Monetary Policy, Inflation, and the Business Cycle

Chapter 2

A Classical Monetary Model

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In the present chapter we lay out a simple model of a classical monetary economy, featuring perfect competition in all markets and fully flexible prices and wages. As stressed below, many of the predictions of that classical economy are strongly at odds with the evidence reviewed in chapter 1. That notwithstanding, we view the analysis of the classical economy as providing a useful benchmark, that we will use in subsequent chapters when more realistic assumptions are introduced. It also allows us to introduce some notation, as well as the assumptions on preferences and technology that are used frquently in the remainder of the book.

Following much of the recent literature, our baseline model attaches a very limited role for money. In the first four chapters the only explicit role played by money is to serve as a unit of account. In that case, and as shown below, whenever monetary policy is specified in terms of an interest rate rule, we do not need to make any reference whatsoever to the quantity of money in circulation in order to determine the economy's equilibrium. When the specification of monetary policy involves the money supply, we postulate a "conventional" money demand equation in order to close the model, without taking a stand on its microfoundations. In the second half of the chapter, we discuss a model in which money plays an explicit role, beyond that of serving as a unit of account. In particular we analyze a model in which real balances are assumed to generate utility to households, and explore the implications for monetary policy of alternative assumptions on the properties of that utility function.

Independently of how money is introduced, the proposed framework assumes a representative household solving a dynamic optimization problem. That problem and the associated optimality conditions are described in the next section. Section 2 introduces the representative's firm's technology and determines its optimal behavior, under the assumption of price and wage-taking. Section 3 characterizes the equilibrium, and shows how real variables are uniquely determined independently of monetary policy. Section 4 discusses the determination of the price level and other nominal variables under alternative monetary policy rules. Finally, section 5 analyzes a version of the model with money in the utility function, and the extent to which the conclusions drawn from the analysis of a cashless economy need to be modified under that assumption.

1 Households

The representative household seeks to maximize the objective function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \; U(C_t, N_t) \tag{1}$$

where C_t is the quantity consumed of the single good, and N_t denotes hours of work or employment.[1]

Maximization of (1) is subject to a sequence of flow budget constraints given by

$$P_t \; C_t + Q_t \; B_t \leq B_{t-1} + W_t \; N_t - T_t \tag{2}$$

for $t = 0, 1, 2, ...$ P_t is the price of the consumption good,. W_t denotes the nominal wage, B_t represents the quantity of one-period nominally riskless discount bonds purchased in period t, and maturing in period $t+1$. Each bond pays one unit of the numéraire at maturity, and its price is Q_t. T_t represents lump-sum additions or subtractions to period income (e.g. lump-sum taxes, dividends, etc.). When solving the problem above, the household is assumed to take as given the price of the good, the wage and the price of bonds.

In addition to (2), we assume that the household is subject to a solvency constraint that prevents it from engaging in Ponzi-type schemes. For our purposes the following constraint is sufficient:

$$\lim_{T \to \infty} E_t\{B_T\} \geq 0 \tag{3}$$

1.0.1 Optimal Consumption and Labor Supply

Let $U_{c,t} \equiv \frac{\partial U(C_t, N_t)}{\partial C_t}$ and $U_{n,t} \equiv \frac{\partial U(C_t, N_t)}{\partial N_t}$ denote the marginal utility of consumption and the marginal disutility of work, respectively. The optimality conditions implied by the maximization of (1) subject to (2) are

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} \tag{4}$$

[1] Alternatively, N_t can be interpreted as the number of household members employed, assuming a large household and ignoring integer constraints.

$$Q_t = \beta \; E_t \left\{ \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right\} \tag{5}$$

for $t = 0, 1, 2, ...$

The previous optimality conditions can be derived using a simple variational argument. Let us first consider the impact on utility of a small departure, in period t, from the household's optimal plan. The departure consists of an increase in consumption dC_t and and increase in hours dN_t, while keeping the remaining variables unchanged (including consumption and hours in other periods). If the household was following an optimal plan to begin with it must be the case that

$$U_{c,t} \; dC_t + U_{n,t}.dN_t = 0$$

for any pair (dC_t, dN_t) satisfying the budget constraint, i.e.

$$P_t \; dC_t = W_t.dN_t$$

for otherwise it would be possible to raise utility by increasing (or decreasing) consumption and hours, thus contradicting the assumption that the household is optimizing to begin with. Note that by combining both equations we obtain the optimality condition (4).

Similarly, we can consider the impact on expected utility as of time t of a reallocation of consumption between periods t and $t+1$, while keeping consumption in any period other than t and $t+1$, and hours worked (in all periods) unchanged. If the household is optimizing it must be the case that

$$U_{c,t} \; dC_t + \beta \; E_t\{U_{c,t+1}.dC_{t+1}\} = 0$$

for any pair (dC_t, dC_{t+1}) satisfying

$$P_{t+1} dC_{t+1} = -\frac{P_t}{Q_t} \; dC_t$$

where the latter equation determines the increase in consumption expenditures in period $t+1$ made possible by the additional savings $-P_t dC_t$ allocated into one-period bonds. Combining the two previous equations we obtain the intertemporal optimality condition (5).

In much of what follows we assume that the period utility takes the form:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

The consumer's optimality conditions (4) and (5) thus become:

$$\frac{W_t}{P_t} = C_t^{\sigma} \; N_t^{\varphi} \tag{6}$$

$$Q_t = \beta \; E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\} \tag{7}$$

Notice, for future reference, that equation (6) can be re-written in log-linear form as follows:

$$w_t - p_t = \sigma \; c_t + \varphi \; n_t \tag{8}$$

where lower case letters denote the natural logs of the corresponding variable (i.e. $x_t = \log X_t$). The previous condition has the interpretation of a competitive labor supply schedule, giving the quantity of labor supplied as a function of the real wage, given the marginal utility of consumption (which under our assumptions is a function of consumption only).

As shown in Appendix 1, a log-linear approximation of (7) around a steady state with constant rates of inflation and consumption growth is given by

$$c_t = E_t\{c_{t+1}\} - \frac{1}{\sigma} \; (i_t - E_t\{\pi_{t+1}\} - \rho) \tag{9}$$

where we have defined $i_t \equiv -\log Q_t$ and $\rho \equiv -\log \beta$. Notice that i_t corresponds to the log of the gross yield on the one-period bond; we henceforth refer to it as the *nominal interest rate.*[2]

While the previous framework does not explicitly introduce a motive for holding money balances, in some cases it will be convenient to postulate a demand for real balances with a log-linear form given by (up to an additive constant):

$$m_t - p_t = y_t - \eta \; i_t \tag{10}$$

[2]The yield on the one period bond is defined by $Q_t \equiv (1 + yield)^{-1}$. Note that $i_t \equiv -\log Q_t = \log(1 + yield_t) \simeq yield_t$ where the latter approximation will be accurate as long as the nominal yield is "small."

A money demand equation similar to (10) can be derived under a variety of assumptions. For instance, in Section 5 we derive it as an optimality condition for the household when money balances yield utility.

2 Firms

We assume a representative firm whose technology is described by a production function given by

$$Y_t = A_t \; N_t^{1-\alpha} \tag{11}$$

where A_t represents the level of technology. We assume $a_t \equiv \log A_t$ evolves exogenously according to some stochastic process.

Each period the firm maximizes profits

$$P_t \; Y_t - W_t \; N_t$$

subject to (11), and taking the price and wage as given.

The optimality condition associated with the firm's problem is given by

$$\frac{W_t}{P_t} = (1-\alpha) \; A_t \; N_t^{-\alpha} \tag{12}$$

i.e. the firm hires labor up to the point where its marginal product equals the real wage. Equivalently, the price P_t must be equal to the marginal cost, with the latter being given by $\frac{W_t}{(1-\alpha)A_t \; N_t^{-\alpha}}$.

In log-linear terms, we have

$$w_t - p_t = a_t - \alpha \; n_t + \log(1-\alpha) \tag{13}$$

which can be interpreted as labor demand schedule, mapping the real wage into the quantity of labor demanded, given the level of technology.

3 Equilibrium

In equilibrium all output must be consumed, thus implying a goods market clearing condition given by

$$y_t = c_t \tag{14}$$

By combining the optimality conditions of households and firms with (14) and the log-linear aggregate production relationship

$$y_t = a_t + (1-\alpha)\ n_t \tag{15}$$

we can determine the equilibrium levels of employment and output, as a function of the level of technology:

$$n_t = \psi_{na}\ a_t + \vartheta_n \tag{16}$$

$$y_t = \psi_{ya}\ a_t + \vartheta_y \tag{17}$$

where $\psi_{na} \equiv \frac{1-\sigma}{\sigma+\varphi+\alpha(1-\sigma)}$, $\vartheta_n \equiv \frac{\log(1-\alpha)}{\sigma+\varphi+\alpha(1-\sigma)}$, $\psi_{ya} \equiv \frac{1+\varphi}{\sigma+\varphi+\alpha(1-\sigma)}$, and $\vartheta_y \equiv (1-\alpha)\vartheta_n$.

Furthermore, given the equilibrium process for output, we can use the Euler equation to determine the implied real interest rate, $r_t \equiv i_t - E_t\{\pi_{t+1}\}$

$$\begin{aligned} r_t &= \rho + \sigma\ E_t\{\Delta y_{t+1}\} \\ &= \rho + \sigma\psi_{ya}\ E_t\{\Delta a_{t+1}\} \end{aligned} \tag{18}$$

Finally, the equilibrium real wage, $\omega_t \equiv w_t - p_t$, is given by

$$\begin{aligned} \omega_t &= y_t - n_t + \log(1-\alpha) \\ &= \psi_{\omega a}\ a_t + \log(1-\alpha) \end{aligned} \tag{19}$$

where $\psi_{\omega a} \equiv \frac{\sigma+\varphi}{\sigma+\varphi+\alpha(1-\sigma)}$.

Notice that the equilibrium dynamics of employment, output, and the real interest rate are determined *independently of monetary policy*. In other words, monetary policy is *neutral* with respect to those real variables. On the other hand, determination of the equilibrium behavior of nominal variables requires that we specify how monetary policy is conducted. Below we consider several monetary policy rules and their implied outcomes.

4 Monetary Policy and Price Level Determination

We start by examining the implications of some interest rate rules. Later we introduce rules that involve monetary aggregates. In all cases we make use of the Fisherian equation:

$$i_t = E_t\{\pi_{t+1}\} + r_t \tag{20}$$

which implies that the nominal rate adjusts one-for-one with expected inflation, given a real interest rate determined exclusively by real factors, as in (18).

4.1 An Exogenous Path for the Nominal Interest Rate

Assume that the nominal interest rate follows an *exogenous* stationary process $\{i_t\}$. Without loss of generality we assume that i_t has mean ρ, which is consistent with a steady state with zero inflation and no secular growth. Notice that a particular case of this rule corresponds to a constant interest rate $i_t = i = \rho$, for all t.

Using (20) we can write,

$$E_t\{\pi_{t+1}\} = i_t - r_t$$

where, as discussed above, $\{r_t\}$ is determined independently of the policy rule.

Notice that while expected inflation is pinned down by the previous equation under the regime considered here, actual inflation is not. Since there is no other condition that can be used to determine inflation, it follows that any path for the price level which satisfies

$$p_{t+1} = p_t + i_t - r_t + \xi_{t+1}$$

is consistent with equilibrium, where ξ_{t+1} is a shock possibly unrelated to economic fundamentals satisfying $E_t\{\xi_{t+1}\} = 0$ for all t. Such shocks are often referred to in the literature as *sunspot* shocks. We refer to an equilibrium in which such non-fundamental factors may cause fluctuations in one or more variables as an *indeterminate* equilibrium. In the example above, we have thus shown how an exogenous nominal interest rate leads to *price level indeterminacy.*

Notice that when (10) is operative the equilibrium path for the money supply (which is endogenous under the present policy regime) is given by

$$m_t = p_t + y_t - \eta\, i_t$$

and hence it inherits the indeterminacy of p_t.

4.2 A Simple Inflation-Based Interest Rate Rule

Suppose that the central bank adjusts the nominal interest rate according to the rule

$$i_t = \rho + \phi_\pi \, \pi_t$$

where $\phi_\pi \geq 0$.

Combining the previous rule with the Fisherian equation (20) we obtain

$$\phi_\pi \, \pi_t = E_t\{\pi_{t+1}\} + \widehat{r}_t \tag{21}$$

where $\widehat{r}_t \equiv r_t - \rho$. We distinguish between two cases.

If $\phi_\pi > 1$, the previous difference equation has only one stationary solution, i.e. a solution that remains in a neighborhood of the steady state. That solution can be obtained by solving (21) forward, which yields

$$\pi_t = \sum_{k=0}^{\infty} \phi_\pi^{-(k+1)} \, E_t\{\widehat{r}_{t+k}\}$$

which fully determines inflation (and, hence, the price level) as a function of the real interest rate path, which in turn is a function of fundamentals, as shown in (18).

On the other hand, if $\phi_\pi < 1$ the stationary solutions to (21) take the form

$$\pi_{t+1} = \phi_\pi \, \pi_t - \widehat{r}_t + \xi_{t+1} \tag{22}$$

where ξ_{t+1} is, again, a sunspot shock satisfying $E_t\{\xi_{t+1}\} = 0$.

Accordingly, any process $\{\pi_t\}$ satisfying (22) is consistent with equilibrium, while remaining in a neighborhood of the steady state. So, as in the case of an exogenous nominal rate, the price level (and, hence, inflation and the nominal rate) are not determined uniquely when the interest rate rule implies a weak response of the nominal rate to changes in inflation. More specifically, the condition for a determinate price level, $\phi_\pi > 1$, requires that the central bank adjust nominal interest rates more than one-for-one in response to any change in inflation, a property known as the *Taylor principle.* The previous result can be viewed as a particular instance of the need to satisfy the *Taylor principle* in order for an interest rate rule to bring about a determinate equilibrium.

4.3 An Exogenous Path for the Money Supply

Suppose that the central bank sets an exogenous path for the money supply $\{m_t\}$. Using (10) to eliminate the nominal interest rate in (20), we can derive the following difference equation for the price level:

$$p_t = \left(\frac{\eta}{1+\eta}\right) E_t\{p_{t+1}\} + \left(\frac{1}{1+\eta}\right) m_t + u_t$$

where $u_t \equiv (1+\eta)^{-1}(\eta\, r_t - y_t)$ evolves independently of $\{m_t\}$.

Assuming $\eta > 0$ and solving forward we obtain:

$$p_t = \frac{1}{1+\eta} \sum_{k=0}^{\infty} \left(\frac{\eta}{1+\eta}\right)^k E_t\{m_{t+k}\} + u'_t$$

where $u'_t \equiv \sum_{k=0}^{\infty} \left(\frac{\eta}{1+\eta}\right)^k E_t\{u_{t+k}\}$ is, again, independent of monetary policy.

Equivalently, we can rewrite the previous expression in terms of expected future growth rate of money:

$$p_t = m_t + \sum_{k=1}^{\infty} \left(\frac{\eta}{1+\eta}\right)^k E_t\{\Delta m_{t+k}\} + u'_t \tag{23}$$

Hence, we see how an arbitrary exogenous path for the money supply always determines the price level uniquely. Given the price level, as determined above, we can then use (10) to solve for the nominal interest rate:

$$\begin{aligned} i_t &= \eta^{-1}\,[y_t - (m_t - p_t)] \\ &= \eta^{-1} \sum_{k=1}^{\infty} \left(\frac{\eta}{1+\eta}\right)^k E_t\{\Delta m_{t+k}\} + u''_t \end{aligned}$$

where $u''_t \equiv \eta^{-1}(u_t + y_t)$.

Example. Consider the case in which money growth follows an AR(1) process.

$$\Delta m_t = \rho_m\, \Delta m_{t-1} + \varepsilon^m_t$$

For simplicity I assume the absence of real shocks, which implies a constant output and real rate; without loss of generality, we set $r = y = 0$. Then it follows from (23) that

$$p_t = m_t + \frac{\eta \rho_m}{1 + \eta(1 - \rho_m)} \, \Delta m_t$$

Hence, in response to an exogenous shock to the money supply, and as long as $\rho_m > 0$ (the empirically relevant case), the price level should respond more than one-for-one with the increase in the money supply, a prediction which contrasts starkly with the sluggish response of the price level observed in empirical estimates of the effects of monetary policy shocks.

The nominal interest rate is in turn given by

$$i_t = \frac{\rho_m}{1 + \eta(1 - \rho_m)} \, \Delta m_t$$

i.e. in response to an expansion of the money supply, an as long as $\rho_m > 0$, the nominal interest rate is predicted to go up. In other words, the model implies the absence of a *liquidity effect*, in contrast with much of the evidence.

4.4 Optimal Monetary Policy

The analysis of the baseline classical economy above has shown that while real variables are independent of monetary policy, the latter can have important implications for the behavior of nominal variables and, in particular, of prices. Yet, and given that the household's utility is a function of consumption and hours only–two real variables that are invariant to the way monetary policy is conducted– it follows that there is no policy rule that is better than any other. Thus, in the basleline model above, a policy that generates large fluctuations in inflation and other nominal variables (perhaps as a consequence of following a policy rule that does not guarantee a unique equilibrium for those variables) is no less desirable that one that succeeds in stabilizing prices in the face of the same shocks.

The previous result, which is clearly extreme and unappealing, can be overcome once we consider versions of the classical monetary model in which a motive to keep part of wealth in the form of monetary assets is introduced explicitly. Next we discuss one such model, in which real balances are assumed to yield utility.

5 Money in the Utility Function (*)

In the model developed above, and in much of the recent monetary literature, the main role played by money is to serve as a numéraire, i.e. unit of account in which prices, wages and securities' payoffs are stated. While we have postulated and used a simple log linear money demand function, we have done so in an ad-hoc manner, without an explicit justification for why agents would want to hold an asset that is dominated in return by bonds, while having identical risk properties. While in much of the analysis in subsequent chapters we will keep the assumption of a *cashless economy*, it is useful to understand how the basic framework can incorporate a role for money other than that of a unit of account and, in particular, how it can generate a demand for money. The discussion in the following section focuses on models that achieve the previous objective by assuming that real balances are an argument of the utility function. [In subsequent sections we discuss justifications for that assumption, as well as alternative models].

The introduction of money in the utility function requires that we modify the households problem in two ways. First, preferences are now given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \; U\left(C_t, \frac{M_t}{P_t}, N_t\right)$$

where M_t denotes holdings of money in period t. Secondly, the flow budget constraint incorporates monetary holdings explicitly, taking the following form:

$$P_t \; C_t + Q_t \; B_t + M_t \leq B_{t-1} + M_{t-1} + W_t \; N_t - T_t$$

Letting $\mathcal{A}_t \equiv B_{t-1} + M_{t-1}$ denote total financial wealth at the beginning of the period t (i.e. before consumption and portfolio decisions are made), we can rewrite the previous flow budget constraint as:

$$P_t \; C_t + Q_t \mathcal{A}_{t+1} + (1 - Q_t) \; M_t \leq \mathcal{A}_t + W_t \; N_t - T_t$$

with the solvency constraint taking now the form $\lim_{T \to \infty} E_t\{\mathcal{A}_T\} \geq 0$.

The previous representation of the budget constraint can be thought of as equivalent to that of an economy in which all financial assets (represented by $\mathcal{A}_t$) yield a gross nominal return Q_t^{-1} ($= \exp\{i_t\}$), and where agents can purchase the utility-yielding "services" of money balances at a unit price $(1 - Q_t) = 1 - \exp\{-i_t\} \simeq i_t$. That unit price for money services, which roughly corresponds to the nominal interest rate, is the opportunity cost of holding one's financial wealth in terms of monetary assets, instead of interest-bearing bonds.

Two of the optimality conditions of the household problem are the same as those obtained for the cashless model, i.e. (6) and (7), with the marginal utility terms being now defined over (and evaluated at) the triplet $(C_t, \frac{M_t}{P_t}, N_t)$. In addition to (6) and (7), there is an addition optimality condition given by

$$\frac{U_{m,t}}{U_{c,t}} = 1 - \exp\{-i_t\} \tag{24}$$

where $U_{m,t} \equiv \frac{\partial U\left(C_t, \frac{M_t}{P_t}, N_t\right)}{\partial (M_t/P_t)} > 0$.

Again, in order to derive that optimality condition we can use a simple variational argument. Suppose that the household is considering deviating from the optimal plan by adjusting consumption and money holdings in period t by amounts dC_t and dM_t respectively, while keeping all other variables unchanged at their optimal values. Optimality of the initial plan requires that utility cannot be raised as a result of the deviation, i.e.

$$U_{c,t}\ dC_t + U_{m,t}\ \frac{1}{P_t}\ dM_t = 0$$

for any pair (dC_t, dM_t) satisfying

$$P_t\ dC_t + (1 - Q_t)\ dM_t = 0$$

which guarantees that the budget constraint without the need to adjust any other variable. Combining the previous two equations and using the definition of the nominal rate $i_t = -\log Q_t$ yields the optimality condition (24).

In order to be able to make any statements about the consequences of having money in the utility function we need to be more precise about the way money balances interact with other variables in yielding utility. In particular, whether the utility function is separable or not in real balances determines the extent to which the neutrality properties derived above for the cashless economy carry over to the economy with money in the utility function. We illustrate that point by considering, in turn, two example economies with separable and non-separable utility.

5.1 An Example with Separable Utility

We specify the household's utility function to have the functional form

$$U\left(C_t, \frac{M_t}{P_t}, N_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma} + \frac{(M_t/P_t)^{1-\nu}}{1-\nu} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

Note that, given the separability of real balances, neither $U_{c,t}$ nor $U_{n,t}$ depend on the level of real balances. As a result, (6) and (7) (as well as their log-linear counterparts, (8) and (9)) continue to hold unchanged. It follows that we can determine the equilibrium values for output, employment, the real rate and the real wage following the same steps as above, and without any reference to monetary policy.

The introduction of money in the utility function, allows us to derive a money demand equation from the household's optimal behavior. Using the above specification of utility we can rewrite the optimality condition (24) as:

$$\frac{M_t}{P_t} = C_t^{\sigma/\nu} \ (1 - \exp\{-i_t\})^{-1/\nu} \tag{25}$$

which can be naturally interpreted as a demand for real balances. The latter is increasing in consumption and inversely related to the nominal interest rate, as in conventional specifications.

Using the first-order Taylor approximation $\log(1 - \exp\{-i_t\}) \simeq const. + \frac{1}{\exp\{i\}-1} \ i_t$, we can rewrite (25) in approximate log-linear form (and up to an uninteresting constant) as:

$$m_t - p_t = \frac{\sigma}{\nu} \ c_t - \eta \ i_t \tag{26}$$

where $\eta \equiv \frac{1}{\nu(\exp\{i\}-1)} \simeq \frac{1}{\nu i}$ is the implied interest semi-elasticity of money demand.

The particular case of $\nu = \sigma$ is an appealing one, since it implies a unit elasticity with respect to consumption. Under that assumption, we obtain a conventional linear demand for real balances

$$\begin{aligned} m_t - p_t &= c_t - \eta \ i_t \\ &= y_t - \eta \ i_t \end{aligned} \tag{27}$$

where the second equality holds in our baseline model economy, in which all output is consumed. The previous specification is often assumed in subsequent chapters, without the need to invoke explicitly its source.

As in the analysis of the cashless economy, the usefulness of (26) (or (27)) is confined to the determination of the equilibrium values for inflation and other nominal variables whenever the description of monetary policy involves the money supply, as it would be the case if the central bank were to follow a money supply rule or in the case of an exogenous process for the money supply considered above. Otherwise, the only use of the money demand equation is to determine the quantity of money that will need to circulate in the economy in order to support in equilibrium the nominal interest rate implied by the policy rule.

5.2 An Example with Non-Separable Utility

Let us consider next an economy in which period utility is given by

$$U\left(C_t, \frac{M_t}{P_t}, N_t\right) = \frac{X_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

where X_t is a composite index of consumption and real balances defined as follows

$$\begin{aligned} X_t &\equiv \left[(1-\vartheta)\ C_t^{1-\nu} + \vartheta\ \left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-v}} \qquad for\ \nu \neq 1 \\ &\equiv C_t^{1-\vartheta}\ \left(\frac{M_t}{P_t}\right)^{\vartheta} \qquad for\ \nu = 1 \end{aligned}$$

with ν represents the (inverse) elasticity of substitution between consumption and real balances, and ϑ the relative weight of real balances in utility.

Notice that the marginal utilities of consumption and real balances are now given, respectively, by

$$U_{c,t} = (1-\vartheta)\ X_t^{\nu-\sigma}\ C_t^{-\nu}$$

$$U_{m,t} = \vartheta\ X_t^{\nu-\sigma}\ \left(\frac{M_t}{P_t}\right)^{-\nu}$$

whereas the marginal (dis)utility of labor is, as before, given by $U_{n,t} = -N_t^{\varphi}$. The optimality conditions of the household's problem, (4), (5) and (24), can now be written as:

$$\frac{W_t}{P_t} = N_t^{\varphi}\ X_t^{\sigma-\nu}\ C_t^{\nu}\ (1-\vartheta)^{-1} \tag{28}$$

$$Q_t = \beta \ E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\nu} \left(\frac{X_{t+1}}{X_t} \right)^{\nu-\sigma} \ \frac{P_t}{P_{t+1}} \right\} \tag{29}$$

$$\frac{M_t}{P_t} = C_t \ (1 - \exp\{-i_t\})^{-\frac{1}{\nu}} \ \left(\frac{\vartheta}{1-\vartheta} \right)^{\frac{1}{\nu}} \tag{30}$$

Notice that in the particular case in which the intertemporal and intratemporal elasticities of substitution coincide (i.e. $\nu = \sigma$), optimality conditions (28) and (29) match exactly those obtained in the case of separable utility, and thus lead to the same equilibrium implications derived for that case and discussed in the previous subsection..

In the general case, however, both the labor supply equation (28) and the Euler equation (29) are influenced by the level of real balances, through the dependence of the index X_t on the latter. The level of real balances depends, in turn, on the nominal rate (as implied by (30)). Those features imply that monetary policy is no longer neutral in the case of non-separable utiillity considered here. In particular, to the extent that different monetary policy rules have different implications for the path of the nominal rate (as will generally be the case), they will also have different effects on real balances and–through the latter's influence on the marginal utility of consumption–on the position of the labor supply schedule and, hence, on employment and output. This mechanism is analyzed formally below.

Notice that the implied money demand equation (30) can be rewritten in log-linear form (and up to an additive constant) as in (27) above, i.e.

$$m_t - p_t = c_t - \eta \ i_t$$

where, again, $\eta = \frac{1}{\nu(\exp\{i\}-1)}$ Thus, the implied interest semi-elasticity of demand η is now proportional to the elasticity of substitution between real balances and consumption, ν^{-1}.

On the other hand, log-linearization of (28) around the zero inflation steady state yields

$$w_t - p_t = \sigma c_t + \varphi n_t + (\nu - \sigma)(c_t - x_t)$$

Log-linearizing the definition of X_t around a zero inflation steady state, and combining the resulting expression with (30) we obtain

$$\begin{aligned} w_t - p_t &= \sigma c_t + \varphi n_t + \chi(\nu - \sigma) \ [c_t - (m_t - p_t)] \\ &= \sigma c_t + \varphi n_t + \chi\eta(\nu - \sigma) \ i_t \end{aligned}$$

where $\chi \equiv \frac{\vartheta^{\frac{1}{\nu}}(1-\beta)^{1-\frac{1}{\nu}}}{(1-\vartheta)^{\frac{1}{\nu}}+\vartheta^{\frac{1}{\nu}}(1-\beta)^{1-\frac{1}{\nu}}} \in [0,1)$, and where the second equality makes use of money demand equation (27).

For future reference it is convenient to rewrite the previous optimality conditions in terms of the steady state ratio $k_m \equiv \frac{M/P}{C}$, i.e. the inverse consumption velocity. Using the money demand equation, we have $k_m = \left(\frac{\vartheta}{(1-\beta)(1-\vartheta)}\right)^{\frac{1}{\nu}}$. Noting that $\chi = \frac{k_m(1-\beta)}{1+k_m(1-\beta)}$, and using the definition of η evaluated at the zero inflation steady state we can rewrite the optimality condition above as

$$w_t - p_t = \sigma c_t + \varphi n_t + \omega \ i_t \tag{31}$$

where $\omega \equiv \frac{k_m \beta(1-\frac{\sigma}{\nu})}{1+k_m(1-\beta)}$. Thus, we see that the sign of the effect of the nominal interest rate on labor supply is determined by the sign of $\nu - \sigma$. When $\nu > \sigma$ (implying $\omega > 0$) the reduction in real balances induced by an increase in the nominal rate lowers the marginal utility of consumption (for any given c_t), lowering the quantity of labor supplied at any given real wage. The opposite effect obtains when $\nu < \sigma$. Note, however, that $\nu \simeq \frac{1}{i\eta}$ is likely to be larger than σ for any plausible values of η and σ. Thus, the case of $U_{cm} > 0$ (and hence $\omega > 0$) appears as the most plausible one, conditional on the specification of preferences analyzed here.

The corresponding log-linear approximation to (29) is given by

$$\begin{aligned} c_t &= E_t\{c_{t+1}\} - \frac{1}{\sigma}\left(i_t - E_t\{\pi_{t+1}\} - (\nu-\sigma)\ E_t\{(c_{t+1}-x_{t+1}) - (c_t - x_t)\} - \rho\right) \\ &= E_t\{c_{t+1}\} - \frac{1}{\sigma}\left(i_t - E_t\{\pi_{t+1}\} - \chi(\nu-\sigma)\ E_t\{\Delta c_{t+1} - \Delta(m_{t+1}-p_{t+1})\} - \rho\right) \\ &= E_t\{c_{t+1}\} - \frac{1}{\sigma}\left(i_t - E_t\{\pi_{t+1}\} - \omega\ E_t\{\Delta i_{t+1}\} - \rho\right) \end{aligned} \tag{32}$$

where, again, the last equality makes use of (27). Thus, when $\nu > \sigma$ (and, hence, $\omega > 0$) the anticipation of a nominal rate increase (and, hence, of a decline in real balances), lowers the expected one period ahead marginal utility of consumption (for any expected c_{t+1}), which induces an increase in current consumption (in order to smooth marginal utility over time).

In order to reflect the changes implied by non-separable utility, we need to modify the economy's log-linearized equilibrium conditions. Thus, combining (31) with the labor demand schedule (13) we obtain the following labor market clearing condition:

$$\sigma c_t + \varphi n_t + \omega \ i_t = y_t - n_t + \log(1-\alpha) \tag{33}$$

which we can rewrite, using the goods market clearing condition (14) and the log-linear production relationship (15) as (ignoring an uninteresting additive constant):

$$y_t = \psi_{ya}\ a_t - \psi_{yi}\ i_t \tag{34}$$

where $\psi_{yi} \equiv \frac{\omega(1-\alpha)}{\sigma+\varphi+\alpha(1-\sigma)}$.

Condition (34) points to a key implication of the property of non-separability ($\omega \neq 0$): equilibrium output is no longer invariant to monetary policy, at least to the extent that the latter implies variations in the nominal interest rate. In other words, monetary policy is not neutral. As a result, equilibrium condition (34) does not suffice to determine the equilibrium level of output, in contrast with the economy with separable utility analyzed above. In order to pin down the equilibrium path of output and other endogenous variables we need to combine (34) with the remaining equilibrium conditions, including a description of monetary policy.

One such additional condition can be obtained by imposing the goods market clearing condition $y_t = c_t$ on Euler equation (32), which yields an equation relating the nominal interest rate to the expected path of output and expected inflation:

$$y_t = E_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - E_t\{\pi_{t+1}\} - \omega\ E_t\{\Delta i_{t+1}\} - \rho\right) \tag{35}$$

Finally, we need an equation which describes how monetary policy is conducted. For the purposes of illustration we assume that the central bank follows the simple inflation-based interest rate rule

$$i_t = \rho + \phi_\pi\ \pi_t + v_t \tag{36}$$

where v_t now represents an exogenous policy disturbance, assumed to follow the stationary AR(1) process

$$v_t = \rho_v\ v_{t-1} + \varepsilon_t^v$$

Similarly, and for concreteness, we assume that the technology parameter follows the AR(1) process

$$a_t = \rho_a\ a_{t-1} + \varepsilon_t^a$$

Using (36) to eliminate the nominal rate in (34) and (35), and combining the resulting two equations we can obtain (after some algebraic manipulation) the following closed form expressions for the equilibrium level of inflation, the nominal rate, and output:

$$\begin{aligned}
\pi_t &= -\frac{\sigma(1-\rho_a)\psi_{ya}}{\phi_\pi(1+\omega\psi)(1-\Theta\rho_a)}\ a_t - \frac{1+(1-\rho_v)\omega\psi}{\phi_\pi(1+\omega\psi)(1-\Theta\rho_v)}\ v_t \\
i_t &= -\frac{\sigma(1-\rho_a)\psi_{ya}}{(1+\omega\psi)(1-\Theta\rho_a)}\ a_t - \frac{\rho_v}{\phi_\pi(1+\omega\psi)(1-\Theta\rho_v)}\ v_t \\
y_t &= \psi_{ya}\left(1+\frac{\sigma(1-\rho_a)\psi_{yi}}{(1+\omega\psi)(1-\Theta\rho_a)}\right)\ a_t + \frac{\rho_v\psi_{yi}}{\phi_\pi(1+\omega\psi)(1-\Theta\rho_v)}\ v_t
\end{aligned}$$

where $\Theta \equiv \frac{1+\omega\psi\phi_\pi}{(1+\omega\psi)\phi_\pi}$ and $\psi \equiv \frac{\alpha+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}$.

A few remarks regarding the impact of monetary policy on the economy's equilibrium are in order. First, note that the interest rate multiplier of output, conditional on an exogenous monetary policy shock is given by $\frac{dy_t}{di_t} = \frac{dy_t/dv_t}{di_t/dv_t} = -\psi_{yi}$. In order to get a sense for the magnitude of that multiplier, recall that $\psi_{yi} \equiv \frac{\omega(1-\alpha)}{\sigma+\varphi+\alpha(1-\sigma)}$. Let us assume parameter values $\sigma = \varphi = 1$ and $\alpha = 1/3$, as in the baseline calibration that will be introduced in chapter 3. Using the definition of ω, and the fact that $\nu = \frac{1}{\eta\rho}$ is "large" for any reasonable values of η, we have $\psi_{yi} \simeq \frac{k_m}{3}$, and so the size of the inverse velocity k_m is a key determinant of the quantitative importance of monetary non-neutralities in the model. Unfortunately, the magnitude of k_m depends crucially on the definition of money used. Thus, and focusing on postwar U.S. data, $k_m \simeq 0.3$ if we take the monetary base as the relevant measure of money.[3] In that case we have $\psi_{yi} \simeq 0.1$, which implies a relative small multiplier: a monetary policy shock that raised the nominal rate by one percentage point (expressed at an annual rates) would generate a decrease in output of about 0.025 percent. By way of contrast, if we use $M2$ as the definition of money, we have $k_m \simeq 3$ and so the impact on output of an analogous monetary policy shock is a 0.25 percent decline. The latter value, while small, appears to be closer to the estimated output effects of a monetary policy shock found in the literature. Yet, even in that case, there are other aspects of the transmission of monetary policy shocks implied by the model that are clearly at odds with the evidence, e.g. the response of inflation and the real interest rate. Thus, note that

$$\frac{d\pi_t}{di_t} = \frac{d\pi_t/dv_t}{di_t/dv_t} = (1+(1-\rho_v)\omega\psi)\ \rho_v^{-1} > 0$$

[3] This is the approach followed in Woodford (2003, chapter 2).

$$\frac{dr_t}{di_t} = 1 - \frac{dE_t\{\pi_{t+1}\}/dv_t}{di_t/dv_t} = -(1-\rho_v)\omega\psi < 0$$

i.e. in response to a monetary policy shock that raises the nominal interest rate and lowers output, inflation tends to increase, and the real rate to go down (as a result of the dominant effect of higher expected inflation). This contrasts with the downward adjustment of inflation and the rise in the real rate observed as part of the economy's response of the economy following a contractionary monetary policy shock.

Finally, there is an additional argument that can be brought up and which calls into question the relevance of the transmission mechanism underlying the classical model with non-separable preferences and which has to do with its implications regarding the long-run effects of monetary policy. To see this, consider an exogenous monetary policy intervention that raises the nominal rate permanently. The implied permanent change in output is determined by (34), and given by $-\psi_{yi}$. Thus, the long-run trade-off between output and the nominal rate is identical to the short-run trade-off. How about the inflation-output trade-off? Equation (35), evaluated at the steady state, requires a long-run increase in inflation of the same size as the increase in the nominal rate. Hence the long-run trade-off between inflation and output is also given by $-\psi_{yi}$. But note that the same coefficient describes the short-run output-inflation trade-off since, in the relevant case of a permanent policy change ($\rho_v = 1$), we have $\frac{dy_t/dv_t}{d\pi_t/dv_t} = -\psi_{yi}$.

As argued above, for a most plausible range of parameter values we have $\psi_{yi} > 0$. Thus, in the present model a permanent increase in inflation will be associated with a permanent decline in output. Given the determinants of ψ_{yi}, whether that long-run trade-off is large or small will largely depend on the size of inverse velocity k_m and, hence, on the relevant measure of money. Thus, the lack of a significant empirical relationship between long-run inflation and economic activity (at least at low levels of inflation), suggests a low value for k_m and ψ_{yi}, as implied by a narrow definition of money. Unfortunately, in the present model, and as argued above, any such calibration with the desirable feature of a negligible long-run trade-off will also be associated with negligible (and hence counterfactual) short run effects of monetary policy.

5.3 Optimal Monetary Policy in a Classical Economy with Money in the Utility Function

In this section we derive the form of the optimal monetary policy in the presence of money in the utility function. We start by laying out and solving the problem facing a hypothetical social planner seeking to maximize the utility of the representative household.

Note that, under our assumptions, there are no aggregate intertemporal links in our simple model: even though each individual household can reallocate its own consumption over time through financial markets, there are no mechanisms that make this possible for the economy as a whole. Thus, the social planner would solve a sequence of static problems of the form

$$\max U\left(C_t, \frac{M_t}{P_t}, N_t\right)$$

subject to the resource constraint

$$C_t = A_t\ N_t^{1-\alpha}$$

The optimality conditions for that problem are given by

$$-\frac{U_{n,t}}{U_{c,t}} = (1-\alpha)\ A_t N_t^{-\alpha} \tag{37}$$

$$U_{m,t} = 0 \tag{38}$$

Condition (37) requires that the marginal rate of substitution between hours of work and consumption be equated to the marginal product of labor. Condition (38) equates the marginal utility of real balances to the "social" marginal cost of producing real balances, which is implicitly assumed to be zero in our setting.

Under what conditions the equilibrium of the decentralized economy satisfies efficiency conditions (37) and (38)? We first note that condition (37) is implied by the combined effect of profit maximization by firms (which equates the real wage to the marginal product of labor; see equation (12)) and the optimal labor supply choice by the household (which equates the real wage to the marginal rate of substitution between hours of work and consumption; see equation (4)). Hence, (37) will be satisfied independently of monetary policy. On the other hand, and as shown above, the household's optimal choice of money balances requires

$$\frac{U_{m,t}}{U_{c,t}} = 1 - \exp\{-i_t\}$$

Accordingly, efficiency condition (38) will be satisfied.if and only if $i_t = 0$ for all t, a policy known as the *Friedman rule*. The rationale for that policy is quite intuitive: while the social cost of producing real balances is zero, the private (opportunity) cost is given by the nominal interest rate. As a result, only when the nominal interest rate is zero are the private and social costs of holding money equated. Note that such a policy implies an average (steady state) rate of inflation

$$\pi = -\rho < 0$$

i.e. prices will decline on average at the rate of time preference. In other words: under the Friedman rule the economy will experience a (moderate) deflation in the long-run.

Implementation of the Friedman rule requires some discussion. As shown earlier a policy rule of the form $i_t = 0$ for all t leaves the price level indeterminate in our model. Even though that indeterminacy should not have any welfare consequences (since (37) and (38) pin down consumption, employment and real balances uniquely), a central bank could avoid that indeterminacy by following a rule of the form

$$i_t = \phi \ (r_{t-1} + \pi_t)$$

for some $\phi > 1$. Combined with (20) that rule implies the difference equation

$$E_t\{i_{t+1}\} = \phi \ i_t$$

whose only stationary solution is $i_t = 0$ for all t. Under that rule, equilibrium inflation is fully predictable and given by

$$\pi_t = -r_{t-1}$$

More generally, any rule that makes the central bank adjust its policy settings (e.g. the money supply) to guarantee that current inflation moves inversely, and one-for-one with the lagged real interest rate will imply a zero nominal interest rate and, thus, an efficient amount of real balances.

6 Notes on the Literature

The modelling approach favored in much of the recent monetary literature, and the one adopted in the present monograph (with the exception of Section 5 of this chapter), does not incorporate monetary assets ("money") explicitly in the analysis. Under that approach the main role played by money is that of a unit of account. Such model economies can be viewed as a limiting case (the cashless limit) of an economy in which money is valued and held by households. Woodford (2003) provides a detailed discussion and a forceful defense of that approach.

Models that introduce monetary assets explicitly rely on one of two alternative formalisms in order to generate a demand for an asset that–as is the case with money–is dominated in its rate of return by alternative assets that have identical risk characteristics: they either assume (i) that real balances generate utility to households or, alternatively, (ii) that the presence of some transaction costs in the purchases of goods can be reduced by household's holding of monetary assets.

The first of those approaches – money in the utility function – traces back to Sidrauski (1967), who introduced that assumption in an otherwise standard neoclassical growth model (with inelastic labor supply). Woodford (2003) offers a detailed analysis of the implications of alternative assumptions on the specification of utility and, in particular, of the likely degree of monetary non-neutralities arising from the non-separability of real balances. Walsh (2003, chapter 2) develops a real business cycle model with money in the utility function, and analyzes the equilibrium properties of a calibrated version of that model. Both analyses conclude, in a way consistent with the discussion above, that even under a utility that is non-separable in real balances, the real effects of monetary policy are quantitatively very small for plausible calibrations of the models.

A common approach to the modelling of a transactions motive for holding money build on the assumption, originally due to Clower (1967), that cash mush be held in advance in order to purchase certain goods. Early examples of classical monetary models in which a demand for money is generated by postulating a cash-in-advance constraint can be found in the work of Lucas (1982) and Svensson (1985). Cooley and Hansen (1989) analyze an otherwise standard real business cycle models augmented with a cash-in-advance constraint for consumption goods, showing that monetary policy is near-neutral for plausible calibrations of that model. Walsh (2003, chapter

3) provides a detailed description of classical monetary models with cash-in-advance constraints and their implications for the role of monetary policy.

The practice, followed in the present monograph, of appending a money demand equation to a set of equilibrium conditions that have been derived in the context of cashless economy is often found in the literature. King and Watson (1995) offers an example of that practice.

The analysis of the form of the optimal monetary policy in classical economy goes back to Friedman (1969), where a case is made for a policy that keeps the nominal interest rate constant at a zero level. More recent treatments of the conditions under which is optimal include Woodford (1990) and Correia and Teles (1999).

Two useful discussions of the notion of monetary neutrality and its evolution in macroeconomic thinking can be found in Patinkin (1987) and Lucas (1996).

Appendix 1: Some Useful Log-Linear Approximations

Euler equation

We can rewrite the consumer's Euler equation as

$$1 = E_t\{\exp(i_t - \sigma \Delta c_{t+1} - \pi_{t+1} - \rho)\} \tag{39}$$

In a perfect foresight steady state with constant inflation π and constant growth γ we must have:

$$i = \rho + \pi + \sigma\gamma$$

with the steady state real rate being given by

$$\begin{aligned} r &\equiv i - \pi \\ &= \rho + \sigma\gamma \end{aligned}$$

A first-order Taylor expansion of $\exp(i_t - \sigma \Delta c_{t+1} - \pi_{t+1} - \rho)$ around that steady state yields:

$$\begin{aligned} \exp(i_t - \sigma \Delta c_{t+1} - \pi_{t+1} - \rho) &\simeq 1 + (i_t - i) - \sigma(\Delta c_{t+1} - \gamma) - (\pi_{t+1} - \pi) \\ &= 1 + i_t - \sigma \Delta c_{t+1} - \pi_{t+1} - \rho \end{aligned}$$

which can be used in (39) to obtain, after some rearrangement of terms, the log-linearized Euler equation

$$c_t = E_t\{c_{t+1}\} - \frac{1}{\sigma}\,(i_t - E_t\{\pi_{t+1}\} - \rho)$$

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Exercises

1. Optimality Conditions under Non-Separable Leisure

Derive the log-linearized optimality conditions of the household problem under the following specification of the period utility function with non-separable leisure (King, Plosser, and Rebelo (1987))

$$U(C_t, N_t) = \frac{1}{1-\sigma}\left[C_t \, (1-N_t)^{\nu}\right]^{1-\sigma}$$

2. Alternative Interest Rules for the Classical Economy

Consider the simple classical economy described in the text, in which the following approximate equilibrium relationships must be satisfied

$$y_t = E_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - E_t\{\pi_{t+1}\} - \rho\right)$$

and

$$\begin{aligned} r_t &\equiv i_t - E_t\{\pi_{t+1}\} \\ &= \rho + \sigma \, E_t\{\Delta y_{t+1}\} \end{aligned}$$

and where y_t and, hence, r_t, are determined independently of monetary policy. Next you are asked to analyze, in turn, three alternative monetary policy rules and their implications. When relevant, we assume that the money market clearing condition takes the form

$$m_t - p_t = y_t - \eta \, i_t + \varepsilon_t^m$$

where ε_t^m is a stochastic money demand disturbance.

a) *Inflation Targeting.*

(i) Derive an interest rate rule that would guarantee full stabilization of inflation, i.e. $\pi_t = \pi^*$ for all t where π^* is an inflation target assumed to be "close to" zero (so that the log-linearized equilibrium conditions remain valid).

(ii) Determine the behavior of money growth that is consistent with the strict inflation targeting policy analyzed in (i).

(iii) Explain why a policy characterized by a constant rate of money growth $\Delta m_t = \pi$ will generally not succeed in stabilizing inflation in that economy.

b) *An Interest Rate Peg*

Derive an interest rate rule that yields a unique equilibrium implying a constant nominal interest rate $i_t = i^*$, for all t

c) *Price Level Targeting.*

(i) Consider the interest rate rule

$$i_t = \rho + \phi_p \; (p_t - p^*)$$

where $\phi_p > 0$, and p^* is a (constant) target for the (log) price level. Determine the equilibrium behavior of the price level under this rule. (hint: you may find it useful to introduce a new variable $\widehat{p}_t \equiv p_t - p^*$ –the deviation of the price level from target–to ease some of the algebraic manipulations).

(ii) Consider instead the money targeting rule

$$m_t = p^*$$

Determine the equilibrium behavior of the price level under this rule.

(iii) Show that the money targeting rule considered in (ii) can be combined with the money market clearing condition and rewritten as a price-level targeting rule of the form

$$i_t = \rho + \psi \; (p_t - p^*) + u_t$$

where ψ is a coefficient and u_t is a stochastic process to be determined.

(iv) Suppose that the central bank want to minimize the volatility of the price level. Discuss the advantages and disadvantages of the interest rate rule in (i) versus the money targeting rule in (ii) in light of your findings above.

3. Nonseparable Preferences and Money Superneutrality

Assume that the representative consumer's period utility is given by:

$$U\left(C_t, \frac{M_t}{P_t}, N_t\right) = \frac{1}{1-\sigma}\left[(1-\vartheta)\; C_t^{1-\nu} + \vartheta \; \left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1-\sigma}{1-\nu}} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

a) Derive the optimality conditions of the associated consumer's problem.

b) Assume that the representative firm has access to a simple technology $Y_t = N_t$ and that the monetary authority keeps a constant money growth γ_m. Derive the economy's steady state equilibrium under the assumption of perfect competition.

c) Discuss the effects on inflation and output of a permanent change in the rate of money growth γ_m, and relate it to the existing evidence.

4. Optimal Monetary Policy in a Classical Economy with an Exact Equilibrium Representation

Consider a version of the classical economy with money in the utility function, where the representative consumer maximizes $E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, \frac{M_t}{P_t}, N_t\right)$ subject to the sequence of dynamic budget constraints

$$P_t C_t + M_t + Q_t B_t \leq M_{t-1} + B_{t-1} + W_t \; N_t - T_t$$

Assume a period utility given by:

$$U\left(C_t, \frac{M_t}{P_t}, N_t\right) = \log C_t + \log \frac{M_t}{P_t} - \frac{N_t^{\;1+\varphi}}{1+\varphi} \qquad (40)$$

Suppose there is a representative perfectly competitive firm, producing the single consumption good. The firm has access to the linear production function $Y_t(i) = A_t \; N_t(i)$, where productivity evolves according to:

$$\frac{A_t}{A_{t-1}} = (1+\gamma_a) \; \exp\{\varepsilon_t^a\}$$

with $\{\varepsilon_t^a\}$ is an i.i.d. random process, normally distributed, with mean 0 and variance σ_a^2.

The money supply varies exogenously according to the process

$$\frac{M_t}{M_{t-1}} = (1+\gamma_m) \; \exp\{\varepsilon_t^m\} \qquad (41)$$

where $\{\varepsilon_t^m\}$ is an i.i.d., normally distributed process with mean 0 and variance σ_u^2 . We assume that $\{\varepsilon_t^m\}$ evolves exogenously, outside the control of the monetary authority (e.g., could reflect shocks in the monetary multiplier that prevent the monetary authority from fully controlling the money supply.). Finally, we assume that all output is consumed, so that in equilibrium $Y_t = C_t$ for all t.

a) Derive the optimality conditions for the problem of households and firms.

b) Determine the equilibrium levels of aggregate employment, output, and inflation (Hint: show that a constant velocity $\frac{P_t Y_t}{M_t} = V$ for all t is a solution)

c) Discuss how utility depends on the two parameters describing monetary policy, γ_m and σ_u^2 (recall that the nominal interest rate is constrained to be non-negative, i.e., $Q_t \leq 1$ for all t). Show that the optimal policy must satisfy the Friedman rule ($i_t = 0$ all t) and discuss alternative ways of supporting that rule in equilibrium.

5. A Shopping Time Model (based on Walsh (2003)).

Assume that the transactions technology is such that consuming C_t requires a quantity of shopping time $N_t^s = s(C_t, \frac{M_t}{P_t})$, where $s_c > 0$ and $s_m \leq 0$. Hence the amount of time diverted from leisure is given by $N_t + N_t^s$, where N_t denotes hours of work. Let the original period utility be given by $V(C_t, L_t)$ where $L_t = 1 - N_t - N_t^s$ denotes leisure.

a) Derive the condition determining the optimal allocation of time.

b) Derive the implied utility function in terms of consumption, hours and real balances, and discuss its properties.

6. A Model with Cash and Credit Goods

Assume that the utility of the representative household is given by:

$$V(C_{1t}, C_{2t}, N_t) \tag{42}$$

where C_{1t} denotes consumption of a "cash-good" (i.e., a good that requires cash in order to be purchased), C_{2t} is consumption of a "credit-good" (which does not require cash), and N_t is labor supply. For simplicity, let us assume that the price of the two goods is identical and equal to P_t (e.g., the production function of the representative firm is given by $Y_{1t} + Y_{2t} = N_t$ and there is perfect competition). Purchases of cash-goods have to be settled in cash, whereas credit goods can be financed by issuing one-period riskless nominal bonds.

The budget constraint is given by

$$P_t \ (C_{1t} + C_{2t}) + Q_t B_t + M_t = B_{t-1} + M_{t-1} + T_t$$

Finally, the CIA constraint is given by

$$P_t\, C_{1t} \leq M_{t-1} + T_t$$

where, in equilibrium, $T_t = \Delta M_t$, i.e. it matches money tranfers made by the central bank, and which consumers take as given. For simplicity we assume no uncertainty.

a) Derive the first order conditions associated with the household's problem

b) Note that whenever the CIA contraint is binding we can define a reduced form period utility:

$$U\left(C_t, \frac{M_t}{P_t}, N_t\right) \equiv V\left(\frac{M_t}{P_t}, C_t - \frac{M_t}{P_t}, N_t\right)$$

where $C_t = C_{1t} + C_{2t}$. Show that $U_m \geq 0$, given the optimality conditions derived in a).