Coordination in Teams

Tridib Sharma*
Centro de Investigación Económica, ITAM
Ricard Torres*
Centro de Investigación Económica, ITAM
and Universitat de Girona

November 2001

Abstract

We consider a model of team production with moral hazard à la Alchian-Demsetz-Holmstrom. To credibly implement the collective punishments required to achieve the first best, an owner (residual claimant) is required. This owner brings about coordination through incentives which are embedded in contracts. We observe that, under borrowing or minimum wage constraints, even the existence of an owner may not guarantee the implementability of the first best. Under such conditions, we show that a welfare improving solution can be achieved if an external agent coordinates the activities of the workers by directing them to certain actions and thereby implementing a correlated equilibrium. In doing so, we are able to extend the theory of the firm by formalizing the notion that coordinating through communication enhances the welfare of the firm over and above what can be achieved through contracts.

Keywords: Moral Hazard, Teams, Correlated Equilibrium, Coordination.

---

*E-mail addresses: `sharma@itam.mx`, `rtorres@itam.mx`. We gratefully acknowledge financial support from the *Asociación Mexicana de Cultura*. We thank Preston McAfee for his comments and suggestions.

1 Introduction.

In a classic contribution to management theory, Barnard (1938) asserted that to understand the theory of the firm two crucial factors have to be recognized. First, the activities of various workers in the firm need to be "coordinated". Second, this coordination has to be brought about by an agent whom Barnard called the "executive". A rationale behind these assertions was proposed by Alchian and Demsetz (1972). In the presence of moral hazard (imperfect observability of other people's actions), the authors argued, a group of workers jointly producing a commodity would free ride on each other.[1] As a result, the market mechanism of relative wages would not always lead to efficiency. In such situations, a supervisor could coordinate the activities of the workers to bring forth a Pareto improvement. The rewards and punishments associated with the action of supervision were explicitly formalized by Holmstrom (1982). A fundamental insight of Holmstrom's paper is that workers by themselves cannot credibly provide these incentives. To ensure credibility, an outside agent in the form of a residual claimant (owner) is needed. Thus the Alchian-Demsetz-Holmstrom model of "moral hazard in teams" provides a reason for the existence of a firm comprising owners and workers. We intend to complement this model by providing a rationale for the existence of a *separate* "coordinator" when the firm faces financial constraints.[2] We model financial constraints in terms of minimum wage and borrowing constraints.

We take Holmstrom (1982) as our point of departure. At times, Holmstrom's contracts may have to impose severe wage cuts or very high bonuses. Large wage cuts, which induce starvation for example, may not be credible in modern societies. Similarly, an owner without "deep pockets" may have to borrow to provide the high bonuses. Papers dealing with the problems generated by these issues are surveyed in Allen and Winton (1995). For example, large borrowings might imply a large number of lenders (Jensen and Meckling, 1976; Woodward, 1985) or a large number of owners (Shleifer and Vishny, 1986; Huddart, 1993; Admati, Pfleiderer and Zechner, 1994). In either case monitoring of the firm may be involved. Such group monitoring may then again lead to the problem of moral hazard in teams. We shall abstract from these recursive problems, not because they are not important but because they are not of immediate concern. Instead

[1] The problem of moral hazard was also recognized by Barnard. But he believed that "morality" (in the ethical, rather than behavioral, sense) should greatly alleviate the problems associated with hidden action.

[2] We use the word "coordinator" instead of "manager" due to the fact that a real world manager may do more than coordinate workers.

we shall directly impose minimum wage and borrowing constraints. Under these assumptions we will show that workers may choose to work under an *owner* who enforces contracts, *and* a *coordinator* who simply communicates to the workers the actions they should take. We shall use the well known solution concept of correlated equilibrium to model such communication.

The Alchian-Demsetz-Holmstrom rationalization of Barnard's assumptions is illuminating. However, in their world, once incentives are declared (say through contracts) there is no reason for any agent to *coordinate* the actions of workers. Yet such coordination forms the very essence of Barnard's *The Functions of the Executive.* Barnard's coordination is brought about by communication. The coordinator emanates authority in the sense that her orders are obeyed. This authority is not bought through "material inducements" and is not backed by threats of punishments. He writes, and we quote:

> "Yet it seems to me to be a matter of common experience that material rewards are ineffective beyond the subsistence level excepting to a very limited proportion of men; that most men neither work harder for more material things, nor can be induced thereby to devote more than a fraction of their contribution to organized effort. It is likewise a matter of both present experience and past history that many of the most effective and powerful organizations are built on incentives in which the materialistic elements, above subsistence, are either relatively lacking or absolutely absent. Military organizations have been relatively lacking in material incentives. The greater part of the work of political organizations is without material incentive. Religious organizations are characterized on the whole by material sacrifice. It seems to me to be definitely a general fact that even in purely commercial organizations material incentives are so weak as to be almost negligible except when reinforced by other incentives, and then only because of wholesale general persuasion in the form of salesmanship and advertising." (Barnard, 1938, page 144)

Though we do not fully subscribe to Barnard's views, we do take them seriously. However, a large part of the present literature in economics ignores the fact that coordination may be brought about through communication which is not *directly* contingent on material incentives. At least from the Alchian-Demsetz-Holmstrom perspective, incentives are provided only through "material inducements". An objective of this paper is to show that Barnard's assertions can be partially, but formally, incorporated in the Alchian-Demsetz-Holmstrom model by invoking the concept of correlated equilibrium.

Correlated equilibria imply a certain notion of explicit communication. We show that, given the optimal incentives à la Holmstrom, such communication may indeed lead to Pareto improvements. More importantly, we show that the coordinator's activity *cannot* be performed by either the owner of the firm or by the employed workers. Hence, we complement the Alchian-Demsetz-Holmstrom model by placing a coordinator in their theory of the firm.

Actually, there is a difference between the vision of Alchian and Demsetz, and Holmstrom's implementation of the idea. Alchian and Demsetz were trying to find an economic justification for the structure and workings of (different types of) actual firms. Therefore, they emphasized that under the term *monitoring* they actually wanted to include a wide variety of tasks:

> "We use the term monitor to connote several activities in addition to its disciplinary connotation. It connotes measuring output performance, apportioning rewards, observing the input behavior of inputs as means of detecting or estimating their marginal productivity and giving assignments or instructions in what to do and how to do it. (It also includes [...] authority to terminate contracts.) Perhaps the contrast between a football coach and a team captain is helpful. The coach selects strategies and tactics and sends in instructions about what plays to utilize. The captain is essentially an observer and reporter of the performance at close hand of the members." (Alchian and Demsetz, 1972, page 782)

Holmstrom partially accomplished the task that Alchian and Demsetz set out to do, in the sense that Holmstrom's owner performs the role of the football captain. We complete the task by bringing in the coach who coordinates actions. In particular, we use Holmstrom's approach to illuminate the fact that, as in the football team, the tasks of monitoring (in a narrow sense) and coordinating must be performed by different people.

To our knowledge, the importance of coordination has been emphasized but not formally modeled. Amongst others Chandler (1977) talks of the importance of managerial coordination in the evolution of firms in the US. Milgrom and Roberts (1992) list several examples to highlight the role of coordination. Segal (2001) is an exception. He explicitly models language as a coordinating device. The concern there is with the cost and complexity of language used for communicating instructions. We abstract from this important issue of complexity. Language, for us, is a set of costless and universally understood instructions. Our objective is to highlight the importance of such instructions over and above contracts in a world where the only friction arises due to moral hazard.

The rest of the paper is organized as follows. In Section 2 we provide a formal but very simple model. We then formalize the notion of implementing actions by means of both Nash and correlated equilibria. In Section 3 we fully characterize all correlated equilibria that result from a symmetric wage profile. Section 4 is devoted to study the implementation problem. We begin by considering Holmstrom's case, and then show that the first best can no longer be implemented as a Nash equilibrium whenever minimum wages or borrowing constraints are present. We then show that, if those constraints prevent the implementation of the first best, but the second best can still be implemented as a Nash equilibrium, then one can do better by resorting to correlated equilibria. We then show that the task of implementing the correlated equilibrium cannot be carried out by either the owner, or the workers. A separate agent is required. In Section 5, given the multiplicity of implementable correlated equilibria, we compute the optimal correlated equilibrium from the welfare viewpoint. Finally, in Section 6 we present the conclusions. Most of the proofs have been transferred to the appendix.

2 The Basic Model

We bring forth the role of the coordinator in the simplest possible model. We do so for two reasons. First, extending a simple model to richer environments is easier. Second, by economizing on the number of parameters we are able to completely characterize the set of possible results in a compact manner. In setting up the model and interpreting the results, we will follow Holmstrom (1982) rather than Alchian and Demsetz (1972). We ignore the subtle differences between the two approaches. These differences are highlighted in Mookherjee (1984). Our discrete model is a simplification of Mookherjee (1984).

There are 2 agents (workers), indexed by $k \in \{1, 2\}$. The choice (action) sets of the agents are $A_1 = A_2 = \{a_1, a_2\}$. Let $A = A_1 \times A_2$ be the set of action profiles, with a as an element. Actions are simultaneously chosen by the agents, and their joint actions result in a random output (revenue) $\tilde{q} \in \{q_1, q_2\} \equiv Q$, where $0 < q_1 < q_2$. The idea is that given an action profile $a \in A$, the output level depends also on a random variable ξ. We can interpret ξ as the vagaries induced by Nature, or the unmodeled larger economy. Accordingly, the production function is

$f(a, \xi) \in Q$. For any action profile $a \in A$, f and the distribution of ξ determine the distribution of $\tilde{q}$ over Q. Let $0 < \pi_{ij} < 1$ be the probability of output level q_1 being obtained, whenever actions (a_i, a_j) are taken by the two players. Thus, the expression $\pi_{ij} q_1 + (1 - \pi_{ij}) q_2$ is the expected output given action profile (a_i, a_j). Actions are costly for the agents. If agent k executes action $a_k \in A_k$, her disutility from working is denoted by $G(a_k)$; note that G has no subscripts, because we are assuming for simplicity that the disutility is the same for both agents. We assume that $0 < G(a_1) < G(a_2)$. The net surplus (welfare function) generated by the agents, when action profile (a_i, a_j) is chosen, is

$$S(a_i, a_j) = \pi_{ij}\, q_1 + (1 - \pi_{ij})\, q_2 - [G(a_i) + G(a_j)]$$

We assume that production is symmetric with respect to the agents.

Assumption 2.1 $S(a_2, a_2) > S(a_1, a_2) = S(a_2, a_1) > S(a_1, a_1) \geq 0$

Together with our initial assumption about interior probabilities, this assumption on net surplus implies that:

Fact 2.1 $1 > \pi_{11} > \pi_{12} = \pi_{21} > \pi_{22} > 0$

Given Assumption 2.1, it is desirable that each agent choose action a_2. To understand how each agent chooses an action, we need to write down their payoffs. The agents are paid wages as a compensation for their actions. Let $w^k = (w_1^k, w_2^k)$ denote a profile of monetary payments for agent k, where w_l^k is the payment of k when the output is q_l. The agents are assumed to be risk neutral (not a necessary assumption for our model) with separable utility. A *wage profile* for the firm (for short, a wage profile without any qualifiers) is a vector $w = (w^1, w^2)$, where w^k is a wage profile for agent k. Given a wage profile w and an action profile (a_i, a_j), the agents' payoffs (utilities) are given by

$$\begin{aligned} u_1(a_i, a_j, w) &= \pi_{ij}\, w_1^1 + (1 - \pi_{ij})\, w_2^1 - G(a_i) \\ u_2(a_i, a_j, w) &= \pi_{ij}\, w_1^2 + (1 - \pi_{ij})\, w_2^2 - G(a_j) \end{aligned} \qquad (1)$$

Thus, a wage profile w induces a normal form game Γ_w among the agents, with the payoffs given by equations 1.

2.1 Implementing Actions in Γ_w.

A solution of Γ_w implies a choice of actions by both agents. The most often used solution concept is that of a Nash equilibrium. Under this solution concept, agents independently choose strategies that are best responses to one another's choice. Note that, as we change the wage profile w, we change the game and hence possibly the Nash equilibrium outcomes. Therefore, it may be possible to influence the agents' equilibrium choice of actions by appropriately choosing the wage profile. As mentioned earlier, one is interested in designing a wage profile w that implements the action profile (a_2, a_2) as an equilibrium outcome. This, in fact, is Holmstrom's exercise. More generally, w can be chosen to implement any action profile (a_i, a_j). Formally:

Definition 2.2 *We say that the wage profile w* Nash implements $(a_i, a_j) \in A$ *if, and only if, the following conditions hold:*

(i) (a_i, a_j) *is a Nash Equilibrium of the game Γ_w.*

(ii) The firm's surplus is nonnegative:

$$\pi_{ij}\,(q_1 - w_1^1 - w_1^2) + (1 - \pi_{ij})\,(q_2 - w_2^1 - w_2^2) \geq 0$$

(iii) The wage profile is individually rational for the players:

$$u_k(a_i, a_j, w) \geq 0, \quad k = 1, 2$$

Condition (i) requires (a_i, a_j) to be a Nash equilibrium of the game Γ_w. Given w, we say that (a_i, a_j) is a (pure-strategy) Nash equilibrium if agent 1 cannot become strictly better off by unilaterally deviating from a_i, and agent 2 cannot become strictly better off by unilaterally deviating from a_j. Condition (ii) restricts the allowable wage profiles to those for which the expected wages do not exceed the expected output, that is, the firm does not operate at a loss. Condition (iii) requires that the workers' expected utility from the project be at least as large as their reservation utility, which is normalized to zero. This constraint imposes no bounds on the state contingent wages. For example, if starvation wages were to be denoted by some number M, then for any $w_1^k < M$ one could still satisfy Condition (iii) by making w_2^k sufficiently large. Also notice that here we are only considering pure-strategy Nash implementation; we will see later on that, in this context, there is no gain in considering mixed strategies.

While it is true that the concept of Nash equilibrium is popular amongst economists, it is perhaps too strong a solution concept to be applied to many types of social interaction. Nash equilibrium requires players to independently choose their strategies. If we think about our model as but one process embedded in a much larger economic context where players interact in other processes also, then correlation of choice is a consideration that one should perhaps not ignore. A solution concept which captures this idea is Aumann's notion of correlated equilibrium. The case in favor of correlated equilibrium has been made by many authors (among them see Aumann, 1974, 1987; Myerson, 1991), and we do not want to repeat their arguments. But one rationale stands out and deserves mention. An appealing feature of correlated equilibria is that boundedly rational players, with simple and intuitive behavioral rules, converge to such equilibria in strategic form games (Foster and Vohra, 1997; Fudenberg and Levine, 1999; Hart and Mas-Colell, 2000).

Correlated equilibria form a superset of Nash equilibria. A simple explanation of correlated equilibria is provided by Hart and Mas-Colell (2000). Suppose that, before playing the game, players receive private signals that do not affect payoffs. Players *may* then choose their actions based on these signals. A correlated equilibrium of the original game is then a Nash equilibrium of the game with signals. If the signals are independent across players then the equilibrium strategies coincide in the two cases. Otherwise, correlated equilibria form a larger set.[3] While it is true that convex combinations of Nash equilibria are always correlated equilibria, Aumann (1974) observes that there may be correlated equilibrium payoffs that are outside of the convex hull of the Nash equilibrium payoffs; as an example, Aumann shows this to be the case in the game of chicken. Therefore, one might think that there are situations in which correlated equilibria allow the implementation of better outcomes than Nash equilibria.[4]

More formally, let θ denote a probability distribution over pure strategy profiles $a \in A$. Given any action a_i, let $\theta(a_i, \cdot)$ denote the marginal probability that player 1 chooses a_i, i.e.

$$\theta(a_i, \cdot) := \theta(a_i, a_1) + \theta(a_i, a_2),$$

[3]There is a one-to-one correspondence between two independently chosen mixed strategies and the correlated distribution which is the product of these two mixed strategies. In this sense, we can compare both sets of equilibria and affirm that correlated equilibria form a larger set.

[4]Not much is known about the nature of the set of correlated equilibria which induce payoffs outside the convex hull of Nash equilibrium payoffs (Fudenberg and Tirole, 1991, page 58). Our setting provides a motivation to pursue this exercise in a natural setting. We carry out this exercise in Sections 4 and 5.

and let $\theta(\cdot, a_i)$ denote the marginal probability that player 2 chooses a_i:

$$\theta(\cdot, a_i) := \theta(a_1, a_i) + \theta(a_2, a_i),$$

Given any two actions a_i and a_j, define also the following completed conditional probabilities

$$\theta^1(a_j|a_i) = \begin{cases} \dfrac{\theta(a_i, a_j)}{\theta(a_i, \cdot)}, & \text{if } \theta(a_i, \cdot) > 0; \\ 0, & \text{if } \theta(a_i, \cdot) = 0. \end{cases}$$

$$\theta^2(a_j|a_i) = \begin{cases} \dfrac{\theta(a_j, a_i)}{\theta(\cdot, a_i)}, & \text{if } \theta(\cdot, a_i) > 0; \\ 0, & \text{if } \theta(\cdot, a_i) = 0. \end{cases}$$

Suppose both players are playing according to the correlated distribution θ, but each player only knows the action she is supposed to play (we will later discuss how this could be implemented). Then $\theta^k(a_j|a_i)$ is the (conditional) probability that player k attaches to action a_j being chosen by the other player, whenever k is playing action a_i. It is convenient to complete the definition of these conditional probabilities by setting them to zero whenever player k never plays a_i according to the distribution θ. It is clear that this completion does not affect our interpretation of the conditional probabilities.

Definition 2.3 *A correlated equilibrium is a probability distribution θ over pure strategy profiles such that, for each two pure strategies a_i and a_j,*

$$\begin{aligned} \theta^1(a_1|a_i)\, u_1(a_i, a_1, w) + \; & \theta^1(a_2|a_i)\, u_1(a_i, a_2, w) \geq \\ & \theta^1(a_1|a_i)\, u_1(a_j, a_1, w) + \theta^1(a_2|a_i)\, u_1(a_j, a_2, w) \\ \theta^2(a_1|a_i)\, u_2(a_1, a_i, w) + \; & \theta^2(a_2|a_i)\, u_2(a_2, a_i, w) \geq \\ & \theta^2(a_1|a_i)\, u_2(a_1, a_j, w) + \theta^2(a_2|a_i)\, u_2(a_2, a_j, w) \end{aligned}$$

Suppose that both players are playing according to the distribution θ, but that, whenever a player is called to choose an action, she does not know exactly which action the other player is going to choose. If player k is called to choose action a_i, her estimate is that the other player has probability $\theta^k(a_l|a_i)$ of choosing a certain action a_l. In this case, the definition tells us that, if θ is a correlated equilibrium, player k should prefer a_i to another choice of action a_j.

Notice also that, whenever $\theta(a_i, \cdot)$ is positive, we can multiply through by it in the definition of correlated equilibrium to get

$$\begin{aligned} \theta(a_i, a_1)\, u_1(a_i, a_1, w) \; + \theta(a_i, a_2)\, u_1(a_i, a_2, w) \geq \\ \theta(a_i, a_1)\, u_1(a_j, a_1, w) + \theta(a_i, a_2)\, u_1(a_j, a_2, w) \end{aligned} \tag{2}$$

Similarly, if $\theta(\cdot, a_i)$ is positive, we multiply through by it to get

$$\begin{aligned} \theta(a_1, a_i)\, u_2(a_1, a_i, w) \ + \theta(a_2, a_i)\, u_2(a_2, a_i, w) \geq \\ \theta(a_1, a_i)\, u_2(a_1, a_j, w) + \theta(a_2, a_i)\, u_2(a_2, a_j, w) \end{aligned} \tag{3}$$

Notice also that the two previous inequalities hold trivially whenever the marginal probabilities are zero. Therefore, we can equivalently express the inequalities defining a correlated equilibrium without conditional probabilities. Actually, this way is more convenient and will be used in the sequel.

Implementing by means of correlated equilibria is similar to doing it with Nash equilibria.

Definition 2.4 *We say that the wage profile w* implements the correlated distribution θ *if, and only if, the following conditions hold:*

(i) θ *is a correlated equilibrium of the game* Γ_w.

(ii) The firm's surplus is nonnegative:

$$\sum_{(a_i,a_j)\in A} \theta(a_i, a_j) \left[\pi_{ij}\,(q_1 - w_1^1 - w_1^2) + (1-\pi_{ij})\,(q_2 - w_2^1 - w_2^2)\right] \geq 0$$

(iii) The wage profile is individually rational for the players:

$$\sum_{(a_i,a_j)\in A} \theta(a_i, a_j)\, u_k(a_i, a_j, w) \geq 0, \quad k = 1, 2$$

We shall see that implementing correlated equilibria in the sense defined above represents no improvement with respect to Nash equilibrium implementation. However, when additional feasibility constraints are imposed, correlated equilibrium implementation will, in general, improve welfare with respect to Nash equilibrium implementation.

A few words about implementing correlated equilibria are in order. As we commented before, correlated equilibria can be viewed as Nash equilibria in a game where players receive signals before choosing their actions, and where the distribution generating those signals is common knowledge. When we consider the problem of implementation, the natural way of generating those signals is by means of a *trusted mediator* who runs a randomizing device according to θ and then tells each player which action she is supposed to play (see Myerson, 1991, chapter 6). The requirements for the distribution to be a correlated equilibrium can then be interpreted as incentive compatibility constraints, in the sense that each player should prefer in each case to choose the action she has been told to play, given the information she has about the overall distribution. We can see that it is essential that both players believe that the mediator will act as prescribed. We will later deal about this key point, but for the moment let us accept it is so.

2.2 Additional Notation and Definitions.

We establish some new notation to simplify our exposition. Let θ_1, θ_2, θ_3, and θ_4 denote the respective probabilities that that the correlated strategies place on the pure strategy combinations (a_1, a_1), (a_1, a_2), (a_2, a_1), and (a_2, a_2).

Since the set up we are considering is perfectly symmetric with respect to the players, there will be no loss of generality in assuming that the total wages paid to the workers are equally split between them. The only problem might arise whenever one tries to implement asymmetric outcomes, as are the action combinations (a_1, a_2) and (a_2, a_1), because then symmetric wages might cause the individual rationality constraint to fail for the player with the high effort level. However, since we are in the world of correlated equilibria, we can easily overcome this difficulty: if both action combinations are Nash equilibria, then the correlated strategy which places equal weight on each Nash equilibrium outcome is a correlated equilibrium which is symmetric for the players. This justifies our claim that restricting our consideration to symmetric wages will represent no loss of generality. As a matter of fact, we may consider only correlated equilibria in which $\theta_2 = \theta_3$, though we will not do so when we describe the set of all equilibria. We formalize the previous discussion in the following definitions.

Definition 2.5 *(a) By w_1 and w_2 we denote the* total wages *paid to the workers in each state, that is, $w_i = w_i^1 + w_i^2$, for $i = 1, 2$. Let $\delta := w_2 - w_1$ be the difference in total wages in the two states.*

(b) We say that a wage profile $w = (w^1, w^2)$ is symmetric *if total wages are equally split between the agents, that is, $w_i^1 = w_i^2 = w_i/2$, for $i = 1, 2$.*

(c) Given a symmetric wage profile w, we say that a correlated distribution θ is symmetric *if $\theta_2 = \theta_3$. If the distribution is an equilibrium, we will refer to it as a* symmetric correlated equilibrium.

Notice that any symmetric correlated distribution results in the same (expected) payoffs for both players. We will see that the relevant decision parameter that determines the different kinds of equilibria in the symmetric case is δ, the difference between total wages in the two states.

We will also use the following notation:

$$\begin{aligned}
\alpha &:= \pi_{11} - \pi_{12} \\
\beta &:= \pi_{21} - \pi_{22} \\
\gamma &:= G(a_2) - G(a_1) \\
\sigma &:= q_2 - q_1
\end{aligned}$$

We will see that equilibria depend on those differences, rather than on the absolute levels. Notice that, except for δ, which is a choice variable for implementation, our assumptions imply that the parameters we have just defined are strictly positive. Also, Assumption 2.1 implies the following, which we keep for further reference.

Fact 2.2 *The following inequalities hold:*

(i) $(\pi_{11} - \pi_{12})(q_2 - q_1) > G(a_2) - G(a_1)$, *i.e.* $\alpha\,\sigma > \gamma$.

(ii) $(\pi_{21} - \pi_{22})(q_2 - q_1) > G(a_2) - G(a_1)$, *i.e.* $\beta\,\sigma > \gamma$.

3 A Characterization of Correlated Equilibria.

Before discussing issues related to feasibility, we would like to characterize the set of correlated equilibria for a given symmetric wage profile and the set of parameters α, β, γ, and σ. Using the simplification we justified in obtaining equations 2 and 3, we can express the inequalities defining correlated equilibria using the absolute probabilities instead of the conditionals and, after a little algebra, regroup terms in the inequalities and simplify them using the parameter differences we defined before. All the correlated equilibria that correspond to a symmetric wage profile w are given by:

$$\begin{cases}
(\alpha\,\delta - 2\,\gamma)\,\theta_1 + (\beta\,\delta - 2\,\gamma)\,\theta_2 \leq 0 \\
(\alpha\,\delta - 2\,\gamma)\,\theta_3 + (\beta\,\delta - 2\,\gamma)\,\theta_4 \geq 0 \\
(\alpha\,\delta - 2\,\gamma)\,\theta_1 + (\beta\,\delta - 2\,\gamma)\,\theta_3 \leq 0 \\
(\alpha\,\delta - 2\,\gamma)\,\theta_2 + (\beta\,\delta - 2\,\gamma)\,\theta_4 \geq 0 \\
\theta_1, \theta_2, \theta_3, \theta_4 \geq 0 \\
\theta_1 + \theta_2 + \theta_3 + \theta_4 = 1
\end{cases} \tag{4}$$

The set of correlated equilibria is a convex and closed set, because it is defined by a system of linear inequalities. The Nash equilibria are correlated equilibria where the probability distribution is a product probability, i.e., corresponds to mixed strategies chosen independently by the players. In this sense, by analyzing the correlated equilibria we are also able to characterize Nash equilibria.

In Table 1,[5] we give a full characterization of all correlated equilibria that correspond to different signs of the coefficients, given a symmetric wage profile w.

	$b < 0$	$b = 0$	$b > 0$
$a < 0$	$\theta_1 = 1$	$\theta_2 = \theta_3 = 0$	$\max\{\theta_2, \theta_3\} \leq \min\left\{\left\|\frac{a}{b}\right\| \theta_1, \left\|\frac{b}{a}\right\| \theta_4\right\}$
$a = 0$	$\theta_4 = 0$	No restrictions	$\theta_2 = \theta_3 = 0$
$a > 0$	$\min\{\theta_2, \theta_3\} \geq \max\left\{\left\|\frac{a}{b}\right\| \theta_1, \left\|\frac{b}{a}\right\| \theta_4\right\}$	$\theta_1 = 0$	$\theta_4 = 1$

Table 1: All correlated equilibria, given a symmetric wage profile.

It is clear from Table 1 that not all distributions are correlated equilibria. For example, if $\alpha < \beta$ then no distribution where $\theta_2 = 1$ can be a correlated equilibrium. The argument in support is simple. Since $\alpha < \beta$, we have to be in either the first row or the third column of Table 1. Note that none of the required restrictions can be satisfied with $\theta_2 = 1$.

Similarly, we can look at the sets of parameter values that are compatible with different pure-strategy Nash equilibria. The *first best* is the Nash equilibrium that maximizes net surplus. Because of Assumption 2.1, this equilibrium is the one where both agents choose action a_2, that is, corresponds to the (symmetric) correlated distribution $\theta_4 = 1$. In table 1 we see that such a Nash equilibrium exists if, and only if, $\delta \geq (2\gamma)/\beta$.

[5] Where, in order to simplify the notation, we have used $a := \alpha\,\delta - 2\,\gamma$ and $b := \beta\,\delta - 2\,\gamma$.

On the other hand, Assumption 2.1 implies that the worst (from the net surplus viewpoint) Nash equilibrium occurs whenever both players choose action a_1, that is, it corresponds to the (symmetric) correlated distribution $\theta_1 = 1$. This equilibrium will exist if, and only if, $\delta \leq (2\gamma)/\alpha$.

Finally, there may be pure-strategy Nash equilibria where both players choose different actions. These equilibria either have the distribution $\theta_3 = 1$, or the distribution $\theta_2 = 1$. In either case, the equilibrium exists if, and only if, $(2\gamma)/\beta \geq \delta \geq (2\gamma)/\alpha$. Note, in particular, that such an equilibrium will exist only if $\alpha \geq \beta$. The conditions for existence of these equilibria are exactly the same that guarantee the existence of any convex combination of them, in particular of the *symmetric* correlated equilibrium that places probability 1/2 on each of the two action combinations: $\theta_2 = \theta_3 = 1/2$.

So far we have simply considered the existence of equilibrium for a given set of parameters and δ. To implement a certain correlated equilibrium we have to choose a δ such that the equilibrium exists and the constraints are satisfied. We now turn to this task.

4 Implementing Correlated Equilibria

Let us define and express a few concepts which will be of use.

Definition 4.1 *Let w be a wage profile and θ a correlated distribution of the game Γ_w. Then:*

(a) The firm's surplus *is defined as:*

$$FS(\theta, w) = \sum_{(a_i,a_j)\in A} \theta(a_i, a_j) \left[\pi_{ij} (q_1 - w_1) + (1 - \pi_{ij}) (q_2 - w_2)\right]$$

(b) The total utility *of the agents is defined as:*

$$TU(\theta, w) = \sum_{(a_i,a_j)\in A} \theta(a_i, a_j) [u_1(a_i, a_j, w) + u_2(a_i, a_j, w)]$$

(c) The welfare function *is defined as:*

$$W(\theta, w) = FS(\theta, w) + TU(\theta, w) = \sum_{(a_i,a_j)\in A} \theta(a_i, a_j) S(a_i, a_j)$$

Simplifying the above expressions, and taking into account our assumption that $\pi_{12} = \pi_{21}$, we obtain the expressions that follow.

Fact 4.1 *The following expressions are equivalent to the given definitions:*

(a) $FS(\theta, w) = \pi_{12}\,(q_1 - w_1) + (1 - \pi_{12})\,(q_2 - w_2) + (\beta\,\theta_4 - \alpha\,\theta_1)\,(\sigma - \delta)$.

(b) $TU(\theta, w) = \pi_{12}\,w_1 + (1 - \pi_{12})\,w_2 - G(a_1) - G(a_2) - (\alpha\,\delta - \gamma)\,\theta_1 + (\beta\,\delta - \gamma)\,\theta_4$.

(c) $W(\theta, w) = S(a_1, a_2) - (\alpha\,\sigma - \gamma)\,\theta_1 + (\beta\,\sigma - \gamma)\,\theta_4$.

We remind the reader that in Fact 2.2 we have shown that the coefficients that multiply θ_1 and θ_4 in the expression for the welfare function are both positive, which shows that θ_1 affects the welfare negatively and θ_4 positively.

Notice also that, since we concentrate on symmetric wage profiles, the individual rationality requirements of Definition 2.3 are equivalent to requiring that $TU(\theta, w) \geq 0$, as long as we restrict our consideration to symmetric correlated equilibria ($\theta_2 = \theta_3$).

Using Fact 4.1, the nonnegativity of the firm's surplus and total utility can be expressed, respectively, as:

$$\begin{aligned} \pi_{12}\,q_1 + (1 - \pi_{12})\,q_2 + (\beta\,\theta_4 - \alpha\,\theta_1)\,(\sigma - \delta) &\geq \pi_{12}\,w_1 + (1 - \pi_{12})\,w_2 \\ \pi_{12}\,w_1 + (1 - \pi_{12})\,w_2 \geq G(a_1) + G(a_2) + (\alpha\,\delta - \gamma)\,\theta_1 &- (\beta\,\delta - \gamma)\,\theta_4 \end{aligned} \tag{5}$$

If we were to combine both inequalities then the resulting inequality would simply state that welfare is nonnegative. But since our primary concern is with implementation, we choose to work with these two inequalities.

The algebraic representation of correlated equilibrium implementation, through 4 and 5 then imply:

Proposition 4.2 *Let α, β, γ, and σ be given. A symmetric wage profile w implements a symmetric distribution θ as a correlated equilibrium if and only if w and θ satisfy 4 and 5.*

Proposition 4.2 identifies the symmetric wage profiles w needed to implement a given set of action profiles as a support of some symmetric correlated equilibrium θ. We shall use this general characterization to focus on some specific issues. We start with Holmstrom's implementation of the first best as a Nash equilibrium.

4.1 Holmstrom's case.

Holmstrom's main concern is whether the two workers can design a wage profile w to implement (a_2, a_2) as a Nash equilibrium. Recall that the total surplus associated with (a_2, a_2) is $S(a_2, a_2)$. Since $S(a_2, a_2)$ is the maximum possible welfare (net surplus) we shall refer to (a_2, a_2) as the *first best*. As noted above, the first best is a Nash equilibrium if, and only if, $\delta \geq (2\gamma)/\beta$. In other words, to ensure existence we need to construct a w such that $\delta := w_2 - w_1$ is large enough. For Nash implementation, however, we need to satisfy 5. Given $x \geq 0$, let

$$w_2 = q_2 + x \quad \text{and} \quad w_1 = q_1 - \frac{1 - \pi_{22}}{\pi_{22}}\, x$$

Notice that, if both workers were to choose the action profile (a_2, a_2), then all output would be distributed as wages in expected terms,[6] as

$$\begin{aligned} &\pi_{22}\, w_1 + (1 - \pi_{22})\, w_2 = \\ &\pi_{22}\, q_1 + (1 - \pi_{22})\, q_2 - \pi_{22}\, \frac{1 - \pi_{22}}{\pi_{22}}\, x + (1 - \pi_{22})\, x = \\ &\pi_{22}\, q_1 + (1 - \pi_{22})\, q_2 \end{aligned}$$

The wage difference that corresponds to this wage profile is:

$$\delta = w_2 - w_1 = q_2 - q_1 + x + \frac{1 - \pi_{22}}{\pi_{22}}\, x = \sigma + \frac{x}{\pi_{22}}$$

Therefore, we can take x large enough so that $\delta \geq (2\gamma)/\beta$. We now claim that, for any $x \geq 0$, both the feasibility constraints given by nonnegativity of the firm's surplus and of the total utility are satisfied. Actually, since all the production is distributed as wages, it follows that the firm's surplus is zero. On the other hand, it is just an algebraic exercise to verify that the total utility inequality can be reduced to $S(a_2, a_2) \geq 0$, which we know is satisfied because of Assumption 2.1. Thus, one of Holmstrom's results is:

Proposition 4.3 *Let α, β, γ, and σ be given. Then there exists a wage profile w that implements the first best as a Nash equilibrium.*

Under Holmstrom's requirements, the first best can always be Nash implemented. To understand the fundamental contribution of Holmstrom, we need

[6] This is different from Holmstrom's requirement of budget–balancing, which means that all output is distributed as wages *in each state.*

to look more closely at the structure of wage profiles which implement the first best. Two (exhaustive) cases are worth noting: (i) $(2\gamma)/\beta > q_2 - q_1$; and (ii) $q_2 - q_1 \geq (2\gamma)/\beta$. Under (i), a wage profile where $w_2 = q_2$ and $w_1 = q_1$ can never implement the first best as then $q_2 - q_1 = \delta < (2\gamma)/\beta$. Furthermore, since we need the firm's surplus to be nonnegative, it cannot be the case that $w_2 \geq q_2$ and $w_1 \geq q_1$ (with at least one strict inequality, as both holding with equality has been ruled out). So it has to be the case that either $w_2 < q_2$ or $w_1 < q_1$. Since total wages are less than the aggregate output in at least one state, such wage profiles cannot be credible if the firm were to have just workers. This is because, *ex post*, the workers will jointly agree to discard the original contract and divide up the remaining surplus amongst themselves. As a result a third party to whom the surplus is legally pledged is needed. This third party is interpreted by Holmstrom as an owner. The owner can always design w where for example we have $w_2 = q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1$ and $w_1 = 0$. It is easy to see that such a w implements the first best. Here, when q_2 occurs, workers get a "bonus" of $\frac{\pi_{22}}{1-\pi_{22}} q_1$ (which comes from the owner's pocket). When q_1 results the owner gets to keep (the residual) q_1. However, such an owner is not always necessary. Under case (ii), a wage profile where $w_2 = q_2$ and $w_1 = q_1$ can always implement the first best, as then $q_2 - q_1 = \delta \geq (2\gamma)/\beta$. Since no output is "wasted" in any state, workers can always design such a w. We would like to add that this result is missing in Holmstrom's paper because of his assumption of a continuum of actions and a concave net surplus function.[7] We shall formally collect these results in Proposition 4.4 below.

In the introduction we have argued that, at times, under Holmstrom's incentives large wage cuts or high bonuses have to be given. This in the above example pertains to Case (i) where $(2\gamma)/\beta > q_2 - q_1$ and a "bonus" of x has to be given when output is q_2 and a "wage cut" of $\frac{1-\pi_{22}}{\pi_{22}} x$ has to be given when output is q_1. Under minimum wage and/or borrowing constraints such wage profiles may no longer possible.

[7] With a continuum of actions and a concave surplus function, if the solution is interior (as Holmstrom's assumptions imply) it is easy to see that the free rider problem implies that there can be no Nash equilibrium in which the first best is implemented with budget–balancing: if there are n workers and one of them reduces the amount of effort with respect to the first best, his reduction in reward is just $1/n$ of the decrease in total production, which with an interior solution is inferior to the worker's reduction in disutility. But the free rider problem appears as well when the solution is not interior, as in our case; the larger the number of workers the more important the free rider problem becomes, and one can show that in this case the possibility of the workers jointly owning the firm is reduced.

4.2 Minimum wages and borrowing constraints.

Under minimum wage constraints, wages have to be higher than some predetermined value M in *all states*. Under borrowing constraints the differential between wages and output cannot exceed some predetermined amount B.

1. *Minimum wages*. In each state, wages must be higher than a certain predetermined value M: $w_i^k \geq M$, for $i = 1, 2$ and $k = 1, 2$.
2. *Borrowing constraints*. In each state, the differential between wages and output cannot exceed some predetermined amount B: $w_i^k - q_i \leq B$, for $i = 1, 2$ and $k = 1, 2$.

These conditions impose changes in our our individual rationality and firm's surplus constraints. However, minimum wage and borrowing constraints are related. Given M, borrowing constraints imply a lower bound on the firm's surplus. Alternatively, given B, minimum wages imply a lower bound on the individual rationality constraint. So in our framework we need to impose only one of these two constraints. We choose to only impose the minimum wage constraint and let the firm's surplus constraint as it is. Imposing a borrowing constraint and letting the individual rationality constraint as it is would not change our results. To simplify our derivations we will assume that the minimum wage M coincides with the reservation utility, which we previously normalized at zero.

Assumption 4.2 *The wage received by each worker in each possible state must be nonnegative: $w_i^k \geq 0$, for $i = 1, 2$ and $k = 1, 2$.*

With this new constraint in place, the first best is not always Nash implementable. As mentioned in the Introduction, this result is also known in the literature. The intuition is as follows. A Nash equilibrium exists if, and only if, $\delta \geq (2\gamma)/\beta$. To find the maximum value that δ can take we can set $w_1 = x \geq 0$. Then the firm's surplus constraint tells us that the maximum value of w_2 under the first best is $q_2 + \frac{\pi_{22}}{1-\pi_{22}}(q_1 - x)$. It is easy to see that, under such wages, individual rationality is satisfied, because nonnegativity of the total utility is equivalent to nonnegativity of the net surplus. The difference between total wages is

$$\delta = q_2 + \frac{\pi_{22}}{1-\pi_{22}}(q_1 - x) - x = q_2 + \frac{\pi_{22}}{1-\pi_{22}}q_1 - \frac{x}{1-\pi_{22}}$$

Thus, δ can at most have a value of $q_2 + \frac{\pi_{22}}{1-\pi_{22}}q_1$ (when $x = 0$). So w can implement the first best as a Nash equilibrium if, and only if, $q_2 + \frac{\pi_{22}}{1-\pi_{22}}q_1 \geq (2\gamma)/\beta$. This, coupled with the discussion in the previous subsection, implies:

Proposition 4.4 *Let α, β, γ, and σ be given, and assume the wage profile must satisfy Assumption 4.2 (nonnegative minimum wages). Then there exists a wage profile w which can implement the first best as a Nash equilibrium if, and only if, $q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1 \geq (2\gamma)/\beta$. This wage profile can always be chosen to be symmetric. Furthermore:*

(a) When $q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1 > q_2 - q_1 \geq (2\gamma)/\beta$, workers can jointly own their firm.

(b) When $q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1 \geq (2\gamma)/\beta > q_2 - q_1$, workers need a third agent to own the firm.

Let us recall again our remark of last subsection, to the effect that case (a) in the previous proposition looses importance with regard to case (b) as the number of workers increases, due to the increasing importance of the free rider effect.

We would now like to know what happens when $q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1 < (2\gamma)/\beta$. The first best is no longer implementable, so our next objective is to see whether (a_1, a_2) or (a_2, a_1) are implementable. In order to maintain symmetry, we will consider the implementation of the correlated distribution that places probability 1/2 on each of those action combinations, that is, $\theta_2 = \theta_3 = 1/2$; we will designate this correlated distribution as the *second best*. Substituting these θ values into the correlated equilibrium characterization (inequalities 4), we find that the distribution is an equilibrium if, and only if, $(2\gamma)/\beta \geq \delta \geq (2\gamma)/\alpha$. Note, in particular, that such an equilibrium will exist only if $\alpha \geq \beta$; assume that this inequality holds. We now check whether the second best is implementable given our minimum wage and firm's surplus constraints. Using the same kind of reasoning as above, we find that for a symmetric w to implement the second best we need $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 \geq (2\gamma)/\alpha$.[8] Conversely, when $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 \geq (2\gamma)/\alpha$ we know that the maximum value of δ is *at least* as large as $(2\gamma)/\alpha$. So, given that $(2\gamma)/\beta \geq (2\gamma)/\alpha$, there always exists a δ such that $(2\gamma)/\beta \geq \delta \geq (2\gamma)/\alpha$.

Proposition 4.5 *Let α, β, γ, and σ be given, and assume the wage profile must satisfy Assumption 4.2 (nonnegative minimum wages). Let $q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1 < (2\gamma)/\beta$ (non feasibility of the first best) and $\alpha \geq \beta$. There exists a wage profile w which can implement the second best $\theta_2 = \theta_3 = 1/2$ as a correlated equilibrium if, and only if, $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 \geq (2\gamma)/\alpha$. This wage profile can always be chosen to be symmetric. Furthermore,*

[8] Note that the feasibility of this condition is not ruled out by $q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1 < (2\gamma)/\beta$ as $\alpha \geq \beta$ and $\pi_{12} > \pi_{22}$ implies $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 > q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1$.

(a) *When* $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 > q_2 - q_1 \geq (2\gamma)/\alpha$, *workers can jointly own their firm.*

(b) *When* $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 \geq (2\gamma)/\alpha > q_2 - q_1$, *workers need a third agent to own the firm.*

Finally, Assumption 2.1 implies that the worst (from the net surplus viewpoint) Nash equilibrium occurs whenever both players choose action a_1, that is, it corresponds to the correlated distribution $\theta_1 = 1$. This equilibrium will exist if, and only if, $\delta \leq (2\gamma)/\alpha$. So, given our earlier propositions, we have:

Proposition 4.6 *Let* α, β, γ, *and* σ *be given, and* σ *be given, and assume the wage profile must satisfy Assumption 4.2 (nonnegative minimum wages). Assume that* $q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1 < (2\gamma)/\beta$. *If either* $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 < (2\gamma)/\alpha$ *or* $\alpha < \beta$, *then only* (a_1, a_1) *might be implemented as a Nash equilibrium. The inequality* $q_2 + \frac{\pi_{11}}{1-\pi_{11}} q_1 \leq (2\gamma)/\alpha$ *guarantees that this equilibrium is feasible.*

Propositions 4.4, 4.5 and 4.6 characterize the implementation of (pure strategy) Nash equilibria, with the restriction of symmetry with respect to the second best. We now come to the main point of our paper and characterize conditions under which non-Nash equilibria are implementable.

4.3 Non Nash Implementation with $\beta > \alpha$.

In this section we consider the optimal correlated equilibrium when $\beta > \alpha$. We will show that, in this case, the optimal correlated equilibrium involves a convex combination between the first best (a_2, a_2) and the worst best (a_1, a_1). We will assume that the first best is not feasible, so $\sigma + \frac{q_1}{1-\pi_{22}} < (2\gamma)/\beta$. We will also assume $\sigma + \frac{q_1}{1-\pi_{11}} > (2\gamma)/\beta$, otherwise there is no possibility of improving upon the worst best. Without loss of generality, we set $w_1 = 0$, so the wage nonnegativity constraint becomes $\delta = w_2 \geq 0$. The set of *symmetric implementable correlated* *equilibria* is given by

$$\begin{cases} \Big[(\alpha - \beta)\,\delta + (\alpha\,\delta - 2\,\gamma)\Big]\,\theta_1 + (2\,\gamma - \beta\,\delta)\,\theta_4 \le 2\,\gamma - \beta\,\delta \\ (\alpha\,\delta - 2\,\gamma)\,\theta_1 + \Big[(\alpha - \beta)\,\delta + (2\,\gamma - \beta\,\delta)\Big]\,\theta_4 \le \alpha\,\delta - 2\,\gamma \\ -\alpha\,(\delta - \sigma)\,\theta_1 + \beta\,(\delta - \sigma)\,\theta_4 \le q_1 - (1 - \pi_{12})\,(\delta - \sigma) \\ (\alpha\,\delta - \gamma)\,\theta_1 - (\beta\,\delta - \gamma)\,\theta_4 \le (1 - \pi_{12})\,\delta - G(a_1) - G(a_2) \\ \theta_1 + \theta_4 \le 1 \\ \theta_1, \theta_4 \ge 0 \\ \delta \ge 0 \\ S(a_1, a_2) - (\alpha\,\sigma - \gamma)\,\theta_1 + (\beta\,\sigma - \gamma)\,\theta_4 \ge S(a_1, a_1) \end{cases} \tag{6}$$

A sufficient condition to guarantee that this set is nonempty is $\sigma + \frac{q_1}{1-\pi_{11}} \le (2\gamma)/\alpha$; in this case, $\delta = (2\gamma)/\alpha$ and $\theta_1 = 1$ satisfy all the inequalities. However, we will not need to impose this assumption.

Our next lemma shows the range of possible values for δ.

Lemma 4.7 *Let $\alpha < \beta$ and let the first best not be feasible*

$$\sigma + \frac{q_1}{1 - \pi_{22}} < \frac{2\,\gamma}{\beta}$$

Then any δ that implements a correlated equilibrium which strictly improves upon the worst best satisfies

$$\frac{2\,\gamma}{\beta} \le \delta \le \frac{2\,\gamma}{\alpha}$$

Proof: If we had

$$\delta < \frac{2\,\gamma}{\beta}$$

then, since $\alpha < \beta$, we would have $\alpha\,\delta < \beta\,\delta < 2\,\gamma$, so the incentive compatibility constraints imply $\theta_1 = 1$. Hence, there can be no improvement upon the worst best.

On the other hand, if we had

$$\delta > \frac{2\,\gamma}{\alpha}$$

then the incentive compatibility constraints imply $\theta_4 = 1$. But we know the first best is not feasible, so this cannot give rise to an implementable correlated equilibrium. □

The previous lemma shows that the values of δ that give rise to implementable correlated equilibria belong to a compact interval. Since all the functions that intervene are continuous and so the set of implementable equilibria is compact, Weierstrass's Theorem guarantees the existence of an optimal correlated equilibrium, provided that the feasible set is nonempty.

Corollary 4.8 *If $\alpha < \beta$, the first best is not feasible and the set of implementable correlated equilibria is nonempty, then there exists an optimal correlated equilibrium.*

Proposition 4.9 *Let $\alpha < \beta$ and let the first best not be feasible*

$$\sigma + \frac{q_1}{1 - \pi_{22}} < \frac{2\,\gamma}{\beta}$$

Assume further that

$$\frac{2\,\gamma}{\beta} < \sigma + \frac{q_1}{1 - \pi_{11}}$$

Then there exists an optimal correlated equilibrium (δ^*, θ^*), given by

$$\theta_2^* = \theta_3^* = 0, \quad \delta^* = \frac{2\,\gamma}{\beta} = \sigma + \frac{q_1}{1 - \pi_{22} - (\pi_{11} - \pi_{22})\,\theta_1^*}, \quad \theta_4^* = 1 - \theta_1^*$$

Proof: The firm's surplus feasibility constraint for δ is

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4 - \alpha\,\theta_1} \geq \delta$$

The expression on the left increases with θ_1 and decreases with θ_4. When $\theta_4 = 1$ we have

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4 - \alpha\,\theta_1} = \sigma + \frac{q_1}{1 - \pi_{22}} < \frac{2\,\gamma}{\beta}$$

And we know δ cannot satisfy this inequality. To make the inequality feasible we have to decrease θ_4 and/or to increase θ_1. Since our objective function is strictly increasing in θ_4 and strictly decreasing in θ_1, we will be interested in the smallest δ that allows the inequality to be satisfied. This happens when we take $\delta = (2\gamma)/\beta$, in which case the incentive constraints imply that $\theta_2 = \theta_3 = 0$. We must show that this solution satisfies individual rationality. When $\theta_2 = \theta_3 = 0$, we have

$\theta_4 = 1 - \theta_1$. The firm's surplus and individual rationality constraints can be written in terms of θ_1 as

$$\sigma + \frac{q_1}{1 - \pi_{12} - (\pi_{11} - \pi_{22})\,\theta_1} \geq \delta \geq \frac{2\,G(a_2) - 2\,\gamma\,\theta_1}{1 - \pi_{12} - (\pi_{11} - \pi_{22})\,\theta_1}$$

Consider the difference between the two bounds for δ:

$$\sigma + \frac{q_1 - 2\,G(a_2) + 2\,\gamma\,\theta_1}{1 - \pi_{12} - (\pi_{11} - \pi_{22})\,\theta_1}$$

The derivative of this expression does not depend on θ_1, so the expression is either strictly increasing, strictly decreasing or constant. The bounds are

$$\sigma + \frac{q_1 - 2\,G(a_2) + 2\,\gamma\,\theta_1}{1 - \pi_{12} - (\pi_{11} - \pi_{22})\,\theta_1} = \begin{cases} \dfrac{S(a_2, a_2)}{1 - \pi_{22}}, & \text{if } \theta_1 = 0; \\ \dfrac{S(a_1, a_1)}{1 - \pi_{11}}, & \text{if } \theta_1 = 1. \end{cases}$$

The first bound is strictly positive and the second weakly so. Hence, for all $\theta_1 < 1$ the expression is strictly positive. We only have to show that the equality of the firm's surplus and $(2\gamma)/\beta$ takes place for $\theta_1 < 1$. But this follows from the hypothesis that $\sigma + \frac{q_1}{1-\pi_{11}} > (2\gamma)/\beta$.

Our previous argument shows that this equilibrium exists and is optimal provided that $\theta_2 = \theta_3 = 0$. We must show that no equilibrium in which those variables are strictly positive improves upon the equilibrium we have described. In order to make those variables positives, we should increase δ further, so we should either increase θ_1 or decrease θ_4, which in either case decreases the objective function. □

4.4 Non Nash Implementation with $\alpha \geq \beta$.

(From now on, the proofs of the statements will be found in the appendix. We make no further mention of this in the text.)

We are particularly interested in the cases in which feasibility constraints prevent the implementation of (a_2, a_2) as a Nash equilibrium. So let it be the case that $q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1 < (2\gamma)/\beta$. If it is also the case that $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 < (2\gamma)/\alpha$ or $\alpha < \beta$, then it follows from Proposition 4.6 that the only correlated (Nash) equilibrium is (a_1, a_1). Therefore, the interesting case to study is when the second best, $\theta_2 = \theta_3 = 1/2$, is implementable. Under this case, from Propositions 4.5 and 4.6, we need $q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1 \geq (2\gamma)/\alpha$ and $\alpha \geq \beta$. So, we would like to see whether one can improve upon the second best, given the following parameter restrictions

$$\begin{cases} q_2 + \dfrac{\pi_{22}}{1-\pi_{22}}\, q_1 < \dfrac{2\,\gamma}{\beta} \\ q_2 + \dfrac{\pi_{12}}{1-\pi_{12}}\, q_1 \geq \dfrac{2\,\gamma}{\alpha} \\ \alpha \geq \beta \end{cases}$$

Without loss of generality, we will assume $w_1 = 0$ (the minimum allowable level), so that $\delta = w_2$, and the minimum wage constraint reduces to $\delta \geq 0$. Thus, the decision variable in the implementation problem is δ. To analyze the implementation problem, it will be convenient to isolate our decision variable in all the constraints. Before doing that, let us observe that our restriction to symmetric correlated equilibria, where $\theta_2 = \theta_3$, eliminates two of the correlated equilibrium constraints. Furthermore, using the probability constraint $\theta_1 + 2\theta_2 + \theta_4 = 1$, we will be able to characterize all correlated equilibria by using just θ_1 and θ_4. The object in which we are interested is the set of *implementable correlated equilibria* that improve upon the second best, which are characterized by the following set of inequalities:

$$\begin{cases} \Big[(\alpha-\beta)\,\delta + (\alpha\,\delta - 2\,\gamma)\Big]\,\theta_1 + (2\,\gamma - \beta\,\delta)\,\theta_4 \leq 2\,\gamma - \beta\,\delta \\ (\alpha\,\delta - 2\,\gamma)\,\theta_1 + \Big[(\alpha-\beta)\,\delta + (2\,\gamma - \beta\,\delta)\Big]\,\theta_4 \leq \alpha\,\delta - 2\,\gamma \\ -\alpha\,(\delta-\sigma)\,\theta_1 + \beta\,(\delta-\sigma)\,\theta_4 \leq q_1 - (1-\pi_{12})\,(\delta-\sigma) \\ (\alpha\,\delta-\gamma)\,\theta_1 - (\beta\,\delta-\gamma)\,\theta_4 \leq (1-\pi_{12})\,\delta - G(a_1) - G(a_2) \\ \theta_1 + \theta_4 \leq 1 \\ \theta_1, \theta_4 \geq 0 \\ \delta \geq 0 \\ S(a_1,a_2) - (\alpha\,\sigma-\gamma)\,\theta_1 + (\beta\,\sigma-\gamma)\,\theta_4 \geq S(a_1,a_2) \end{cases} \qquad (7)$$

Note that this set of inequalities is well defined, because setting $\theta_1 = \theta_4 = 0$ we get the second best, which we know from Proposition 4.5 can be implemented. However, we are concerned with implementable equilibria which *strictly* improve upon the second best and, moreover, in finding whether there is an optimal equilibrium among this class. Proposition 4.10 below shows that this exercise is not possible for all parameter values.

Proposition 4.10 *Let $\alpha \geq \beta$ and*

$$q_2 + \frac{\pi_{12}}{1 - \pi_{12}} q_1 = \frac{2\gamma}{\alpha}.$$

Then the maximum welfare attainable is $S(a_1, a_2)$, the one that corresponds to the second best.

The hypotheses of the last proposition hold for a non-generic case. There is another non-generic case, favorable to our objectives, which we state next.

Proposition 4.11 *Assume that the first best is not attainable*

$$\frac{2\gamma}{\beta} > q_2 + \frac{\pi_{22}}{1 - \pi_{22}} q_1.$$

Let $\alpha = \beta$, and

$$q_2 + \frac{\pi_{12}}{1 - \pi_{12}} q_1 > \frac{2\gamma}{\alpha}.$$

Then there exists a symmetric correlated equilibrium which strictly improves upon the second best welfare $S(a_1, a_2)$.

Given the two previous propositions, we will be interested from now on in the cases that satisfy the following parameter restrictions:

Assumption 4.3 *(a) The first best is not feasible:*

$$q_2 + \frac{\pi_{22}}{1 - \pi_{22}} q_1 < \frac{2\gamma}{\beta}$$

(b) The second best is feasible with genericity:

$$\begin{cases} q_2 + \dfrac{\pi_{12}}{1 - \pi_{12}} q_1 > \dfrac{2\gamma}{\alpha} \\ \alpha > \beta \end{cases}$$

The inequalities in Assumption 4.3 imply:

$$\min\left\{\frac{2\,\gamma}{\beta},\ q_2+\frac{\pi_{12}}{1-\pi_{12}}\,q_1\right\} > \max\left\{\frac{2\,\gamma}{\alpha}\,q_2+\frac{\pi_{22}}{1-\pi_{22}}\,q_1\right\}$$

Proposition 4.12 *Let Assumption 4.3 hold, i.e. the first best is not feasible and the second best is feasible with genericity. Consider the set of implementable correlated equilibria given by the system of inequalities 7. We have that:*

1. *For any (δ,θ) that satisfy all the inequalities, the following is true:*

$$\min\left\{\frac{2\,\gamma}{\beta},\ q_2+\frac{\pi_{12}}{1-\pi_{12}}\,q_1\right\} \geq \delta \geq \frac{2\,\gamma}{\alpha} \tag{8}$$

2. *The set of implementable correlated equilibria which improve upon the second best is nonempty and compact, and there exists (δ^*,σ^*) in that set that maximizes the welfare.*

Remark: In the previous proposition we are not claiming that for any δ within the given bounds there is a correlated equilibrium θ such that (δ,θ) is implementable. We just say that, if there is such an implementable pair, then δ must satisfy the condition.

In the previous proposition we show that there is an optimal correlated equilibrium. Next we show that this optimal correlated equilibrium strictly improves upon the second best.

Proposition 4.13 *Let Assumption 4.3 hold, i.e. the first best is not feasible and the second best is feasible with genericity. Then there exist implementable correlated equilibria which strictly improve upon the welfare associated with the second best.*

Taken together, the last two propositions imply that the optimal implementable correlated equilibrium strictly improves upon the second best.

The proof of the last proposition (found in the appendix) shows that, for almost all δ that belong to the specified range one can get (uncountably) many implementable correlated equilibria with a welfare larger than the second best. The darkly shaded area in the left panel of figure 1 represents this set. As δ increases this set changes, as we see in the right panel of figure 1.

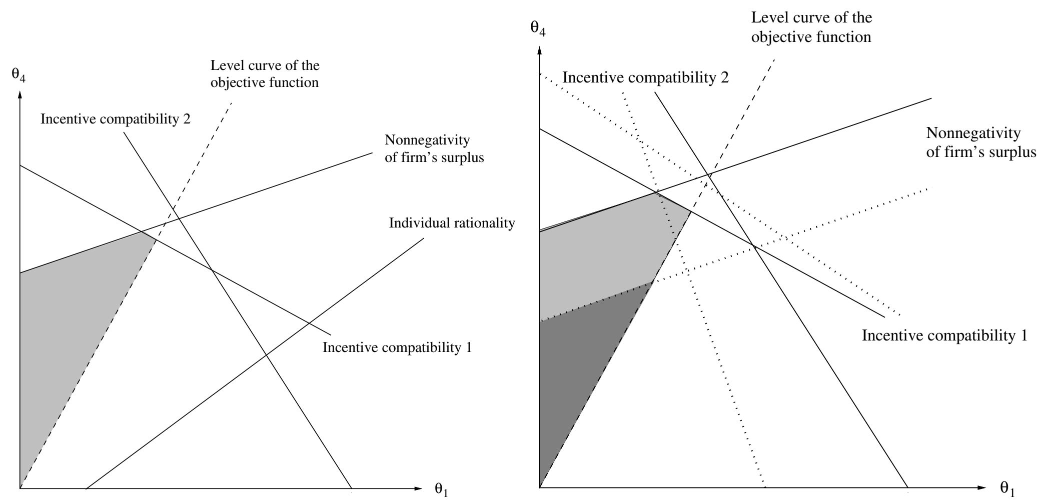

Figure 1: Implementable correlated equilibria which improve upon the second best. In the right, the effects of an increase in δ are shown.

4.5 Who Implements?

Consider the set of implementable correlated equilibria which: (i) improve upon the second best; (ii) assign strictly positive weight on only θ_2 ($= \theta_3$) and θ_4; and (iii) δ satisfies the inequalities 8. We shall later see that, for optimality reasons, this is the set of interest. Call this set C. Since $\theta_1 = 0$ in C, wages are symmetric, and $G(a_2) > G(a_1)$, none of these equilibria are Nash. Furthermore, though the distribution is common knowledge, signals to individual workers are private knowledge. Otherwise, if signals were publicly observed then either all equilibrium strategies would ignore the signals (which is not true in C) or signals would reveal strategies and players would best respond to each other and we would have a Nash equilibrium (which does not belong to C), or a convex combination of Nash equilibria, as in the second best.

Given $\{w, \theta\}$, $\theta \in C$, we now ask how an equilibrium is implemented. Recall that θ is a distribution over the action profiles. Once θ generates an outcome profile (a_i, a_j), the resulting actions are communicated to the respective workers by a *coordinator* (or Myerson's mediator). Following on Alchian-Demsetz-Holmstrom, our firm consists of two workers and an owner (residual claimant). We ask whether any of these players can act as the coordinator. Since signals have to be privately communicated in all equilibria in C, neither the workers nor the owner can act as the coordinator.

If worker i were to be the coordinator, then she would know the signal that she sends to j (i.e. the action that j takes). Credibility would then require that j best responds to that signal. Player i would know this. But then (a_2, a_2) will never result with positive probability as it is not a Nash equilibrium.

Now, consider the owner as the coordinator and let this owner maximize welfare. Since no worker can be the coordinator and since signals cannot be public, it must be the case that only the owner observes the outcome generated by θ. Furthermore, since θ is a correlated equilibrium, all outcomes (a_i, a_j) are such that, when the respective actions are privately communicated to the worker, it is in the best interest of the worker to carry it out. Since (a_2, a_2) results with positive probability, the owner would ignore the outcome profile generated by θ and always ask each worker to carry out action a_2. Thus, the (credible) θ would have $\theta_4 = 1$ and this distribution would be common knowledge. But such a distribution does not belong to C.

Suppose now that the coordinating owner (residual claimant) were to maximize her payoff. Since any δ that satisfies 8, we have in particular $\delta > q_2 - q_1$, so the owner strictly prefers the state with the low output q_1. Thus, for similar reasons as above, the credible θ would have $\theta_1 = 1$. But such an equilibrium does not belong to C.

Hence, we need a coordinator who is neither a worker nor the owner. For example, in our framework, a third worker with the same reservation utility of zero can be hired at a fixed wage of zero to act as the *coordinator*.[9]

5 Characterization of the Optimal Correlated Equilibrium.

Under the continuing assumption 4.3, that is, the first best is not feasible and the second best is feasible with genericity, we now turn to the problem of finding the optimal (from a welfare viewpoint) correlated equilibrium. As in subsection 4.4 we impose without loss of generality that $w_1 = 0$. We have seen before that the feasible set in this optimization program is nonempty, and that there exists an optimal solution for which objective function is strictly larger than the second

[9]We would like to add that, like in Alchian-Demsetz-Holmstrom, we have to rule out side contracts between players.

best welfare $S(a_1, a_2)$. For a fixed value of δ, the optimization problem is linear. However, in trying to characterize the global optimum things are not so simple because critical problems appear when we consider the implementation variable δ. This problem is highlighted in Figure 2, where the lightly shaded parts indicate correlated distributions that satisfy nonnegativity of the firm's surplus; note that in the left panel this area lies above the constraint whereas in the right one it lies below. To deal with such problems and characterize the solution, we will use a series of lemmas. In all the statements that follow, we will be assuming that Assumption 4.3 holds, that is, that the first best is not attainable, and that the second best is feasible with genericity.

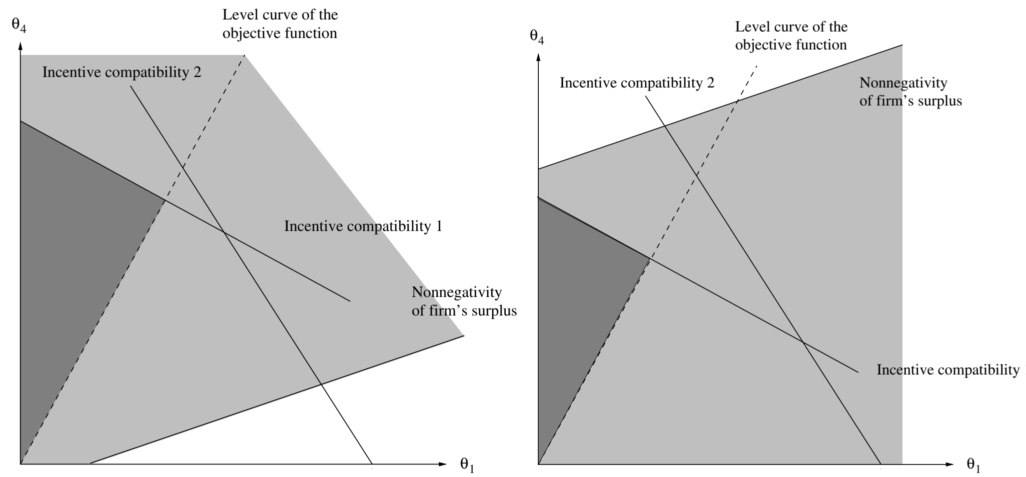

Figure 2: Feasibility of the firm's surplus constraint for a small (left) and a large (right) value of δ.

Lemma 5.1 *Let Assumption 4.3 hold. In any optimal correlated equilibrium* $\{\delta^*, \theta^*\}$, δ^* *satisfies:*

$$\frac{2\gamma}{\alpha} < \delta^* < \frac{2\gamma}{\beta}$$

Lemma 5.2 *Let Assumption 4.3 hold. In any optimal correlated equilibrium* $\{\delta^*, \theta^*\}$, $\theta_1^* = 0$.

In spite of these lemmas, it is still difficult to verify where δ^* will fall. The next lemma takes care of this problem. In it, the first statement implies that cases like the left panel of Figure 2 are not relevant, and the second statement implies that the right hand side is not relevant. In particular, the first statement is implied by the second, but it has interest in its own, since the feasible set changes drastically depending on whether δ is smaller or larger than σ. Additionally, proving the first statement is a step toward the proof of the more general one.

Lemma 5.3 *Let Assumption 4.3 hold. In any optimal correlated equilibrium* $\{\delta^*, \theta^*\}$*,* δ^* *satisfies the inequalities:*

1. $\delta^* > \sigma$.
2. $\max\left\{\frac{2\gamma}{\alpha},\ q_2 + \frac{\pi_{22}}{1-\pi_{22}} q_1\right\} < \delta^* < \min\left\{\frac{2\gamma}{\beta},\ q_2 + \frac{\pi_{12}}{1-\pi_{12}} q_1\right\}$

We can finally characterize the optimal implementable correlated equilibrium.

Proposition 5.4 *Let Assumption 4.3 hold. At the optimal correlated equilibrium the only binding constraints are the firm's surplus and one of the incentive constraints. This optimal equilibrium is unique and strictly improves upon the second best.*

The second statement in Lemma 5.3 demarcates the relevant range for δ^*. The domain of the optimization problem under this region is shown by the shaded area in the left panel of Figure 3. Note that optimization in this domain implies that the dotted level curve of the objective function must shift to the left, and the optimum occurs at the intercept of the firm's surplus constraint. This does not mean that for a given δ no other correlated equilibria can be implemented: all of the equilibria in the shaded region can be implemented. At the optimal δ^*, the firm's surplus and the incentive compatibility constraints meet precisely at the optimum θ_4^*, as shown in the right panel of Figure 3.

6 Conclusions.

The importance of some agents coordinating the activities of others in organizations has been emphasized by classical authors in the theory of the firm. However, it had not been formalized. In this paper, we find a natural framework in which to carry out this formalization. In a team production setting, the introduction of an external agent who implements a correlated equilibrium by coordinating the *productive* activities of the team members improves the social welfare.

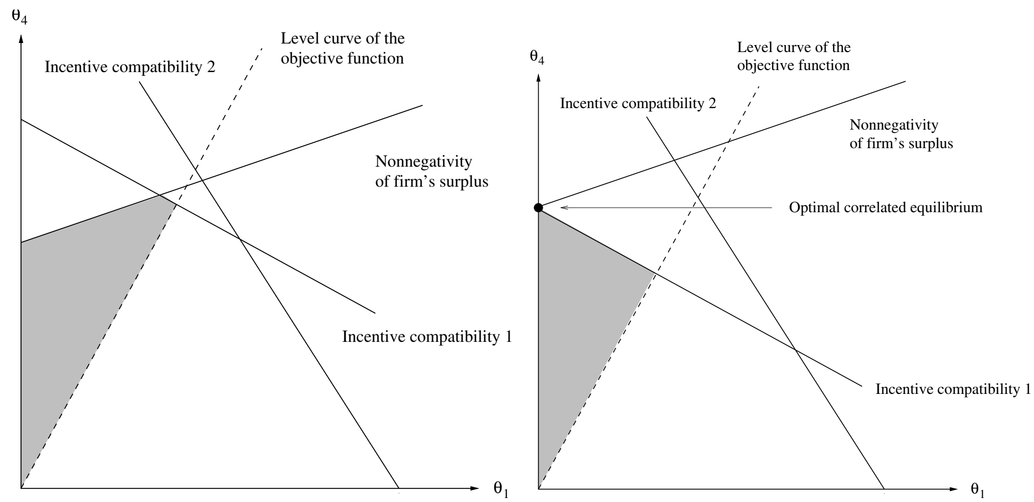

Figure 3: The relevant optimization problem and its solution.

While correlated equilibrium is recognized as an important solution concept, it seems that this importance has not been fully reflected in the literature, where the Nash equilibrium solution concept prevails by a wide margin. Some authors believe this is due to the necessity of using (in Myerson's terminology) a *mediator*, or substituting it by some complicated procedures (see, e.g., Ben-Porath, 1998, especially its introductory discussion). Our result highlights the importance of the mediator in implementation problems which use correlated equilibrium as the solution concept. In our model, the mediator has a natural interpretation as any agent in the firm whose task is to direct the activities of others.

We would like to end by pointing out that in our model communication plays a very important role. There are models in the literature (Forges, 1990) where people communicate and the outcome of this communication process is some (depending on the beliefs of people about the communication process itself) correlated equilibrium. But our communication here is very different, because it has a very definite aim, a socially optimal equilibrium. This is the distinguishing feature of using correlated equilibria in an implementation problem.

Appendix

It will sometimes be convenient to isolate δ in some of the implementability constraints. Nonnegativity of the firm's surplus becomes:

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4 - \alpha\,\theta_1} \geq \delta \tag{9}$$

where the denominator is always positive.

In the same manner, individual rationality (nonnegative total utility) can be expressed as:

$$\delta \geq \frac{G(a_1) + G(a_2) + \gamma\,(\theta_4 - \theta_1)}{1 - \pi_{12} + \beta\,\theta_4 - \alpha\,\theta_1} \tag{10}$$

where the denominator is exactly the same as in the previous constraint.

The following is an auxiliary result which will be useful to show that individual rationality is satisfied.

Lemma 6.1 *The following inequalities are true:*

$$\frac{2\,G(a_1)}{1-\pi_{11}} > \frac{G(a_1) + G(a_2)}{1-\pi_{12}} > \frac{2\,G(a_2)}{1-\pi_{22}}$$

Proof: By Assumption 2.1, we know that:

$$\begin{aligned} & q_2 + \frac{\pi_{22}}{1-\pi_{22}}\,q_1 - \frac{2\,G(a_2)}{1-\pi_{22}} > \\ & q_2 + \frac{\pi_{12}}{1-\pi_{12}}\,q_1 - \frac{G(a_1) + G(a_2)}{1-\pi_{12}} > \\ & q_2 + \frac{\pi_{11}}{1-\pi_{11}}\,q_1 - \frac{2\,G(a_1)}{1-\pi_{11}} \geq 0 \end{aligned}$$

Since $1 > \pi_{11} > \pi_{12} > \pi_{22} > 0$, we have

$$q_2 + \frac{\pi_{11}}{1-\pi_{11}}\,q_1 > q_2 + \frac{\pi_{12}}{1-\pi_{12}}\,q_1 > q_2 + \frac{\pi_{22}}{1-\pi_{22}}\,q_1$$

This concludes the proof. □

Proof of Proposition 4.10:

Assume $\theta_1 < 1$, then we can isolate δ in the first incentive compatibility constraint inequality in 7, to obtain

$$\delta \geq \frac{2\,\gamma\,(1-\theta_1+\theta_4)}{\alpha\,(1-\theta_1-\theta_4)+2\,\beta\,\theta_4}$$

Differentiating the fraction, we can find that it is strictly increasing in θ_4, and the extremes are:

$$\frac{2\,\gamma\,(1-\theta_1+\theta_4)}{\alpha\,(1-\theta_1-\theta_4)+2\,\beta\,\theta_4} = \begin{cases} \dfrac{2\,\gamma}{\alpha}, & \text{if } \theta_4 = 0; \\ \dfrac{2\,\gamma}{\beta}, & \text{if } \theta_4 = 1. \end{cases}$$

On the other hand, the fraction that appears in the firm's surplus inequality 9 is strictly decreasing in θ_4, and attains its maximum value

$$q_2 + \frac{\pi_{12}}{1-\pi_{12}}\,q_1$$

when $\theta_4 = 0$.

Therefore, the following inequalities hold whenever $\theta_1 < 1$:

$$q_2 + \frac{\pi_{12}}{1-\pi_{12}}\,q_1 \geq \sigma + \frac{q_1}{1-\pi_{12}+\beta\,\theta_4-\alpha\,\theta_1} \geq$$
$$\delta \geq \frac{2\,\gamma\,(1-\theta_1+\theta_4)}{\alpha\,(1-\theta_1-\theta_4)+2\,\beta\,\theta_4} \geq \frac{2\,\gamma}{\alpha}$$

Therefore, under the hypothesis of the proposition all inequalities hold as equalities. In particular, it must be the case that $\theta_4 = 0$, and our claim follows from the characterization of the welfare function given in Fact 4.1. □

Proof of Proposition 4.11:

If we set $\hat{\delta} = (2\,\gamma)/\alpha = (2\,\gamma)/\beta$, then the incentive compatibility constraints are void ($a = b$), so we just need to check that there is a symmetric correlated distribution that satisfies feasibility. Let $\hat{\theta}_1 = 0$. The feasibility constraints then become:

$$\sigma + \frac{q_1}{1-\pi_{12}+\beta\,\theta_4} \geq \frac{2\,\gamma}{\beta} \geq \frac{G(a_1)+G(a_2)+\gamma\,\theta_4}{1-\pi_{12}+\beta\,\theta_4}$$

Notice that the leftmost expression is strictly decreasing with respect to θ_4, and moves between the following extremes:

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4} = \begin{cases} q_2 + \dfrac{\pi_{12}}{1 - \pi_{12}}\, q_1, & \text{if } \theta_4 = 0; \\ q_2 + \dfrac{\pi_{22}}{1 - \pi_{22}}\, q_1, & \text{if } \theta_4 = 1. \end{cases}$$

By hypothesis:

$$q_2 + \frac{\pi_{12}}{1 - \pi_{12}}\, q_1 > \frac{2\,\gamma}{\beta} > q_2 + \frac{\pi_{22}}{1 - \pi_{22}}\, q_1$$

Therefore, there is a unique $\hat{\theta}_4$ such that $0 < \hat{\theta}_4 < 1$ and

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\hat{\theta}_4} = \frac{2\,\gamma}{\beta}$$

We claim that the distribution thus induced satisfies individual rationality. To see this, just consider the difference:

$$\frac{2\,\gamma}{\beta} - \frac{G(a_1) + G(a_2) + \gamma\,\hat{\theta}_4}{1 - \pi_{12} + \beta\,\hat{\theta}_4} = \sigma + \frac{q_1 - G(a_1) - G(a_2) - \gamma\,\hat{\theta}_4}{1 - \pi_{12} + \beta\,\hat{\theta}_4}$$

View the expression on the right as a function of θ_4. Then the sign of its derivative does not depend on θ_4, so the function is either strictly increasing, or strictly decreasing, or (non-generically) constant. In any case, in the extremes it takes the values:

$$\sigma + \frac{q_1 - G(a_1) - G(a_2) - \gamma\,\theta_4}{1 - \pi_{12} + \beta\,\theta_4} = \begin{cases} \dfrac{S(a_1, a_2)}{1 - \pi_{12}}, & \text{if } \theta_4 = 0; \\ \dfrac{S(a_2, a_2)}{1 - \pi_{22}}, & \text{if } \theta_4 = 1. \end{cases}$$

Both the extreme values are strictly positive by Assumption 2.1. Therefore, it must be the case that the expression is strictly positive for all $0 \leq \theta_4 \leq 1$, i.e. individual rationality is satisfied with strict inequality for $\hat{\theta}_4$. Finally, since $\hat{\theta}_4 > 0$ and $\hat{\theta}_1 = 0$, Fact 4.1 implies that the welfare under this correlated equilibrium is strictly larger than under the second best. □

Proof of Proposition 4.12:

As we noticed in the main text, Assumption 4.3 implies

$$\min\left\{\frac{2\,\gamma}{\beta},\; q_2+\frac{\pi_{12}}{1-\pi_{12}}\,q_1\right\} > \max\left\{\frac{2\,\gamma}{\alpha},\; q_2+\frac{\pi_{22}}{1-\pi_{22}}\,q_1\right\}$$

so the bounds for δ are consistent.

Non-emptiness of the set of implementable equilibria can be shown by taking:

$$\delta = \min\left\{\frac{2\,\gamma}{\beta},\; q_2+\frac{\pi_{12}}{1-\pi_{12}}\,q_1\right\}$$
$$\theta_1 = \theta_4 = 0$$

This feasibly implements the second best.

Let (δ,θ) be an implementable correlated equilibrium. Since $\alpha > \beta$, if we had

$$\delta < \frac{2\,\gamma}{\alpha} < \frac{2\,\gamma}{\beta}$$

then the incentive compatibility constraints as their appear in table 1 imply (using the notation in that table) that both $a < 0$ and $b < 0$, so only $\theta_1 = 1$ is possible, yielding a welfare strictly smaller than the second best.

Assume now that

$$\delta > \frac{2\,\gamma}{\beta}$$

then we have $a > 0$ and $b > 0$, so only $\theta_4 = 1$ is possible. But we know the first best is not implementable. This proves that

$$\frac{2\,\gamma}{\alpha} \le \delta \le \frac{2\,\gamma}{\beta}$$

Suppose now that the optimal δ satisfies

$$\delta > q_2+\frac{\pi_{12}}{1-\pi_{12}}\,q_1$$

Then, using the firm's surplus constraint, we will have

$$\sigma + \frac{q_1}{1-\pi_{12}+\beta\,\theta_4-\alpha\,\theta_1} \ge \delta > q_2+\frac{\pi_{12}}{1-\pi_{12}}\,q_1$$

This implies, operating with the inequality between the leftmost and the rightmost terms, that

$$\alpha\,\theta_1 > \beta\,\theta_4$$

The implementable equilibria improve upon the second best, i.e.

$$(\beta\,\sigma-\gamma)\,\theta_4 \geq (\alpha\,\sigma-\gamma)\,\theta_4$$

This implies

$$\gamma\,(\theta_1-\theta_4) \geq \sigma\,(\alpha\,\theta_1-\beta\,\theta_4)$$

Combining this inequality with the strict inequality we just found, we get that $\theta_1 > \theta_4$. But this in turn implies that

$$(\beta\,\sigma-\gamma)\,\theta_4 < (\alpha\,\sigma-\gamma)\,\theta_1$$

A contradiction.

Now, given part 1 in this proposition, in all implementable correlated equilibria the three variables belong to compact intervals. All the functions defining the inequalities are continuous in those variables, so the set that satisfies those inequalities is closed. Therefore, the implementable equilibria form a compact set. Since the welfare function is continuous in the variables, Weierstrass's Theorem implies that there exists a welfare-maximizing equilibrium. □

Proof of Proposition 4.13:

Given $\varepsilon > 0$, consider

$$\delta'(\varepsilon) = \min\left\{\frac{2\,\gamma}{\beta},\ q_2 + \frac{\pi_{12}}{1-\pi_{12}}\,q_1\right\} - \varepsilon$$

Assume ε is small enough so that

$$\delta'(\varepsilon) > \max\left\{\frac{2\,\gamma}{\alpha},\ q_2 + \frac{\pi_{22}}{1-\pi_{22}}\,q_1\right\}$$

In particular, $\delta'(\varepsilon) > \sigma = q_2 - q_1$. Now let $\theta_1' = 0$, and define $\theta_4'(\varepsilon)$ to be such that equality holds for the firm's surplus constraint:

$$\sigma + \frac{q_1}{1-\pi_{12}+\beta\,\theta_4} = \delta'(\varepsilon)$$

By the same argument we use in the proof of proposition 4.11, we have that $0 < \theta_4'(\varepsilon) < 1$, since by construction

$$q_2 + \frac{\pi_{22}}{1-\pi_{22}}\,q_1 < \delta'(\varepsilon) < q_2 + \frac{\pi_{12}}{1-\pi_{12}}\,q_1.$$

We remark that $\delta'(\varepsilon)$ is strictly decreasing in ε, and thus θ_4' is strictly increasing in ε.

We can show that individual rationality (10) is satisfied using an argument similar to the one used in the proof of Proposition 4.11. That is, we show that the difference

$$\delta'(\varepsilon) - \frac{G(a_1) + G(a_2) + \gamma\,\theta_4'}{1 - \pi_{12} + \beta\,\theta_4'} = \sigma + \frac{q_1 - G(a_1) - G(a_2) - \gamma\,\theta_4'}{1 - \pi_{12} + \beta\,\theta_4'}$$

is strictly positive. We refer the reader to the aforementioned proof and do not repeat the argument here.

Finally, we must show that, for some ε small enough, the distribution induced by $\{\delta'(\varepsilon), \theta_1', \theta_4'(\varepsilon)\}$ satisfies the incentive compatibility constraints. Since $\theta_1' = 0$, the only incentive compatibility constraint that matters given our assumptions is

$$\Big(\alpha\,[1 - \theta_4'(\varepsilon)] + 2\,\beta\,\theta_4'(\varepsilon)\Big)\,\delta'(\varepsilon) \geq 2\,\gamma\,[1 + \theta_4'(\varepsilon)]$$

Therefore, to prove feasibility we must show that, for some ε small enough, the following inequality holds

$$\delta'(\varepsilon) = \sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4'(\varepsilon)} \geq \frac{2\,\gamma\,[1 + \theta_4'(\varepsilon)]}{\alpha\,[1 - \theta_4'(\varepsilon)] + 2\,\beta\,\theta_4'(\varepsilon)}$$

Consider the difference between both expressions as a function of θ_4:

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4} - \frac{2\,\gamma\,(1 + \theta_4)}{\alpha\,(1 - \theta_4) + 2\,\beta\,\theta_4}$$

This difference is strictly decreasing with respect to θ_4. When $\theta_4 = 0$ it equals

$$q_2 + \frac{\pi_{12}}{1 - \pi_{12}}\,q_1 - \frac{2\,\gamma}{\alpha} > 0$$

And when $\theta_4 = 1$ it equals

$$q_2 + \frac{\pi_{22}}{1 - \pi_{22}}\,q_1 - \frac{2\,\gamma}{\beta} < 0$$

Therefore, there is a $0 < \hat{\theta}_4 < 1$ such that it equals 0, and for all $\theta_4 < \hat{\theta}_4$ it is strictly greater. In particular, we can see that, whenever $\varepsilon = 0$, then whether $\delta'(0)$ takes any of the two extremes in its definition the inequality is strict, because the maximum value that the coefficient of the incentive compatibility constraint can take is $(2\gamma)/\beta$. By continuity, there is $\hat{\varepsilon} > 0$ such that feasibility is satisfied, and so is it for all $0 < \varepsilon < \hat{\varepsilon}$.

Finally, since $\theta_1' = 0$ and $\theta_4'(\varepsilon) > 0$, the expression of the welfare function written in Fact 4.1 implies that the welfare under this feasible correlated equilibrium is strictly larger than under the second best. □

Proof of Lemma 5.1:

Since $\alpha > \beta$, if we had

$$\delta = \frac{2\,\gamma}{\alpha} < \frac{2\,\gamma}{\beta}$$

then $a = 0$ and $b < 0$, so $\theta_4 = 0$. In this case, there can be no improvement over the second best welfare level, so δ cannot be optimal. This last claim is justified because we have shown in Proposition 4.13 that there are feasible correlated equilibria that improve upon the second best.

Assume now that

$$\delta = \frac{2\,\gamma}{\beta}$$

then $a > 0$ and $b = 0$, which implies $\theta_1 = 0$ and $\delta = (2\,\gamma)/\beta$. Let us consider, in this last case, the firm's surplus constraint:

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4} \geq \delta = \frac{2\,\gamma}{\beta}$$

We know that the expression on the left is strictly decreasing with respect to θ_4. Consider the following cases. Firstly, if

$$q_2 + \frac{\pi_{12}}{1 - \pi_{12}}\,q_1 < \frac{2\,\gamma}{\beta}$$

then the given δ is not feasible. On the other hand, if there is equality, then the only solution is $\theta_4 = 1$, which we know is not feasible. Therefore, the only possibility left is

$$q_2 + \frac{\pi_{12}}{1 - \pi_{12}}\,q_1 > \frac{2\,\gamma}{\beta}$$

In this case, there is some $0 < \hat{\theta}_4 < 1$ that satisfies the firm's surplus constraint with equality. If $\theta_4 > \hat{\theta}_4$, then θ_4 violates the constraint. But we claim that one can improve welfare by slightly lowering δ, because this would render feasible some $\theta_4 > \hat{\theta}_4$. Notice that we do not have to deal with the individual rationality constraint because by the argument used in the proof of Proposition 4.11 we know that, whenever the firm's surplus constraint is satisfied as an equality, the individual rationality constraint is automatically satisfied. This proves that the optimal δ^* cannot satisfy $\delta^* \geq (2\,\gamma)/\beta$. $\square$

Proof of Lemma 5.2:

For fixed δ, the optimization problem is linear with a compact feasible set and therefore, if the feasible set is nonempty, it attains the optimal value at one of its extreme points.

The objective function, which does not depend on δ, has gradient

$$\Big(-(\alpha\,\sigma-\gamma),\,(\beta\,\sigma-\gamma)\Big)$$

We know from Fact 2.2 that the terms inside the parentheses are strictly positive, i.e. the gradient points toward the Northwest in the (θ_1,θ_4) space.

We are assuming there is a δ^* that gives rise to an optimal solution in the optimization problem (let us drop the asterisk in what follows to make the notation simpler). The incentive compatibility constraints in the optimization problem are

$$\begin{aligned}
\Big[(\alpha-\beta)\,\delta+(\alpha\,\delta-2\,\gamma)\Big]\,\theta_1+(2\,\gamma-\beta\,\delta)\,\theta_4 \;&\leq\; 2\,\gamma-\beta\,\delta\\
(\alpha\,\delta-2\,\gamma)\,\theta_1+\Big[(\alpha-\beta)\,\delta+(2\,\gamma-\beta\,\delta)\Big]\,\theta_4 \;&\leq\; \alpha\,\delta-2\,\gamma
\end{aligned}$$

By Lemma 5.1, all the coefficients in the two constraints are strictly positive. In particular, the gradients (with respect to the θ variables) of both constraints have strictly positive components. The feasible region in the (θ_1,θ_4) space delimited by the two constraints and the probability constraints is nonempty, compact and convex. Since the gradient of the objective function points toward the Northwest direction, if neither the firm's surplus nor the individual rationality constraints bind at the optimum, then the solution is at the corner where the variables are

$$\begin{cases}
\theta_1=0\\
\theta_4=\dfrac{\alpha\,\delta-2\,\gamma}{(\alpha-\beta)\,\delta+(2\,\gamma-\beta\,\delta)}
\end{cases}\tag{11}$$

Consider now the firm's surplus constraint:

$$-\alpha\,(\delta-\sigma)\,\theta_1+\beta\,(\delta-\sigma)\,\theta_4\leq q_1-(1-\pi_{12})\,(\delta-\sigma)$$

We showed before (9) that it can be written:

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4 - \alpha\,\theta_1} \geq \delta$$

Whenever $\delta \leq \sigma$ the inequality is trivially satisfied and can never be binding. Therefore, in order for the constraint to bind at the optimum it must be the case that $\delta > \sigma$. In this case the gradient of the constraint is $(-\alpha, \beta)$. This gradient points toward the Northwest, as the one of the objective function, but it is easy to see that *it lies to the right* of the latter. That is, if the constraint is binding at the optimum, then either $\theta_4^* = 0$ of $\theta_1^* = 0$. The former option can be discarded, as we know that there are feasible points which improve upon the second best, and any of those must satisfy $\theta_4 > 0$. Hence, we can conclude that, if the firm's surplus constraint binds at the optimum, then $\theta_1^* = 0$.

Consider finally the individual rationality constraint. Because of what we have just shown, in order to prove that if the constraint binds at the optimum θ_1^* must be zero, it is sufficient to consider the case in which neither the firm's surplus nor the incentive compatibility constraints bind.

$$(\alpha\,\delta - \gamma)\,\theta_1 - (\beta\,\delta - \gamma)\,\theta_4 \leq (1 - \pi_{12})\,\delta - G(a_1) - G(a_2)$$

Lemma 5.1 implies that the coefficient of θ_1 is strictly positive. Therefore, the gradient of the constraint points either toward the Southeast or toward the Northeast. Let us consider the different cases separately.

Assume $\beta\,\delta < \gamma$, so that the gradient points toward the Northeast. Since the objective function improves toward the Northwest, if the constraint binds at the optimum, but neither the firm's surplus nor the incentive compatibility constraints bind, then the highest welfare point occurs when $\theta_1^* = 0$.

If $\beta\,\delta = \gamma$, then if the constraint binds at the optimum it means that the feasible set is a subset of the vertical axis, and hence $\theta_1^* = 0$.

So suppose that $\beta\,\delta > \gamma$, so that the gradient points toward the Southeast, and assume that this constraint binds at the optimum. Since welfare improves by moving toward the Northwest, the only possibility is that either the firm's surplus or the incentive compatibility constraints bind also, and this means that $\theta_1^* = 0$. □

Proof of Lemma 5.3:

We showed before (10) that the individual rationality constraint can be written as:

$$\delta \geq \frac{G(a_1) + G(a_2) + \gamma\,(\theta_4 - \theta_1)}{1 - \pi_{12} + \beta\,\theta_4 - \alpha\,\theta_1}$$

By Lemma 5.2 we know that, if there exists an optimum, then $\theta_1^* = 0$. Therefore, this constraint becomes:

$$\delta \geq \frac{G(a_1) + G(a_2) + \gamma\,\theta_4}{1 - \pi_{12} + \beta\,\theta_4}$$

Let us consider the expression on the right. If we differentiate it with respect to θ_4, we find that the sign of the derivative does not depend on θ_4, and therefore it is either strictly increasing, strictly decreasing or constant. We have that:

$$\frac{G(a_1) + G(a_2) + \gamma\,\theta_4}{1 - \pi_{12} + \beta\,\theta_4} = \begin{cases} \dfrac{G(a_1) + G(a_2)}{1 - \pi_{12}}, & \text{if } \theta_4 = 0; \\ \dfrac{2\,G(a_2)}{1 - \pi_{22}}, & \text{if } \theta_4 = 1. \end{cases}$$

By Lemma 6.1, we know that the first expression is strictly larger than the second, and therefore the term on the right of the individual rationality constraint is strictly decreasing with respect to θ_4.

The firm's surplus constraint can be written (9):

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4 - \alpha\,\theta_1} \geq \delta$$

Suppose now that $\delta \leq \sigma$. In this case the inequality is trivially satisfied and can never be binding. Since the individual rationality constraint is strictly decreasing in θ_4, we can set this variable to a value as large as the incentive compatibility constraints allow, which we saw in the proof of Lemma 5.2 is:

$$\theta_4 = \frac{\alpha\,\delta - 2\,\gamma}{(\alpha - \beta)\,\delta + (2\,\gamma - \beta\,\delta)} \tag{12}$$

By differentiation, we can see that the value of θ_4 in this expression is strictly increasing with respect to δ, which means that by increasing δ slightly all the constraints would still be satisfied and the welfare increased. Our last remark depends on the fact that if δ is only slightly above σ the firm's surplus constraint is still trivially satisfied.

This proves the first part of our lemma: $\delta > \sigma$.

Suppose now that the optimal δ^* satisfies

$$\sigma < \delta^* \leq q_2 + \frac{\pi_{22}}{1 - \pi_{22}}\,q_1$$

Since we know $\theta_1^* = 0$, we have that the firm's surplus constraint is again trivially satisfied because whenever $\theta_4 < 1$ we have

$$\sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4} > q_2 + \frac{\pi_{22}}{1 - \pi_{22}}\,q_1 \geq \delta$$

The same argument given above shows that individual rationality places no upper bounds on θ_4, and therefore the welfare maximizing value is the one given by the above equation 12. Since this expression is strictly increasing in δ, we might improve welfare by slightly raising θ_4.

Finally, if we had

$$\delta \geq q_2 + \frac{\pi_{12}}{1 - \pi_{12}}\,q_1$$

then we can see, using the firm's surplus constraint, that any $\theta_4 > 0$ would satisfy

$$\delta \geq q_2 + \frac{\pi_{12}}{1 - \pi_{12}}\,q_1 > \sigma + \frac{q_1}{1 - \pi_{12} + \beta\,\theta_4} \geq \delta$$

A contradiction. This completes the proof. □

Proof of Proposition 5.4:

Given the previous lemmas, we can restrict our attention to the case in which $\theta_1 = 0$ and δ satisfies the bounds stated in Lemma 5.3. Then one of the incentive compatibility constraints vanishes, and the optimization problem is equivalent to:

$$\begin{aligned}
\max_{\delta,\theta_4} \quad & \theta_4 \\
\text{s.t.} \quad & \theta_4 \leq \frac{\alpha\,\delta - 2\,\gamma}{(\alpha - \beta)\,\delta + (2\,\gamma - \beta\,\delta)} \\
& \theta_4 \leq \frac{q_1}{\delta - \sigma} - (1 - \pi_{12}) \\
& \theta_4 \geq \frac{-(1 - \pi_{12})\,\delta + G(a_1) + G(a_2)}{\beta\,\delta - \gamma} \\
& \max\left\{\frac{2\,\gamma}{\alpha},\ q_2 + \frac{\pi_{22}}{1 - \pi_{22}}\,q_1\right\} < \delta < \min\left\{\frac{2\,\gamma}{\beta},\ q_2 + \frac{\pi_{12}}{1 - \pi_{12}}\,q_1\right\} \\
& 0 \leq \theta_4 \leq 1
\end{aligned}$$

Even though the feasible set is not closed in δ, we know it never pays to attain the boundaries.

Let us find the maximum θ_4^* that satisfies

$$\theta_4 \leq \min\left\{\frac{\alpha\,\delta - 2\,\gamma}{(\alpha-\beta)\,\delta + (2\,\gamma - \beta\,\delta)},\ \frac{q_1}{\delta-\sigma} - (1-\pi_{12})\right\}$$

The first term is strictly increasing with respect to δ and the second is strictly decreasing. Therefore, let δ^* cause the two terms to equate:

$$\frac{\alpha\,\delta^* - 2\,\gamma}{(\alpha-\beta)\,\delta^* + (2\,\gamma - \beta\,\delta^*)} = \frac{q_1}{\delta^*-\sigma} - (1-\pi_{12})$$

We claim that there exists such a δ. We have that

$$\frac{q_1}{\delta-\sigma} - (1-\pi_{12}) = \begin{cases} 0, & \text{if } \delta = q_2 + \dfrac{\pi_{12}}{1-\pi_{12}}\,q_1; \\ \beta, & \text{if } \delta = q_2 + \dfrac{\pi_{22}}{1-\pi_{22}}\,q_1. \end{cases}$$

On the other hand

$$\frac{\alpha\,\delta - 2\,\gamma}{(\alpha-\beta)\,\delta + (2\,\gamma - \beta\,\delta)} = \begin{cases} 1, & \text{if } \delta = \dfrac{2\,\gamma}{\beta}; \\ 0, & \text{if } \delta = \dfrac{2\,\gamma}{\alpha}. \end{cases}$$

Since both expressions are monotonic, there is δ^* where both are equal, and this value must be strictly between the bounds in both cases, that is

$$\max\left\{\frac{2\,\gamma}{\alpha},\ q_2 + \frac{\pi_{22}}{1-\pi_{22}}\,q_1\right\} < \delta^* < \min\left\{\frac{2\,\gamma}{\beta},\ q_2 + \frac{\pi_{12}}{1-\pi_{12}}\,q_1\right\}$$

So δ^* satisfies the conditions of Lemma 5.3. Set now $\theta_1^* = 0$. We claim that

$$\theta_4^* = \frac{\alpha\,\delta^* - 2\,\gamma}{(\alpha-\beta)\,\delta^* + (2\,\gamma - \beta\,\delta^*)} = \frac{q_1}{\delta^*-\sigma} - (1-\pi_{12})$$

satisfies the individual rationality constraint as a strict inequality. First of all, since

$$\theta_4^* = \frac{\alpha\,\delta^* - 2\,\gamma}{(\alpha-\beta)\,\delta^* + (2\,\gamma - \beta\,\delta^*)}$$

and δ^* is strictly between $(2\,\gamma)/\beta$ and $(2\,\gamma)/\alpha$, it follows that $0 < \theta_4^* < 1$ (actually, θ_4^* cannot exceed the upper bound of the firm's surplus constraint, which is β). Now in the proof of Proposition 4.11 we have shown that, for $\theta_1^* = 0$ and $\theta_4^* > 0$ as we have here, the individual rationality constraint is strictly satisfied (i.e. it does not bind). This shows that the $\{\delta^*, \theta^*\}$ we have constructed is feasible. Furthermore, we saw that θ_4^* is the maximum value that this variable (which coincides with the objective function) can attain for any δ, and therefore we have found the (unique) optimal solution. The fact that this solution improves upon the second best follows from $\theta_1^* = 0$ and $\theta_4^* > 0$. $\square$

References

Admati, A., Pfleiderer, P. and Zechner, J. (1994). Large shareholder activism,risk sharing, and financial market equilibrium, *Journal of Political Economy* **102**: 1097–1130.

Alchian, A. and Demsetz, H. (1972). Production, information costs, and economic organization, *American Economic Review* **62**: 777–795.

Allen, F. and Winton, A. (1995). Corporate financial structure, incentives and optimal contracting, *in* R. Jarrow *et al* (ed.), *Handbooks in OR and MS*, Vol. 9, Elsevier, pp. 693–720.

Aumann, R. (1974). Subjectivity and correlation in randomized strategies, *Journal of Mathematical Economics* **1**: 67–96.

Aumann, R. (1987). Correlated equilibrium as an expression of bayesian rationality, *Econometrica* **55**: 1–18.

Barnard, C. (1938). *The functions of the executive*, Harvard University Press, Cambridge, Mass.

Ben-Porath, E. (1998). Correlation without mediation: Expanding the set of equilibrium outcomes by “cheap” pre-play procedures, *Journal of Economic Theory* **80**: 108–122.

Chandler, A. (1977). *The visible hand: the managerial revolution in American business*, Harvard University Press, Cambridge, Mass.

Forges, F. (1990). Universal mechanisms, *Econometrica* **58**: 1341–1364.

Foster, D. and Vohra, R. V. (1997). Calibrated learning and correlated equilibrium, *Games and Economic Behavior* **21**: 40–55.

Fudenberg, D. and Levine, D. (1999). Conditional universal consistency, *Games and Economic Behavior* **29**: 104–130.

Fudenberg, D. and Tirole, J. (1991). *Game Theory*, The MIT Press, Cambridge, Mass.

Hart, S. and Mas-Colell, A. (2000). A simple adaptive procedure leading to correlated equilibrium, *Econometrica* **68**: 1127–1150.

Holmstrom, B. (1982). Moral hazard in teams, *Bell Journal of Economics* **13**: 324–340.

Huddart, S. (1993). The effect of a large shareholder on corporate value, *Management Science* **39**: 1407–1421.

Jensen, M. C. and Meckling, W. H. (1976). Theory of the firm: Managerial behavior, agency cost and ownership structure, *Journal of Financial Economics* **3**: 305–360.

Milgrom, P. and Roberts, J. (1992). *Economics, Organization, and Management*, Prentice–Hall, Englewood Cliffs, NJ.

Mookherjee, D. (1984). Optimal incentive schemes with many agents, *Review of Economic Studies* **51**: 433–446.

Myerson, R. (1991). *Game Theory: Analysis of Conflict*, Harvard University Press, Cambridge, Mass.

Segal, I. (2001). Communication complexity and coordination by authority. Working paper, University of California at Berkeley.

Shleifer, A. and Vishny, R. (1986). Large shareholders and corporate control, *JPE* **94**: 461–488.

Woodward, S. (1985). Limited liability in the theory of the firm, *Z. Gesamte Staatswiss. J. Inst. Theor. Econ.* **141**: 601–611.