# Collective choice under dichotomous preferences

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**Abstract**

Agents partition deterministic outcomes into *good* or *bad*. A direct revelation mechanism selects a lottery over outcomes – also interpreted as time-shares.

The *utilitarian* mechanism averages over all deterministic outcomes "good" for the largest number of agents. It is efficient, strategyproof and treats equally agents and outcomes.

Insist that for each agent, a good outcome is selected during at least a *fair share* of total time: efficiency, strategyproofness and equal treatment are then incompatible.

Three mechanisms guaranteeing fair share and equal treatment are *random priority* (strategyproof, but efficient only "ex-post"); the *Nash bargaining* solution, and a *fair* variant of the *utilitarian* mechanism (both efficient, but not strategyproof). The latter mechanism, unlike Nash and random priority, is computed in linear time.

*Keywords:* collective choice, dichotomous preferences, randomization, time-sharing, strategyproofness, fair share.

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# Collective choice under dichotomous preferences

## 1 The problem and the punch line

Randomization is surely the oldest and most practical tool to achieve equity in collective decision making. The idea is to design a lottery choosing one of several mutually exclusive outcomes: the weights of the lottery (the probability distribution over the deterministic outcomes) are adjusted to capture a certain notion of fair compromise. Say that we have three outcomes $a, b, c$, and ten concerned agents split into 2 supporters of $a$, 5 of $b$ and 3 of $c$ : a prima facie fair compromise is the lottery $p_a = .2$, $p_b = .5$, $p_c = .3$.

An alternative interpretation of the convex combination of outcomes is time-sharing. Think of our agents sharing a "source" such as a TV or radio in a public space (a gym, the living room of the house they share), and allocating timeshares to the available channels.

In this paper we revisit the classic negative result on collective choice using randomization/time-sharing, under the assumption that the preferences are dichotomous. That is, each agent sorts the outcomes as either "good" or "bad" (possibly "all good" or "all bad"); her preferences over deterministic outcomes have at most two indifference classes.

This is a considerable restriction of the standard domain of rational preferences (complete and transitive). Yet dichotomous preferences arise naturally in a number of time-sharing problems. In the TV example, the three channels may broadcast the same news program but in English, French and Spanish respectively, and each agent understands a certain subset of these languages. Or think of an antenna broadcasting to geographically dispersed agents; for each orientation of the antenna, only a certain subset of agents receive an adequate signal, hence their preferences over the feasible set of orientations are dichotomous. Or the source may be choosing the software to run the email server that a number of machines are sharing, and each machine is compatible only with certain softwares.

Scheduling problems provide another natural example where preferences are dichotomous: a user's only concern is that his job be completed before a certain deadline. When the server cannot complete all jobs before their respective deadlines, randomization is the preferred device for fair compromise (see Shenker [1995], Demers et al. [1990]). We discuss a couple of scheduling examples in Section 3 (Examples 1,2).

We investigate the compatibility, in the dichotomous domain, of the three perennial goals of mechanism design: efficiency, incentive compatibility and fairness.

Incentive compatibility takes the standard form of *strategyproofness*: truthtelling is a dominant strategy in the direct mechanism where agents report their preferences.

Efficiency is the usual Pareto optimality of the lottery/time-sharing. In the case of lotteries, a weaker property is also of interest: it only requires that each deterministic outcome chosen with positive probability be Pareto optimal. To distinguish these two properties, we speak of *ex-post efficiency* for the latter, and of *ex-ante efficiency* for the former. We stress that ex-post efficiency has no convincing interpretation in the time-sharing context.

Fairness takes two forms. First, we wish to rule out any systematic bias between agents, or between (deterministic) outcomes: this corresponds to the familiar properties of *anonymity* and *neutrality*. A second and more interesting constraint guarantees to each participant a certain share of the collective benefits: specifically, for each one of the $n$ agents, the time share (probability) of the set of her good outcomes is at least $\frac{1}{n}$. One interpretation of this Fair Share property, in the spirit of bargaining theory, is that each participant can veto any lottery/time-sharing and claim his fair share of control over the final outcome. The property nicely captures the idea that the collective choice is a compromise, taking into account to a significant degree each participant's preferences.

We find that the above properties reach the impossibility frontier. It is easy to find anonymous, neutral and strategyproof mechanism which are also ex-post efficient and guarantee the fair share, or which are ex-ante efficient. But none of them can be ex-ante efficient and guarantee the fair share: Theorem 1 in Section 7. We also propose several anonymous, neutral and ex-ante efficient solutions guaranteeing the fair share. One of them is particularly easy to compute: Theorem 3 in Section 8.

## 2 Overview of the results

Recall first the central "negative" result of the literature on collective choice with lotteries/time-sharing, in the standard preference domain, i.e., when agents have complete and transitive preferences over deterministic outcomes, and Von Neumann-Morgenstern utilities over lotteries.

Consider the *random dictator* mechanism: an agent is randomly selected, with equal probability for each agent, and chooses freely the final outcome. This mechanism is fair (anonymous and neutral) and strategyproof (ignoring tie-breaking issues). It is also efficient in the weak sense of ex-post efficiency, namely every (deterministic) outcome selected with a positive probability is Pareto optimal. However, it fails the stronger requirement of ex-ante efficiency: the resulting lottery over outcomes may be Pareto inferior[1].

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[1] In the above example with 10 agents, imagine a fourth outcome $d$ that is a good compromise between $a, b$, and $c$. For instance, VNM utilities are:

| # of agents | $u(a)$ | $u(b)$ | $u(c)$ | $u(d)$ |
|---|---|---|---|---|
| 2 | 11 | 0 | 0 | 10 |
| 5 | 0 | 11 | 0 | 10 |
| 3 | 0 | 0 | 11 | 10 |

The utility of the random priority lottery is 2.2, 5.5, and 3.3 for the three types of agents

Nandeibam [2001] shows the following converse statement. The random dictator mechanism is essentially characterized by anonymity (equal treatment of agents), strategyproofness, and ex-post efficiency[2],[3]. If we insist on ex-ante efficiency, then a strategyproof mechanism involves a full time dictator, namely an agent such that, for any utility profile, any outcome chosen with positive probability is among the dictator's top outcomes. Gibbard [1977], the seminal paper on collective choice with lotteries as outcomes, restricts attention to mechanisms eliciting only ordinal preferences over deterministic outcomes; the same is true of Barbera [1979], who describes the fair and strategyproof – yet inefficient – mechanisms in this class. The tiny literature on mechanisms eliciting complete Von Neumann-Morgenstern utilities consists of Gibbard [1978], Hylland [1980], Freixas [1984], Barbera, Bogomolnaia and Van der Stel [1998], and Nandeibam [2001].

When preferences are dichotomous over deterministic outcomes, first order stochastic dominance is a complete preference ordering of the lotteries, and the probability that a certain lottery assigns to an outcome good for agent $i$ is her canonical utility. The VNM axiomatic construction is pointless in this preference domain, and the mechanisms eliciting ordinal preferences are the only ones we need to consider. An easy consequence of this considerable simplification is this: the incompatibility between anonymity, strategyproofness and ex-ante efficiency disappears. A mechanism meeting all three requirements is the one we call *utilitarian*, inspired by plurality voting[4]: choose randomly and with equal probability any one of the outcomes deemed "good" by the largest number of agents (see Section 2, in particular Proposition 1 for details). Yet the utilitarian mechanism is as uncompromising as any voting rule: if a single outcome has the largest support, it is chosen with probability one and all agents who do not like this outcome are left out in the cold (in our numerical example, those are the five supporters of $a$ and $c$). The utilitarian mechanism violates the Fair Share property.

A natural mechanism meeting Fair Share simply adapts the random dictator idea to the dichotomous domain. For each ordering of the agents, we select a deterministic outcome maximizing lexicographically the utility profile (i.e., the set of agents for whom the outcome is good). The *random priority* mechanism obtains by randomizing uniformly over all $n!$ priority orderings. A fixed priority mechanism is clearly strategyproof, and this property is preserved by averaging: therefore random priority is a strategyproof mechanism. On the other hand averaging does not preserve efficiency ex-ante: random priority is only ex-post efficient.

respectively, well below the 10 utils from choosing $d$.

[2] A much weaker form of efficiency is sufficient for the result: when a certain outcome is the top for every agent, it is chosen with probability one. Moreover, if we drop the anonymity requirement, the only new mechanisms choose a dictator according to an arbitrary fixed probability distribution.

[3] When preferences over lotteries (or any other convex set of outcomes) are strictly convex, a similar characterization result obtains: Dutta, Peters and Sen [1999].

[4] Or approval voting (Brams and Fishburn [1978]): they are the same thing given dichotomous preferences.

Notice that in the time-sharing interpretation the random selection of a priority ordering is only virtual. We only care about the actual time shares of the different outcomes: the time share of outcome $a$ equals the relative frequency of these priority orderings for which $a$ is a lexicographic maximum. The computation of these shares turns out to be hard (more on this below).

Thus, within the large family of strategyproof, anonymous and neutral mechanisms, we have one ex-ante efficient but uncompromising (utilitarian), and one guaranteeing fair share but inefficient (random priority). It turns out that this trade-off is inescapable: Fair Share and ex-ante efficiency cannot both be met by an anonymous, neutral and strategyproof mechanism: Theorem 1 in Section 7. The incompatibility is preserved if we strenghten strategyproofness to *Preference Monotonicity* (when the set of agents for whom outcome $a$ is good grows, the time-share of any other outcome cannot increase) and weaken Fair Share to *Positive Share* (the time-share of any agent's good outcomes must be strictly positive): Theorem 2 in Section 7.

The next task is to discover some simple mechanisms meeting Fair Share and ex-ante efficient (hence not strategyproof). The most natural approach is to maximize a collective utility function. This is especially natural in our problem, where the canonical utility representation of preferences over lotteries is simply the probability $u_i(p)$ that lottery $p$ chooses one of agent $i$'s good outcomes (i.e., we normalize $u_i(a) = 1$ if $a$ is good and $u_i(a) = 0$ if it is bad).

The Nash collective utility function $\sum_i \log\ (u_i)$ is interesting on two accounts. For any profile of dichotomous preferences, its maximization yields a unique lottery $p^*$; moreover $p^*$ guarantees his fair share to every agent. This *Nash solution* even satisfies a stronger lower bound property that we call *Fair Group Share:* for any subset $S$ of $k$ agents in an $n$ agents problem, the probability that (time-share when) the outcome selected is good for at least one agent in $S$ is no less than $\frac{k}{n}$. Fair Group Share rewards groups of agents with identical preferences: each one of $k$ identical agents is guaranteed the utility level $\frac{k}{n}$. Note that the random priority solution meets Fair Group Share as well. We show in Section 6 that, among all solutions maximizing a separably additive collective utility function, only the Nash solution meets Fair Group Share: Proposition 6.

In the last Section 8, we compare the computational complexity of the three solutions, utilitarian, random priority and Nash. We find that the utilitarian solution is by far the easiest to compute, and the random priority by far the hardest.

Our main positive result (Theorem 3) proposes an efficient and fair mechanism that is as easy to compute as the utilitarian one. We call it the *fair utilitarian* mechanism because it identifies the linear combination of individual utilities maximized on its support. It is anonymous, neutral, ex-ante efficient and meets Fair Group Share.

The companion piece Bogomolnaia and Moulin [2001] develops a related model of random assignment, under the same assumptions of dichotomous preferences. The problem is to assign to each agent at most one from a set of heterogeneous objects, when each agent partitions the objects as "good" or "bad".

Randomization (time-sharing) restores fairness. The main simplification of the assignment problem, relative to the general decision problem discussed here, is that a random assignment is efficient if and only if it is utilitarian (maximizes the sum of utilities); in particular ex-post and ex-ante efficiency coincide. It is then easy to find an efficient and strategyproof mechanism guaranteeing a "fair" share to every participant: the random priority mechanism is an example, the revelation mechanism selecting the Nash solution is another. In fact, the utility profile of the Nash solution dominates every other feasible profile with respect to the Lorenz partial ordering; moreover the associated revelation mechanism is group-strategyproof as well.

The model is defined in Section 3. In Section 4, we discuss the utilitarian solution, as well as the familiar property of *Group Strategyproofness*: we show that this strengthening of strategyproofness allowing for coalitional manipulations, is simply too demanding in our problem: Propositions 2, 3. Sections 5 and 6 are devoted to, respectively, the random priority and the Nash mechanisms. The impossibility of combining ex-ante efficiency, strategyproofness and fair share is the subject of Section 7. The final Section 8 discusses the issue of computational complexity and defines the fair utilitarian mechanism. All non trivial proofs are gathered in the Appendix.

# 3 The model

The set $N$ of agents is finite, and so is the set $A$ of (deterministic) outcomes. These two sets are fixed for most of our results, with the exception of the complexity results in Section 8, and of a couple of "minor" axioms[5].

A dichotomous preference on $A$ is described by a row vector $v \in \{0,1\}^A$ with the interpretation that outcome $a$ is good if $v^a = 1$ and bad if $v^a = 0$. We also say that an agent likes $a$ if $v^a = 1$ and dislikes it if $v^a = 0$. Note that $v^a = 0$ for all $a$, and $v^a = 1$ for all $a$, are the same preference: agents who are thus indifferent will not matter in any of the mechanisms discussed below.

A $N$-profile of dichotomous preferences is a $N \times A$ matrix $U$ with entry $u_i^a = 0$ or $1 : U \in \{0,1\}^{N \times A}$. We abuse notations slightly by identifying the $i-$ row $U_i$ with agent $i$'s preference, i.e., the subset of outcomes that she likes, and the $a$-column $U^a$ with the set of agents who like outcome $a$.

A *problem* is a triple $(N, A, U)$ and a lottery (vector of time-shares) is a column vector $p$ in the $A$-simplex: $p_a \geq 0$ for all $a$ and $\sum_A p_a = 1$. The canonical utility of agent $i$ for the lottery $p$ is thus written as $u_i(p) = U_i \bullet p = \sum_A u_i^a . p_a$.

A *solution* to the problem $(N, A, U)$ is a lottery $p$ deemed desirable according to certain properties of efficiency and fairness (defined below). When we discuss properties relating the solutions of different problems, we speak of a *mechanism.*

Given $N$ and $A$, a *mechanism* is a mapping $\pi$ selecting for each problem $(N, A, U)$ a solution $p = \pi(U)$. The following three properties of a mechanism are standard, and require no further comments:

[5] Namely independence of clones in Section 4, and outcome monotonicity in Section 7.

*Anonymity:* for any profile $U$, and permutation $\sigma$ of $N$
$\pi(U) = \pi({}^{\sigma}U)$ where ${}^{\sigma}U_i = U_{\sigma(i)}$
*Neutrality:* for any profile $U,$ and permutation $\tau$ of $A$
$\pi({}^{\tau}U) = {}^{\tau}\pi(U)$ where ${}^{\tau}p_a = p_{\tau(a)}$ and ${}^{\tau}U^a = U^{\tau(a)}$
*Strategyproofness:* for any $i \in N$, and profiles $U$, $U'$
$\{U_j = U'_j$ for all $j \neq i\} \Rightarrow \{U_i \bullet \pi(U) \geq U_i \bullet \pi(U')\}$

An outcome $a, a \in A$, is efficient at profile $U$ if for any other outcome $b, \{U^a \leq U^b\} \Rightarrow \{U^a = U^b\}$. We distinguish two notions of efficiency for a lottery $p$ :

*Ex-Post Efficiency* (EXP): for all $a$ in $A$ $\{p_a > 0\} \Rightarrow \{a$ is efficient$\}$
*Ex-Ante Efficiency* (EXA): for all lotteries $p'$ $\{U \bullet p \leq U \bullet p'\} \Rightarrow \{U \bullet p = U \bullet p'\}$

Ex-ante efficiency coincides with ex-post efficiency for a deterministic outcome $a$, but for lotteries the latter is a strictly weaker requirement than the former. The simplest example involves five outcomes and six agents: see Figure 1. All five outcomes are efficient, thus any lottery is ex-post efficient. However $p$, $p_c = p_d = p_e = \frac{1}{3}$, is strictly Pareto inferior to $q$, $q_a = q_b = \frac{1}{2}$.

# 4 The utilitarian solution

Given a problem $(N, A, U)$ we denote by $A_1$ the subset of outcomes liked by the largest number of agents:

$$a \in A_1 \Leftrightarrow |U^a| \geq \left|U^b\right| \text{ for all } b \in A$$

One could define the utilitarian solution as the uniform lottery with support $A_1$. However, the corresponding mechanism is not independent of "clones".

Two outcomes $a, b$ are *clones* at $U$ if $U^a = U^b$. A mechanism $\pi$ is *independent of clones* if the utility vector $U \bullet \pi(U)$ depends only upon the subset in $\{0,1\}^N$ of feasible utility profiles $\{U^a \mid a \in A\}$. In other words, the utility profile does not change if we replace a subset of clones by a single outcome. All solutions discussed in the subsequent sections are independent of clones.

We define the utilitarian solution by first *decloning* the set $A_1$, namely keeping only one copy of each set of clones, which leaves the set $A_1^*$, then taking for $p$ the uniform lottery over $A_1^*$.

This defines unambiguously the total probability/time-share of any set of clones, but leaves the distribution among clones unspecified. The utility profile is, however, uniquely defined. We say that the utilitarian solution is defined "modulo clones". The obvious proof of our first results is omitted.

**Proposition 1** *The utilitarian mechanism is anonymous, neutral, strategyproof and ex-ante efficient.*

The statement is preserved for the "clone-conscious" version of the utilitarian mechanism, namely the uniform lottery over $A_1$.

The utilitarian mechanism is robust against strategic misreporting by a single agent, but it is eminently vulnerable to joint misreporting by several agents. This is already the case in the simple example with three agents and three outcomes depicted on Figure 2, where $1^*$ stands for a misreport: any two agents benefit strictly by pretending to like each other's good outcome!

The familiar strengthening of strategyproofness to coalitions writes as follows for an arbitrary mechanism $\pi$ :

*Group-Strategyproofness* (GSP): for all $S \subseteq N$, and profiles $U, U'$

$\{U_j = U'_j$ for all $j \in N \diagdown S$, and $U_i \bullet \pi(U') \geq U_i \bullet \pi(U)$ for all $i \in S\} \Rightarrow U_i \bullet \pi(U') = U_i \bullet \pi(U)$ for all $i \in S$.

The next result shows that this property is not even compatible with neutrality and the weak version of efficiency.

**Proposition 2** *Assume $|N| \geq 4$ and $|A| \geq 4$. No mechanism is both ex-post efficient and group-strategyproof.*

A weaker requirement than GSP only rules out misreporting that strictly improves the utility of each member of the deviating coalition.

*Weak Group-Strategyproofness* for all $S \subseteq N$,and profiles $U, U'$ :

$\{U_j = U'_j$ for all $j \in N \diagdown S\} \Rightarrow \{U_i \bullet \pi(U') \leq U_i \bullet \pi(U)$ for some $i \in S\}$

Weak Group-Strategyproofness is not incompatible with efficiency, even in the ex-ante sense. Fix an arbitrary ordering $\sigma$ of $N$, say $i_1 > i_2 > ... > i_n$, and a preference profile $U$. Say that outcome $a$ is a *$\sigma$-priority outcome* at $U$ if $U^a$ is a lexicographic maximum for the ordering $\sigma$ : for any $b \in A$, there exists an integer $m, 0 \leq m \leq n$, such that:

$u^a_{i_k} = u^b_{i_k}$ for $k = 1, ..., m$ and $u^a_{i_{k+1}} > u^b_{i_{k+1}}$ (where $m = 0$ if $u^a_{i_1} > u^b_{i_1}$ and $m = n$ if $U^a = U^b$).

Notice that two $\sigma$-priority outcomes at $U$ must be clones. A *$\sigma$-priority mechanism* selects at profile $U$ a $\sigma$-priority outcome, or a lottery over these outcomes. Such a mechanism is defined modulo clones, hence its utility profile is uniquely defined.

The $\sigma$-priority mechanism is obviously weakly group-strategyproof, as well as ex-ante efficient. But if an ex-ante efficient mechanism treats equally agents and outcomes, it must violate weak GSP, just like the utilitarian mechanism does in the example of Figure 2.

**Proposition 3** *Assume $|N| \geq 4$ and $|A| \geq 6$. An anonymous and neutral mechanism cannot be both ex-post efficient and weakly group-strategyproof.*

Proposition 3 is a tight statement. A $\sigma$-priority mechanism shows that we cannot drop anonymity. The following non neutral utilitarian mechanism shows that we cannot drop neutrality either. Fix an arbitrary ordering of $A$ and select – with probability one – the smallest outcome in $A_1$ according to this ordering: this mechanism is efficient ex-ante, weakly groupstrategyproof and anonymous (we omit the easy proof).

## 5 The random priority solution

We introduce two fairness properties conveying the idea that collective choice is a compromise awarding to each participant a fair share of the collective benefits. In the two definitions below, a problem $(N, A, U)$ and a solution $p$ are given:

*Fair Share* (FS): for any $i \in N$

$\{U_i \neq 0\} \Rightarrow U_i \bullet p \geq \frac{1}{|N|}$

*Fair Group Share* (FGS): for any $S \subseteq N$

$\{U_i \neq 0 \text{ for all } i \in S\} \Rightarrow U_S \bullet p \geq \frac{|S|}{|N|}$, where $U_S^a = \max_{i \in S} u_i^a$

Fair Group Share is especially advantageous for agents with identical preferences: when $U_i = U_j$ for all $i, j \in S$, this common preference equals $U_S$ as well, therefore everyone in coalition $S$ is guaranteed the utility $\frac{|S|}{|N|}$. For instance consider a profile where each agent likes at most one outcome: given FGS, we must share the time between the outcomes in proportion to the number of their supporters.

The *random priority* solution selects a priority ordering $\sigma$ of $N$, with uniform probability over all orderings, and implements a $\sigma$-priority outcome $a(\sigma)$(defined at the end of Section 4). Formally, let $\mathcal{S}_N$ be the set of permutations of $N$, then for all profiles $U$, this solution $p$ is:

$$p_a = \frac{1}{|N|!} |\{\sigma \in \mathcal{S}_N : a(\sigma) = a\}| \qquad \text{for all } a \in A$$

Recall that the $\sigma$-priority outcome $a(\sigma)$ is defined "modulo clones": the same is true of the random priority solution.

**Proposition 4** *The random priority mechanism is anonymous, neutral and meets Fair Group Share; it is strategyproof and ex-post efficient. It is not ex-ante efficient.*

Anonymity and neutrality are obvious. For any fixed ordering $\sigma$, the $\sigma$-priority mechanism is strategyproof, and this property is preserved by convex combinations with fixed coefficients. To check Fair Group Share, observe that for a priority ordering $\sigma$ such that the highest priority agent is in $S$, the outcome $a(\sigma)$ is such that $U_S^{a(\sigma)} = 1$.

Finally the example of Figure 1 shows why random priority fails ex-ante efficiency. Here the $\sigma$-priority outcome is $c$ if $\sigma$ starts with agent 1 and agent 4 has a higher priority than 2 and 3, or if $\sigma$ starts with 4 and agent 1 is higher than 5 and 6. Thus $p_c = p_d = p_e = \frac{1}{9}$, $p_a = p_b = \frac{1}{3}$, and $u_i = \frac{4}{9}$ for all $i$, whereas $q_a = q_b = \frac{1}{2}$ yields $u_i = \frac{1}{2}$ for all $i$.

It is easy to generalize this example so that the random priority lottery wastes an arbitrary large fraction of total surplus.

**Example 1** *There are $n = 2m^2$ agents, each requiring a job of length 1. There are $m$ types of job $k = 1, 2, ..., m$, and $2m$ agents in each type of job. The server is available for $2m$ periods during which it can process all $2m$ agents* *of any given type. However, it takes the server $2m-2$ periods to switch from one type of job to another. Thus it can only process two (any two) agents of different types.*

*The fair and efficient solution chooses one of the $m$ types with uniform probability, and serves all agents of this type, resulting in utility $u_i = \frac{1}{m}$ for all $i$. In the random priority solution, on the other hand, for every ordering $\sigma$ where the first two agents are of different types the server ends up processing only these two. Hence the expected utility $u'_i = \frac{4m-3}{m(2m^2-1)} \simeq \frac{2}{m^2}$, for all $i$.*

In the next family of scheduling problems, on the contrary, the difference between ex-ante and ex-post efficiency is nil or small, therefore random priority is ex-ante efficient, or nearly so.

**Example 2** *The job of agent $i, i \in N$, requires $x_i$ units of the server's time. The server is available for $T$ periods, and can switch instantly between jobs. Utility of agent $i$ is 1 if her job is completed within $T$ periods, 0 otherwise. A vector $a \in \{0,1\}^N$ is a feasible (deterministic) utility vector if and only if $x \bullet a \leq T$ where $x = (x_i)$. By efficiency we only need to consider the maximal feasible vectors $a$, i.e., those for which the set of satisfied agents is maximal. Denote by $A$ the set of these vectors, and simply take the column vector $a$ to be the corresponding column of $U$.*

*In the general problem just described, all outcomes of $A$ are efficient by construction, but a lottery over $A$ may well be inefficient.*

*However, suppose that all efficient outcomes $a$ exhaust the $T$ available periods: $x \bullet a = T$ for all $a$. (This holds true for instance if there are at least $T$ agents with a 1-period job). Then every feasible utility vector $u$ (every convex combination of the column vectors $a$) has $x \bullet u = T$ as well, hence is ex-ante efficient: the two notions of efficiency coincide.*

*Another interesting case is when the length of the different jobs are not too different. Then the efficiency loss incurred at any (ex-post efficient) lottery will be small* [6]*.*

# 6 The Nash and other welfarist solutions

Given an increasing and concave function $f$ defined on $[0,1]$, any lottery $p$ maximizing the separably additive utility function $\sum_N f(u_i) = \sum_N f(U_i \bullet p)$ is ex-ante efficient. If $f$ is strictly concave, the utility profile of any such lottery is unique for all $U$: this defines, modulo clones, an anonymous, neutral, and ex-ante efficient mechanism. If $f$ is not strictly concave, so that there may be more than one optimal utility profile, it is clearly possible to select a solution within the convex optimal set in such a way that the resulting mechanism is anonymous and neutral.

[6] Set $\alpha = \max_i x_i$ and $\beta = \min_i x_i$. Efficiency of the deterministic outcome $a$ implies $T < x \bullet a + \alpha$ (ignoring the trivial case where all agents can be served at the same time). Therefore $T < x \bullet u + \alpha$ for any feasible utility vector. If $u'$ is feasible and Pareto superior to $u$, inequalities $x \bullet u' \leq T < x \bullet u + \alpha$ imply easily $\sum_i u'_i - \sum_i u_i < \frac{\alpha}{\beta}$.

We call a $f$-solution any lottery maximizing the $f$-collective utility, and speak of a $f$-mechanism if a $f$-solution is chosen for any problem. In this section, we look for $f$-mechanisms satisfying Fair Share *or* strategyproofness.

**Proposition 5** *If $f$ takes the form $f(u) = g(\log u)$, where $g$ is increasing and concave, then any $f$-solution meets Fair Share.*

Note that we must remove "empty" agents ($U_i = 0$) when computing the above $f$-collective utility function.

Two important members of the family of solutions described in Proposition 5 are the Nash solution corresponding to $f(u) = \log u$, and, abusing language, the egalitarian solution corresponding to the limit of $f$-mechanisms where $f$ is increasingly concave, e.g., $f(u) = -|\log u|^t$ and $t$ goes to infinity. The latter solution maximizes the leximin ordering (the lexicographic ordering over utility profiles rearranged increasingly).

**Proposition 6** *The Nash solution meets Fair Group Share. No other $f$-solution meets this property for all $N, A$.*

Recall from Section 4 that the utilitarian mechanism is strategyproof but fails Fair Share. Now the Nash mechanism, and a number of other $f$-mechanisms, meet Fair Share (Proposition 5) but neither of them is strategyproof.

The four agents, three outcomes example depicted in Figure 3 establishes the latter claim for the Nash mechanism[7]: the Nash solution is $p_a = p_b = \frac{1}{2}$ with utility $u_i = \frac{1}{2}$ for all $i$; if agent 2 (who likes $a$ and $b$) denies that she likes $a$, the Nash solution in the new problem is $p_a = \frac{1}{4}$, $p_b = p_c = \frac{3}{8}$ and agent 2's true utility is $\frac{5}{8}$.

In fact the utilitarian mechanism is (up to the choice of the tie breaking rule) the only strategyproof $f$-mechanism:

**Proposition 7** *If the $f$-mechanism is strategyproof for all N, A, then $f$ is linear, so the mechanism chooses with probability one an outcome that is liked by the largest number of agents.*

Within the family of $f$-mechanisms, a clear trade-off between strategyproofness and Fair Share is the message of Propositions 5, 6, 7. The results of the next section demonstrate that this trade-off affects a much broader set of mechanisms.

# 7 Two impossibility results

**Theorem 1** *Assume $|N| \geq 5$ and $|A| \geq 17$. An anonymous and neutral mechanism cannot be ex-ante efficient, strategyproof, and meet Fair Share.*

[7] In this example the same misreport by agent 2 shows that the egalitarian mechanism is not strategyproof.

In the above statement, we cannot dispense with either one of ex-ante efficiency, strategyproofness, or Fair Share: think of the random priority, Nash or utilitarian mechanism respectively. However, we have not been able to determine if one of the anonymity or neutrality property (or both) can be dropped.

Another challenging open question is to find an anonymous, neutral, ex-ante efficient and strategyproof mechanism that is not utilitarian i.e., at some profile a positive weight goes to an outcome $a$ that does not have the largest support $|U^a|$.

Our second impossibility results bears on a natural monotonicity property reminiscent of the positive responsiveness axiom in classic social choice. The property is defined as follows for a given pair $N, A$ and mechanism $\pi$ :

*Preference Monotonicity* (PM): for any $i \in N$ and profiles $U, U'$
$\{U_j = U'_j$ for all $j \neq i$ and $U_i \underset{\neq}{\subset} U'_i\} \Rightarrow \pi_b(U) \geq \pi_b(U')$ for all $b \in U'_i \diagdown U_i$

The PM property states that whenever a certain outcome $a$ becomes more popular (in the sense that one or more agents for whom $a$ was bad, now see it as good) the probability/time-share of every other outcome cannot increase. It is easy to check that the utilitarian and random priority mechanisms are both preference monotonic.

Preference Monotonicity is interesting in its own right: it rules out cross-subsidization between outcomes (lobbying to increase the support of $a$ cannot increase the time-share of $b$). Moreover, PM implies strategyproofness.

We check this claim in the simple case where agent $i$ with true utility $U_i$, reports instead $U'_i$ where $U'_i \diagdown U_i = \{a\}$ and $U_i \diagdown U'_i = \{b\}$ (the general case is just as easy). Set $U''_i = U_i \diagdown \{b\}$ and write $U', U''$ for the profiles where $U_i$ is replaced by $U'_i$ and $U''_i$ respectively. By $PM$ and $U'_i = U''_i \cup \{a\}$ we have successively:

$$\begin{aligned} \pi_x(U) &\leq \pi_x(U'') \text{ for all } x \notin U_i \Rightarrow U_i \bullet \pi(U'') \leq U_i \bullet \pi(U) \\ \pi_y(U') &\leq \pi_y(U'') \text{ for all } y \in U_i \Rightarrow U_i \bullet \pi(U') \leq U_i \bullet \pi(U'') \end{aligned}$$

and strategyproofness follows.

On the fairness side, we consider a much weaker lower bound on individual utilities. Given a problem $(N, A, U)$ and solution $p$ :

$$\textit{Positive Share} \text{ (PS)}: \text{ for any } i \in N \ \{U_i \neq 0\} \Rightarrow U_i \bullet p > 0$$

Positive Share merely rules out giving no benefit at all to some agent. Having thus strengthened the incentive compatibility requirement (to PM) and weakened the fairness one (to PS), a striking incompatibility comes up.

**Theorem 2** *Assume* $|N| \geq 6$ *and* $|A| \geq 8$. *There is no mechanism meeting ex-ante efficiency, Preference Monotonicity, and Positive Shares.*

**Remark 1**

In the statement of Theorem 2, we could replace Preference Monotonicity by the Outcome Monotonicity (OM) property. OM compares two problems $(N, A, U)$ and $(N, A', U')$ where $A \subsetneq A'$ and $U$ is the restriction of $U'$ to $A$. It requires $\pi_a(A', U') \le \pi_a(A, U)$ for all $a \in A$: the appearance of new outcomes in the feasible set cannot result in a larger time share for any old outcome.

The interpretations of PM and OM are similar. The two properties are closely related logically as well. See the Appendix for details.

# 8 Complexity and the fair utilitarian solution

We start this section by an informal discussion of the the computational complexity of our three leading solutions, namely utilitarian, Nash and random priority.

Working on small size examples such as the one depicted in Figure 4 and involving 14 agents and 6 outcomes, one sees that the utilitarian solution is the easiest to compute of the three. We remove first all but one copy of clone outcomes, next identify the columns with the largest sum of coordinates and finally take the average of the columns with the largest sum. In Figure 1 we get at once $A_1 = \{a, b\}$ hence the utilitarian solution is $p_a = p_b = \frac{1}{2}$.

Computing the Nash solution is harder because we must discover the support of the optimal lottery, thereby identifying the linear system of first order optimality conditions. In the example the support of the Nash solutions $p$ is $\{a, b, c, e, f\}$. Notice that Fair Group Share guarantees $p_f = \frac{3}{14}$, therefore we only need to solve the reduced system of first order conditions involving agents 1,..., 11 and outcomes $a, b, c, e$ :

$$\frac{2}{p_{abc}} + \frac{1}{p_{ac}} + \frac{3}{p_{ab}} = \frac{2}{p_{abc}} + \frac{3}{p_{ab}} + \frac{1}{p_b} = \frac{2}{p_{abc}} + \frac{1}{p_{ac}} + \frac{2}{p_{ce}} = \frac{2}{p_{ce}} + \frac{2}{p_e}$$

from which we deduce easily $p_a = \frac{4}{15}$, $p_b = \frac{1}{3}$, $p_c = \frac{1}{15}$, $p_e = \frac{1}{3}$.

In general, the Nash solution can be approximated by solving a linear program. One way to see this is simply to approximate the Log function by a piecewise linear function, thus transforming our maximization problem into a linear program, known to be solvable in polynomial time[8]. Another way is to observe that, up to the non-negativity constraints on the lottery $p$, the Nash maximization problem amounts to find the analytic center of a polytope[9], and the latter is approximated in polynomial time. However the degree of the polynomial rises with the pace of approximation, so that an exact computation of the Nash solution may be prohibitively long.

The random priority mechanism appears to be even harder to compute than the Nash one. A probable approximation obtains by a simple Monte Carlo

[8] We thank Rakesh Vohra for this argument.

[9] See e.g., Ye [1997], Chapter 2, Section 2. We thank Eric Friedman for this observation.

method: draw a priority ordering of $N$ at random, compute the corresponding priority outcome, and take the average of these outcomes. But a deterministic approximation is a much more challenging computation. We give in the Appendix an exact computation in exponential time. We conjecture that the random priority solution cannot even be approximated in polynomial time.

In the example, the algorithm described in the Appendix gives the random priority lottery: $p_a = 0.177$, $p_b = 0.214$, $p_c = 0.105$, $p_d = 0.100$, $p_e = 0.190$, $p_f = 0.214$. As with the Nash solution, we take advantage of the fact that $p_f = \frac{3}{14}$, thus reducing the number of agents to 11 and that of outcomes to 5.

We now construct a mechanism as easy to compute as the utilitarian one –indeed, its very definition consists of a sequence of "utilitarian" computations–, and sharing our most demanding properties of fairness and efficiency. In problems of large size, we submit this mechanism as a reasonable alternative to the Nash mechanism.

Starting with a problem $(N, A, U)$ in which there are no clones, we construct first a finite increasing sequence $1 = \lambda_1 < \lambda_2 < ... < \lambda_T$, a partition $N_1, N_2, ..., N_T$ of $N$, and a sequence $A_1, A_2, ..., A_T$ of pairwise disjoint subsets of $A$ such that:

$$u_i^a = 0 \quad \text{if } a \in A_t, i \in N_{t'},\ 1 \le t < t' \le T \tag{1}$$

$$\arg\max_{a \in A} \{u_{N_1}^a + \lambda_2 u_{N_2}^a + ... + \lambda_t u_{N_t}^a\} = \underset{1 \le k \le t}{\cup} A_k, \text{ for all } t = 1, ..., T \tag{2}$$

where we use the notation $u_S^a = \sum_S u_i^a$.

Define first $\lambda_1 = 1$, $A_1 = \arg\max u_N^a$ and $N_1 = \underset{a \in A_1}{\cup} U^a$: $N_1$ contains all agents who like at least one outcome in $A_1$. If $N_1 = N$, the algorithm stops here and $T = 1$. Otherwise, set $\mu = u_N^a$ for any $a \in A_1$ and check $\mu = u_{N_1}^a$ for all $a \in A_1$ and $\arg\max u_{N_1}^a = A_1$.

Next we define for any $a$ the weight $\lambda_2(a)$ as the solution of:

$$u_{N_1}^a + \lambda_2(a) u_{N_1^c}^a = \mu, \text{ or } \lambda_2(a) = +\infty \text{ if } u_{N_1^c}^a = 0$$

Observe that $\lambda_2(a) = 0$ if $a \in A_1$ and $\lambda_2(a) > 1$ otherwise. Define:

$$\lambda_2 = \min_{A_1^c} \lambda_2(a);\ A_2 = \arg\min_{A_1^c} \lambda_2(a);\ N_2 = \{\underset{A_2}{\cup} U^a\} \diagdown N_1$$

By construction $u_i^a = 0$ whenever $i \in N_2$ and $a \in A_1$, and $\arg\max\{u_{N_1+}^a \lambda_2 u_{N_2}^a\} = A_1 \cup A_2$.

The algorithm stops here if $N_1 \cup N_2 = N$. Otherwise we define $\lambda_3(a)$ as the solution of:

$$u_{N_1}^a + \lambda_2 u_{N_2}^a + \lambda_3(a) u_{(N_1 \cup N_2)^c}^a = \mu \quad \text{or } \lambda_3(a) = 0 \quad \text{if } u_{(N_1 \cup N_2)^c}^a = 0$$

and:

$$\lambda_3 = \min_{(A_1 \cup A_2)^c} \lambda_3(a), \; A_3 = \arg\min_{(A_1 \cup A_2)^c} \lambda_3(a), \quad N_3 = \{\underset{A_3}{\cup} U^a\} \diagdown (N_1 \cup N_2)$$

One checks the announced properties (1), (2) at every step of this algorithm. It stops whenever $N_1, ..., N_t$ partition $N$, which must happen after at most $\frac{n+1}{2}$ steps, where $n = |N|$ (see below).

We are now ready to define the *fair utilitarian* solution $p^*$. The idea is that an agent $i \in N_t$ splits the weight $\frac{1}{n}$ equally between all outcomes of $U_i \cap A_t$. Thus:

$$p^*_a = \sum_{i \in N_t \cap U^a} \frac{1}{n\,|U_i \cap A_t|} \text{ if } a \in A_t, \text{ and } p^*_a = 0 \text{ otherwise} \tag{3}$$

**Theorem 3** *The fair utilitarian mechanism is anonymous, neutral, ex-ante efficient, and it satisfies the Fair Group Share.*

From property (2) at $t = T$, every outcome in $\underset{1 \le k \le T}{\cup} A_k$ maximizes a certain linear combination of the utility functions where all coefficients are strictly positive. Any lottery with support in this set is therefore efficient ex ante.

Check now that the fair utilitarian solution meets Fair Group Share. Fix a coalition $S$. As the $i$-th row $U_i$ of the preference matrix is identified with the subset of good outcomes for agent $i$, so are $U_S$ and $\underset{S}{\cup} U_i$, namely the set of outcomes liked by at least one agent in $S$. Writing $S_t = S \cap N_t$, by construction $U_{S_t} \cap A_t$ receives a weight $\frac{1}{n}$ from each $i \in S_t$ therefore:

$$\frac{|S_t|}{n} \le (U_{S_t} \cap A_t) \bullet p \le (U_S \cap A_t) \bullet p$$

Because the subsets $A_t$ are pairwise disjoint, the desired inequality follows by summation over $t$.

In the example of Figure 1 the fair utilitarian algorithm gives:

$A_1 = \{a, b\}$, $N_1 = \{1, 2, ..., 7\}$;

$\lambda_2 = \frac{3}{2}$, $A_2 = \{c, d, e\}$, $N_2 = \{8, 9, 10, 11\}$;

$\lambda_3 = 2$, $A_3 = \{f\}$, $N_3 = \{12, 13, 14\}$

hence the lottery $p_a = p_b = \frac{1}{4}$, $p_c = p_d = \frac{1}{21}$, $p_e = \frac{4}{21}$, $p_f = \frac{3}{14}$

The utility profiles corresponding to our four solutions are given in our last table. Taking away the utilitarian outlier, we see that for all agents the relative variation in utility $\frac{\max\ u - \min u}{\max\ u}$ is below 30%, and for half of the agents it is below 22%.

| | $u_{1,2}$ | $u_3$ | $u_4$ | $u_{5,6}$ | $u_7$ | $u_{8,9}$ | $u_{10,11}$ | $u_{12}$ |
|---|---|---|---|---|---|---|---|---|
| utilitarian | 1 | 0.5 | 1 | 1 | 05 | 0 | 0 | 0 |
| fair utilitarian | 0.596 | 0.298 | 0.548 | 0.5 | 0.25 | 0.286 | 0.190 | 0.214 |
| Nash | 0.524 | 0.262 | 0.472 | 0.472 | 0.262 | 0.314 | 0.262 | 0.214 |
| random priority | 0.596 | 0.282 | 0.491 | 0.391 | 0.214 | 0.395 | 0.190 | 0.214 |

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# APPENDIX: Proofs

## 1. Proposition 2

We assume $N = \{1, 2, 3, 4\}$ and $A = \{a, b, c, d\}$. The generalization to more agents or more outcomes is clear, by "neutralizing" all but four agents and outcomes (i.e., neutral agents like no outcome and neutral outcomes are liked by no one). We fix an ex-post efficient mechanism meeting GSP and derive a contradiction.

Let, when agent 1 likes only $a$, agent 2 likes only $b$, agent 3 likes only $c$, and agent 4 likes only $d$, the solution be $p = (p_1, p_2, p_3, p_4)$. Without loss of generality we can suppose that $p_1 > 0$. We now fix preferences of agents 2, 3, 4 and consider changes in preferences of agent 1 only.

When agent 1 likes all four outcomes, EXP implies that $a$ gets zero probability and hence the solution is $q = (0, q_2, q_3, q_4)$, where $q_2 + q_3 + q_4 = 1$. Note that when agent 1 likes all outcomes she always gets utility 1. Therefore she can deny liking some outcomes without decreasing her utility. GSP implies then that such a lie by agent 1 cannot benefit any other agent. Hence, if agent 1 announces to like $a$ and any non-empty subset of $\{b, c, d\}$ and the preferences of agents 2, 3, 4 remain fixed, the solution must still be $q = (0, q_2, q_3, q_4)$.

Thus, if agent 1 likes $\{a, b, c\}$, then she gets $1 - q_4$. However in this case, if she lies and denies $b$ and $c$, she receives utility $1 - p_4$. Hence, strategyproofness requires $p_4 \geq q_4$. Analogously, we obtain $p_3 \geq q_3$ and $p_2 \geq q_2$. Therefore, $p_2 + p_3 + p_4 \geq q_2 + q_3 + q_4 = 1$, and so $p_1 = 0$, a desired contradiction.

## 2. Proposition 3

We assume $|A| = 6$, $N = \{1, 2, 3, 4\}$, and fix a mechanism satisfying the four properties listed in the statement. We use the same notations as in the previous proof, namely outcome $a$ is identified (by neutrality) with the coalition of agents who like $a$, and a profile is written as a list of at most six coalitions, with the understanding that the list is completed by empty coalitions.

We compute first the lottery selected at a handful of simple profiles, and write such a statement as $[12, 34] \to (\frac{1}{2}, \frac{1}{2})$, where the right-hand side lists the probabilities of all the efficient outcomes in the order in which they appear in the profile on the left-hand side. The above statement follows from anonymity.

Next consider $[12, 34, 3, 3, 4] \to (p, 1 - p)$, where the shares are for the two efficient outcomes 12 and 34. Notice that $p < \frac{1}{2}$ allows a profitable misreport by coalition $\{3, 4\}$ at $[12, 34]$, whereas $p > \frac{1}{2}$ allows such a move by $\{3, 4\}$ at $[12, 34, 3, 3, 4]$. Therefore $[12, 34, 3, 3, 4] \to (\frac{1}{2}, \frac{1}{2})$.

Our next profile is $[12, 3, 4]$, where by anonymity (combined with neutrality), $[12, 3, 4] \to (y, y', y')$ with $y + 2y' = 1$. Next $[12, 3, 3, 4, 4] \to (y, \frac{y'}{2}, \frac{y'}{2}, \frac{y'}{2}, \frac{y'}{2})$ follows, as in the previous paragraph, from weak GSP applied to coalition $\{3,4\}$, anonymity and neutrality. Moreover $[12, 3, 4, 2] \to (y, y', y')$ because, if outcome 12 receives $z > y$, agent 2 has a profitable misreport at $[12, 3, 4]$, whereas $z < y$ gives him such a move at $[12, 3, 4, 2]$. Finally, agent 2 denying outcome "12" changes the latter profile to $[1,3,4,2]$, where by symmetry, each outcome gets probability $\frac{1}{4}$. Therefore strategyproofness implies $y \geq \frac{1}{2}$. The following just proven property will be used below:

$$[12,3,3,4,4] \rightarrow (y, y'', y'', y'', y'') \text{ with } y \geq \frac{1}{2} \text{ and } y + 4y'' = 1$$

Consider next $[12,23,13,4]$ where by symmetry the lottery selected is $(w,w,w,w')$. By strategyproofness applied to agent 4, $[12,23,13,4,4,4] \rightarrow (w,w,w,\frac{w\prime}{3},\frac{w\prime}{3},\frac{w\prime}{3})$, and similarly $[12,23,13,4,4] \rightarrow (w,w,w,\frac{w\prime}{2},\frac{w\prime}{2})$.

At the profile $U = [12,23,13,14,24,34]$ every agent gets utility $\frac{1}{2}$ by symmetry. If each agent $i, i = 1,2,3$, denies outcome $i4$, the reported profile is $[12,23,13,4,4,4]$. Therefore weak GSP for coalition $\{1,2,3\}$ requires $2w + \frac{w'}{3} \leq \frac{1}{2} \Leftrightarrow w \leq \frac{1}{6}$.

Finally we go back to profile $V = [12,23,13,4,4]$ where $(w,w,w,\frac{w\prime}{2},\frac{w\prime}{2})$ is selected. If each agent $i, i = 1,2$ denies $i3$, the reported profile is $[12,3,3,4,4]$ and agents 1,2 each get utility $y + y'' \geq \frac{1}{2}$. This is a profitable manipulation by coalition $\{1,2\}$ : we showed in the previous paragraph that agents 1 and $2's$ common utility at $V$ is $2w \leq \frac{1}{3}$.

# 3. Proposition 5

Assume that $g$ is continuously differentiable. Fix a problem $(N,A,U)$ where $U_i \neq 0$ for all $i$. Then the $f$-collective utility is strictly concave over the set of feasible utility profiles $U$.

Let the (unique!) $\arg\max \sum_N f(u_i) = v = (v_1, ..., v_n)$ where $v_1 \leq ... \leq v_n$. Concavity of $g$ implies that $g'(v_1) \geq ... \geq g'(v_n) \geq 0$.

Since $f$ is increasing, the convex set of feasible utility vectors $u$ lies entirely below the tangent hyperplane at $v$, i.e.,

$$\sum_N \frac{g'(v_i)}{v_i} u_i \leq \sum_N \frac{g'(v_i)}{v_i} v_i = \sum_N g'(v_i) \leq n g'(v_1)$$

Applying this inequality to a feasible utility vector $w$ with $w_1 = 1$, we obtain

$$\frac{g'(v_1)}{v_1} = \frac{g'(v_1)}{v_1} w_1 \leq \sum_N \frac{g'(v_i)}{v_i} w_i \leq n g'(v_1)$$

and hence $v_1 \geq 1/n$, implying all agents get at least Fair Share $1/n$.

The proof remains true if we drop the differentiability assumption. We denote then by $g'$ the multi-valued increasing function s.t. $g'(u_i)$ is the interval between the left and the right derivatives of $g$ at $u_i$, and notice that the coordinates of the normal vector to the tangent hyperplane at $v$ must be selections from $g'(v_i)$.

# 4. Proposition 6

Check first that the Nash solution meets Fair Group Share. Suppose to the contrary that there is a set $S = \{1, ..., m\}$ of agents who like outcomes $1, ..., k$, (i.e., $U_S^a = 1$ for $a = 1, ..., k$) and the Nash solution $p$ is such that $t = p_1 + ... + p_k < k/n$. Consider a small $\varepsilon > 0$ and the probability vector $q$, where $q_a = p_a \exp(\frac{n-k}{n}\varepsilon)$ for $a \leq k$, and $q_a = p_a \exp(-\frac{k}{n}\varepsilon)$ for $a > k$.

Let $N(r) = \sum_i \log U_i \bullet r$ be the collective utility at lottery $r$. We compare $N(p)$ and $N(q)$. Note that the utility of agents 1 to $k$ increased by $\frac{n-k}{n}\varepsilon$ from $p$ to $q$, while the utility of any of the remaining agents could not decrease by more than $\frac{k}{n}\varepsilon$. Thus, we have $N(q) \geq N(p) + \frac{n-k}{n}\varepsilon k - \frac{k}{n}\varepsilon(n-k) = N(p)$.

On the other hand, $\sum_a q_a = t\exp(\frac{n-k}{n}\varepsilon) + (1-t)\exp(-\frac{k}{n}\varepsilon) = t(1 + \frac{n-k}{n}\varepsilon) + (1-t)(1 - \frac{k}{n}\varepsilon) + o(\varepsilon^2) = 1 + (t\frac{n-k}{n} - (1-t)\frac{k}{n})\varepsilon + o(\varepsilon^2) < 1$ for $\varepsilon$ small enough, since it follows from $t < k/n$ that the expression in parentheses is negative. If we now proportionally increase all $q_a$ to make their sum, $\sum_a q'_a$, equal to 1, we obtain that $N(q') > N(q) \geq N(p)$, and hence $p$ cannot be the Nash solution.

To prove the converse statement, assume that $f$ is continuously differentiable.

Consider $n$ agents, an integer $k$, such that $n > k \geq 2$, and $1 + \binom{n}{k-1}$ outcomes. Fix a profile at which there is one outcome $a$, liked by agent 1 only, and for any subset of $k$ agents among $2, ..., n$ there exists an outcome, liked by this subset exactly. Let $p$ be the probability the outcome $a$ gets under the $f$-solution. If we spread the probability $1 - p$ over the outcomes other then $a$, the sum $\sum_{n-\{1\}} f(u_i)$ will be maximized when all $u_i$ are equal, i.e., when each outcome receives probability $(1-p) \Big/ \binom{n}{k-1}$ and the resulting utility vector is $u_1 = p$, $u_2 = ... = u_n = k(1-p)/(n-1)$. By Fair Group Share, $p = u_1 \geq 1/n$ and the total utility of the remaining agents $1 - p \geq (n-1)/n$. Thus, $\max \sum_N f(u_i) = \max_p f(p) + (n-1)f(k(1-p)/(n-1))$ is reached at $p = 1/n$. Assuming $f$ to be differentiable, this implies $f'(1/n) - kf'(k/n) = 0$, or $(1/n)f'(1/n) = (k/n)f'(k/n)$. Since $k$ and $n$ were arbitrary, it follows that $uf'(u)$ is a constant. Hence $uf(u) = a\ln u + b$, and our $f$-solution is the Nash one.

Without the differentiability assumption, $f$ must be continuous on $[0, 1)$ and its derivative $f'$ will be a decreasing interval-valued function. The argument follows similar lines, but becomes more technically involved.

## 5. Proposition 7

Note that by concavity $f$ is continuous on $(0, 1]$.

Fix the integers $k, m$, $k > m \geq 2$ and consider $m+k-1$ agents $i_1, ..., i_k, j_1, ..., j_{m-1}$. Suppose that for any $S \subset I = \{i_1, ..., i_k\}$ of size $m$ there exist an outcome $a(S)$, liked exactly by the agents from $S$, and in addition there is only one outcome $b$, liked by $\{j_1, ..., j_{m-1}\}$. Thus we have $\binom{k}{m} + 1$ outcomes in total. Let $p$ be the total probability allocated to all outcomes $a(S)$ and suppose without loss of generality that agent $i_1$ gets the maximal utility in $I$. Then $u_{i_1} \geq pm/k$, and hence the total probability allocated to the outcomes $a(S)$ with $i_1 \notin S$ is at most $p - pm/k \leq 1 - m/k$.

Assume that the preferences of agent $i_1$ change, and now he also likes $b$. If he announces his old preferences, he gets at least $m/k$. Therefore announcing his new preferences must give him at least $m/k$. Since every outcome is now liked by exactly $m$ agents, the sum of all utilities will be $m$ and hence the sum of the utilities of the remaining $m+k-2$ agents will be at most $m-m/k = m(k-1)/k$, and the least happy of them would have utility at most $m(k-1)/(k(m+k-2))$.

Note that it is possible to find a lottery at which each agent gets utility $m/(m+k-1)$. Choose outcome $b$ with probability $m/(m+k-1)$ and divide the remaining probability equally between all outcomes $a(S)$ with $i_1 \notin S$. Our $f$-solution picks a utility vector maximizing $\sum_i f(u_i)$, therefore $\sum_i f(u_i) \geq \sum_i f(m/(m+k-1)) = (m+k-1)f(m/(m+k-1))$.

On the other hand, concavity of $f$ implies $\sum_i f(u_i) \leq \sum_i f(m/(m+k-1))$ (recall that $\sum_i u_i = m$), with equality if and only if $f$ is linear on $[\min u_i, \max u_i]$. Thus, for any $k, m$, we have equality and $f$ is linear on $[\min u_i, \max u_i] \supseteq [m(k-1)/(k(m+k-2)), m/k]$. Since $m(k-1)/(k(m+k-2)) < m/(m+k-2) \leq m/(k+1)$, these intervals for the same $m$ and consecutive $k$ have intersecting interiors. It follows that $f$ is linear on $(0, m/(m+1)]$ and, given that $m$ is arbitrary, on $(0,1)$ and hence on $(0,1]$.

## 6. Theorem 1

Assume to the contrary that there exists a mechanism $\pi$, satisfying the premises of the theorem. We will need the following two lemmas.

**Lemma 1** *Suppose that $p = \pi(U)$, $p_a = 0$, and an agent $i$ likes the outcome $a$. If $i$ changes her message by denying liking $a$, her utility remains unchanged and the outcome $a$ still gets zero probability.*

**Proof.** Let $u_i$ and $u_i'$ be agent's $i$ utility, respectively under her true and falsified preferences, and $p_a'$ be the probability of $a$ under the latter. SP at the two corresponding profiles $U$, $U'$ implies $u_i \geq p_a' + u_i' \geq p_a' + (u_i - p_a) = p_a' + u_i$. Hence, $u_i = u_i'$ and $p_a' = 0$. ■

**Lemma 2** *Suppose that an outcome $a$ is the only one liked by all agents from $M$, $U^a = M = \{i_1, ..., i_m\}$, for any $i_k \in M$ there exists an outcome $b_k$ liked by $i_k$ only, and there is no outcome liked by some agents from $M$ and some from $N - M$. Then $p_a \geq m/n$.*

**Proof.** We proceed by induction on $m$. The case $m = 1$ is a special case of FS. Assume $m > 1$. Note that EXP implies $u_i = p_a$ for all $i \in M$.

Suppose that there is no outcome liked by all agents in $M - i_m$. If agent $i_m$ claims to only like outcome $b_m$ (with resulting solution $p'$), then by FS $p'_{b_m} \geq 1/n$, and by induction $p_a' \geq (m-1)/n$. Hence by SP, $p_a = u_{i_m} \geq p'_{b_m} + p_a' \geq m/n$.

Suppose now that there are some outcomes liked by all agents in $M - i_m$. By EXP they all must get zero probability. Hence by the previous lemma agent $i_1$ can lie and deny liking those outcomes without changing her utility $u_{i_1} = p_a$. The above argument applied to the new preference profile gives $p_a \geq m/n$ again. ■

$>$From here on assume $N = \{i, j, k, l, m\}$. For any outcome $\alpha$ call the set $U^\alpha$ a "coalition". By EXP, only outcomes liked by inclusion maximal coalitions receive positive probability. We will concentrate on the preference profiles under which there is an outcome $\alpha$, such that $U^\alpha = \{i, j, k\}$, and all other coalitions are of size at most 2. We will represent such profiles graphically, with vertices for the agents, a shaded triangle to denote outcome $\alpha$ and edges to denote maximal coalitions of size 2. The six instances of interest are shown on Figure 5.

We will assume that all non-empty non-maximal coalitions are singletons, i.e. coming from outcomes liked by only one agent. We will further insist that each agent likes exactly 4 outcomes. Thus each diagram on the Figure 5 fully specifies the non-maximal coalitions as well. In the argument below we will often imagine an agent lying, claiming to like or dislike a certain outcome. In each of those cases we will also assume the agent who lies to keep a total of 4 outcomes of which she likes, either by denying one of the outcomes only she likes or claiming to like an outcome no one likes. We will not explicitly specify these fixes below. Our assumption $|A| \geq 17$ allows in each case to construct a preference profile with the above features. For instance for the profile corresponding to the case B we need 11 singleton-liked outcomes in addition to the 4 outcomes liked by at least two agents; two of the remaining "empty" outcomes will be used when we consider below a possibility that the agent 1 denies outcomes $[1, 13]$ and $[1, 3]$ (and thus implicitly assume he claims to like instead another two outcomes, previously not liked by anybody).

In Figure 5 each maximal coalition is labelled by the probability it receives under our mechanism $\pi$. We will derive relations between those probabilities below. Note that we have used anonymity and neutrality assumptions to label certain edges with the same variable. To reference certain agents we have labelled some vertices by circled positive integers. We will denote the outcome liked by coalition $(x, y)$ as $[x, y]$.

Note also that certain edges were labelled by zero. This follows from neutrality and EXA as follows. Consider case F. By neutrality the four outcomes $[9, 11]$, $[9, 12]$, $[10, 11]$, and $[10, 12]$ must all have the same probability. However if $[9, 11]$, and $[10, 12]$ have positive probability, then we can obtain a Pareto superior solution by splitting this probability equally between the outcomes $\alpha$ and $[11, 12]$. A similar remark gives zeroes in case D and shows that in case E either $i$ or $k$ is zero.

Since certain edges have zero probability, we can now apply Lemma 1.

Looking at 10 denying $[10, 11]$, we see $m = h + k$.

Looking at 11 denying $[10, 11]$, we see $1 - m = 1 - h - j - k$, or $m = h + j + k$, hence $j = 0$.

Looking at 10 denying $[10, 11]$ and $[10, 12]$, we see $m = b$.

Looking at 12 denying $[9, 12]$ and $[10, 12]$, we see $1 - m = 1 - e - 2f$, or $m = e + 2f$.

Looking at 7 denying $[7, 8]$, we see $g = h + i$ (since $j = 0$).

Now suppose 3 denies $[1, 3]$ and simultaneously claims to approve an outcome $[2, 3]$. In this case 3 gets $1 - g$, hence by SP we must have $1 - b - c \geq 1 - g$ or $g \geq b + c$. Next suppose 4 denies $[4, 5]$ and simultaneously claims to approve an outcome $[4,6]$. In this case 4 gets $g$, hence by SP we must have $e+f \geq g$. Combining these we see that $e+f \geq h+i = g \geq b+c \geq b = m = h+k = e+2f$.

Thus $c = f = 0$ and $i = k$, but since either $i$ or $k$ must be zero, this says $i = k = 0$. Hence $b = e = g = h = m$. Applying Lemma 1 to 1 denying $[1,13]$ and $[1,3]$, we see $a = b$. Applying Lemma 2 to case A twice we see that $a \geq 3/5$ and $1 - a \geq 2/5$, hence $a = b = e = g = h = m = 3/5$.

Consider now what happens if the true preferences are given by case B from Figure 5. An agent 1 will have utility $b = 2/5$. Suppose he lies, denying the outcome $\alpha$, liked by three agents coalition. The resulting profile is shown on Figure 6. By Lemma 2, $\alpha$ will receive at least 2/5 and by neutrality the remaining outcomes liked by maximal coalitions of size 2 will each receive a third of whatever is left, i.e. at most 1/5. But that means that agent 1 will have utility at least 4/5. Thus SP is violated and no proposed mechanism exists.

# 7. Theorem 2

We prove first the variant of Theorem 2 where Outcome Monotonicity (OM, Remark 1) replaces Preference Monotonicity (PM). The proof of Theorem 2 and the close logical relation between OM and PM are discussed below.

Suppose we have a mechanism meeting OM, EXA and PS. We will look for outcomes that get zero probability. By OM if an outcome gets zero probability and we add additional outcomes (with any approval sets) then that outcome still gets zero probability. We will use this remark to find an instance in which all outcomes approved by a particular agent get zero probability, contradicting PS.

Consider 6 agents $\{0,1,2,3,4,5\}$. Suppose $(i,j,k,m)$ is an (ordered) quadruple of distinct elements of $\{1,2,3,4,5\}$. We will first look at sets of 5 outcomes. The first four will be liked respectively by coalitions $\{i,j\}$, $\{j,k\}$, $\{k,m\}$, and $\{m,i\}$ (forming a "square"), and the fifth one by a coalition obtained by adding 0 to one of these four. For example, consider coalitions $\{i,j\}$, $\{j,k\}$, $\{k,m\}$, $\{m,i\}$, and $\{0,m,i\}$. For this example EXA requires either $\{i,j\}$ or $\{k,m\}$ to get zero probability (and $\{m,i\}$ must of course get zero probability). If $\{i,j\}$ gets zero probability then we will say $\{m,i\}$ zeroes $\{i,j\}$ and write $\{m,i\} \longrightarrow \{i,j\}$. Note that each edge of the square zeroes at least one of its "neighbors".

Suppose $\{m,i\} \longrightarrow \{i,j\}$ and $\{k,m\} \longrightarrow \{j,k\}$. Then it follows from OM that with six outcomes and coalitions $\{i,j\}$, $\{j,k\}$, $\{k,m\}$, $\{m,i\}$, $\{0,m,i\}$, and $\{0,k,m\}$ both $\{i,j\}$ and $\{j,k\}$ get zero probability and hence $u(j) = 0$, contradicting PS. Thus the square must be "oriented", either $\{m,i\} \longrightarrow \{i,j\} \longrightarrow \{j,k\} \longrightarrow \{k,m\} \longrightarrow \{m,i\}$, or the reverse. In particular, each edge zeroes only one of its neighbors.

Consider 6 outcomes with respective coalitions $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,1\}$, $\{1,5\}$, and $\{5,3\}$. These 6 outcomes form 3 squares $(1,2,3,4)$, $(1,2,3,5)$, and $(1,5,3,4)$. It is impossible to orient all 3 squares in such a way that the orientations of any two squares coincide on their two common ages. Therefore without loss of generality we may assume $(1,2,3,4)$ is oriented $\{4,1\} \longrightarrow \{1,2\} \longrightarrow \{2,3\} \longrightarrow \{3,4\} \longrightarrow \{4,1\}$ and $(1,2,3,5)$ is oriented $\{1,5\} \longrightarrow \{5,3\} \longrightarrow$

$\{3,2\} \longrightarrow \{2,1\} \longrightarrow \{1,5\}$. Suppose we add two outcomes with coalitions $\{0,1,4\}$ and $\{0,3,5\}$ respectively. From the square $(1,2,3,4)$ and $\{0,1,4\}$ we see $\{1,2\}$ gets zero probability. >From the square $(1,2,3,5)$ and $\{0,3,5\}$ we see that $\{2,3\}$ gets zero probability. Therefore $u_2 = 0$ contradicting PS.

The above proof is easily turned into a proof of Theorem 2. We fix $A$ with 8 outcomes, and we complete each one of the profiles just discussed by "empty" outcomes. This fix is used everywhere, except in the last profile of the previous paragraph, where all 8 outcomes are non-empty. By PM, if an outcome gets zero probability and the coalition liking certain other outcomes increases, then the former outcome still gets zero probability. Thus the above proof remains valid provided "adding new outcomes" is replaced by "filling an empty outcome".

## Remark

We note that the two properties PM and OM are closely related. First, if we make the assumption that adding "empty" outcomes does not change the solution, then PM clearly implies OM. Conversely, if the mechanism is neutral and EXP, then OM implies PM.

Indeed, suppose an agent $i$ does not like an outcome $a$. Add a new outcome $b$, such that $U^b = U^a \cup \{i\}$. By OM, the probabilities for all outcomes except $b$ do not increase. By EXP outcome $a$ now gets zero probability. Again by OM, deleting outcome $a$ will not decrease the probabilities of other outcomes. Since these probabilities sum to one, deleting $a$ will not change the probabilities. Thus we can replace $a$ by $b$, which is equivalent to changing agent's $i$ preferences in favor of $a$, and the probability of other outcomes will not increase. By neutrality outcomes are interchangeable, and PM follows.

## 8. An algorithm computing the random priority solution.

Given a problem $(N, A, U)$, we denote by $(M, B)$ a subproblem where $M \subseteq N$, $B \subseteq A$, and preferences are the $M \times B$ restriction of $U$. We call a problem $(N, A, U)$ *clean* if it has neither clones or non-efficient outcomes, nor any agent with a single indifference class:

$U^a \subseteq U^b \Rightarrow a = b$ for all $a, b \in A$

$\emptyset \neq U^i \neq A$ for all $i \in N$

Given a clean problem $(N, A, U)$ and a subproblem $(M, B)$, we set $S(M, B) = \{i \in M \mid \emptyset \neq U_i \cap B \neq B\}$and check that $S(M, B) = \emptyset$ if and only if $|B| = 1$, whereas $(S(M, B), B)$ is a clean subproblem if $|B| \geq 2$.

The algorithm starts from a "cleansed" problem $(N, A, U)$ and relies on the following recursive formula for the random priority solution over any subproblem $(M, B)$ :

$$\pi_a(M, B) = \frac{1}{|M|} \sum_{i \in U^a \cap M} \pi_a(S(M, U_i \cap B), U_i \cap B) \text{ for all } a \in B$$

where we set $\pi_a(\emptyset, B) = 1$. To check the formula, notice that $a$ can only be chosen in problem $(M, B)$ if the hightest priority agent is in $U^a \cap M$; given that such an agent $i$ is selected first, observe that all outcomes outside $U_i \cap B$ become irrelevant and all agents outside $S(M, U_i \cap B)$ can similarly be dropped.

Next we fix an outcome $a$ and an ordering $\sigma$ of $U^a$. Writing for simplicity $S(B)$ instead of $S(N, B)$ we construct a sequence $i_i, ... i_{k^*}$ of length at most $|U^a|$ :

$$i_1 \text{ is } \sigma\text{-first in } U^a;\ i_2 \text{ is } \sigma\text{-first in } S(U_{i_1}) \cap U^a$$

$$; ...;\ i_k \text{ is } \sigma\text{-first in } S(V_{k-1}) \cap U^a, \text{ where we set } V_t = \underset{1 \leq k \leq t}{\cap} U_{i_k}$$

The sequence stops at the first integer $k^*$ such that $V_k = \{a\}$, which happens in at most $|U^a|$ steps, because cleanliness implies $\underset{i \in U^a}{\cap} U_i = \{a\}$. Moreover, as long as $V_k$ contains one or more outcomes other than $a$, $S(V_k) \cap U^a \neq \emptyset$ (by cleanliness again). Now we set:

$$\theta(a; \sigma) = (|S(V_1)| \times |S(V_2)| \times ... \times |S(V_{k^*})|)^{-1}$$

and the share of $a$ under random priority is finally given by:

$$\pi_a = \frac{1}{|N|} \cdot \{ \sum_{\sigma \in \mathcal{S}_{U^a}} \theta(\sigma) \}$$

where $\mathcal{S}_{U^a}$ is the set of permutations of $U^a$.

| $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|
| 1 | | 1 | | |
| 1 | | | 1 | |
| 1 | | | | 1 |
| | 1 | 1 | | |
| | 1 | | 1 | |
| | 1 | | | 1 |

Figure 1

| $a$ | $b$ | $c$ |
|---|---|---|
| 1 | $1^*$ | |
| $1^*$ | 1 | |
| | | 1 |

Figure 2

| $a$ | $b$ | $c$ |
|---|---|---|
| 1 | | |
| 1 | 1 | |
| | 1 | 1 |
| | | 1 |

Figure 3

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| 1,2 | 1 | 1 | 1 | 1 | | |
| 3 | 1 | | 1 | | | |
| 4 | 1 | 1 | | 1 | | |
| 5,6 | 1 | 1 | | | | |
| 7 | | 1 | | | | |
| 8,9 | | | 1 | 1 | 1 | |
| 10,11 | | | | | 1 | |
| 12,13,14 | | | | | | 1 |

Figure 4

Figure 1: