G022: MSc Core Macroeconomics Exam

You have THREE HOURS. Answer ALL questions. There are 100 points on the exam.

Part B: Mean variance behavior and portfolio choice (10 points)

B.1 Recall the definition of a mean-variance efficient portfolio. State the two-fund theorem.

B.2 Consider an economy where there is a risk free asset of gross return $R_0 = 1$ and two risky assets of return $\tilde{R}_1$ and $\tilde{R}_2$, such that

$$E\tilde{R}_1 = E\tilde{R}_2 = \mathrm{var}\tilde{R}_1 = \mathrm{var}\tilde{R}_2 = 1 \qquad \mathrm{cov}(\tilde{R}_1, \tilde{R}_2) = 0.$$

State and solve the program that yields the set of mean variance efficient portfolios (x_0, x_1, x_2).

All the assets have the same expected return. Looking for the mean-variance amounts to look for the risky portfolio $x_1 + x_2 = 1$ of minimal variance, i.e.

$$\begin{cases} \min x_1^2 + x_2^2 \\ x_1 + x_2 = 1 \end{cases}$$

The solution is $x_1 = x_2 = 1/2$. All the mean variance efficient portfolio have to be of the form

$$(x_0, \frac{1-x_0}{2}, \frac{1-x_0}{2}),$$

so that they have expected return equal to 1, and variance equal to $(1-x^0)^2/2$. The unique mean variance efficient portfolio therefore is the risk free asset, $x_0 = 1$.

B.2 resit Consider an economy with a representative agent with intertemporal utility function

$$\sum_{t=0}^{\infty} \delta^t \ln(C_t),$$

with $\delta = 0.99$, and time is measured in years. Draw the yield curve when agggregate consumption is constant, when it increases of 1% per year (derive your result from an explicit computation).

By definition

$$\frac{1}{(1+r_t)^t} = \delta^t \frac{C_0}{C_t}.$$

When consumption is constant, $r_t = 0.01$ for all t. When it increases at a constant 1% rate, $r_t = 0.02$.

Part C: Overlapping generations model with physical capital and financial assets (40 points)

Consider an economy with generations of consumers who all live for two periods and reproduce identically. They produce inelastically a fixed quantity L of good when young. At date t, the young consumer sells her production on the market, and saves the proceeds to consume C_{t+1} when old. Savings takes the form of physical capital K_t and/or of financial assets B_t. Financial assets serve as the numeraire, and their overall quantity B, positive or negative, stays constant over time. Letting P_t denote the price of good at date t, the value of savings is $P_t K_t + B_t$. At any date savings physical capital K (in a frim) yields a quantity of good $A\sqrt{K}$ one period later, where A is a positive number. The program of the young consumer born at date t is

$$\begin{cases} \max C_{t+1} \\ P_t K_t + B_t = P_t \\ P^e_{t+1} C_{t+1} = P^e_{t+1} A\sqrt{K_t} + B_t \end{cases}$$

C.1 Describe the situation of the old consumer at date t. Give the expression of her demand for consumption good as a function of her past physical investment K_{t-1}, her cash balance B_{t-1} and the current price level P_t.

The old consumer has cash B_{t-1} and a quantity of good $A\sqrt{K_{t-1}}$. She consumes

$$C_t = A\sqrt{K_{t-1}} + \frac{B_{t-1}}{P_t}. \tag{1}$$

C.2 Show that, in the absence of borrowing constraints, the typical consumer maximizes

$$C = (1+\rho)L + A\sqrt{K} - (1+\rho)K,$$

where $(1+\rho)$ is the expected real interest rate, equal to P_t/P^e_{t+1} for the consumer born at date t. Solve for the optimal demand for physical capital as a function of $(1+\rho)$, $\chi(1+\rho)$.

Eliminating B among the current and future budget constraints yields the expression for C. Maximizing with respect to K gives

$$\chi(1+\rho) = \left(\frac{A}{2(1+\rho)}\right)^2$$

C.3 Define a temporary competitive equilibrium at date t. Write down the equations that characterize it. Give the equation linking the temporary equilibrium price to the initial conditions at date t and the expected future price P^e_{t+1}.

A temporary competitive equilibrium, given expectations and initial endowments, is a price P_t and quantities C_t, K_t, B_t such that the consumers maximize their utilities under their budget constraints, and demand is equal to supply for both good and assets:

$$C_t + K_t = L + A\sqrt{K_{t-1}},$$

$$B_t = B_{t-1}.$$

By Walras' law, from the budget constraints, the first equality implies the second one.

The consumption of the young consumer at date $t+1$ is

$$C_{t+1} = A\sqrt{K} - \frac{P_t}{P^e_{t+1}} K + \frac{P_t}{P^e_{t+1}} L.$$

The demand for physical capital therefore is

$$K_t = \left(A \frac{P^e_{t+1}}{2P_t} \right)^2 . \tag{2}$$

Plugging (1) and (2) into the supply-demand equation yields

$$A^2 P^{e2}_{t+1} = 4LP_t^2 - 4B_{t-1}P_t. \tag{3}$$

C.4 resit We consider a special case where expectations are functions of past observations, with the simple rule

$$P^e_{t+1} = P_{t-1}.$$

Given the numerical values $L = A = 1$, describe the temporary equilibrium trajectories and their limits. Comment.

Prices satisfy the following equation

$$4P_t^2 - 4BP_t - P_{t-1}^2 = 0.$$

For any value of P_{t-1}, there is a unique temporary equilibrium price, which is the positive root of the above second degree equation:

$$P_t = \frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{P_{t-1}^2}{4}}.$$

It follows that P_t is an increasing function of P_{t-1}. The equation has two stationary points, $P = 0$ and $P = 4B/3$, and it easy to see that

$$P_{t-1} < \frac{4B}{3} \Longrightarrow P_{t-1} < P_t < \frac{4B}{3}$$

and

$$P_{t-1} > \frac{4B}{3} \Longrightarrow P_{t-1} > P_t > \frac{4B}{3}.$$

All the trajectories are monotone and converge to $4B/3$.

C.4 Draw the offer curve of the typical consumer in the plan ($x = K - L, y = C - A\sqrt{K}$), that relates the demand for good when old ($C - A\sqrt{K}$) to the demand for good when young ($K - L$), when ρ varies between -1 and $+\infty$. Locate the offer curve with respect to the 45^0 line $x + y = 0$ (discuss depending on whether A is smaller or greater than $4L$).

From the budget constraint, for all consumer choices, we identically have

$$y + (1 + \rho)x = 0.$$

The offer curve is obtained by eliminating ρ using the expression for the demand of capital:

$$1 + \rho = \frac{A}{2\sqrt{K}} = \frac{A}{2\sqrt{x + L}},$$

which yields

$$y + \frac{A}{2} \frac{x}{\sqrt{x + L}} = 0.$$

The derivative is

$$y' = -\frac{A}{2\sqrt{x + L}} \frac{x + 2L}{2(x + L)}.$$

The function is decreasing on all its domain of definition, $x \geq -L$ or $K \geq 0$. It cuts the 45^0 line $y + x = 0$ at the origin and at the point where ρ is equal to zero, i.e. $K = A^2/4$, or $x = A^2/4 - L$.

C.5 Define a competitive equilibrium with perfect foresight, from date 0 to the end of times. Explain how the offer curve of the preceding section can be used to determine the competitive equilibria with perfect foresight. Describe all the equilibria when $L = 1$ and $A = 1$. Comment.

Given a stock of capital K_{-1} and a financial asset holding B_{-1}, a perfect foresight equilibrium is a sequence (P_t, C_t, K_t) of prices and quantities such that:

(a) The optimal behavior of consumer born at date t, for any $t \geq 0$, given prices, is to chose the stock of capital K_t, the stock of financial assets B_{-1} and the consumption C_{t+1}. Moreover

$$C_0 = \frac{B_{-1}}{P_0} + A\sqrt{K_{-1}}.$$

(b) At each date t, $t \geq 0$, supply is equal to demand:

$$C_t + K_t = L + A\sqrt{K_{t-1}}.$$

For $A = L = 1$, the offer curve lies below the 45^0 line for $-3/4 \leq x \leq 0$, or y in the same interval. For an initial value of $C_0 - A\sqrt{K_{-1}} = B_{-1}/P_0$ in this interval, one can build an equilibrium sequence, iterating between the offer curve and the 45^0 line. Any starting point outside of this interval leads out of the domain in finite time, and does not correspond to an equilibrium. All trajectories, except the one that starts at -3/4 (stationary optimal equilibrium), converge to the origin, which is the autarkic equilibrium.

C.6 Describe all the perfect foresight equilibria when $L = 1$ and $A = 4$.

For $A = 4$ and $L = 1$, the offer curve lies below the 45^0 line for $0 \leq x \leq 3$, or y in the same interval. For an initial value of $C_0 - A\sqrt{K_{-1}} = B_{-1}/P_0$ in this interval, one can build an equilibrium sequence, iterating between the offer curve and the 45^0 line. Any starting point outside of this interval leads out of the domain in finite time, and does not correspond to an equilibrium.

C.7 The government introduces a pay as you go pension scheme. Wages are subject to a pension contribution, equal to a fraction $\theta = 0.20$ of the wage bill. This contribution is transferred to the current old consumer. How does this pay as you go scheme affect the equations of the model? How does it change the offer curves in the two cases where L is equal to 1 and A equals 1 or 4? What is its long run impact? Comment.

The pay as you go scheme acts as if the consumer was providing $(1 - \theta)L$ when young and θL when old. Consumption becomes

$$C = (1 + \rho)(1 - \theta)L + \theta L + A\sqrt{K} - (1 + \rho)K.$$

The offer curve must now be drawn in the plan $\tilde{x} = K - (1-\theta)L$ and $\tilde{y} = C - \theta L - A\sqrt{K}$, but the demand for physical capital as a function of the real interest rate is unchanged. Therefore

$$1 + \rho = \frac{A}{2\sqrt{K}} = \frac{A}{2\sqrt{\tilde{x} + (1 - \theta)L}},$$

and the equation of the offer curve becomes:

$$\tilde{y} + \frac{A\tilde{x}}{2\sqrt{\tilde{x} + (1 - \theta)L}} = 0.$$

In the long run, the pension scheme has no real effect if the economy goes to the efficient golden rule equilibrium with a zero real interest rate. This is not the case if the economy goes to the autarkic equilibrium: $\tilde{x} = 0$ gives

$$K = (1 - \theta)L,$$

so that the stock of capital is reduced by the introduction of the pension scheme. This is an improvement if the capital stock was too high (negative interest rate or $L > A^2/4$), a bad outcome when it was too low (positive interest rate or $L < A^2/4$).