same notation $q_t$ for the price of land. Eliminating land holdings $T$ between the two budget constraints, the program becomes:

$$\begin{cases} \max\ U(C^y, C^o) \\ (1 + \rho_t) C_t^y + C_{t+1}^o = (1 + \rho_t) Y^y + Y^o \end{cases}$$

where:

$$1 + \rho_t = \frac{1 + q_{t+1}}{q_t} \quad C_t^y + q_t T_t = Y^y.$$

We can use again the supply curve of the preceding section. The difference comes from the scarcity constraint which becomes here:

$$z^y(\rho_t) + z^o(\rho_{t-1}) = T,$$

while before the right hand side was equal to zero. The previous graph is modified by translating the second bissector of a length $T$ towards the east. There is only one equilibrium left, which is similar but different from the previous efficient stationary equilibrium. The net real interest rate is strictly positive, too high for efficiency. If there were production using physical capital, the capital stock would be smaller than at the golden rule. A few classical economists (Maurice Allais) have advocated a collective ownership of land to correct this defect of capitalism.

## 2.3 Economies with population growth

It is easy to accommodate a growing population, at a constant rate $n$, $n > 0$, in the base model.

Indeed the consumer's programs are unchanged. The differences appear in the scarcity constraints, which can now be written when the nominal asset is in constant quantity:

$$(1+n)^t B_t = B_0,$$

$$(1+n)z^y + z^o = 0.$$

Geometrically, one does not describe the dynamics any more by going from the supply curve to the second bissector, but to the line of slope $-(1+n)$. One gets a continuum of equilibria as in the case of a constant population. The balanced growth equilibrium, associated with the point $E$ where the supply curve intersects the scarcity line, corresponds to per capita consumptions that are constant over time. As the per capita quantity of nominal asset decreases at rate $n$, there is deflation at the same rate: the real interest rate $p_t/p_{t+1} - 1$ is equal to the growth rate of population, and this equilibrium is the golden rule equilibrium.

# Chapter 3

# Nominal assets in an infinite horizon model

Definition of the *long run*: limit state of the short run dynamics in an infinite horizon setup... We often suppose (but we shall see some variants) no systematic forecast errors (i.e. rational expectations), stationary allocation, or at least stationary growth rates of the allocation. Competitive markets.

Two non exclusive types of models, with very different properties as far as nominal assets are concerned: finite horizon (overlapping generations) vs. infinite horizon agents.

There is a single all purpose commodity.

## 3.1 Model without nominal asset

Population of identical consumers with an infinite horizon. They supply labor inelastically, $L_t$ hours at date $t$. The typical consumer maximizes

$$\sum_{t=1}^{\infty} \beta^t U(C_t)$$

subject to the budget constraints, for $t = 1, \dots$ :

$$p_t C_t + p_t K_t = w_t L_t + \Pi_t + \rho_t K_{t-1} + p_t K_{t-1},$$

given her initial capital stock $K_0$. One can choose an arbitrary numéraire at each date to measure the price $p_t$ and wage $w_t$. We imagine that there is a good, storable without cost, which serves as *intertemporal* numéraire. The manager of the (aggregate) firm maximizes his short run profit:

$$\Pi_t = p_t Q_t - w_t N_t - \rho_t K_{t-1}$$

subject to the technical constraint (production function) :

$$Q_t = F(K_{t-1}, N_t)$$

and distributes the profits to the owners of the capital stock. There is no technical progress. To simplify notations, I assume that the production function is *net*, i.e. that it describes output after repairs and expenses to put the capital stock back in its original state before undertaking production. Similarly, the interest rate $\rho_t$ is net. The production function is concave and exhibits constant returns to scale, so that profits are equal to zero at the equilibrium; furthermore $F(0, L) = 0$, $F$ is twice continuously differentiable and satisfies the Inada conditions at the frontier of the positive orthant.

An equilibrium is a sequence of prices and quantities $(w_t/p_t, \rho_t/p_t, Q_t, C_t, N_t, L_t, K_t)$ such that, every agent maximizing his/her objective subject to his/her constraints given prices, the corresponding allocations satisfy the scarcity constraints:

$$C_t + K_t = Q_t + K_{t-1},$$

$$N_t = L_t.$$

We first write down the necessary first order conditions satisfied at any equilibrium where the capital stocks and labor supplies are strictly positive. Profit maximization yields at each date:

$$F'_K = \frac{\rho}{p} \qquad F'_N = \frac{w}{p}.$$

Furthermore, eliminating $K_t$ among the budget constraints of dates $t$ and $t+1$ :

$$\beta \frac{U'(C_{t+1})}{U'(C_t)} = \frac{p_{t+1}}{p_{t+1} + \rho_{t+1}}.$$

One can then study conditions, bearing in particular on the initial value of the capital stock, under which the dynamics of the economy has some stationary properties, i.e. the economy stays in a stationary state, or all quantities increase at the same constant rate.

If the stock of capital per labor unit stays constant, a property observed along any equilibrium of the type just described in the absence of technical progress, $F'_K$ is constant, which implies that

$$\beta \frac{U'(C_{t+1})}{U'(C_t)} = \frac{1}{1 + F'_K}$$

is also constant. If consumption stays constant, at any equilibrium one has

$$\beta = \frac{1}{1 + F'_k}.$$

The marginal productivity of capital is equal to the psychological discount rate of the consumers: this is the simplest version of the *golden rule*. This is only possible when labor supply is constant.

If labor supply $L$ exogenously grows at rate $n$, consumption and the capital stock must grow at the same rate, if one looks for a constant growth equilibrium (again stressing the absence of technical progress in the model). In general, one cannot expect the first order conditions to hold unless $U$ is a *homogenous function* of $C$. If utility is logarithmic, one again gets the golden rule :

$$\beta \frac{1}{1+n} = \frac{1}{1+F'_k}.$$

These conditions on the marginal rates of substitution determine the ratio $K/N$, and as a consequence the initial capital stock that allows to remain from then on on the constant growth path. Consumption can then be computed from the scarcity constraint.

To check that the allocation that we have just determined from the first order conditions is indeed an equilibrium allocation, one must make sure that it maximizes the objective of the consumers. One potential difficulty is that there may not *exist* a maximum to the consumer's program. With a constant labor supply, a maximum only exists for $\beta < 1$, i.e. with some *impatience*. (The condition depends on the precise shape of the utility function when labor supply grows: impatience is a sufficient requirement for a logarithmic utility).

## 3.2 The role of nominal assets

With a finite number of agents and an infinite number of goods, any competitive equilibrium is Pareto optimal. There is therefore no room for government intervention, for instance to better adjust saving to investment. Any change of investment away from the equilibrium trajectory will hurt some consumer, here where there is a single representative agent, will decrease her welfare.

This can be checked directly. The optimum allocation maximizes $\sum \beta^t U(C_t)$ subject to the scarcity constraints:

$$C_t + K_t = F(K_{t-1}, L_t) + K_{t-1},$$

and indeed is characterized by the above first order conditions when the functions are concave[1]. The limit case where impatience is minimal can be reached by letting $\beta$ go to 1 (also called *overtaking*), and gives the *capitalistic optimum* which maximizes consumption per head.

[1]Let $\beta^t \lambda_t$ be the multiplier associated with the scarcity constraint of period $t$. Taking the derivative of the Lagrangian with respect to $C_t$ yields

$$U'(C_t) = \lambda_t,$$

If one adds a nominal asset, say a constant positive quantity $M$, the only equilibrium will give it zero value. Indeed, consider the consumer's program subject to the budget constraints

$$p_t C_t + p_t K_t + M_t = w_t L_t + \Pi_t + \rho_t K_{t-1} + p_t K_{t-1} + M_{t-1}.$$

To yield an equilibrium, it should be such that $M_t = M$ for all $t$. But this is inconsistent with utility maximization as soon as $M > 0$: the plan which consumes $M$ at the first date and keeps a zero money balance from then on dominates the reference plan.

Impatience rules out long term savings. The fact that there is a single consumer makes irrelevant any transfers.

In more complicated models, the possible existence of bubbles, i.e. of an asset whose equilibrium price today differs from the the sum of present expected value of the dividends it distributes, is studied through arbitrage conditions. One way to rule out bubbles is to remark that the price of the asset under study goes to infinity with time, which is inconsistent with consumption behavior in a finite economy if the quantity of asset is strictly positive. This type of argument also can be used here: having simultaneously some physical capital and nominal asset in the consumers' portfolios yields the arbitrage condition for all $t \geq 0$ :

$$p_t = p_{t+1} + \rho_{t+1}.$$

Whenever $\rho$ is positive, the price of the good in numéraire terms decreases (the price of the nominal asset in terms of good goes to infinity), and for constant positive $\rho$ becomes negative in finite time! Note that this argument has less power than the previous one: it does not work for non positive $\rho$ (see Santos Woodford).

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while the derivative with respect to $K_t$ gives

$$\beta(1 + F'_{K_{t+1}})\lambda_{t+1} = \lambda_t.$$

# Chapter 4

# Fiscal and monetary policy

## 4.1 The quantity theory of money

The study of monetary policy is closely linked to the financing modes of the Treasury. It nevertheless is useful to first consider the impact of a fictitious monetary manipulation, close to a simple change of unit, to see how it makes its way through the equilibrium equations. More specifically, we are going to see that the so-called *quantity theory of money* is not a *long run* property, and we shall link it to the celebrated Lucas critique on the role of economic policy in expectations formation.

Consider the simple two periods, one good, one asset, temporary equilibrium model of Chapter 1.

The quantity theory of money is a consequence of the *absence of money illusion* of the agents, a property closely associated with the shape of expectations.

Consumers are said to be *free of money illusion* if, when their nominal asset holdings and all nominal prices are multiplied by a positive scalar $\lambda$, their demand for good stays unchanged and their demand for nominal asset is multiplied by $\lambda$. This operation looks very much like a change in monetary unit (for instance, going from the pound to the euro!), and for this reason classical economists find it natural to postulate the absence of money illusion, a kind of minimal rationality requirement on the part of the economic agents.

The assumption of absence of money illusion implies the *quantity theory of money.* This theory formally expresses a comparative statics property of the equilibrium, or of the set of equilibria when the equilibrium is not unique. If initial nominal holdings (or debts) are multiplied by $\lambda$, all government actions staying unchanged in real terms, the set of equilibria after transformation can be obtained from the initial equilibria as follows: real quantities are unchanged, while all nominal quantities, prices, wages, monetary injections,.. are multiplied by $\lambda$. Does the theory hold in the models that are commonly used? I shall focus most of the attention here on the temporary equilibrium model, leaving it to the reader to adapt the analysis to other specifications.

To formally write the condition of *no money illusion*, one must be careful on how expectations are specified. We are not any more in the situation where we studied the existence of a temporary equilibrium, where everything but the current price was held fixed. Following Lucas (1976), the expectations may depend on all the agents' informations, including endogenous and *exogenous* variables. In particular, expectations may be directly influenced by policy measures.

In the interest of the current discussion, let us therefore add the initial money holding as an argument of the expectation function, since we are interested in the effect of changes in initial money holdings, everything else kept equal. The condition for no money illusion can be written on the consumer program as:

$$\begin{cases} \max U(C_1,\ C_2) \\ \lambda p_1 C_1 + B_1 = \lambda p_1 Y_1 + \lambda B_0 \\ p_2(\lambda) C_2 = p_2(\lambda) Y_2 + B_1 \\ C_1,\ C_2 \geq\ 0, \end{cases}$$

has a solution $(C_1, C_2)$ independent of $\lambda$, with a $B_1$ proportional to $\lambda$, when $\lambda$ varies in $\mathbb{R}_{++}$.

For this property to hold in general, except for very special utility functions, one needs $p_2(\lambda) = \lambda p_2(1)$ for all positive $\lambda$.

The expected price is an homogenous function of degree 1 of the couple $(p_1, B_0)$. Does this condition make economic sense? The answer to this question is important to judge of the pertinence of the quantity theory of money.

**Remark:** The above discussion only is meaningful in a more general setup than the base model, where dichotomy implies the invariance of all real quantities, whatever the level of the quantity of money! This property is very specific. One way to see this is to consider a similar model, but with several consumers: one can thus approach the distributive aspects of monetary policy. Then the homogeneity condition on expectations bears on $\psi(p_1, B_0^i, i = 1, \ldots, I)$, where $(B_0^i, i = 1, \ldots, I)$ is the vector of initial money holdings (or debts when negative) of all the consumers.

### 4.1.1 Is the quantity theory of money valid?

To simplify, consider two extreme cases of homogeneity, one where the expected price does not depend on $B_0$, and therefore is proportional to the current price, the other where the expected price is independent of the current price and proportional to the initial money holdings.

a) *The expected price is independent of monetary policy*

Let us denote $\epsilon_\psi$ the elasticity of the expected price with respect to the current price, that is the partial derivative of $\log \psi(p)$ with respect to $\log p$

$$\epsilon_\psi = p_1 \psi_1'(p_1)/\psi(p_1)$$

where $\psi_1'$ is the derivative of $\psi$ with respect to its first argument. $\psi$ is homogenous of degree 1 with respect to $p_1$ if and only if the elasticity of the expected price with respect to the current price is equal to 1. To summarize:

*If the expected price is independent of the level of initial money holdings, except for specific shapes of utility functions, a unit elasticity of the expected price with respect to the current price is a necessary and sufficient condition for the validity of the quantity theory of money.*

An immediate consequence is an apparent inconsistency between the existence of a temporary competitive equilibrium and the quantity theory: indeed the *sufficient* conditions for existence that we have obtained imply some money illusion.

b) *The expected price is independent of the current price and homogenous of degree 1 with respect to* $(B_0^i)_{i=1,\ldots,I}$.

This is the situation considered by Lucas: the expectations depend on the exogenous policy. The preceding line of reasoning shows that in this circumstance there is no money illusion: the quantity theory holds. One should stress that an individual change of money holding of a single agent, that of the others being unchanged, even if accompanied with a proportional change of the current prices, does not entail in general a 'no money illusion' type of behavior. The no money illusion only requires a degree 1 homogeneity with respect to the *whole vector* of individual money debts or holdings.

**Interest rate policy**

The previous analysis allows to study some of the short run effects of an interest rate policy. Assume that the government, or the central bank, decides to pay interest to the money holders, while in parallel charging the same rate on the debtors (this is a kind of *helicopter* money, where all quantities are multiplied by $(1+r)$). The program of the typical consumer becomes:

$$\begin{cases} \max U(C_1,\ C_2) \\ p_1 C_1 + B_1 = p_1 Y_1 + B_0 \\ \psi(p_1, r) C_2 = \psi(p_1, r) Y_2 + (1+r) B_1 \\ C_1,\ C_2 \geq\ 0. \end{cases}$$

Then if the expectations satisfy $\psi(p_1, r) = (1+r)\psi(p_1, 0)$, the interest rate policy is neutral in the short run. The change of interest rate is akin to a change of monetary unit in period 1, with perfect expectations from the agents. The interest rate policy only has real effects when the inflation expectations are not adjusted in a way that leaves the real interest rate unchanged. One channel for the effect in practice comes from the various maturities of the nominal assets: a rate change can only be neutral if it bears simultaneously on the nominal assets of all maturities, leaving the term structure of the real interest rates unchanged.

**Long run**

In the long run, *by definition*, the relative prices and interest rates are constant. In most models with nominal assets, one can write the equilibrium system of equations in a dichotomic form, an autonomous subsystem yielding all the real magnitudes. The nominal equilibrium quantities are then determined from the real ones in a system of equations which is homogenous of degree 1 in the aggregate quantity of money. This is the quantity theory of the classics.

## 4.2 Public deficit and inflation tax

We now proceed from models with a single nominal asset to a situation closer to day-to-day practice, where we try to separate cash, which mostly serves for transactions (Clower constraint), from a generic nominal asset which is used to transfer wealth across (longer periods of) time.

### 4.2.1 The setup

We introduce the government and the monetary authorities (or central bank). The actions of the government are designated with an upper index $g$, those of the central bank with $b$, and those of the representative private consumer with $c$. In the economy, there are a non storable physical good, a short run nominal asset, and cash which serves for transactions. Cash is the numeraire. A unit of asset bought at date $t$ entails its owner to receive $(1 + r_t)$ money units at the next date. One can hold long or short positions on the nominal asset, but all the assets have the same price whatever the default risk of the debtors, because the central bank is a lender of last resort and substitutes to the debtors in the short run in case of bankruptcy, while in the long run which is our focus there are no bankruptcies. The price of the physical good is noted $p$.

The central bank initially owns $B^b_{t-1}$ bonds, whose value is equal to its debt, the quantity of money held by the public $M_{t-1}$. The budget constraint of the bank is:

$$M_t = M_{t-1} - (1 + r_{t-1})B^b_{t-1} + B^b_t - r_{t-1}B^b_{t-1}.$$

The current quantity of money is equal to that of yesterday, minus the redeem of the previous loans including interests (here the assumption there are no bankruptcies plays a role), plus the new loans granted during the period and the profits of the bank which I assume to be immediately transferred to its owner, the government. The equilibrium of the balance sheet at date $t-1$ then automatically insures its equilibrium at date $t$.

The government, at the beginning of period $t$, has a debt $B^g_{t-1}$. Its budget constraint is:

$$p_t G_t = p_t T_t + r_{t-1} B^b_{t-1} - (1 + r_{t-1}) B^g_{t-1} + B^g_t.$$

$G_t$ and $T_t$ respectively are lump-sum expenditures and taxes, measured in units of good. The profits of the bank are an income, while the last two terms on the right hand side correspond to the rolling over of the public debt.

We pursue the analysis in an overlapping generation model. To simplify notations, assume that all taxes are paid when young.

$$\begin{cases} \max \ U(C^y_t, C^o_{t+1}) \\ p_t C^y_t + B^c_t + M^c_t = p_t(Y^y - T_t) \\ p_{t+1} C^o_{t+1} = p_{t+1} Y^o + (1 + r_t) B^c_t + M^c_t \\ M^c_t \geq k p_t (Y^y - T_t), \end{cases}$$

where the final constraint, the Clower constraint, is a reduced form expression of cash needed for transaction purposes during period $t$. The scalar $k$ is strictly positive and strictly smaller than 1.

The economy enters period $t$ with a set of initial conditions which balance holdings and debts:

$$M_{t-1} = M^c_{t-1},$$

$$B^g_{t-1} = B^b_{t-1} + B^c_{t-1}.$$

An allocation is feasible if the period scarcity constraints are satisfied:

$$C^y_t + C^o_t + G_t = Y^y + Y^o,$$

$$M_t = M^c_t,$$

$$B^g_t = B^b_t + B^c_t.$$

Given the initial conditions, using Walras' law, one of the equalities is redundant.

Private nominal wealth is equal to the sum $M^c_{t-1} + B^c_{t-1}$. By definition, it is equal to the public debt, since under the current accounting convention, the wealth of the bank is equal to zero:

$$M^c_{t-1} + B^c_{t-1} = B^b_{t-1} + B^c_{t-1} = B^g_{t-1}.$$

The aggregate quantity of money, following Gurley and Shaw, can be decomposed into outside money $B^g$ and inside money $-B^c$. Only outside money counts in the aggregate wealth effects.

### 4.2.2 Transaction constraints and inflationary tax

There is not a sign constraint on $B^c$ in the consumer's program. Thus there is no solution to the consumer's program unless it is not profitable to get indebted in order to buy cash. In other words, at any equilibrium, one must have:

$$r_t \geq 0.$$

When $r_t$ is strictly positive, the Clower constraint binds: holding cash is costly, and in the absence of a transaction motive, the consumers would reduce their money holdings to zero. When $r_t = 0$, cash and bonds are perfect substitute. To avoid complicated notations, we assume then that the cash balance of the consumer is equal to $kp_t(Y^y - T_t)$.

Eliminating $B^c$, the two budget constraints yield:

$$\frac{(1+r_t)p_t}{p_{t+1}}(C_t^y + T_t - Y^y) + C_{t+1}^o - Y^o + r_t \frac{M_t^c}{p_{t+1}} = 0.$$

The intertemporal choices of the consumer are governed by a real gross interest rate $1 + \rho_t = (1 + r_t)/(1 + i_t)$, where $r_t$ is the nominal interest rate and $i_t$ the inflation rate. Cash is costly to the consumer. The cost can be designated as an inflation tax for reasons that will appear below, and it is proportional to the nominal interest rate.

The role of inflation for financing public spending clearly comes out when one considers a long run equilibrium path, along which real quantities stay constant, while the nominal interest rate and the rate of inflation are kept fixed.

As previously, there typically are two kinds of long run equilibria, which satisfy the scarcity constraint:

$$C^y + C^o + G = Y^y + Y^o.$$

To see this property, we use the budget constraints to get a necessary condition for an equilibrium. From the consumers' budget constraints:

$$(1+\rho)(C^y + T - Y^y) + C^o - Y^o + \frac{r}{1+r}(1+\rho)k(Y^y - T) = 0.$$

The consumers' supply curve can be deduced from that of the preceding chapter by withdrawing from the income the tax $T$ and the inflation tax $r/(1+r)k(Y^y - T)$. For the government, let us write that current expenditure is covered by tax receipts and banking profits together with a deficit $D$. In other words, the deficit $D$ is defined through:

$$G = T + \frac{r}{1+r}k(Y^y - T) + D.$$

Using the budget constraints to eliminate $(C^o - Y^o)$ and $T$, one gets:[1]

$$\rho(C^y + T + \frac{r}{1+r}k(Y^y - T) - Y^y) = D.$$

[1] I thank Marc Jourdain de Muizon for pointing out an error here in an earlier version of this text.

When $D = 0$, the *autarkic* equilibrium is obtained when

$$C^y = Y^y - T - \frac{r}{1+r}k(Y^y - T) \quad C^o = Y^o.$$

The golden rule equilibrium is associated with a zero real interest rate, i.e. a rate of inflation equal to the nominal interest rate. This *ex post* justifies the denomination *inflationist* given to the opportunity cost of holding cash.

When public spending differs from government revenue, one gets long run equilibria involving public debt (or holdings). The supply curve is drawn in the plan $(C^y - Y^y + T + \frac{r}{1+r}k(Y^y - T), C^o - Y^o)$, and the stationary equilibria are at its intersection with the scarcity constraint $z^y + z^o + D = 0$. When the deficit is positive, it cannot be too large in absolute value. Equilibria associated with negative real interest rates (inflation) correspond to positive deficits.

### 4.2.3 The neutrality of Treasury policy in the long run

In the overlapping generations model, the mode of financing, through debt or through taxes, may change the resource allocation, and in particular the real interest rate. This cannot occur in the neoclassical growth model, since the stock of nominal assets is zero in the long run in this model. One can easily link the two approaches through an altruistic motive of the current generations with respect to their descendants, following Barro.

Formally, suppose that, for all $t$, the utility of the generation born at $t$ is as follows:

$$V_t(C_t^y, C_{t+1}^o, V_{t+1}),$$

where $V_\tau$ represents the utility of the $\tau$ generation. An often used specification, on which I shall focus here, is :

$$V_t(C_t^y, C_{t+1}^o, V_{t+1}) = U(C_t^y, C_{t+1}^o) + \delta V_{t+1},$$

which, when the function $U$ is bounded and $0 < \delta < 1$, yields :

$$V_t = \sum_{\tau=t}^{\infty} \delta^{\tau-t} U(C_\tau^y, C_{\tau+1}^o).$$

Assume furthermore that gifts between generations are allowed (and not subject to tax), and let $H_t$ be the (monetary) bequest received by generation $t$ from his parents. To fix ideas, suppose that the bequest is transferred at the end of the lives of the parents, and therefore received by the beneficiaries at the end of their first period of life. The budget constraints of generation $t$ then become:

$$p_t C_t^y + B_t^c = p_t(Y^y - T) + p_t H_t,$$

$$p_{t+1} C_{t+1}^o + p_{t+1} H_{t+1} = p_{t+1} Y^o + (1 + r_t) B_t^c,$$

where I have ignored cash balances to make things simpler.

We now study the equilibrium properties of such an economy, assuming perfect foresight (the current generation knows exactly the living conditions of all their lineage, the bequests that they will receive in turn, the future prices, etc...). If there are no sign constraints on $H_t$, (when negative, its absolute value is the aid the children give to their old parents), one can consider that the generation $t$, who receives the exogenous transfer $H_t$, faces the sequence of budget constraints:

$$p_t C_t^y + B_t^c = p_t(Y^y - T) + p_t H_t,$$

and for $\tau \geq t+1$ the sum of the budget constraints of the young and old:

$$p_\tau(C_\tau^y + C_\tau^o) + B_\tau^c = p_\tau(Y^y + Y^o - T) + (1 + r_{\tau-1})B_{\tau-1}^c.$$

The possibility of transfers between generations imply the following first order condition:

$$\delta U_y'(C_\tau^y, C_{\tau+1}^o) = U_o'(C_{\tau-1}^y, C_\tau^o).$$

Intertemporal optimization of the typical generation also yields:

$$\frac{U_y'(C_\tau^y, C_{\tau+1}^o)}{U_o'(C_\tau^y, C_{\tau+1}^o)} = \frac{p_\tau(1+r_\tau)}{p_{\tau+1}}.$$

If one only considers *stationary* equilibria, the real interest rate $\rho$ is constant along time. The two above equalities then imply:

$$\delta = \frac{1}{1+\rho}.$$

The sum of the discounted budget constraints from date $t+1$ onwards is:

$$\sum_{\tau=t+1}^{T} \delta^{\tau-t}[C_\tau^y + C_\tau^o + T - Y^y - Y^o] = -\delta^{T-t}\frac{B_T^c}{p_T}.$$

The discounted value of debt at infinity cannot be strictly positive along an optimal program: otherwise, under impatience ($\delta < 1$), one could build a preferred consumption path by consuming more today by spending the long run wealth, while keeping consumption unchanged in the later periods... Therefore, $\lim_{t\to\infty} \delta^t B_t^c / p_t \leq 0$. The scarcity constraints then imply that along the equilibrium path

$$\sum_{\tau=t+1}^{\infty} \delta^{\tau-t}[T - G] \geq 0.$$

The government satisfies an intertemporal budget constraint: no equilibrium exists with persistent deficits. Any transitory deficit is backed by future taxes, and by assumption this is perfectly anticipated by the private agents. Public debt (or government bonds) is not net wealth.

**Remark :** The analysis depends on the assumption that the optimal intergenerational transfers are unconstrained: there are no corner solutions. It does not accord well with the available econometric evidence on bequests and gifts within the family. A typical result gives a derivative of the transfer with respect to an increase in the parents' incomes, keeping constant the discounted wealth of the lineage, equal to 0.17 instead of the 1 predicted by theory. It is nevertheless a worthwhile exercise. It stresses that the standard Keynesian multiplier is likely to depend on the (change of) expectation on future taxes: during the life of a person, if not across generations, intertemporal smoothing is very likely, barring liquidity constraints. A change in the tax profile, keeping fixed their discounted value, would only be met with a compensatory change in the savings profile, without any real effect on consumption, provided the household does not face liquidity constraints during the transformation.

# Chapter 5

# Pensions: pay as you go or fully funded

## 5.1 The model

The model builds on the overlapping generations model with production of Diamond [1965], where generations have a two period lifetime. Is a pay as you go pension scheme inefficient? Does it lead to less capital accumulation than a fully funded scheme (this is different from the inefficiency question)? These questions are studied here in a long run environment.

Population increases at a constant rate $n$, $n \geq 0$, and we focus on balanced growth trajectories along which quantities per head stay constant.

*Production and prices*

Production exhibits constant returns to scale, and is made out of the stock of capital inherited from the previous period and from the labor supplied by the young generation. Constant returns to scale allows to work with quantities per head: let $k$ be the stock of capital per head of the young generation at the beginning of the period; $f(k)$ is gross production per head of worker[1]. As often, we assume:

**Assumption 1 :** *Gross production per head $f$ is a concave differentiable function of capital per head. Moreover* $\lim_{k\to 0} f'(k) = \infty, \lim_{k\to\infty} f'(k) = 0$.

During production, capital gets consumed at rate $\delta$, $\delta > 0$, per period: for an input of $k$ entering the productive activities, one gets back $f(k) + (1-\delta)k$ at the end of the productive process.

We suppose perfect competition, so that the wage and interest rate are respectively equal to the marginal productivities of labor and capital

$$w_t = f(k_t) - k_t f'(k_t),\ r_t = f'(k_t) - \delta. \tag{5.1}$$

[1] With standard notations $k_t = K_{t-1}/L_t$ and $f(k_t) = F(K_{t-1}, L_t)/L_t$ where $F$ is the gross aggregate production production.

Constant returns to scale implies

$$f(k_t) = w_t + (r_t + \delta)k_t. \tag{5.2}$$

*Consumers*

Consumers can put their savings either in physical capital or in Treasury bonds. Absent randomness or liquidity constraints, these two assets have the same return $r$. The pay as you go pension is proportional to contributions paid while working. It is parameterized by the replacement rate $\theta$: along a balanced growth path, the contribution rate is $\theta/(1+n)$. Denoting by $c_t^y$ and $c_{t+1}^o$ the consumptions of generation $t$ during the two periods of its life, the program of the typical generation is

$$\begin{cases} \max \quad u(c_t^y, c_{t+1}^o) \\ c_t^y + s_t + b_t = \left[w_t - \dfrac{\theta w_t}{1+n}\right] \\ c_{t+1}^o = s_t + b_t + \theta w_{t+1} + r_{t+1}(s_t + b_t), \end{cases} \tag{5.3}$$

where $s_t$ and $b_t$ are holdings respectively in physical capital and Treasury bonds. If there are no debt constraints (the agents can be short in Treasury bonds), the two budget constraints are equivalent to a single intertemporal budget constraint

$$c^y + \frac{c^o}{1+r_{t+1}} = W_t$$

with

$$W_t = \left[w_t + \theta\left(-\frac{w_t}{1+n} + \frac{w_{t+1}}{1+r_{t+1}}\right)\right]. \tag{5.4}$$

*The government*

Let $g_t$ be public spending per head of young agent. The government budget constraint, per head of young agent is

$$g_t = -\frac{b_{t-1}(1+r_t)}{1+n} + b_t.$$

Once the parameter $\theta$ which describes pensions is fixed, there remains one degree of freedom for government policy, which chooses a couple $(g_t, b_t)$ subject to its budget constraint.

*Equilibrium*

The scarcity constraints, per head of young agents, can be written at date $t$

$$g_t + c_t^y + \frac{c_t^o}{1+n} + s_t = f(k_{t-1}) + (1-\delta)k_{t-1}. \tag{5.5}$$

### 5.1.1 Balanced equilibrium trajectories

We keep public spending $g$ and the replacement rate $\theta$ constant over time. We want to study the associated equilibrium trajectories and are particularly interested in their comparative statics properties when $\theta$ varies.

Along a balanced equilibrium, prices and quantities per head stay constant. The government budget constraint simplifies into

$$g = b\frac{n-r}{1+n}. \tag{5.6}$$

It is useful to define the consumption functions of the young and old agents $C^y(r, W)$ and $C^o(r, W)$, solutions of

$$\begin{cases} \max \quad u(c^y, c^o) \\ c^y + \dfrac{c^o}{1+r} = W \end{cases} \tag{5.7}$$

**Assumption 2 :** *Consumptions at the two dates are normal goods: they are increasing functions of the wealth of the consumer.*

Consider a stationary trajectory along which capital per head is equal to $k$. The rate of interest and wage then are given by (5.1), and (5.4) yields lifetime wealth as a function of the capital stock $k$ and of the replacement rate $\theta$. Substituting $r$ and $W$ with their expressions as functions of

$k$ and $\theta$ into the consumption functions $C^y$ and $C^o$, we get what we will call the *long run demands* $D^y(k, \theta)$ and $D^o(k, \theta)$. These are the expressions for demands that are the most useful to study the balanced growth equilibria.

A balanced growth trajectory is an equilibrium if it satisfies the scarcity constraints, when the agents take their optimal competitive decisions, given the factor prices associated with the level of capital stock. Since consistency requires $s = k(1+n)$, this leads to the following definition

**Definition.** *The capital stock level $k$ corresponds to an equilibrium balance path associated with public expenditure $g$ if*

$$g + D^y(k, \theta) + \frac{D^o(k, \theta)}{1+n} + (n+\delta)k = f(k). \tag{5.8}$$

The levels of debt coming from the consumers and government budget constraints are equal, by Walras' law: using the constant returns assumption ($f(k) = w + (r+\delta)k$), the sum of the budget constraints of the young agents, of the old agents (multiplied by $1/(1+n)$) and of the government is identical to the overall scarcity constraint.

### 5.1.2 The golden rule

The golden rule capital stock, $k^*$, plays a fundamental role. It is defined through the equality between the capital net return and the population growth rate:

$$n + \delta = f'(k^*).$$

At the golden rule, the economy is at a long run optimum. Any change in the pension contribution rate which does not affect labor supply is undone by the private agents if there are no liquidity constraints. Indeed, at the golden rule the present value of lifetime income is independent of the replacement rate $\theta$ of the pay as you go scheme. This property has a simple interpretation. The contribution to the PAYG pension is a type of forced savings, whose return is equal to the population growth rate. At the golden rule, it is identical to that of the financial assets. When there are no liquidity constraints, the system is neutral since forced savings can be compensated by contracting a debt of equal magnitude.

When is the golden rule capital stock $k^*$ an equilibrium ? The government budget constraint does not depend on the value of the debt, as seen from equation (5.6) : since the amount of debt per head is constant, the increase of debt associated with the demographic change exactly pays for the interest charges. The primary deficit of the government is equal to zero and the level of public expenditures $g^*$ is determined by tax receipts, here equal to zero, independently of the PAYG contribution rate[2].

These neutrality properties hold more generally, whenever the government can replicate some of the PAYG transfers through taxes or subsidies, as discussed by Belan and Pestieau [1997] or Pestieau and Possen [1997]. One then can replace a PAYG scheme with a tax on young agents together with a subsidy for the old while increasing public debt by the appropriate amount.

Apart from its neutrality properties, the golden rule capital stock plays an important role in the study of the equilibria. In particular, the fact that an increase in the contribution rate increases or decreases lifetime income for a given stock of capital (and the associated factor prices) only depends on the position of this capital stock relative to the golden rule $k^*$:

**Proposition 0 :** *Life time discounted income is independent of $\theta$ when the*

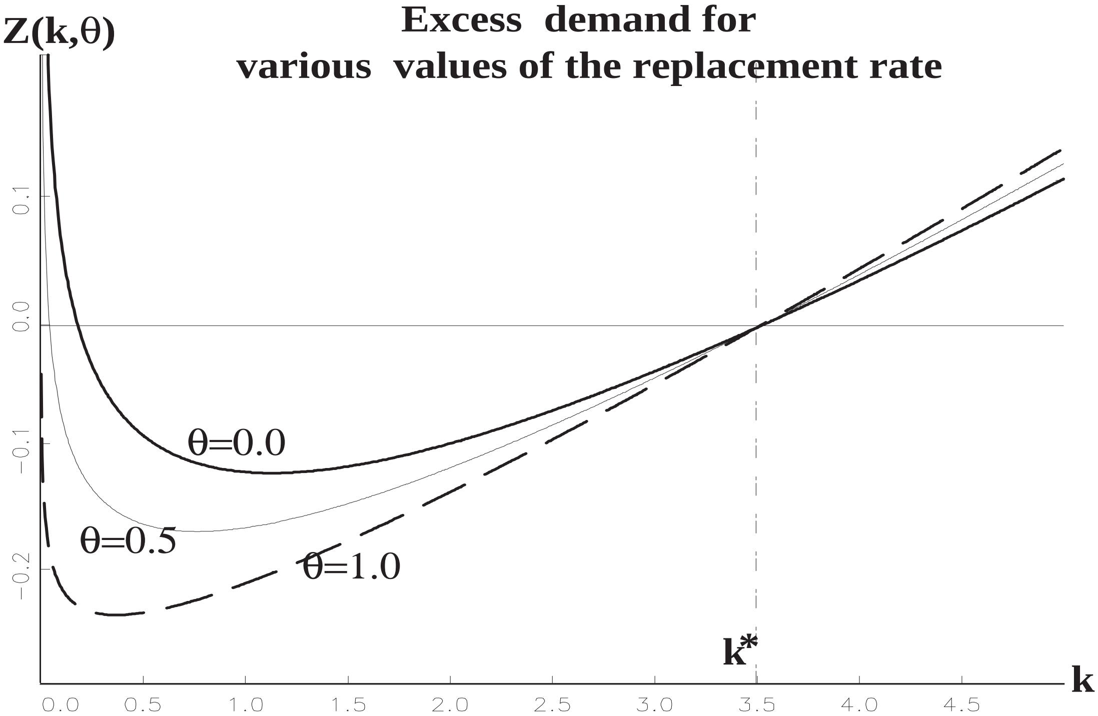
[2]This can be checked analytically. Aggregate demand is equal to lifetime income, here $w^*$. The equilibrium equation becomes

$$g^* + w^* + (\delta + n)k^* = f(k^*)$$

i.e., using the equality $f(k) = w + (r + \delta)k$ :

$$g^* = 0.$$

**Figure 1**
**Excess demand for**
**various values of the replacement rate**




*capital stock is at its golden rule value, $k = k^*$. It is decreasing in $\theta$ when the capital stock is smaller ($k < k^*$) and increasing when it is larger ($k > k^*$).*

Proposition 0 is easy to prove. If the capital stock is smaller than the golden rule stock, $k < k^*$, then $r > n$ : from (5.4), an increase in $\theta$ is equivalent to a decrease in lifetime income, due to the difference between the return of the financial asset and the population growth rate. This is a familiar argument in the pension debate in favor of fully funded schemes and against PAYG. The usual argument, however, implicitly assumes a constant capital stock. Our long run analysis will account for the behavioral reactions and the change in the equilibrium capital stock following a change in the contribution rate.

## 5.2 Existence and multiplicity

Before studying the comparative statics properties of the equilibria, we first have to examine their existence and possible multiplicity. To this end, it is useful to define aggregate demand $D$ and aggregate excess demand $Z$ through:

$$D(k,\theta) = D^y(k,\theta) + \frac{D^o(k,\theta)}{1+n} \text{ and } Z(k,\theta) = D(k,\theta) + (n+\delta)k - f(k).$$

By construction, $k$ is a capital stock associated with a balanced equilibrium path for public expenditure $g$ if and only if:

$$Z(k, \theta) + g = 0.$$

The existence and properties of the equilibria come from the shape of the function $Z$. Under assumptions A1 and A2 the behavior of $Z(., \theta)$ at the boundary of the domain is simple: $Z$ is non negative for $k$ close to 0 and when $k$ goes to infinity. When public expenditure $g$ is positive, $Z + g$ is positive at both boundaries of the domain. Generically there is an even (possibly null) number of roots to the equilibrium equation. When public expenditures are not too large, there are typically at least two equilibria[3].

## 5.3 Analysis of the balanced equilibrium paths

A change in the pension scheme should be evaluated *in fine* through the induced variation in welfare. We proceed in two steps: first we look for the change in the equilibria capital stocks, holding public expenditure constant, when the replacement rate $\theta$ varies. Then we discuss the impact on welfare.

### 5.3.1 How do the equilibrium paths depend on the replacement rate

We consider a marginal change of $\theta$, and follow its effect on the equilibria by continuity. The derivative with respect to $\theta$ of one of the solutions of the equilibrium equation

$$Z(k, \theta) + g = D(k, \theta) + (n + \delta)k - f(k) + g = 0,$$

can be computed through a straightforward application of the implicit function theorem, $Z_\theta d\theta + Z_k dk = 0$, where the subscript refers to the differentiation variable.

The direction of change of $k$ depends on the sign of the derivatives of $Z$ with respect to $\theta$ and $k$. The sign of $Z_\theta$ follows from Proposition 0: only $D$ depends on $\theta$, and only through lifetime income. Under A1, demand is increasing in income. From Proposition 0, $Z$ is decreasing[4] in $\theta$ if and only if $k < k^*$. As far as $Z_k$ is concerned, some more work is needed. For the time being, let us state the comparative statics properties of the equilibria as a function of the sign of $Z_k$.

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[3] A sufficient condition under which there are at most two equilibria is that the excess demand function $Z(k, \theta)$ be first decreasing and then increasing in $k$. Under A1, the net production per head $f(k) - (n + \delta)k$ is concave. It would suffice that the demand $D$ be a convex function of the capital stock, but I do not know of simple conditions on the utility function that warrants this property.

[4] One can be more precise on the magnitude of the derivative of the equilibrium capital stock with respect to the replacement rate $\theta$, under differentiability assumptions. Adding the budget

**Proposition 1 :** *Let $k(\theta)$ be a root of the equation $Z(k,\theta) = g$. A marginal decrease of the contribution rate $\theta$*

1. *increases the equilibrium capital stock if ($k(\theta) < k^*$ and $Z_k < 0$), or if ($k(\theta) > k^*$ and $Z_k > 0$);*

2. *decreases the capital stock if ($k(\theta) < k^*$ and $Z_k > 0$), or if ($k(\theta) > k^*$ and $Z_k < 0$).*

**Proof :** This si a direct consequence of the fact that $Z$ is decreasing (resp. increasing) with respect to $\theta$ if $r > n$ (resp. $r < n$), i.e. when $k(\theta) < k^*$ (resp. $k(\theta) > k^*$). ∎

Figure 1 shows in a class room case[5] the graphs of the excess demand function $k \to Z(k,\theta)$ for various values of $\theta$. The curves all go through the same golden rule point $(k^*, -g^*)$. When $\theta$ increases, the curves move downwards in the interval $[0, k^*]$, and upwards when the capital stock is larger than $k^*$. At the smallest root, say $k_{\min}$, the derivative of $Z$ with respect to $k$ is negative, while it is positive at the largest root $k_{\max}$. At $k_{\min}$, which is smaller than the golden rule capital stock, a marginal decrease of the replacement rate $\theta$ increases the capital stock. The variation is in the same direction at the largest root $k_{\max}$, when it is larger than the golden rule capital stock.

Proposition 1 is somewhat surprising. While one decreases the share of pensions financed through PAYG and the workers savings are to increase, the overall capital stock in the economy may decrease. The debates on how to finance pensions do mention a crowding out effect, the physical assets bought by the pension funds coming in part from preexisting investments, but to the best of my knowledge, do not evoke the possibility that the crowding effect be so strong as to reduce the overall stock of capital! We stress the crucial role of the aggregate consumption demand of both the workers and pensioners: that is the shape of

constraints of the young and old generations, the latter multiplied by $1/(1+n)$, one gets:

$$\frac{\partial Z}{\partial \theta} = \frac{r-n}{1+n}\frac{\partial (s+b)}{\partial \theta}.$$

At an equilibrium where capital is smaller than the golden rule level (under investment), $(r-n)$ is positive. The derivative of saving $(b+s)$ with respect to $\theta$ is also proportional to $(r-n)$, so that the derivative of $Z$ with respect to $\theta$ is proportional to the *square* of the difference between the return on capital and the population growth rate: the impact of a reduction of the PAYG scheme on the capital stock is much larger in absolute value at an equilibrium far from the golden rule.

[5] Lifetime lasts two periods and population increases by 2% per period. The representative consumer utility function is $\log c^j + \log c^v$. The production function is Cobb-Douglas, $f(k) = k^\alpha$, with $\alpha = 0.3$. Capital depreciates at a 10% rate per period. Capital income is taxed at a 20% rate. The $Z$ curve is represented in three situations: no PAYG pension ($\theta = 0$), the pension is equal to a half ($\theta = 0.5$) or to all ($\theta = 1$) of the wage when working.

this demand that determines the change in the equilibrium capital stock, and the extent of the eventual crowding out.

As already mentioned, to properly appreciate the reform, the level of the capital stock is not the end of the story: the level of utility of the agents, which we now study, at the equilibrium is what matters.

### 5.3.2 The impact of the replacement rate on welfare

The following proposition justifies to qualify as *under investment* the capital stocks smaller than the golden rule.

**Proposition 2** : *Assume that the utility function is intertemporally separable. Let $k < k^*$ an equilibrium value of the capital stock associated with public expenditure $g$. Any change in $\theta$ which, keeping public expenditure fixed, increases the stock of capital also increases the equilibrium utility level.*

**Proof** : In the interest of generality, we prove the property in the case where the agents live during $a = 1, \ldots, A$ periods. Let $c^a$ be the consumption of the agent of age $a$, $u'^a$ her instantaneous marginal utility. Consider an equilibrium with public spending $g$ and capital stock $k$. It is characterized by the scarcity constraint

$$g + \sum_{a=1}^{A} \frac{c^a}{(1+n)^{a-1}} = f(k) - (n+\delta)k,$$

and the first order conditions

$$u'^a = (1+r)u'^{a+1}, \text{ où } r = f'(k) - \delta. \tag{5.9}$$

Consider a change in $\theta$, inducing a change in $k$, keeping $g$ fixed. The change in utility is given by

$$\triangle u = \sum_{a=1}^{A} u'^a \triangle c^a = u'^1 \sum_{a=1}^{A} \frac{\triangle c^a}{(1+r)^{a-1}}$$

where the changes in consumptions satisfy

$$\sum_{a=1}^{A} \frac{\triangle c^a}{(1+n)^{a-1}} = \{f'(k) - (\delta+n)\} \triangle k = (r-n) \triangle k. \tag{5.10}$$

Suppose that $\triangle k > 0$. We show that if $r > n$ ($k$ smaller than the golden rule $k^*$) consumption increases at the beginning of the life, up to an age $\overline{a} > 1$, and then eventually decreases until death. Indeed, from (5.9), if $\triangle c^a$ is negative, as the interest rate decreases, $\triangle c^{a+1}$ also is negative: whenever consumption decreases at a given age, it decreases at all larger ages. From the scarcity constraint, the discounted sum at rate $(1+n)$ of the changes in life time consumptions is positive, which implies $\triangle c^1 > 0$.

Let $\lambda = (1+n)/(1+r)$. The assumption $k < k^*$ implies $\lambda < 1$. The change in utility can then be written as:

$$\begin{aligned} \frac{\triangle u}{u'^1} &= \sum_{a=1}^{A} \frac{\triangle c^a}{(1+n)^{a-1}} \lambda^{a-1} \\ &= \lambda^{\overline{a}} \sum_{a=1}^{A} \frac{\triangle c^a}{(1+n)^{a-1}} \lambda^{a-1-\overline{a}} \end{aligned}$$

But for all $a$, one has $\triangle c^a(\lambda^{a-1-\overline{a}} - 1) \geq 0$. Indeed, if $a < \overline{a}+1$, the two terms $\triangle c^a$ and $(\lambda^{a-1-\overline{a}} - 1)$ are positive and conversely, if $a \geq \overline{a}+1$, they are both negative. It follows that

$$\frac{\triangle u}{u'^1} \geq \sum_{a=1}^{A} \lambda^{\overline{a}} \{ \frac{\triangle c^a}{(1+n)^{a-1}} \}$$

The term between brackets is equal to the change in aggregate consumption which, from the scarcity constraint (5.10), is equal to $(r-n) \triangle k > 0$, which is positive. ■

A simple adaptation of the proof shows that a capital stock larger than that of the golden rule corresponds to an *over investment*: reducing the capital stock increases the utility level at the stationary equilibrium.

We are now in a position with the help of Propositions 1 and 2 to come back to the economic policy question of interest: in which situations a reduction of the PAYG replacement rate is beneficial?

*Suppose that the economy is at an equilibrium where the capital stock is smaller than the golden rule $k^*$. A reduction of the replacement rate increases the long run welfare if $Z_k > 0$, decreases it if $Z_k < 0$.*

## 5.4 Is going from PAYG to a fully funded scheme beneficial in the long run?

The theoretical analysis has given us the tools to study the long run effects of a move from PAYG to a fully funded scheme. Technically, we have to evaluate the partial derivatives in the equality:

$$Z_\theta d\theta + Z_k dk = 0.$$

In economic terms, we must know how the aggregate excess demand varies with the replacement rate and with the capital stock. We shall calibrate the economy in a model where the agents live a number of periods (generalizing the two period case used up to now for simplicity). All workers earn the same wage, whatever their ages, and pensions are indexed on wages. The unit of time is the year, $L$ is the length of life, $\ell$ the length of the working life. Adapting the notations of (5.7), we note $C^a(r,W)$, $a=1,\dots,L$, the solution of the consumption program:

$$\begin{cases} \max \sum_{a=1}^{L} \frac{1}{(1+\rho)^{a-1}} u(c^a) \\ \sum_{a=1}^{L} \frac{c^a}{(1+r)^{a-1}} = \sum_{a=1}^{\ell} \frac{w(1-\tau_r)}{(1+r)^{a-1}} + \sum_{a=\ell+1}^{L} \frac{\theta w}{(1+r)^{a-1}} = W, \end{cases}$$

where $\tau_r$ is the contribution rate to the PAYG regime which balances its budget. The aggregate consumption demand at the date of reference is the sum of the demands of the living agents, accounting for the different sizes of the generations:

$$D(r,W) = \sum_{a=1}^{L} \frac{1}{(1+n)^a} C^a(r,W).$$

First it is important to locate the level of the capital stock with respect to the golden rule, since, from Proposition 0 which extends easily to the multiperiod case, it determines the sign of the derivative of life time income with respect to $\theta$, i.e. $\partial W/\partial\theta$. We shall then study the two terms $Z_\theta$ and $Z_k$ whose expressions are

$$Z_\theta = \partial D/\partial W \times \partial W/\partial\theta, \text{ and } Z_k = D_k - (r-n).$$

### 5.4.1 Going from PAYG to fully funded, everything else equal, increases the household discounted life time income

There are many studies that argue that the net marginal productivity of physical capital is much larger than the sum of the rate of population growth and of the rate of technical progress (cf. Abel and al. [1989] for the USA). The capital stock therefore is smaller than its *golden rule* value. Even when one accounts for the fact that capital is taxed, it is widely believed that the inequality still holds for the after tax marginal productivity of capital. We will therefore focus on the case where the capital stock is smaller than at the golden rule.

More concretely, according to Feldstein and Samwick [1996], the average before tax return on capital $r$ would have been equal to 9% per year in the USA on the previous century. Its mathematical expectations is much larger than the return on PAYG contributions[6], 2.5%, a number close to the GNP gross rate (which

[6]The pension administrations use a slightly smaller estimate, close to 2%, in the USA.

in our model is akin to the sum of the population growth rate and of the rate of technical progress). A simple calibration, with the help of a macroeconomic Cobb Douglas production function, yields similar conclusions.

Therefore, all other things equal, without taking into account the agents behavioral reactions, a transfer from PAYG to a fully funded regime increases in the long run the discounted life time wealth of the agents. This change comes from the difference $r - n$: through the play of exponential discounting it is substantial. This effect is much quoted by the advocates of fully funded scheme. In Feldstein and Samwick [1997], the mechanism is reinforced by the fact that the subscription to the fully funded regime is *mandatory*, and that the pension funds are *not subject to taxes on capital*. Nevertheless, this increase in wealth must induce changes in demand and savings. In the long run, the extra demand of goods will induce an increase in production, and an associated accumulation of capital, which in turn may reduce the interest rate. The increase in production and income, the decrease in interest rate will modify consumption demand. We now turn to the analysis of these mechanisms through the variations of the excess demand with respect to $\theta$ and $k$.

### 5.4.2 How excess demand varies with the replacement rate and the capital stock

The variation of excess demand with respect to the replacement rate follows from the partial derivative $Z_\theta$ :

$$Z_\theta = \partial D/\partial W \times \partial W/\partial \theta.$$

The second term on the right hand side has just been studied. The first one would be equal to 1 if the population growth rate were equal to the after tax growth rate, from the budget constraint. Under Assumption A2 that consumption is a normal good at all dates, when the after tax interest rate is larger than the population growth rate, it is larger than 1:

$$\sum_{a=1}^{L} \frac{1}{(1+n)^a} C_W^a(r, W) > \sum_{a=1}^{L} \frac{1}{(1+r)^a} C_W^a(r, W) = 1.$$

As far as the variation of excess demand with respect to the capital stock is concerned, it is given by $Z_k = D_k - (r - n)$. $D_k$ comes from the aggregate demand function $D(r, W)$, after substitution of $W$ and $r$ with their expressions as a function of $k$ in (5.1). One gets

$$D_k = D_r \frac{\partial r}{\partial k} + D_W \frac{\partial W}{\partial k}.$$

We have seen that $D_W$ is larger than 1. The term $\partial W/\partial k$ is a complex combination of the changes in wages and interest rate: an increase in $k$ increases wages, lowers the interest rate and therefore increases discounted life time income $W$. As a consequence, the second term is positive. The derivative of consumption with respect to the interest rate $D_r$ can be decomposed in a negative substitution effect and an income effect, which is negative for the debtors and positive for the lenders. It is multiplied by a negative term. All things considered, the sign of $Z_k$ cannot be assessed with any degree of confidence.

We are left with the task of putting forward conjectures on the sign and magnitude of $Z_k$. The following illustrative computation is a class room example, calibrated on French data. The production function is Cobb Douglas with a capital exponent of 0.3. The depreciation rate is 8%, so that the net before tax interest rate is 16-8=8%. The length of the working life is 40 years, that of retirement is 15 years. The population growth rate is 0.5% per year, the rate of technical progress is 1%. The replacement rate of the PAYG scheme is 55%, while the contribution rate has been set so as to balance the accounts of the pension regime. Labor and capital income are respectively taxed at rates 30% and 20%. Finally the utility functions are logarithmic, with various discount rates $\rho$. The derivatives, computed at the currently observed capital stock, are shown in the following table:

| $\rho$ | 5% | 10% | 15% |
|---|---|---|---|
| $W_\theta$ | | -420 | |
| $D_\theta$ | -1145 | -765 | -619 |
| $D_W$ | 2.7 | 1.8 | 1.5 |
| $D_W \frac{\partial W}{\partial k}$ | 0.40 | 0.27 | 0.22 |
| $D_k$ | -0.19 | 0.03 | 0.10 |
| $Z_k$ | -0.25 | -0.03 | 0.04 |