Risk sharing, multiple assets and uncertainty

Plan

1. Pricing by arbitrage
   1.1 The model: states, assets
   1.2 Application: intertemporal arbitrage
   1.3 Risk neutral ‘probability’
2. Asset markets and risk sharing
   2.1 Uncertainty and demand for assets
   2.2 Mean-variance efficiency
   2.3 Beyond mean variance
3. The yield curve
   3.1 The representative agent model
   3.2 The yield curve

Pricing by arbitrage

The model: states

We consider the world from an initial date $t = 0$.
All the future and uncertainty is described through a finite set of possible states e, the *events* or *states of nature*. The number of such states is E.
For the time being, and for all that concerns arbitrage, we do **not** put a probability on the state space.

The model: assets

There are $k = 1, \ldots, K$ assets.

A unit of asset k is defined by the received earnings (interest, dividends, possibly liquidation value at the end of the game) or due payments in currency associated with its ownership in the various states of nature.

For asset k in state e this is denoted by $a_{ke} = a_k(e)$, positive for a gain, negative for a payment.

The matrix of general term a_{ke}, row index k, column index e, is denoted $\tilde{a}$. Its dimension is $(K \times E)$.

The model: portfolios and markets

Let z_k denote a (positive or negative) number of units of asset k. A portfolio is a column vector $z = (z_k)_{k=1,\dots,K}$, with possibly long and short positions.

Assets are exchanged on a (competitive) market: the selling price is equal to the buying price, and is independent of the traded quantity.

The price today of asset k is denoted p_k, and the price vector p is a column vector of dimension K.

$$\sum_{k=1}^{K} p_k z_k = p'z.$$

The earnings accruing to the owner of z in state e are

$$c_z(e) = \sum_{k=1}^{K} z_k a_k(e),$$

$$\tilde{c}_z = z'\tilde{a}.$$

Arbitrage opportunities

Definition 1 : *Arbitrage opportunity*

An arbitrage opportunity is a portfolio z such that

$$z'\tilde{a} \geq 0 \quad \text{and} \quad z'p \leq 0,$$

or equivalently

$$\sum_{1}^{K} z_k a_k(e) \geq 0 \;\; \text{for all } e \quad \text{and} \quad z'p \leq 0,$$

with at least a strict inequality among the $E+1$'s.

Absence of arbitrage opportunities

Theorem 1 : State prices

A market is without arbitrage opportunities if and only if there exists a vector $q = (q(e))_{e \in \mathcal{E}}$ *with all components* strictly *positive such that*

$$p_k = \sum_e q(e) a_k(e) \quad \textit{for all } k.$$

The vector q is called a *state price* (or Arrow-Debreu price) vector.
Consequence: valuation by arbitrage ('duplication', 'redundancy')

Corollary 1 : valuation by state prices

Let q *be* a *state price vector,* z *a portfolio. Then*

$$p'z = \sum_e q(e) c_z(e) \tag{1}$$

Complete markets

Definition 2 : Complete markets

Markets are complete if for all $\tilde{c} = (c(e))$ *in* $\mathbb{R}^E$ *there exists a portfolio* z *such that* $\tilde{c} = z'\tilde{a}$, *i.e.*

$$c(e) = \sum_{1}^{K} z_k \tilde{a}_k(e) \quad \textit{for all } e.$$

Theorem 2 :

1) Markets are complete if and only if $\tilde{a}$ *has rank* E*.*

2) A complete market without arbitrage opportunities has a unique *state price vector* $(q(e))$, *and* *any* *future income stream* $\tilde{c}$ *has a current value given by the present value formula :*

$$\sum_{e} q(e)c(e).$$

Complete and incomplete markets

Corollary 2 : Arrow Debreu prices

Let q be a state price vector. Consider the income stream made of £1 in state e and nothing in any other state.
If there exists a portfolio replicating this income stream, its value is $q(e)$.

Application : dynamics

Intertemporal model without uncertainty: *spot (or yield) curve*.

'Zero coupons bonds': they give one pound at the term (or maturity), and nothing otherwise.

$q_t(\tau)$: price today at date t of a zero coupon bond whose term is in τ years.

If markets span zero coupons for all terms, any risk free asset yielding $a(t+\tau)$ in τ periods, $\tau = 1, \ldots,$ can be valued through the formula

$$p(t) = \sum_{\tau} q_t(\tau) a(t+\tau).$$

Spot rate curve

Definition 3 : *Let* $q_t(\tau)$ *be the price today of a zero coupon of maturity* τ*. The interest rate* $\bar{r}_\tau$ *for risk free trades at maturity* τ *is defined by*

$$\frac{1}{(1+\bar{r}_\tau)^\tau} = q_t(\tau).$$

The spot rate curve *of date* t *is the curve that graphs* $\bar{r}_\tau$ *as a function of* τ*.*

Risk adjusted probability

Risk-free asset between dates 0 and 1, bearing interest r:

$$1 = \sum_e q(e)(1+r).$$

Probability (??) distribution $\bar{\pi}$ on $\mathcal{E}$:

$$\bar{\pi}(e) = q(e)(1+r).$$

Valuation of asset k :

$$p_k = \frac{1}{1+r} \sum_e \bar{\pi}(e) a_k(e)$$

Intertemporal valuation

Corollary 3 : Risk-adjusted probability. *A market at date 0, where a risk-free asset and other risk bearing assets on date 1 are traded, is without arbitrage opportunities if and only if there exists a* strictly positive *probability distribution* $\bar{\pi}$ *on* $\mathcal{E}$ *such that the price of any asset is equal to the discounted (at the risk free rate) mathematical expectation under* $\bar{\pi}$ *of the future incomes it generates:*

$$p_k = \frac{1}{1+r} \sum_e \bar{\pi}(e) a_k(e) \text{ for any asset } k.$$

or equivalently

$$\sum_e \bar{\pi}(e) \frac{a_k(e)}{p_k} = 1 + r$$

The mathematical expectations of the returns of all assets are equal.

Pricing by arbitrage: lessons

Under complete markets, it is possible to reproduce any risky income flow, and to price it by duplication. This seems to have fueled the finance industry of derivatives.
Paradox: pure duplication should not bring much benefit! Profits should come from the invention of products that allow a better allocation of risks. But then pricing by arbitrage is not valid...

Asset markets and risk sharing

Mean variance portfolio choice: Plan

1. Uncertainty and the demand for assets
2. Mean-variance efficiency
3. Beyond mean variance

Households balance-sheet (USA)

Trillions of dollars

	2001		2007	
	A	L	A	L
Tangible assets	**15.5**		**24.1**	
Real estate	12.4		20.0	
Consumer durable goods	3.1		4.1	
Financial assets	**31.8**	**7.6**	**45.4**	**13.7**
Deposits	4.8		7.3	
Mortgages	0.1	5.3	0.2	10.5
Other credit market instruments	2.4	2.0	3.9	2.6
Life insurance reserves	0.9		1.2	
Pension fund reserves	8.8		12.8	
Corporate equities	6.5		5.5	
Mutual fund shares	2.6		5.1	
Equity in non corporate businesses	4.8		7.9	
Others	0.8	0.2	1.6	0.4
All assets	**47.3**	**7.6**	**69.5**	**13.7**
Net worth		**39.7**		**55.8**

A simple model: setup

Two dates $t = 0$, $t = 1$.

States of nature e, $e = 1, \ldots, E$, at date 1.

(Subjective?) probability of the realization of these states : $\pi(e)$, $\pi(e) > 0$ and $\sum_{e=1}^{E} \pi(e) = 1$.

$k = 0$: risk free asset $a_0(e) = 1$, for all e, i.e. $a_0 = \mathbb{1}_E$
Risky assets k, $k = 1, \ldots, K$.

Typical investor

Initial wealth ω_0, which serves as the numeraire
Portfolio: (z_0, z), where $z = (z_k)_{k=1,\dots,K}$.
Budget constraint at date 0

$$p_0 z_0 + p' z = \omega_0.$$

Random endowment available at date 1 $\omega(e)$.
Consumption at date 1 is

$$c_z(e) = \omega(e) + z_0 + \sum_{k=1,\dots,K} z_k a_k(e)$$

Assumption :

There are no redundant assets. The matrix $\begin{pmatrix} \mathbb{1}_E \\ \tilde{a} \end{pmatrix}$ *is of full rank.*

Interest rate and prices

By definition the (net risk-free) interest rate r between date 0 and date 1 satisfies

$$\frac{1}{1+r} = p_0.$$

Ranking of income prospects

Von Neumann Morgenstern assumption:

$$Ev(\tilde{c}_z) = \int v(x) dF_z(x) = \sum_e \pi(e) v(c_z(e))$$

Risk aversion, risk tolerance

One assumes some risk aversion, formally a concave v.

$$Ev(c + d\tilde{c}) = v(c) + v'(c)Ed\tilde{c} + \frac{1}{2}v''(c^*)Ed\tilde{c}^2$$

Local measures of risk aversion: $Ed\tilde{c} = 0$, $\mathrm{var}(d\tilde{c}) = \sigma^2$ decreases utility by $|v''|\sigma^2/2$.
Certainty equivalent $-\frac{v''(c)}{v'(c)}\frac{\sigma^2}{2}$.
Absolute risk aversion:

$$R_a(c) = -\frac{v''(c)}{v'(c)}$$

Risk tolerance :

$$T(c) = \frac{1}{R_a(c)} = -\frac{v'(c)}{v''(c)}$$

$$\begin{cases} \max Ev(\tilde{c}) = Ev(\tilde{\omega} + z_0 + z'\tilde{a}) = \displaystyle\sum_{e=1}^{E} \pi(e) v \left[\omega(e) + z_0 + \sum_{k=1}^{K} z_k a_k(e) \right] \\ p_0 z_0 + p' z = \omega_0 \end{cases}$$

First order conditions:

$$Ev'(\tilde{c}) = \lambda p_0$$

$$Ev'(\tilde{c})\tilde{a}_k = \sum_{e=1}^{E} \pi(e) v' \left[\omega(e) + z_0 + \sum_{k=1}^{K} z_k a_k(e) \right] a_k(e) = \lambda p_k$$

Asset demand as a function of prices is the solution of the $K+2$ equations, made of the $K+1$ first order conditions and of the budget constraint, in the $K+2$ unknowns (z_0, z, λ).

What of arbitrage opportunities?

Definition 4 : *An* arbitrage opportunity *is a portfolio* (z_0, z) *that yields a non negative income stream, which is positive with a non zero probability. Formally:*

$$p_0 z_0 + p' z \leq 0 \qquad z_0 + \sum_{k=1}^{K} \tilde{a}_k z_k \geq 0,$$

with positive probability of a strict inequality.

This is equivalent to the previous definition when all states have positive probability.

State prices

Define

$$\pi(e)v'(c_z(e)) = \lambda q(e).$$

Then, for all k, when the FOC hold :

$$p_k = \sum_e q(e)a_k(e).$$

Under the FOC, there are no arbitrage opportunities. Marginal utility for income in state e is a possible state price, after suitable normalization.

Mean variance efficiency

Assumption :
The investor portfolio ranking is increasing in the expectation of financial income $z_0 + Ez'\tilde{a}$ *and decreasing in its variance* $\mathrm{var}(z'\tilde{a})$.

$$\left\{\begin{array}{l} \min \operatorname{var}(z'\tilde{a}) \\ z_0 + Ez'\tilde{a} \geq M \\ p_0 z_0 + p'z = \omega_0, \end{array}\right.$$

Homogeneity (of degree 2) in the couple (M, ω_0).

Finance notations

Counting in terms of *composition in pounds* of the portfolio, instead of a *number of shares of each asset*.

$$x_k = \frac{p_k z_k}{\omega_0}, \quad k \in \mathcal{K}$$

Gross return of asset k = random income at date 1 in pounds, which is brought by one pound invested in this asset today.

$$\tilde{R}_k = \frac{\tilde{a}_k}{p_k}, \quad k = 1, \ldots, K.$$

$\tilde{R}_0$ is constant over the states of nature and equal to $(1 + r)$.

$$z' \tilde{a} = \omega_0 x' \tilde{R}$$

$$z' p = \omega_0 \mathbb{1}'_K x$$

$$\begin{cases} \min \operatorname{var}(z'\tilde{a}) \\ z_0 + Ez'\tilde{a} \geq M \\ p_0 z_0 + p'z = \omega_0, \end{cases}$$

Mean variance efficient portfolios:

$$\begin{cases} \min \operatorname{var}(x'\tilde{R}) \\ x_0(1+r) + Ex'\tilde{R} \geq \frac{M}{\omega_0} \\ x_0 + \mathbb{1}'x = 1. \end{cases}$$

Risk is identified with the variance of returns

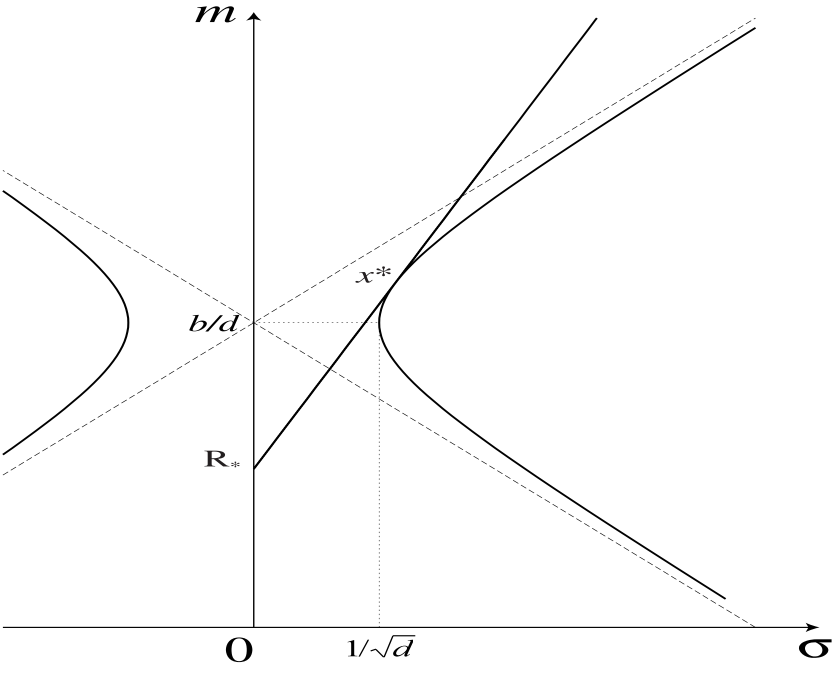
For the risk free asset $k = 0$, by definition, $\tilde{R}_0(e) = E\tilde{R}_0$ for all e. $\tilde{R}$, as well as $\tilde{a}$, are random column vectors of dimension K. One lets:

$$\gamma_{hk} = \sum_{e=1}^{E} \pi(e)(\tilde{R}_h(e) - E\tilde{R}_h)(\tilde{R}_k(e) - E\tilde{R}_k).$$

Γ is the $K \times K$ matrix associated with the risky assets.

Finance and graphical approach: solution in two steps

1) Search of portfolios only made of risky assets which minimize variance, for a given expectation: *hyperbola in the plan (standard-error, expectation)*
One first works on risky assets only.


m
x*
b/d
R*
O
1/√d
σ

2) Graphical solution for the general case with a risk free asset. With one pound to invest, one can put x_0 pound on the risk free asset (standard-error 0, expectation $(1+r)$), and $(1-x_0)$ on a risky portfolio of arbitrary composition (standard-error σ, expectation m).
Graphically, in the plan
(standard-error, expectation) :

$$[|1-x_0|\sigma, \quad x_0(1+r)+(1-x_0)m]$$

$$x^* = \alpha\Gamma^{-1}(E\tilde{R} - R_0\mathbb{1}_K)$$

There exists a fixed composition of risky assets x^*, such that all mean variance efficient portfolios are linear combinations of x^* and of the risk free asset:

Two funds theorem: *All mean variance efficient portfolios are made of two mutual funds, the risk free fund and a unique risky fund.*

Why do households have different investments

Possible explanations

- Different *after tax* returns: $E\tilde{a}$ varies with the investor
- Information: the moments of $\tilde{a}$ are computed *conditionally on the available information.*

A glance at the practical implementation of mean variance analysis

Static model vs. dynamic world: assumptions on the change in market conditions.
Data: computation of past returns.
Investment horizon.

$$\tilde{R}(t) = \frac{\tilde{p}(t+1) + \tilde{d}(t, t+1)}{p(t)}$$

Naive forecast: for risky assets, the $\tilde{R}(t)$ are i.i.d..
Then $E\tilde{R}$ can be estimated as the average of past observations, and the variance-covariance matrix Γ with empirical covariances.

$$ER = \frac{1}{T}\sum_{1}^{T} R(t)$$

$$\mathrm{cov}(R_h, R_k) = \frac{1}{T-1}\sum_{1}^{T}(R_h(t) - ER_h)(R_k(t) - ER_k)$$

Given the risk free interest rate, one has all the relevant data to invest in the current period!

More realistic model, using updated information...

$$\tilde{R}_k(t) = \sum_{i=1,\tau}^{I} \theta_{ki\tau} f_i(t-\tau) + \varepsilon_k(t)$$

f 'factors': exchange rate, interest rates, state of the economy, firm specific data (size, financial rations, price earning ratio, etc..)
Assumptions on the time varying structure of the residuals.
Forecast of factors and covariance matrix for the current date.

Anomalies

Size increases average return.
Week end effects.
Increase in volatility after market crashes.

Asset demand: beyond mean-variance

When does utility maximization yield a mean variance efficient portfolio?

The CARA-gaussian case.

$$v(c) = -\exp(-\beta c) \quad \beta > 0$$

Mathematical expectation of a log-normal variable:

$$Ev(\tilde{c}) = -\exp\left[-\beta(E\tilde{c} - \frac{\beta}{2}\mathrm{var}\tilde{c})\right] = v(E\tilde{c} - \frac{\beta}{2}\mathrm{var}\tilde{c}).$$

Utility is increasing in the expectation of future consumption $E\tilde{c}$ and decreasing in its variance $\mathrm{var}\tilde{c}$.

This does not imply that the investor will choose a mean-variance efficient portfolio!

$$E\tilde{c} = E\tilde{\omega} + z_0 + z' E\tilde{a}$$

$$\mathrm{var}\tilde{c} = z' \ \mathrm{var}(\tilde{a}) \ z + 2z'\mathrm{cov}(\tilde{a}, \tilde{\omega}) + \mathrm{var}(\tilde{\omega})$$

The portfolio is mean-variance efficient if $\mathrm{cov}(\tilde{a}, \tilde{\omega}) = 0$.

Hedging demand
The *hedging* portfolio is the portfolio that minimizes the variance of future consumption.

$$\mathrm{var}\tilde{c} = z' \ \mathrm{var}(\tilde{a}) \ z + 2z'\mathrm{cov}(\tilde{a}, \tilde{\omega}) + \mathrm{var}(\tilde{\omega})$$

FOC:

$$2\mathrm{var}(\tilde{a}) \ z + 2\mathrm{cov}(\tilde{a}, \tilde{\omega}) = 0$$

$$z_c = -\mathrm{var}(\tilde{a})^{-1}\mathrm{cov}(\tilde{a}, \tilde{\omega}).$$

Portfolio choice

$$\begin{cases} \max\{E\tilde{\omega} + z_0 + z'E\tilde{a} \\ \qquad -\frac{\beta}{2}[z' \operatorname{var}(\tilde{a})\ z + 2z'\operatorname{cov}(\tilde{a},\tilde{\omega}) + \operatorname{var}(\tilde{\omega})]\} \\ \dfrac{z_0}{1+r} + p'z = \omega_0 \end{cases}$$

Since $T(c) = 1/\beta$:

$$\boxed{z = \operatorname{var}(\tilde{a})^{-1}\left\{-\operatorname{cov}(\tilde{a},\tilde{\omega})\ + T(E\tilde{c})\left[E\tilde{a} - (1+r)p\right]\right\}.}$$

Assets and risk sharing: conclusion

In the CARA gaussian setup, it is easy to compute the financial equilibrium and to make explicit the allocation of risks in the economy.
On the space spanned by the existing assets, the agents first hedge their incomes. All the idiosyncratic risks are pooled, and shared proportionately to risk tolerance.

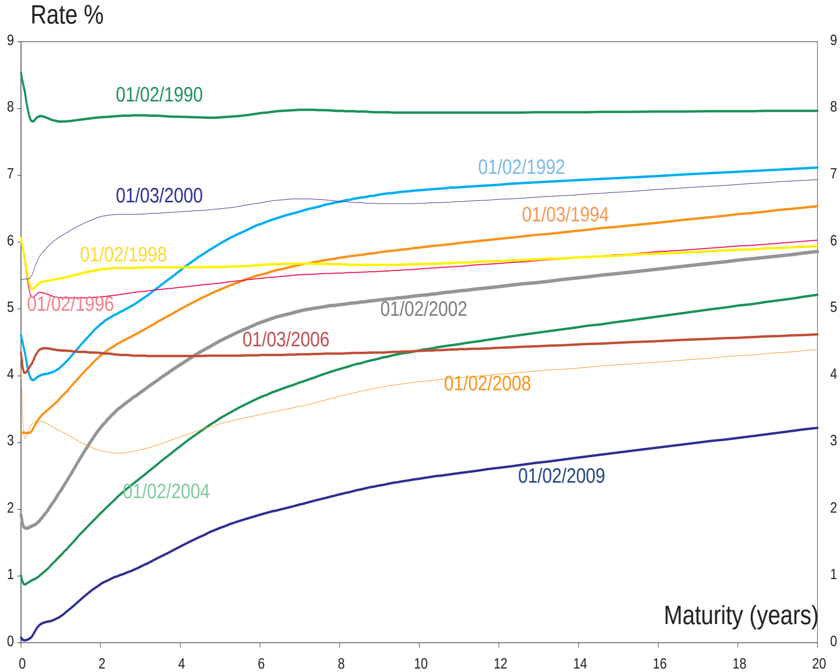
The yield curve

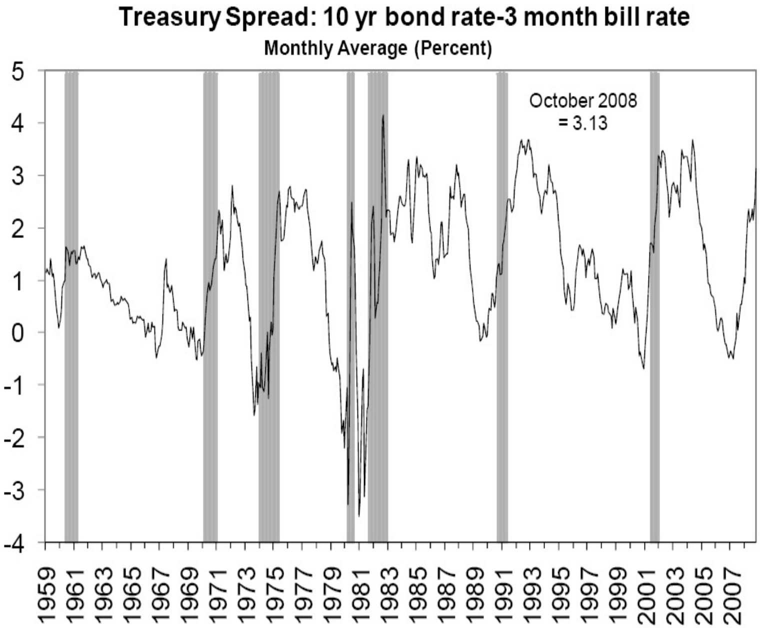
Yield structure and asset prices

Plan

1. The representative agent model
2. The yield structure

Selected US spot rate curves

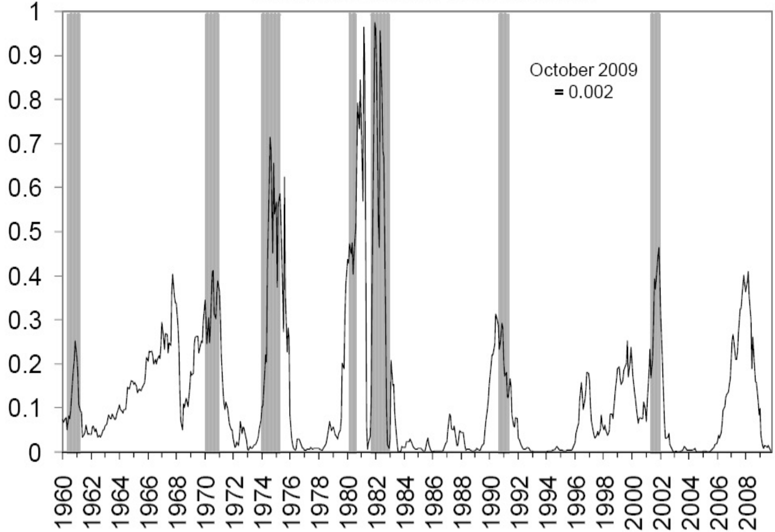




Treasury Spread: 10 yr bond rate-3 month bill rate
Monthly Average (Percent)
October 2008
= 3.13
5
4
3
2
1
0
-1
-2
-3
-4
1959
1961
1963
1965
1967
1969
1971
1973
1975
1977
1979
1981
1983
1985
1987
1989
1991
1993
1995
1997
1999
2001
2003
2005
2007

Probability of US Recession Predicted by Treasury Spread*

Twelve Months Ahead (month averages)



* Parameters estimated using data from January 1959 to December 2005, recession probabilities predicted using data through October 2008

The representative agent

Infinite horizon economy, in discrete time, $t = 0, \ldots, \infty$.
Single consumer:
consumption plan $\mathbf{c} = (c_t)_{t=0,\ldots,\infty}$

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t E_0 u(c_t)$$

ρ psychological discount **rate**, associated with psychological discount **factor** δ:

$$\delta = \frac{1}{1+\rho}.$$

The mathematical expectation E is conditional on available information at date 0.

Temporal consistency of tastes: at all dates, preferences are represented with the same function, the only difference coming from the fact the mathematical expectation is conditional on the current information.

Non storable resources : $\omega_0, \ldots, \omega_t, \ldots$.

Intertemporal equilibrium: $\mathbf{c} = \omega$.

The yield curve

At date 0, assets ('zero coupons') promise of getting one numeraire unit at date t, $t = 1, \dots$
Price $q(t)$ on date 0 spot market, where the current good is the numeraire.

$$q(t) = \frac{1}{(1 + r(t))^t}.$$

Rate measures for a period length equal to the model time unit.
Yield structure at date 0: curve $r(t)$ as a function of t.
Used to evaluate firms, investment projects, etc...

Demand z_t in date t zero coupon maximizes U subject to:

$$\begin{cases} c_0 = \omega_0 - q(t)z_t, \\ c_t = \omega_t + z_t, \end{cases}$$

all other consumptions being kept unchanged.
The FOC is:

$$\frac{\delta^t E_0 u'(c_t)}{u'(c_0)} = q(t).$$

And we have : $c_t = \omega_t$.

Pricing in the representative agent model

$$q(t) = \frac{\delta^t u'(\omega_t)}{u'(\omega_0)},$$

$$\frac{1 + r(t)}{1 + \rho} = \left[\frac{u'(\omega_0)}{u'(\omega_t)}\right]^{1/t}.$$

Therefore (concavity of Von Neumann Morgenstern utility):

$$\omega_t > \omega_0 \iff r(t) > \rho.$$

Concluding thoughts

The purpose of finance is to allocate risks in the economy.

It is an intermediary between savers (pension funds, etc..) and investors.

As a consequence, it is at the crux of the macroeconomic equilibrium determination, balancing demand and supply of current goods.