Models of inflation

Dynamic stochastic general equilibrium models

Random shocks to generate trajectories that look like the observed national accounts.

Rational expectations; representative agents; often calibration and solution by numerical techniques.

1. The classical or real business cycle model: households, firms, equilibrium
2. Monopolistic competition: households, firms, equilibrium
3. Forward looking price fixation
4. Solving for linear rational expectations equilibria

Basic building blocks.

The real business cycle

Description of consumer's behavior

Discrete time, infinite horizon.

Nominal risk-free bonds of one period maturity, no bankruptcy, incomplete markets

Non storable consumption good

Endogenous labor supply

Important notation: upper case letters are for levels; lower case letters for the natural logarithms

Description of consumer's behavior

The household has a competitive behavior: s/he considers prices, wages and interest rates as given.

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

Budget constraints:

$$P_t C_t + Q_t B_t = W_t L_t + B_{t-1} + \text{ lump sum transfers net of taxes}$$

End of game:

$$\lim_{T \to \infty} B_T \geq 0$$

In practice we shall take

$$U(C, L) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{L^{1+\phi}}{1+\phi}$$

First order conditions

Intra period

$$-\frac{U_L}{U_C} = C^{\sigma} L^{\phi} = \frac{W}{P}$$

or in logarithms

$$w_t - p_t = \sigma c_t + \phi \ell_t \qquad \text{labor supply}$$

between period t and $t + 1$

$$Q_t \frac{U_{C,t}}{P_t} = \beta E_t \frac{U_{C,t+1}}{P_{t+1}} \qquad \text{Euler equation (1)}$$

Transformation of Euler equation

Definitions:

nominal interest rate $i_t = -\ln Q_t$

discount rate $\rho = -\ln \beta$

inflation rate $\pi_{t+1} = \ln(P_{t+1}/P_t)$

(expected) real interest rate $r_t = i_t - E_t\pi_{t+1}$

Approximation in a neighborhood of a perfect foresight constant growth path at rate γ: $\gamma = \ln(C_{t+1}/C_t) = c_{t+1} - c_t$. Along the constant growth path, the Euler equation is

$$i = \rho + \pi + \sigma\gamma$$

Log linearization of Euler equation

The Euler equation can be rewritten as:

$$E_t \exp[-\sigma \Delta c_{t+1} - \pi_{t+1} - \rho + i_t] = 1$$

A first order Taylor expansion in a small neighborhood of the reference trajectory gives

$$\exp[-\sigma \Delta c_{t+1} - \pi_{t+1} - \rho + i_t] \approx 1 - \sigma \Delta c_{t+1} - \pi_{t+1} - \rho + i_t,$$

so that, at a first order approximation

$$i_t = \rho + E_t \pi_{t+1} + \sigma E_t \Delta c_{t+1},$$

or

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}[\underbrace{i_t - E_t \pi_{t+1}}_{r_t} - \rho]$$

Euler equation (2)

Firms

Exogenous stochastic technical progress, no capital (or fixed capital stock)

Competitive representative firm with production function

$$Y_t = A_t N_t^{1-\alpha}$$

or in logarithm

$$y_t = a_t + (1-\alpha) n_t$$

Maximization of profit $P_t Y_t - W_t N_t$ in each period, given prices and wages

$$\frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha}$$

which yields the labor demand schedule in logarithm

$$w_t - p_t = a_t - \alpha n_t + \ln(1-\alpha) \quad \text{labor demand}$$

Production and costs

The cost of producing Y is

$$\Psi_t(Y) = W_t \left(\frac{Y}{A_t}\right)^{1/(1-\alpha)} .$$

The real marginal cost is (the wage argument is implicitly in the index t)

$$\mathrm{MC}_t(Y, P_t) = \frac{\Psi_t'(Y)}{P_t} = \frac{W_t}{P_t}\frac{1}{1-\alpha}\left(\frac{Y}{A_t}\right)^{1/(1-\alpha)}\frac{1}{Y} = \frac{W_t}{P_t}\frac{1}{1-\alpha}\frac{\bar{N}(Y)^{\alpha}}{A_t}$$

Here $\bar{N}(Y)$ stands for the quantity of labor required to produce Y.

Marginal costs

In logarithm:

$$mc_t = w_t - p_t - \ln(1-\alpha) - a_t + \alpha n.$$

The labor demand schedule is defined by the first order condition that price is equal to marginal cost, that is real marginal cost equals 1, or the logarithm of real marginal cost is equal to zero.

Equilibrium allocation

Equalizing labor supply with labor demand, and good demand with production yields

$$w_t - p_t = a_t - \alpha n_t + \ln(1-\alpha) = \sigma y_t + \phi n_t,$$

where $y_t = a_t + (1-\alpha)n_t$. A straightforward computation gives

$$n_t = \frac{1-\sigma}{\sigma(1-\alpha)+\phi+\alpha} a_t + \frac{\ln(1-\alpha)}{\sigma(1-\alpha)+\phi+\alpha}$$

$$y_t = \frac{1+\phi}{\sigma(1-\alpha)+\phi+\alpha} a_t + \frac{(1-\alpha)\ln(1-\alpha)}{\sigma(1-\alpha)+\phi+\alpha}$$

Employment and output are determined independently of nominal quantities (dichotomy).

Real interest rate

The Euler equation gives the real interest rate that supports the allocation:

$$r_t = i_t - E_t \pi_{t+1} = \rho + \sigma E_t \Delta y_{t+1},$$

so that

$$r_t = \rho + \sigma \frac{1+\phi}{\sigma(1-\alpha)+\phi+\alpha} E_t \Delta a_{t+1}$$

The real interest rate is determined by the real path of consumption (or output) together with the impatience of the agents.

Price indeterminacy, or the path of assets

Rational expectations (or expectations are chosen to rationalize the equilibrium path)

In the absence of bank interventions ($Q_t = 1$ or $i_t = 0$), there seems to be no motive to hold nominal assets in the long run (the aggregate stock of assets has to be zero, and the general price level indeterminate).

Short run models: fixed capital stock, no motives to hold nominal balances (but only productivity shocks?).

How to have unemployment and a non neutral monetary policy, at least in the short run

- Equilibria with quantity rationing, with slow adjustment of nominal wages and prices (Walras tatonnement). Who sets prices? The only one with 'true' unemployment.
- Confusion between real and nominal shocks may give scope for monetary policy.
- Equilibria with monopolistic competition (local monopoly power), underemployment compared with perfect competition. Some nominal inertia is added:
  - Delayed arrival of information
  - Menu costs in changing prices : the new Keynesian model

Monopolistic competition

Variety of goods: households

Continuum of goods, designated with an index i which takes its valued in the interval $[0, 1]$.

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

where

$$C_t = \left(\int_0^1 C_t(i)^{1-1/\varepsilon} \mathrm{d}i \right)^{\varepsilon/(\varepsilon-1)}, \quad \varepsilon > 1$$

Budget constraints:

$$\int_0^1 P_t(i) C_t(i) \mathrm{d}i + Q_t B_t = W_t L_t + B_{t-1} + \text{ lump sum transfers}$$

Allocation of total consumption among varieties

The choice among varieties follows from

$$\max \int_0^1 \frac{C_t(i)^{1-1/\varepsilon}}{1-1/\varepsilon} \mathrm{d}i$$

subject to

$$\int_0^1 P_t(i) C_t(i) \mathrm{d}i = Z$$

First order condition

$$C_t(i)^{-1/\varepsilon} = \lambda P_t(i)$$

Aggregation

Define

$$P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon} \mathrm{d}i \right)^{1/(1-\varepsilon)}$$

We want to show that with this definition, whatever the individual prices,

1. $$P_t C_t = \int_0^1 P_t(i) C_t(i) \mathrm{d}i = Z$$
2. The demand for good i satisfies

$$C_t(i) = C_t \left[\frac{P_t(i)}{P_t} \right]^{-\varepsilon}$$

Allocation among varieties: aggregation works

Manipulating the FOC and the identities defining the aggregates:

$$Z = \lambda^{-\varepsilon} \int_0^1 P_t(i)^{1-\varepsilon} \mathrm{d}i = \lambda^{-\varepsilon} P_t^{1-\varepsilon}$$

$$Z = \lambda^{-1} \int_0^1 C_t(i)^{1-1/\varepsilon} \mathrm{d}i = \lambda^{-1} C_t^{1-1/\varepsilon}$$

Raising the first equation to the power $-1/\varepsilon$ and multiplying by the second eliminates λ and gives the aggregate budget constraint.

Substituting $Z = P_t C_t$ in the second equality in turn yields

$$C_t^{-1/\varepsilon} = \lambda P_t.$$

Households: concluded

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

subject to

$$P_t C_t + Q_t B_t = W_t L_t + B_{t-1} + \text{ lump sum transfers}$$

First order conditions yield (as in the real business cycle case)

$$w_t - p_t = \sigma c_t + \phi \ell_t$$

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}[i_t - E_t \pi_{t+1} - \rho].$$

Firms

One firm per variety, all with identical production functions

$$Y_t(i) = A_t N_t(i)^{1-\alpha}$$

Common aggregate productivity shock.
Firms are local monopolists who take the aggregates (C_t, P_t) as given, and face identical isoelastic demand schedules.
They choose optimally their price, given the possible rigidities.

Price fixing

Profit of the local monopolist is $P(i)C(i) - \Psi_t(C(i))$.

The revenue $P(i)C(i)$ can be rewritten as $C(i)^{1-1/\varepsilon}\frac{P}{C^{-1/\varepsilon}}$.

Assuming that the individual firm considers aggregate magnitudes constant when it fixes its own price and quantity, marginal revenue is

$$\frac{\varepsilon - 1}{\varepsilon} P \left(\frac{C(i)}{C}\right)^{-1/\varepsilon} = \frac{\varepsilon - 1}{\varepsilon} P,$$

since at the equilibrium, all agents produce the same quantity and choose the same price.

The FOC, marginal revenue = marginal cost, yields

$$1 = \frac{\varepsilon}{\varepsilon - 1} \mathrm{MC}_t(C(i), P_t).$$

In the competitive model, real marginal cost is equal to 1. Here the product of real marginal cost by the markup is equal to 1.

The labor demand equation is modified accordingly

$$w_t - p_t = a_t - \alpha n_t + \ln(1-\alpha) + \ln\left(\frac{\varepsilon - 1}{\varepsilon}\right)$$

Equilibrium with flexible prices

The only difference with the classical labor demand schedule demand comes from the markup term $(\varepsilon - 1)/\varepsilon$: firms reduce their output and labor demand to increase their profits.

The dichotomy still is present and there are no real effects of monetary policy.

Imperfect competition in itself is not enough to create ‘involuntary’ unemployment á la Keynes and a role for monetary policy.

The main step is to introduce some sand in the wheels of the system, so that prices do not adjust instantaneously to the shocks in the economy.

Forward looking price fixing

Price rigidities

- Taylor: wage contracts last for an exogenously fixed number of periods; the wage is fixed in nominal terms during the length of the contract, in view of the expected future changes in prices.
- Rotemberg: changing nominal prices is costly ('menu costs'), say proportional to the square of the change $(\Delta p_t)^2$.
- Calvo: each firm has probability $(1 - \theta)$ to be allowed to reset its price in any period, independently of duration since last adjustment.

The algebra is pretty heavy even in the simplest setup (Calvo), and I shall just give some of the intuitions (see Gali for a complete treatment).

Optimal price setting

The firm chooses the price level P^* which maximizes its expected discounted profits during the period where it will stay unchanged:

$$\max \sum_{\tau=0}^{\infty} \theta^{\tau} E_t Q_{t,t+\tau} [P^* Y_{t+\tau|t} - \Psi_{t+\tau}(Y_{t+\tau|t})]$$

Here

- $Q_{t,t+\tau}$ is the discount factor for nominal transfers between periods t and $t+\tau$;
- $Y_{t+\tau|t}$ is the output at date $t+\tau$ of a firm whose price has been fixed at date t; from the consumer demand $Y_{t+\tau|t} = C_{t+\tau}(P^*/P_{t+\tau})^{-\varepsilon}$;
- $\Psi_{t+\tau}(Y) = W_{t+\tau} N_{t+\tau} = W_{t+\tau}(Y/A_{t+\tau})^{1/(1-\alpha}$ is the cost of producing Y at date $t+\tau$.

Main steps

- First order condition for the firm: the discounted sum of expected real future marginal costs at fixed price multiplied by the markup is equal to 1.
- Individual firm productions $Y_{t+\tau|t}$, of the firms that have changed price τ periods before, are linked to the aggregate production $Y_{t+\tau}$.
- Write the first order condition at $t+1$, and manipulate with the first order condition at t to eliminate all the $t+\tau$ terms for $\tau > 1$.

Aggregate labor demand

Let

$$\overline{\mathrm{MC}} = \frac{\varepsilon - 1}{\varepsilon}$$

be the real marginal cost (inverse of the markup) that would prevail at the steady state, under zero inflation.

Then log-linearization in a neighborhood of the steady state gives:

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\beta\theta)(1-\theta)}{\theta} \frac{1-\alpha}{1-\alpha+\alpha\varepsilon} [\mathrm{mc}_t - \overline{\mathrm{mc}}].$$

Linking marginal cost to output

Recall that the expression for the marginal cost is

$$\mathrm{mc}_t = w_t - p_t - \ln(1-\alpha) - a_t + \alpha n_t.$$

The labor supply of the household satisfies

$$w_t - p_t = \sigma y_t + \phi n_t$$

Eliminating the real wage and employment using
$y_t = a_t + (1-\alpha)n_t$,

$$\mathrm{mc}_t = \sigma y_t - \ln(1-\alpha) - a_t + \frac{\alpha+\phi}{1-\alpha}(y_t - a_t).$$

The output gap

$$\mathrm{mc}_t = \left(\sigma + \frac{\alpha+\phi}{1-\alpha}\right) y_t - \frac{\alpha+\phi}{1-\alpha} a_t - \ln(1-\alpha)$$

The *natural* level of output y_t^n is defined as the one that would prevail under flexible prices, i.e. the value of production such that $\mathrm{mc}_t = \overline{\mathrm{mc}}$. It only depends on the technology shocks.

$$\overline{\mathrm{mc}} = \left(\sigma + \frac{\alpha+\phi}{1-\alpha}\right) y_t^n - \frac{\alpha+\phi}{1-\alpha} a_t - \ln(1-\alpha)$$

The *output gap* $\tilde{y}_t$ is the difference between output y_t and its natural level y_t^n.

The New Keynesian Phillips curve

It follows from expressing the marginal cost in terms of the output gap:

$$\pi_t = \beta E_t \pi_{t+1} + \underbrace{\frac{(1-\beta\theta)(1-\theta)}{\theta}\frac{1-\alpha}{1-\alpha+\alpha\varepsilon}\left(\sigma + \frac{\alpha+\phi}{1-\alpha}\right)}_{k}\tilde{y}_t.$$

The dynamic IS equation

The Euler equation is:

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}[i_t - E_t \pi_{t+1} - \rho].$$

Define the natural rate of interest r_t^n as the rate that would prevail on the reference trajectory, with zero expected inflation:

$$y_t^n = E_t y_{t+1}^n - \frac{1}{\sigma}[r_t^n - \rho].$$

The difference between these two equalities gives the dynamic IS equation

$$\tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma}[i_t - E_t \pi_{t+1} - r_t^n].$$

The equilibrium

Two equations to determine the output gap and the inflation rate.

One needs to make precise the monetary policy to determine the nominal interest rate i_t and close the model.

We solve an example with an interest rate rule.

Solving for linear rational expectations equilibria

Equilibrium under an interest rate rule

Assume

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + v_t$$

The coefficients of inflation and the output gap are non negative.
The constant term is chosen so as to allow a zero inflation rate.
The random term v_t is exogenous with zero mean.

Substituting into the dynamic IS equation gives

$$\left(1 + \frac{\phi_y}{\sigma}\right) \tilde{y}_t + \frac{\phi_\pi}{\sigma} \pi_t = E_t \tilde{y}_{t+1} + \frac{1}{\sigma} [E_t \pi_{t+1} + r_t^n - \rho - v_t]$$

The system of equilibrium equations

In matrix form, the Phillips curve and IS equations become

$$\begin{pmatrix} -k & 1 \\ \sigma + \phi_y & \phi_\pi \end{pmatrix} \begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \sigma & 1 \end{pmatrix} \begin{pmatrix} E_t \tilde{y}_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} 0 \\ r_t^n - \rho - v_t \end{pmatrix}$$

This yields

$$\begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \frac{1}{\sigma + \phi_y + k\phi_\pi} \begin{pmatrix} -\phi_\pi & 1 \\ \sigma + \phi_y & k \end{pmatrix} \left[\begin{pmatrix} 0 & \beta \\ \sigma & 1 \end{pmatrix} \begin{pmatrix} E_t \tilde{y}_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} 0 \\ r_t^n - \rho - v_t \end{pmatrix} \right.$$

$$\begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \frac{1}{\sigma + \phi_y + k\phi_\pi} \left[\begin{pmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma k & \beta(\sigma + \phi_y) + k \end{pmatrix} \begin{pmatrix} E_t \tilde{y}_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} 1 \\ k \end{pmatrix} (r_t^n - \rho - v_t) \right]$$

Finding linear rational expectations equilibria

- Checking for uniqueness
- Finding the coefficients
- Impulse response functions

Uniqueness (determinacy) of the equilibrium

Properties of linear systems of difference equations (e.g. Blanchard-Kahn).

Position of eigenvalues in the complex plane relative to the unit circle and number of predetermined variables.

The matrix eigenvalues

The eigenvalues are the roots of the characteristic polynomial

$$f(\lambda) = \lambda^2 - S\lambda + P$$

The sum of the roots S is equal to the trace

$$S = \frac{\sigma + \beta(\sigma + \phi_y) + k}{\sigma + \phi_y + k\phi_\pi}$$

while their product is equal to the determinant

$$P = \frac{\beta\sigma}{\sigma + \phi_y + k\phi_\pi}$$

Position of the eigenvalues relative to the unit circle

The determinant is smaller than 1: by assumption the coefficients of the interest rate rule are non negative and k is non negative. If the roots are complex, they are both inside the unit circle. If they are real, at least one of them is inside the unit circle. A necessary and sufficient condition for both roots to be inside the unit circle then is $f(-1) > 0,\ f(1) > 0$. Now

$$f(-1) = 1+S+P > 0, \quad f(1) = 1-S+P = \frac{(1-\beta)\phi_y + k(\phi_\pi - 1)}{\sigma + \phi_y + k\phi_\pi}$$

A necessary and sufficient condition for both roots to be inside the unit circle is

$$(1-\beta)\phi_y + k(\phi_\pi - 1) > 0$$

Computing the rational expectations equilibrium, monetary shocks

Suppose v_t is AR(1)

$$v_t = \rho_v v_{t-1} + \varepsilon_t^v$$

One looks for a linear solution in v_t,

$$\tilde{y}_t = \psi_y v_t \quad \pi_t = \psi_\pi v_t$$

where the unknowns are the coefficients ψ_y and ψ_π.

$$E_t \tilde{y}_{t+1} = \psi_y \rho_v v_t \quad E_t \pi_{t+1} = \psi_\pi \rho_v v_t$$

Finding the rational expectations equilibrium

Substituting into the IS and Phillips curves yield two equations in the two unknowns

$$\psi_\pi = \beta \rho_v \psi_\pi + k \psi_y$$

$$\psi_y = \rho_v \psi_y - \frac{1}{\sigma}[\phi_\pi \psi_\pi + \phi_y \psi_y + 1 - \psi_\pi \rho_v]$$

A similar computation can be done for the impact of technology shocks, supposing a AR(1) structure for technical progress

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a$$

Calibration

$\beta = 0.99$	Period: quarter
$\sigma = 1$	log utility
$\phi = 1$	unit Frisch elasticity of labor supply
$\theta = 2/3$	average price duration of three quarters
$\rho_v = 0.5$	
$\rho_a = 0.9$	

Greenspan time $\phi_\pi = 1.5$, $\phi_y = 0.5/4$

Contractionary monetary shock: increase of 0.25% of ε^v at initial quarter. Responses expressed in annualized terms.

Figure 3.1: Effects of a Monetary Policy Shock (Interest Rate Rule))