Networks - Fall 2005

Chapter 1

Network formation

October 25, 2005

Summary

WHAT IS A NETWORK? *(1/3)*

- A *collection of "entities" (nodes) and bilateral relationships (links).*

The *links/relationships* can be:

Directed : Not necessarily reciprocal.

Undirected : Always reciprocal.

Weighted : Some links are more "equal" than others.

Stochastic : The links are realized with some probability.

WHAT IS A NETWORK? *(2/3)*

Two crucial characteristics of networks:

A : Interactions are not anonymous (as opposed to standard "market" transactions.)

B : The particular place agents occupy in the set of relationships is important.

WHAT IS A NETWORK? *(3/3)*

Network does potentially two things:

1. Production $\Longrightarrow$ Efficiency.

2. Allocation$\Longrightarrow$ Stability.

The interaction between the two produces a tension for network formation.

Q1 Which is the efficient productive network?

Q2 What is the stable network?

Q3 Are efficient networks stable and vice versa?

JACKSON-WOLINSKY MODEL(S) *(1/4)*

THE GENERAL MODEL

Let $N = \{1, 2, ..., n\}$ be the set of all individual nodes.

We denote by ij a potential link between players i, $j \in N$.

A graph g is a collection of *undirected* links ij.

We assume $ii \notin g$.

Let $N(g) = \{j \in N : \exists ij \in g\}$, and $n(g)$ the cardinality of $N(g)$.

Let $N_i(g) = \{j \in N : ij \in g\}$, and $n_i(g)$ the cardinality of $N_i(g)$.

Payoff functions for each player: $u_i : g \to \Re$.

Distance: We denote by $d_{ij}(g)$ the shortest (geodesic) distance between i and j in g.

Components: The graph $g' \subset g$ is a *component* of g if for all $i, j \in N(g')$ $(i \neq j)$, there exists a path in g' connecting i and j, and for any $i \in N(g')$, $j \in N(g)$ if $ij \in g$, then $ij \in g'$.

JACKSON-WOLINSKY MODEL(S) *(3/4)*

PARTICULAR MODELS

MODEL 1-CONNECTIONS:

$u_i(g) = \sum_{j \notin i} \delta^{d_{ij}(g)} - c \cdot n_i(g)$, $0 < \delta < 1$, $c \geq 0$.

- Never detrimental to third parties if two agents creates a link between them (positive externality.)

- Two connections can have different effects on a player.

MODEL 2-CO-AUTHOR:

$$u_i(g) = \sum_{ij\in g}\left[\frac{1}{n_i(g)} + \frac{1}{n_j(g)} + \frac{1}{n_i(g)n_j(g)}\right].$$

$u_i(g) = 0$ if $n_i(g) = 0$.

$$u_i(g) = 1 + \left(1 + \frac{1}{n_i}\right)\sum_{ij\in g}\left[\frac{1}{n_j(g)}\right].$$

Never beneficial to third parties if two agents creates a link between them (negative externality.)

STABILITY AND EFFICIENCY *(1/17)*

- *Efficiency:* Let $W(g) = \sum_{i \in N} u_i(g)$. We say g^* is *efficient* iff $W(g^*) \geq W(g)\ \forall g$.

  Notice that this notion is *utilitarian* not *Paretian.*

- *Stability:* We say that a network g' is *pairwise stable* iff:

  1. $u_i(g') \geq u_i(g' - ij)$ and $u_j(g') \geq u_j(g' - ij)$, $\forall ij \in g$.
  2. $u_i(g' + ij) > u_i(g') \Rightarrow u_j(g' + ij) < u_j(g')$, $\forall ij \notin g$.

- Notice that:
  - Only checks single link deviation.
  - Checks bilateral creation and unilateral cutting.

EFFICIENCY IN CONNECTIONS MODEL

$$u_i(g) = \sum_{j \notin i} \delta^{d_{ij}(g)} - c \cdot n_i(g), 0 < \delta < 1, c \geq 0.$$

1. The *complete* graph is *efficient* if $c < \delta - \delta^2$.

   $\delta - \delta^2$ is *minimum* increased benefit from a new direct link.

   Cost of a direct link c

2. A *star* encompassing N is *efficient* if $\delta - \delta^2 < c < \delta + ((N-2)/2)\delta^2$.

3. *No links* are efficient if $\delta + ((N-2)/2)\delta^2 < c$.

4. Proof of 2+3:

   - Let a component g' with m nodes and k links.
   - Value of direct links is $k(2\delta - 2c)$.
   - Maximum value of indirect links $(m(m-1)/2 - k)2\delta^2$.
   - So $W(g') \leq \overline{W} = k(2\delta - 2c) + (m(m-1) - 2k)\delta^2$.
   - $W(m-star) = (m-1)(2\delta - 2c) + (m-1)(m-2)\delta^2$.
   - Thus $\overline{W} - W(m-star) = (k-(m-1))(2\delta - 2c - 2\delta^2) \leq 0$.
     (since $k \geq m-1$ and $\delta - \delta^2 < c$).
   - Thus every *component* of *efficient* graph must be a star. A star of $m+n$ is more efficient than two separate stars.
   - And $W(star) \geq 0 \Leftrightarrow \delta + \frac{m-2}{2}\delta^2 \geq c$.

STABILITY IN CONNECTIONS MODEL

1. The *complete* graph is *pairwise stable* if $c < \delta - \delta^2$.

   Same reason as before, argument was pairwise.

2. *Pairwise stable* networks are always *fully connected*.

- For a contradiction, assume g has pw-stable subcomponents g', g''.
- Let $ij \in g'$, and $kl \in g''$.
- Then pw-stability of $g' \Rightarrow u_i(g) - u_i(g - ij) \geq 0$.
- But, $u_k(g + kj) - u_k(g) > u_i(g) - u_i(g - ij)$, since any new benefit that i gets from j, k also gets and in addition k gets δ^2 times the benefits of i's connections.
- Similarly, $u_j(g + jk) - u_j(g) > u_l(g) - u_l(g - lk) \geq 0$.
- This contradicts pw-stability since $jk \notin g$.

3. For $\delta - \delta^2 < c < \delta$ star is pw-stable, but not always uniquely so.

- Deleting means losing at least δ and gaining c.
- Adding ij : net gain $\delta - \delta^2$, cost c.
- For $N = 4$, and $\delta - \delta^3 < c < \delta$, the line is also pw-stable.
- For $N = 4$, and $\delta - \delta^3 > c > \delta - \delta^2$, the circle is also pw-stable.

4. For $\delta < c$, any non-empty network is inefficient.

   - For $\delta < c$, connection ij is unprofitable to i if $N_j(g) = i$ (cost to i is c, benefit δ).

   - Star is not stable.

   - For $N = 5$, and $\delta - \delta^4 + \delta^2 - \delta^3 > c$, the circle is pw-stable (deleting one link benefit is $\delta - \delta^4 + \delta^2 - \delta^3$, cost is c; adding one ling benefit is $\delta - \delta^2$, cost is c).

EFFICIENCY IN CO-AUTHOR MODEL

1. For n even, the efficient network is $n/2$ pairs.

$$W(g) = \sum_{i \in N} u_i(g) = \sum_{i:n_i(g)>0} \sum_{ij \in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right]$$

But since $\sum_{i:n_i(g)>0} \sum_{ij \in g} \left[\frac{1}{n_i} \right] \leq n$ (equality only if $n_i > 0$ for all i)

$$W(g) \leq 2n + \sum_{i:n_i(g)>0} \sum_{ij \in g} \left[\frac{1}{n_i n_j} \right]$$

But

$$\sum_{i:n_i(g)>0} \sum_{ij \in g} \left[\frac{1}{n_i n_j}\right] = \sum_{i:n_i(g)>0} \frac{1}{n_i} \sum_{ij \in g} \left[\frac{1}{n_j}\right] \leq n$$

(since $\sum_{ij \in g} \left[1/n_j\right] \leq n_i$) and equality can only be achieved if $n_j = 1$ for all $j \in N$.

STABILITY IN CO-AUTHOR MODEL

1. *Pairwise stable* networks are composed of fully intra-connected components of different sizes.

   Let i and j not linked.

$$u_i(g+ij) = 1 + \left(1 + \frac{1}{n_i+1}\right)\left[\frac{1}{n_j+1} + \sum_{ik \in g} \frac{1}{n_k}\right].$$

   A new link ij is beneficial to i iff:

$$\begin{aligned}
\left(1 + \frac{1}{n_i+1}\right)\frac{1}{n_j+1} &> \left(\frac{1}{n_i} - \frac{1}{n_i+1}\right)\sum_{ik \in g}\frac{1}{n_k} \\
\left(\frac{n_i+2}{n_i+1}\right)\frac{1}{n_j+1} &> \left(\frac{1}{n_i(n_i+1)}\right)\sum_{ik \in g}\frac{1}{n_k} \\
\frac{n_i+2}{n_j+1} &> \frac{1}{n_i}\sum_{ik \in g}\frac{1}{n_k}
\end{aligned}$$

(a) If $n_i = n_j$ i wants j and vice versa.
$\frac{1}{n_i}\sum_{ik\in g}\frac{1}{n_k} \leq 1$ (average of fractions.)
So if $n_i \geq n_j$ linking to j is beneficial for i. When $n_i = n_j$ this is reciprocal.

(b) If $n_h \leq \max\{n_k | ik \in g\}$ then i wants a link to h.

Let j such that $ij \in g$ and $n_j = \max\{n_k | ik \in g\}$.

Case 1 $n_i \geq n_j - 1$

$$\frac{n_i + 2}{n_h + 1} \geq \frac{n_i + 2}{n_j + 1} \geq 1 \begin{cases} \frac{n_i+2}{n_h+1} > 1 \Rightarrow i \text{ wants } h \\ \frac{n_i+2}{n_h+1} = 1 \Rightarrow n_h \geq 2 \Rightarrow n_j \geq 2 \\ \Rightarrow \frac{1}{n_i} \sum_{ik \in g} \frac{1}{n_k} < 1 \Rightarrow i \text{ wants } h \end{cases}$$

Case 2 $n_i < n_j - 1$

$$\frac{n_i+2}{n_h+1} \geq \frac{n_i+2}{n_j+1} = \frac{n_i+1+1}{n_j+1} > \frac{n_i+1}{n_j}$$

Since $ij \in g$ this implies

$$\frac{n_i+1}{n_j} \geq \frac{1}{n_i-1} \sum_{\substack{ik\in g \\ k\neq j}} \frac{1}{n_k} \geq \frac{1}{n_i} \sum_{ik\in g} \frac{1}{n_k}$$

The last inequality holds since the extra term $1/n_j$ is smaller than other in the average. Thus,

$$\frac{n_i+2}{n_h+1} \geq \frac{1}{n_i} \sum_{ik\in g} \frac{1}{n_k}$$

(c) If m is the number of members in one component, and n in the next largest, then $m > n^2$.

Let j in a component and i in the next largest. i does not want j iff:

$$\frac{n_i + 2}{n_j + 1} \leq \frac{1}{n_i} \Rightarrow n_j + 1 \geq (n_i + 2)\, n_i \Rightarrow n_j \geq n_i^2$$

The first inequality is true since all connections of i have n_i connections.

Remark a) implies that all i with maximal n_i have to be inter-linked.

b) implies that if j is linked to one i with maximal n_i, j wants to be linked to all other k with maximal n_k and those with whom they are themselves connected.

So fully intra-connected components at maximum. Then, iterate.

- Evidence of "connectedness" in science in:
    - Newman (2004) PNAS.
    - Goyal, van der Leij, Moraga (2004).
- Seems like over-connected.
- Tension between stability and efficiency is well-captured by pw-stability.
- Positive issues in pw-stability: *Existence*.

EXISTENCE AND PW-STABILITY *(1/5)*

Trading networks

- Set of players $N = \{1, ..., n\}$, players are nodes of a network g.
- Endowments for player i stochastic: $(x_i, y_i) \in \{(1,0),(0,1)\}$ equally likely.
- Production function: $f(x,y) = x \cdot y$.
- Trade is possible between agents i and j if they belong to the same component.
- Let $P = \{i_0, i_1, ..., i_p\} \subset N$, such that $g|_P$ is a component of g.

EXISTENCE AND PW-STABILITY *(2/5)*

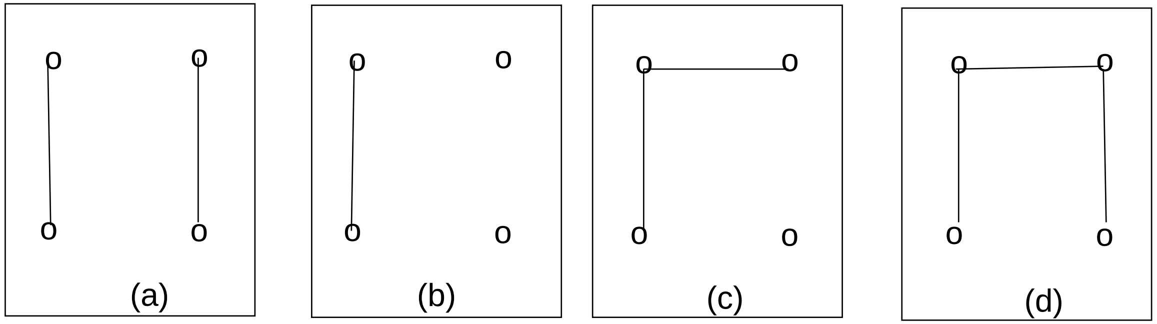
- Trading outcome for a player $i \in P$ is: $\omega_i = \frac{1}{p+1}\left(\sum_{k=0}^{p} x_{i_k}, \sum_{k=0}^{p} y_{i_k}\right)$.

- That is, endowments are aggregated within connected component and shared equally.

- Cost of every link is c.

- Network formation is done *before* endowments are realized (need to use expected payoffs.)

$n = 4$



1.(a) $Eu_i = \frac{1}{2} f\left(\frac{1}{2}, \frac{1}{2}\right) - c = \frac{1}{8} - c$, for all $i \in N$.

(b) $Eu_i = \frac{1}{2} f\left(\frac{1}{2}, \frac{1}{2}\right) - c$ for $i \in \{1, 2\}$ and $Eu_i = 0$ for $i \in \{3, 4\}$.

(c) $Eu_i = \frac{6}{8} f\left(\frac{2}{3}, \frac{1}{3}\right) - c = \frac{1}{6} - c$ for $i \in \{1, 3\}, Eu_i = \frac{1}{6} - 2c$ for $i = 2$, and $Eu_i = 0$ for $i = 4$.

(d) $Eu_i = \frac{8}{16} f\left(\frac{3}{4}, \frac{1}{4}\right) + \frac{6}{16} f\left(\frac{2}{4}, \frac{2}{4}\right) - c = \frac{3}{16} - c$ for $i \in \{1, 4\}$, and $Eu_i = \frac{3}{16} - 2c$ for $i \in \{2, 3\}$.

2. (b) is not stable for $c \leq \frac{1}{8}$ since players 3 and 4 would like to create a link.

3. (a) is not stable for $c \leq \frac{3}{16} - \frac{1}{8} = \frac{1}{16}$ since players 2 and 3 would like to create a link.

4. (d) is not stable for $c \geq \frac{3}{16} - \frac{1}{6} = \frac{1}{48}$ since player 3 would like to delete link 34.

5. (c) is not stable for $c \geq \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$ since player 2 would like to delete link 23.

6. All other configurations are unstable since links are redundant.

These observations together imply that for $\frac{1}{24} \leq c \leq \frac{1}{8}$there is no stable *trading network.*

DYNAMIC STABILITY

- For many parameters/payoff functions (e.g. co-author) there are multiple pw-stable networks.
- In games one approach to decrease multiplicity is evolutionary dynamics.
- In particular - *stochastic stability*
  - Young, or, Kandori, Mailath and Rob, both 1993 Econometrica

MULTIPLICITY AND PW-STABILITY *(2/3)*

- Stochastic process:
  - State variable - past actually played strategies (perhaps time-averaged.)
  - Updating rule/transition probabilities:
    - Best-response (or better-response) to state - with prob. $1 - \varepsilon$.
    - Anything else - with probability ε.
- Stochastic process reaches all states with positive probability.
- Thus, it is ergodic and has a stationary distribution μ^{ε}.
- *Stochastically stable states* are those with positive probability in $\overline{\mu} = \lim_{\varepsilon \to 0} \mu^{\varepsilon}$.

MULTIPLICITY AND PW-STABILITY *(3/3)*

- *Stochastically stable networks*
  - State variable: network g.
  - Updating rule: one-link deviation possibility.
  - Example: co-author model - two pw-stable networks.
  - More mistakes are needed to do one transition than the other.

THE MYERSON GAME *(1/9)*

- Set of players: $N = \{1, ..., n\}$.
- Strategy set: $S_i = \{0, 1\}^{n-1}$.
- Let strategy $s_i = (s_{i1}, s_{i2}, ..., s_{in}) \in S_i$
  - $s_{ij} = 0$ if i does not want to link to j,
  - $s_{ij} = 1$ if i wants to link to j.
- $s = (s_1, ..., s_n) \in S$ is a strategy profile.
- Let $g(s)$ be the network that arises from s.
- For $g(s)$, let $g_{ij}(s) \in \{0, 1\}$ denote the presence of absence of link ij.

- One-sided link formation (directed networks): $g_{ij}(s) = s_{ij}$
- Two-sided link formation (undirected): $g_{ij}(s) = s_{ij} * s_{ji}$.

- Example of one- sided: Bala and Goyal (2000) *Econometrica*.

$$u_i(g) = \sum_{j \notin i} \delta^{d_{ij}(g)} - c \cdot n_i(g), 0 < \delta < 1, c \geq 0.$$

MULTIPLICITY IN MYERSON GAMES: REFINEMENTS

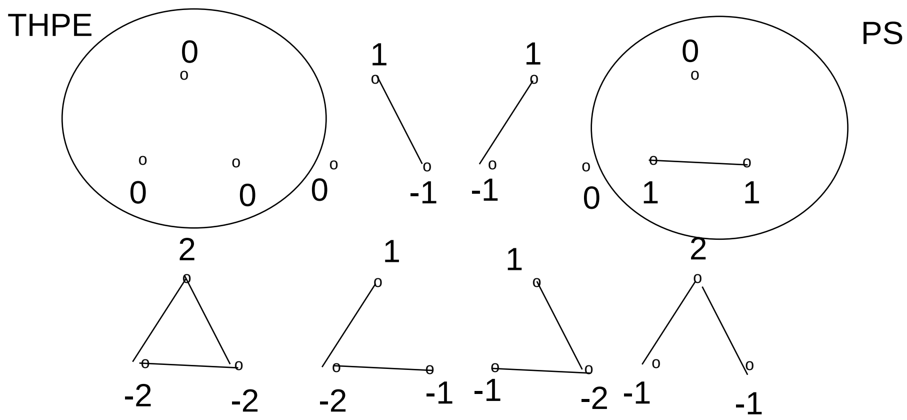
- Let:

$s_1 \backslash s_2$	s_{21}	s_{22}
s_{11}	-2,-2	-2,-2
s_{12}	-2,-2	0,0

- *Trembling-hand perfect equilibrium* (THPE):
  - σ^{ε} is a $\varepsilon-$constrained equilibrium if it is:
    1. Completely mixed.
    2. $\sigma_i^{\varepsilon} \in \arg\max\{u_i(\sigma_i, \sigma_{-i}^{\varepsilon}) | \sigma_i(s_i) \geq \varepsilon(s_i)\}$.
  - σ is a THPE iff $\sigma = \lim_{\varepsilon \to 0} \sigma^{\varepsilon}$ where σ^{ε} is some sequence of $\varepsilon-$constrained eq.

- (s_{11}, s_{21}) in the example is NE but not THPE.
- Unfortunately that is not general.

Claim 1 *THPE does not eliminate all "unwanted" Nash equilibria in the following Example.*



It is easy to see that the null graph is a Nash equilibrium, but not stable. We will now show it is a THPE.

Represent a mixed strategy $\sigma \in \triangle\{0,1\}^2$ as in:

$\sigma_1 =$

	$s_{13} = 0$	$s_{13} = 1$
$s_{12} = 0$	a	b
$s_{12} = 1$	c	$1 - a - b - c$

Then we will check that the following in an $\varepsilon-$ constrained equilibrium (for sufficiently small ε.)

$\sigma_1^\varepsilon =$

	$s_{13} = 0$	$s_{13} = 1$
$s_{12} = 0$	ε	ε
$s_{12} = 1$	ε	$1 - 3\varepsilon$

, $\sigma_2^\varepsilon =$

	$s_{23} = 0$	$s_{23} = 1$
$s_{21} = 0$	$1 - 2\varepsilon^4 - \varepsilon$	ε^4
$s_{21} = 1$	ε^4	ε

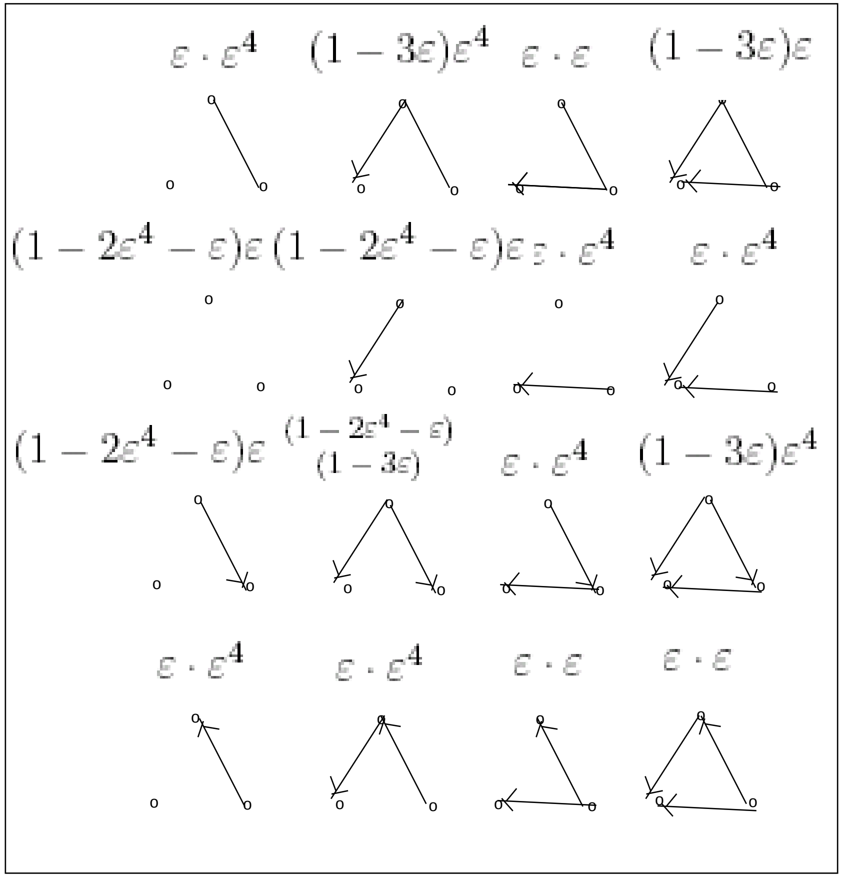
,

$\sigma_2^\varepsilon =$

	$s_{32} = 0$	$s_{32} = 1$
$s_{31} = 0$	$1 - 2\varepsilon^4 - \varepsilon$	ε^4
$s_{31} = 1$	ε^4	ε

- Easy to check σ_1^ε is optimal. Player 1 has a dominant strategy to create as many links as possible.

- Why is σ_2^ε optimal against $\sigma_{-2}^\varepsilon = (\sigma_1^\varepsilon, \sigma_3^\varepsilon)$?



Let $u_2((s_{31} = 0,\ s_{32} = 1), \sigma^{\varepsilon}_{-2})$ and disregard terms of order ε^2 or more. Then

$$u_2((s_{31} = 0, s_{32} = 1), \sigma^{\varepsilon}_{-2}) \approx ((1 - 3\varepsilon)\varepsilon + \varepsilon^2) \cdot (-1) + 2\varepsilon^2 \cdot 1 < 0$$

whereas

$$u_2((s_{31} = 0, s_{32} = 0), \sigma^{\varepsilon}_{-2}) = 0$$

- Notice that it is crucial that the "mistake" of sending links to both 1 and 2 by player 3 is ε, whereas the (less serious) of sending only to 3 is ε^2.

- Thus *proper equilibrium* may be better.

  - σ^{ε} is a $\varepsilon-$proper equilibrium if it is:

  1. Completely mixed.

2. $u_i(s_i, \sigma^{\varepsilon}_{-i}) < u_i(s'_i, \sigma^{\varepsilon}_{-i}) \Rightarrow \sigma_i(s_i) < \varepsilon \cdot \sigma_i(s'_i)\}$.

- σ is a *proper equilibrium* iff $\sigma = \lim_{\varepsilon \to 0} \sigma^{\varepsilon}$ where σ^{ε} is some sequence of $\varepsilon-$proper eq.

- (s_{11}, s_{21}) in the example is NE but not THPE.

- Unfortunately that is not general.

PAIRWISE NASH EQUILIBRIA *(1/3)*

- Let again the (Myerson) network formation game.

- We say that g is *pairwise Nash* iff:
  - g is a Nash equilibrium of the Myerson game.
  - $u_i(g+ij) > u_i(g) \Rightarrow u_j(g+ij) > u_j(g)$.

- This is a Nash equilibrium for which every mutually beneficial link is created.

- A pairwise Nash network is robust to:
  - Bilateral single link creation.
  - Unilateral *multi*-link destruction.

- For the latter reason, this is more demanding than pw-stability.

- Pairwise stability:

$$g \in PS \Rightarrow u_i(g - ij) - u_i(g) \leq 0 \quad \forall i \in N, ij \in g \quad (*).$$

- Pairwise Nash:

$$g \in PN \Rightarrow u_i(g - ij_1 - ij_2 ... - ij_p) - u_i(g) \leq 0 \;\; \forall i \in N, ij_1, ij_2, ..., ij_p \in g \;\; (**).$$

- Obviously $(**) \Rightarrow (*)$. If $(*) \Rightarrow (**)$, then Pairwise stability and Pairwise Nash are equivalent.

- A condition guaranteeing this is $u_i(.)$ being $\alpha-$ *convex*.

PAIRWISE NASH EQUILIBRIA *(3/3)*

- $u_i(.)$ is $\alpha-$ *convex* iff

$$u_i(g - ij_1 - ij_2 ... - ij_l) - u_i(g) \geq \alpha \sum_{k=1}^{p} \left(u_i(g - ij_k) - u_i(g) \right).$$

- To find α take the

$$\min_{g' \subset g} \left\{ u_i(g - ij_1 - ij_2 ... - ij_l) - u_i(g) \right\} / \max_{ij_k} \left\{ u_i(g - ij_k) - u_i(g) \right\}.$$

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Chapter 1

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