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Optimal Information Transmission in Organizations: Search and Congestion

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Summary

- Introduction
- Related literature
- The model
- Analysis
- Optimal Networks
- Summary and extensions

Introduction *(1/3)*

- Problem: Optimal information transmission in organizations.

- Focus: Increasing knowledge forces specialization. We deal with problems where knowing others' knowledge is a scarce resource.

- The organization is modelled as a network:

Introduction *(2/3)*

1. Individuals are specialized problem-solving nodes
2. Problems arrive at random nodes, with random (independent) destinations.
3. The (mutual) communication abilities and knowledge of other's knowledge are the links.
4. Search must respect this knowledge constraint.
5. Aim: Find best way to connect, given fixed number of links and local algorithm.

Introduction *(3/3)*

- Findings: We show tradeoff between distance and congestion.

  1. We solve for smallest arrival rate or problems that collapses network.

  2. Below critical rate, we find its average stock of floating problems (thus, length of time to solve them).

  3. Then we solve for optimal organizational form: either very centralized or very decentralized.

Related literature *(1/3)*

- Economics of organizations: Radner (1992), Bolton Dewatripoint (1994), or van Zandt (1999). Abstract from search. Tradeoff: Benefit of parallel processing vs. coordination problem of communication.

- Sah and Stiglitz (1986) and Visser (2000) focus on contrast between hyerarchic and polyarchic organizations.

- Closer in is Garicano (2000). Each individual specializes. If she cannot solve a problem, there is another person to deal with it. Task of the designer: assign knowledge sets and design the routes.

- Crucial difference between Garicano's (2000) model and ours. We abstract from the knowledge acquisition problem.

- We feel that our model is relevant for firms in which endowments of knowledge are not easy to replicate in a standardized fashion.

Related literature *(2/3)*

- Watts and Strogatz (1998) - small-worlds. Many local links and a few long-range links, but low average distance. Abstracts from search. Albert and Lászlo-Barabási (2002) survey.

- Kleinberg (1999, 2000), addresses search. Helped by knowledge of topology: effective in small-world, not so in random net. Abstracts from congestion.

- Arenas, Díaz-Guilera and Guimerà (2001) similar to us. They restrict, organizational forms, so no genuine search.

The model *(1/5)*

- Our organization is modeled as an undirected graph.
- Nodes are the individuals. $N = \{1, 2, ..., n\}$.
- A link between i and j implies both know each others' knowledge and can communicate.
- We define $g_{ij} \in \{0, 1\}$. Graph is undirected, $g_{ij} = 1$ if and only if $g_{ji} = 1$.
- Let $\Gamma = \{N, (g_{ij})^n_{i,j=1}\}$ be a given network. Neighborhood $N_i = \{j \in N : g_{ij} = 1\}$.

The model *(2/5)*

- The mission of this organization is to solve problems.

- Problems first appear in an organization with independent probability ρ at each node.

- Each problem has an "address" indicating the node k where it is to be solved. Let us then refer to "problem k ".

The model *(3/5)*

- Rules by which the problem travels:

  1. If the arrival node can solve it, then it will do so.

  2. Problems that are chosen to travel further:

     - If $k \in N_i$, the problem is sent to k with $p_{ik}^k = 1$ and it is solved.

     - If $k \notin N_i$, the problem is sent to some $j \in N_i$ with some probability p_{ij}^k . (Of course, $\sum_{j \in N_i} p_{ij}^k = 1$.)

The model *(4/5)*

The network plus search protocol leads to:

$$\{P^k \equiv (p^k_{ij})_{i,j \in N}\}_{k \in N}. \quad (1)$$

Stochastic process governing steps:

$$\begin{array}{rcll} p^k_{ij} & = & 0 & \text{if} \quad j \notin N_i \\ p^k_{ik} & = & 1 & \text{if} \quad k \in N_i \\ p^k_{kj} & = & 0 & \forall j \in N_i. \end{array}$$

We may compute, for each $r \in N$:

$$q^k_{ij}(r) = \sum_{l_1, l_2, \ldots, l_{r-1}} p^k_{il_1} p^k_{l_1 l_2} \cdots p^k_{l_{r-1} j}$$

as the probability of a problem k arising in i to be in node j after r steps. Or simply,

$$Q^k(r) = (P^k)^r = P^k \overset{(r \text{ times})}{\cdots} P^k$$

No-congestion

- First, assume no congestion. Then, $q_{ij}^k(r)$ reinterpreted as the probability that, at any given time $t(\geq r)$,a problem k originated r periods ago in i is faced by j.

- Then

$$b_{ij}^k \equiv \sum_{r=1}^{\infty} q_{ij}^k(r)$$

  steady-state expected number of problems k which arose in i currently passing through j.

- Let B^k denote the matrix $(b_{ij}^k)_{i,j\in N}$ for any given k. Then, compactly:

$$B^k = \sum_{r=1}^{\infty} Q^k(r) = \sum_{r=1}^{\infty} (P^k)^r = (I - P^k)^{-1} P^k$$

Define *notional betweenness* of node j by:

$$\beta_j \equiv \sum_{i,k \in N} b_{ij}^k,$$

Interpret β_j as the expected number of problems going through node j in the long run.

- Effective betweenness:

$$\tilde{\beta}_j(\rho) \equiv \frac{\rho \beta_j}{n-1},$$

Congestion and collapse

- Nodes behave as statistical queues (departures assumed to follow exponential distribution, so arrivals are Poisson) - More on this later.

- Length of queue grows without bound when arrival rate higher than delivery rate (normalized to one). Thus, a node j saturates/collapses, provided no other does, iff $\tilde{\beta}_j(\rho) > 1$,

- Implies that the maximum ρ consistent with no node collapsing in the network is:

$$\rho_c = \frac{n-1}{\beta^*} \tag{2}$$

where $\beta^* \equiv \max_j \beta_j$ is the maximum effective betweenness.

CONCRETE EXAMPLE

(a) For all $i, j, k \in N$, such that $i \neq k$ and $k \notin N_i$,

$$p_{ij}^k = \frac{1}{|N_i|}.$$

(b) Every problem k at node i, is processed with prob $\frac{1}{q_i}$, and q_i the number in the queue.

Below the point of collapse

- Arrivals and departures from each node i follow a Poisson processes with rates equal to $\nu_i = \rho \frac{\beta_i}{n-1}$ and unity, respectively.

- Below the critical ρ_c, well-defined steady state probabilities.

- Denote by p_{im} the steady state probability of a queue of size m in node i. The induced distribution $(p_{im})_{m=0}^{\infty}$ must satisfy:

$$\begin{aligned} \nu_i p_{i,m-1} + p_{i,m+1} &= (\nu_i + 1) p_{im} \\ p_{i1} &= \nu_i p_{i0} \end{aligned}$$

Analysis *(6/7)*

- Left-hand side of first equation is the flow rate into the state m. No other possible transitions, since two simultaneous events do not happen.

- Right-hand side is the departure rate from state m, it adds the rates at which a queue that has m problem receives one more, or solves one.

- The second equation is like the first one, except it notes that a queue in state zero cannot go to state minus one.

- The solution to the system:

$$p_{im} = (1 - \nu_i)\nu_i^m, \; m = 0, 1, 2, \ldots$$

Analysis *(7/7)*

- Given this, the expectation for the length of the queue at i, denoted by λ_i, is:

$$\lambda_i = \sum_{m=0}^{\infty} m(1-\nu_i)\nu_i^m = \frac{\nu_i}{1-\nu_i}.$$

- Over the whole network, the stock of floating problems is

$$\lambda(\rho) = \sum_{i \in N} \lambda_i(\rho) = \sum_{i \in N} \frac{\rho \frac{\beta_i}{n-1}}{1 - \rho \frac{\beta_i}{n-1}}. \quad (3)$$

- This magnitude, implies average delay, denoted $\Delta(\rho)$, by Law of Little,

$$\Delta(\rho) = \frac{1}{n\rho}\lambda(\rho).$$

- Given any network Γ, denote by λ^Γ, ρ_c^Γ, β_i^Γ. Then:

$$\begin{aligned} \lambda^\Gamma(0) &= 0 \\ \lim_{\rho \uparrow \rho_c^\Gamma} \lambda^\Gamma(\rho) &= \infty. \end{aligned}$$

- Let U be the set of all networks with a fixed number of nodes and links, by λ^* the lower envelope of $\{\lambda^\Gamma\}_{\Gamma \in \mathcal{U}}$, i.e.

$$\lambda^*(\rho) \equiv \min_{\Gamma \in \mathcal{U}} \lambda^\Gamma(\rho)$$

with

$$\mathcal{B}^*(\rho) \equiv \arg\min_{\Gamma \in \mathcal{U}} \lambda^\Gamma(\rho).$$

- Our aim is to characterize the topological properties of networks in $B^*(\rho)$. We shall focus on their polarization.

Optimal Networks *(2/9)*

- We first define the *topological betweenness* and denote it by γ_i: It considers minimum distance paths between nodes.

- Now define *polarization:*

$$\theta(\Gamma) = \frac{\max_{i \in N} \gamma_i - \langle \gamma_i \rangle}{\langle \gamma_i \rangle}$$

- For networks associated to a $B^*(\rho)$ denote their polarization $\theta^*(\rho)$.

Optimal Networks *(3/9)*

1. For ρ low, optimality should involve minimizing distance, which is achieved with high polarization: a star network. We expect $\theta^*(\rho)$ to take the highest possible value.

2. As ρ draws close to the maximum $\bar{\rho}_c$, congestion becomes crucial, and optimality should involve a balanced network. $\theta^*(\rho)$ should take the smallest possible value.

- Note that, for low ρ, the performance of Γ can be approximated:

$$\lambda^\Gamma(\rho) = \sum_{i \in N} \frac{\rho \frac{\beta_i^\Gamma}{n-1}}{1 - \rho \frac{\beta_i^\Gamma}{n-1}} \approx \frac{\rho}{n-1} \sum_{i \in N} \beta_i^\Gamma .$$

Therefore, finding the optimal $\Gamma^*(\rho)$ involves minimizing the aggregate betweenness. This, happens for a star-like network.

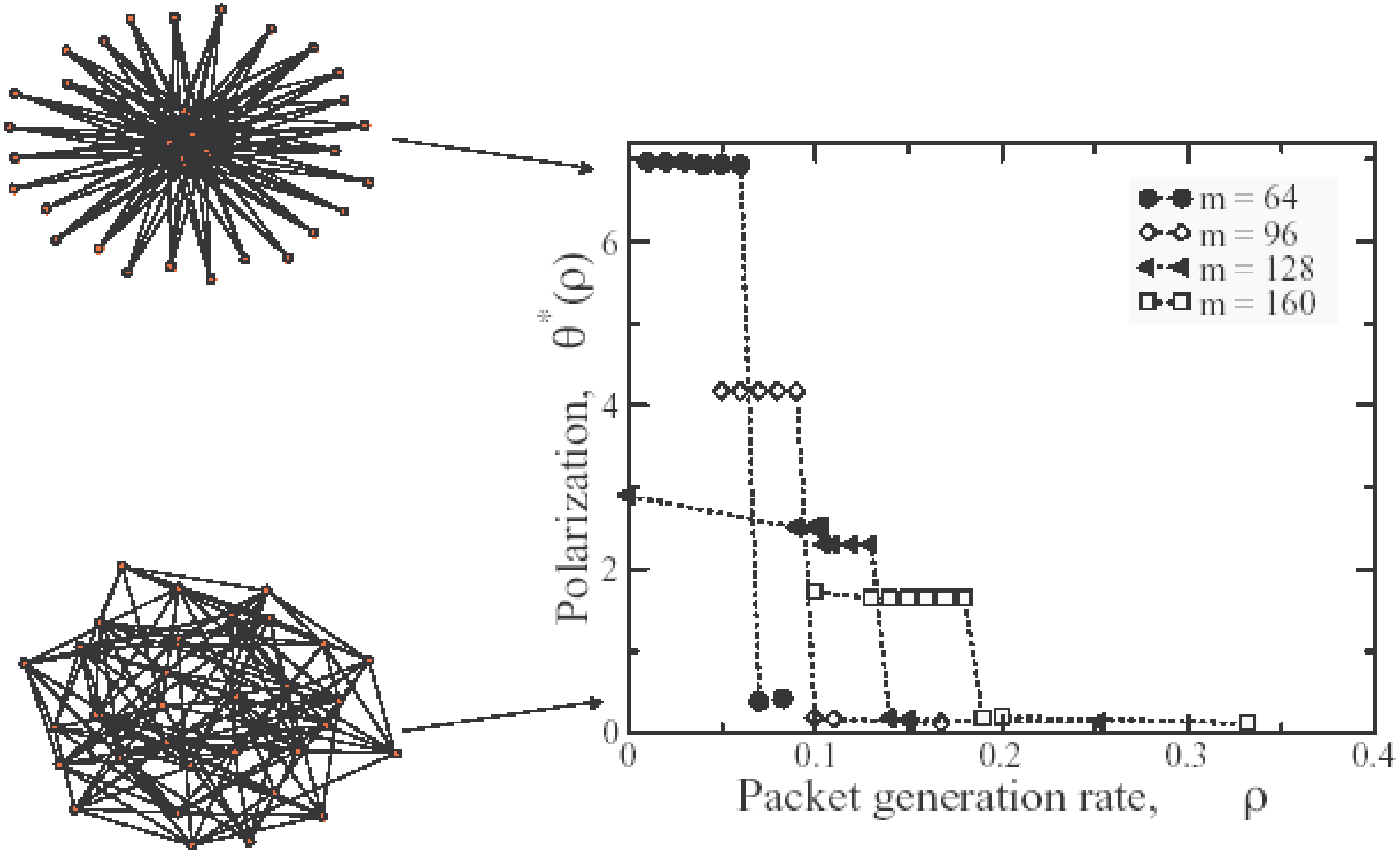
- Instead, for high ρ, we have that, as the stock of floating problems rises its order of magnitude satisfies:

$$\begin{aligned}\lambda^{\Gamma}(\rho) &\sim \mathcal{O}\left(\max_{i\in N} \frac{1}{1-\rho\frac{\beta_i^{\Gamma}}{n-1}}\right)\\ &= \mathcal{O}\left(\frac{1}{1-\frac{\rho}{n-1}\max_{i\in N}\beta_i^{\Gamma}}\right).\end{aligned}$$

  This implies that optimizing $\Gamma^*(\rho)$ involves minimizing the maximal $\beta^* \equiv \max_i \beta_i$.Such a maximal β^* obtains in a homogenous network.

- Confirmed by the simulations.

Optimal Networks *(5/9)*



- Two further interesting features:

  1. First, there is an abrupt (threshold) change between the two extreme topologies (i.e. star-like and symmetric) as ρ varies.

  2. The larger is the number of links, the lower is the threshold for change and the larger the magnitude of this change.

EXPLANATION

- Optimize over vector of betweenness:

$$\min_{\beta} \sum_{i \in N} \frac{\rho \frac{\beta_i}{n-1}}{1 - \rho \frac{\beta_i}{n-1}}$$

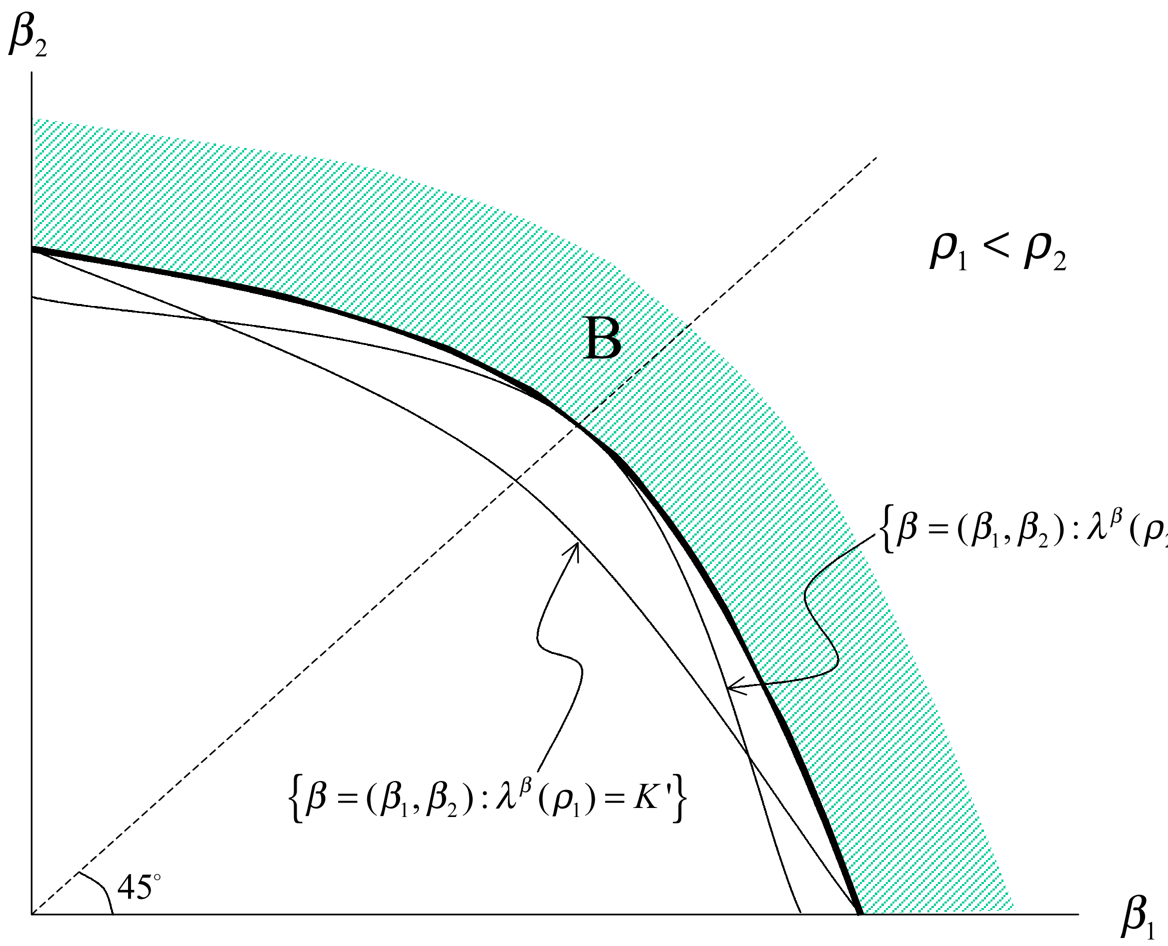
subject to

$$(\beta_1, \beta_2 \ldots, \beta_n) \in H$$

where H is the feasible set.

- Symmetry forces homogeneous (interior) vector of β in a concave problem.

- But objective function is convex and H does not depend on ρ


β_2
$\rho_1 < \rho_2$
B
$\{\beta = (\beta_1, \beta_2) : \lambda^{\beta}(\rho_2) = K\}$
$\{\beta = (\beta_1, \beta_2) : \lambda^{\beta}(\rho_1) = K'\}$
45°
β_1

Summary and extensions *(1/2)*

- We propose an abstract model of a problem solving organization which:

  1. Operates through local communication,

  2. Is forced to search restricted by local information

  3. Is subject to the effects of congestion.

- We provide an analytical characterization of the threshold of collapse and the stock of floating problems and we then find the network which optimizes performance.

Summary and extensions *(2/2)*

- A number of extensions could be explored. One is effect of a larger "information radius":

  1. Concerning the analytical approach used to characterize the collapse threshold and average delay, may be applied unchanged for any information radius.

  2. The optimal network becomes less polarized as the information radius expands.

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