Unequal connections

Sanjeev Goyal* Sumit Joshi†

Preliminary version: February 4, 2002

Abstract

Connections seem to matter both at the individual level, in terms how well a person, a firm, or a country does, as well as for the aggregate performance of the system. In this paper we develop a simple model of networks to study the conditions which give rise to differences in the level of connections across individual entities.

We suppose that links are costly to form but also generate benefits for the involved individuals. Our analysis shows that increasing returns from links facilitate the emergence of asymmetric architectures, i.e., networks with an unequal distribution of connections across individuals. Moreover, positive spillovers across links reinforce this tendency and can accentuate inequality in connections. By contrast, decreasing returns from links (and negative spillovers across links), allow for a wider range of architectures, which include symmetric networks as well as asymmetric networks. Several social and economic applications are presented to illustrate the scope of the analysis.

*Department of Economics, Queen Mary College, London, and Econometric Institute, Erasmus University, 3000 DR Rotterdam. E-mail: s.goyal@qmul.ac.uk

†Department of Economics, 624 Funger Hall, George Washington University, 2201 G Street N.W., Washington D.C. 20052. E-mail: sumjos@gwu.edu

1 Introduction

Connections seem to matter both at the individual level as well as in the aggregate. For example, it has been argued that better connected managers are higher achievers in organizations, that firms use collaborations to gain competitive advantage vis-a-vis competitors, and countries which are better linked may exploit their ties to bargain for better terms in particular contexts.[1] Similarly, the aggregate effects of differences in connections can be substantial; for example, the outcome of competition among firms which have very different number of collaborative ties, and hence different cost structures, is different from the outcome where all firms are symmetric. In this paper, we develop a simple model of networks to study the conditions which give rise to differences in the level of connections across individuals.

We consider a strategic game of link formation. There are two stages; in the first stage ex-ante identical players choose to form links with other players. Links are bilateral and costly and are formed only if both players agree to put in the required resources. The links of all players together defines a a network. A network in turn defines a interaction structure within which the second stage action takes place. We suppose that for every network, the second stage game generates a well defined payoff. We then focus on the reduced form payoffs of a single stage game. We use a solution concept that is a refinement of Nash equilibrium: a pair-wise stable equilibrium network is one which is supported by a Nash equilibrium strategy profile, and in addition satisfies the property that there is no pair of players which would be better off by forming an additional link. To keep matters simple, we assume throughout that the costs of link formation are the same for every link and that they are linearly increasing in the number of links. The analysis focuses on the marginal returns from links as a function of the links of a player and the links of his partners.

We start with an analysis of situations in which marginal returns from links are increasing in the number of links a player has. A special case of this situation is one where the returns

[1]There is a large body of theoretical and empirical work on this subject. See e.g., Burt (1992) on careers of professional managers, Delapierre and Mytelka (1998) and Hagedoorn and Schakenraad (1990) on collaborations among firms, and Siedman (2001) on international trading ties. There is an extensive literature in sociology on issues of power and inequality in social networks; see e.g., Wellman and Berkowitz (1988).

from links can be expressed as a function of the number of links a player has and the number of links of the population at large. In this setting, marginal returns to a player from an additional link are the same irrespective of whom he connects with; thus the identity of the partners is not critical per se. We first consider the case of *increasing returns*: given any network, the marginal returns to an additional link are increasing in the number of links formed by the player.[2] In this case we find that equilibrium networks have the *dominant group* architecture (see Figure 1 in section 3, below). This is a structure in which there are essentially two groups of players, one group consists of a completely connected group of players (which is referred to as a complete component) and another group which consists of isolated players. This architecture is thus quite asymmetric, with some players having many links, while others have no links at all. The intuition behind this result is as follows: Consider a network in which two players i and j have formed some links. This means that the marginal returns from the links are larger than the marginal costs of forming them. Given that the costs remain constant across links and marginal returns from links are increasing, it then follows that the two players have an incentive to form a link with each other. Thus every pair of players who have any links must be also linked with each other. This argument does not apply to isolated players who may not have an incentive to link with anyone. Hence, every non-empty network will have at most one non-singleton component, and the rest of the players will be isolated.

We then consider the case of *decreasing* returns: the marginal returns from an additional link are decreasing in the number of links a player has. In this case, it is difficult to provide a general analysis without referring to the effects of a bilateral link on third parties. If these effects are positive (or zero) then equilibrium networks have the following property: *if there are three players i, j and k, such that player i is less linked than player j who in turn is less linked than player k, then it must be the case that players i and j have a link with each other.* This property implies in particular that dominant group architectures cannot arise in equilibrium (see Figure 2 in section 3 below for an illustration of asymmetries that can be sustained). By contrast, if these effects across links are negative then a wide variety of

[2] Cost-reducing collaboration links between Cournot firms and between local monopolies (examples 1 and 2 in section 5 below) satisfy the property of increasing returns.

symmetric as well as asymmetric architectures can arise.[3] The intuition behind this finding is as follows: as a set of players build up links, their marginal returns from further links go down and they may stop forming additional links. The returns to others from forming links, however, may go down even more sharply, due to negative effects, and thus an unequal distribution of links across players can be sustained. This argument also suggests why sharp asymmetries are difficult to sustain when effects across links are small or positive.

We then examine a class of games in which the returns to a player from an additional link depend crucially on the position of the potential partner in the network; specifically, we consider the case where marginal returns to a player i from a link with player j depend on the number of links of player i as well as the number of links of player j. In this analysis we first take up the case of *positive spillovers*: marginal payoffs are increasing in the number of links that players i and j have.[4] In this setting, we find that equilibrium networks have the *dominant group* architecture or the *inter-linked stars* architecture (see Figure 3 in section 4 below). Consider a partition of players into groups corresponding to the number of links players have. An inter-linked stars architecture refers to a partition in which the maximally linked group is linked with all players (and hence to each other), while the minimally linked group consists of players who are only linked to the maximally connected players.[5] Architectures such as interlinked stars (of which the star is a special case) can display greater variation in the number of links across individual playes as compared to dominant group architectures. Our results suggest, therefore, that positive spillovers may sustain and enhance the pressure toward unequal connections and asymmetric architectures that are present in games with increasing marginal returns.

Finally, we study the case of *negative spillovers*: where marginal returns for player i from a link with a player j are decreasing in the number of links that players i and j have.[6] In

[3] Knowledge sharing links between firms engaged in a patent race (example 3 in section 5) satisfy this decreasing returns and negative effects.

[4] Information sharing links for the provision of a pure public good (example 4 in section 5) satisfy this positive spillovers property.

[5] Formal definitions of different network architectures are given in section 2.

[6] Bilateral free-trade agreement links between countries satisfy the negative spillovers property.

this setting, we find that symmetric networks arise naturally. In addition, a moderate level of inequality in links across individuals is also sustainable. In particular, we find that if a pair of players i and j do not have a link with each other, then two players l and k who have more links than i and j cannot have a link either. This property implies that networks such as dominant group architectures (with 2 or more isolated players) and inter-linked stars cannot arise in equilibrium.

The conditions on payoffs we identify are simple and appealing from an economic point of view. In section 5 we apply the general results to several economic and social examples to derive the architecture of equilibrium networks. These applications illustrate the scope of the analysis and also demonstrate existence of equilibrium networks in these settings.

Our paper is intimately related to the research on social structure and economic performance (see e.g., Bala and Goyal (1998), Benabou (1993), Ellison and Fudenberg (1993), Glaezer, Sacerdote and Schienkman (1996), Goyal (1996), and Morris (2000). Also refer to Granovetter (1985) for a discussion of the role of embeddedness in economic transactions and Schelling (1975) for early work in economics on the role of social interaction.) This literature on social structure and economic performance has highlighted the different ways in which structure, broadly construed, effects key economic outcomes. This research is one of the main motivations underlying the attempt to develop a better understanding of the factors which shape the interaction structure. In this paper we take the view that in many interesting settings, individual entities consciously decide on whether to form links/ties with others. Thus social and economic structure arise out of individual incentives. In making these choices, individuals trade off the benefits of forming links against the costs of doing so. The main contribution of the present paper to this literature is to illustrate that this approach yields sharp predictions about the architecture of sustainable networks. In particular, our results highlight the circumstances which facilitate the emergence of unequal distribution of connections across individuals (who are ex-ante identical).

The paper is closely related to the recent literature on the theory of network formation. This is currently a very active field of research; see e.g., Bala and Goyal (2000), Boorman (1975), Calvo (2000), Dutta, van den Nouweland and Tijs (1995), Kranton and Minehart (2001), and Jackson and Wolinsky (1996)). The main contribution of the present paper to the theory of network formation is the use of 'local conditions' on payoffs in explaining differences in levels of connections across players. This approach focuses on how marginal returns for a player i from an additional link with player j can be expressed as a function of two variables, the number of links of the player i and the number of links of the potential partner j. The examples we present in section 5 illustrate that individual incentives are appropriately reflected in the dependence of marginal returns on the local network structure.

Issues relating to group formation and cooperation have long been a central concern of economics in general, and game theory in particular. The traditional approach to these issues has been in terms of coalitions. In recent years, there has been considerable work on coalition formation in games; see e.g., Bloch (1997) and Ray and Vohra (1997). In this literature, group formation is modeled in terms of a *coalition structure* which is a partition of the set of firms. Each player, therefore, can belong to one, and only one, element of the partition, referred to as a *coalition*. In our paper, we consider two-player relationships. In this sense, our model is somewhat restrictive as compared to the work referred to above, which allows for groups of arbitrary size. However, the principal distinction concerns the nature of collaboration structures we examine. Our approach accommodates collaborative relations that are *non-exclusive* (or *non-transitive*). From a conceptual point of view, this distinction is substantive. It means that we allow for relationships across coalitions. Thus, we consider a class of cooperative structures which are significantly different from those studied in the coalition formation literature. Our analysis suggests that these structures – such as arbitrary symmetric networks, or asymmetric networks like stars and interlinked stars – arise quite naturally in a wide class of economic and social games. Further, these structures are also empirically common but are ruled out in the coalition literature.[7]

The paper is organized as follows. Section 2 presents the model. Section 3 studies situations in which marginal returns depend only on the number of links of a player. Section 4 analyzes situations in which marginal returns from a link depend crucially on the number of links of

[7]For a discussion of the relationship between networks and coalitions, see Jackson and Wolinsky (1996).

the potential partner as well. Section 5 illustrates the scope of the results by applying the results to a variety of social and economic examples. Section 6 concludes.

2 The Model

We envision a two-stage model in which players form bilateral links with each other in the first stage. Formation of a link entails a commitment to invest resources in the relationship and, therefore, imposes a fixed cost on the two players involved in the link. The structure of bilateral links between the players defines a network. In the subsequent stage, the players play a non-cooperative game contingent on the links they have established in the first stage. We assume there is a unique second stage Nash equilibrium corresponding to each possible network structure. The Nash equilibrium payoffs can therefore be written as a function of the network structure. We then characterize the set of equilibrium networks under general restrictions on the reduced form payoffs.

2.1 Networks

Let $N = \{1, 2, ..., n\}$ denote a finite set of ex-ante identical players. We shall assume that $n \geq 3$. Every player makes an announcement of intended links. An intended link $s_{i,j} \in \{0, 1\}$, where $s_{i,j} = 1$ means that player i intends to form a link with player j, while $s_{i,j} = 0$ means that player i does not intend to form such a link. Thus a strategy of player i is given by $s_i = \{\{s_{i,j}\}_{j \in N \setminus \{i\}}\}$. Let S_i denote the strategy set of player i. A link between two players i and j is formed if and only if $s_{i,j} = s_{j,i} = 1$. We denote the formed link by $g_{i,j}$. A strategy profile s therefore induces a network $g(s)$. In what follows, for expositional simplicity we shall often omit the dependence of the network on the underlying strategy profile. A *network* $g = \{(g_{i,j})\}$, is a formal description of the pair-wise relationships that exist between the players. We let $\mathcal{G}$ denote the set of all networks (the set of all undirected networks with n vertices.)

$\mathcal{G}$ is a partially ordered set with the ordering relation $\geq$ defined as follows: for $g = \{(g_{i,j})\}$, $g' = \{(g'_{i,j})\}$ in $\mathcal{G}$, $g \geq g'$ if $g_{i,j} \geq g'_{i,j}$ $\forall i, j \in N$. The strict ordering relation is defined as follows: for $g = \{(g_{i,j})\}$, $g' = \{(g'_{i,j})\}$ in $\mathcal{G}$, $g > g'$ if $g_{i,j} \geq g'_{i,j}$ $\forall i, j \in N$ and $g_{i,j} > g'_{i,j}$ for some $i, j \in N$.

A *path* in g connecting players i and j is a distinct set of players $\{i_1, \ldots, i_n\}$ such that $g_{i,i_1} = g_{i_1,i_2} = g_{i_2,i_3} = \cdots = g_{i_n,j} = 1$. We say that a network is *connected* if there exists a path between any pair $i, j \in N$. A network, $g' \subset g$, is a *component* of g if for all $i, j \in g'$, $i \neq j$, there exists a path in g' connecting i and j, and for all $i \in g'$ and $k \in g$, $g_{i,k} = 1$ implies $k \in g'$. We will say that a component $g' \subset g$ is *complete* if $g_{i,j} = 1$ for all $i, j \in g'$. Finally, let $\mathcal{N}_i(g) = \{j \in N : j \neq i,\ g_{i,j} = 1\}$ be the set of players with whom player i has a link in the network g, and let $\eta_i(g) = |\mathcal{N}_i(g)|$ be the cardinality of this set. Given a network g, we will let $g + g_{i,j}$ denote the network obtained by replacing $g_{i,j} = 0$ in network g by $g_{i,j} = 1$. Similarly, we will let $g - g_{i,j}$ denote the network obtained by replacing $g_{i,j} = 1$ in network g by $g_{i,j} = 0$. Let $\eta_i(g)$ be the number of links that player i has in network g and let $n_i(g) = \eta_i(g) + 1$.

Networks in which all players have the same number of links are referred to as *symmetric*. In a symmetric network $\eta_i(g) = \eta$ $\forall i \in N$, and we refer to η as the *degree* of the network. The *complete* network, g^c, is a network in which $\eta = n - 1$, $\forall i \in N$, while the *empty* network, g^e, is a network in which $\eta = 0$, $\forall i \in N$.

If two or more players have an unequal number of links then we say that the network is *asymmetric*. Let $N_1(g), N_2(g), ..., N_m(g)$ be a partition of players, corresponding to the number of links that players have. In particular, if $i, j \in N_k(g)$, $k = 1, 2, ..m$, then $\eta_i(g) = \eta_j(g)$. We note that k here refers to the order in the partition and not the precise number of the links that players have. An *inter-linked stars* architecture is one in which there are at least two members in this partition and the minimum and maximum linked groups satisfy the following two conditions: one, there exist players who have links with all the other players, in other words, $\eta_i(g) = n - 1$ for all $i \in N_m(g)$ and two, there are players who have links with only the universally linked players and no one else, $\mathcal{N}_i(g) = N_m(g)$ for all $i \in N_1(g)$. We note that since players in $N_m(g)$ are linked to all other players, they are also linked to each other; the network, therefore, resembles a set of inter-linked stars. We note that the star network is a special case of this architecture, with $|N_m(g)| = 1$ and $|N_1(g)| = n - 1$. A *dominant group* architecture is characterized by one complete non-singleton component and a set of singleton players. Thus there are two groups, $N_1(g)$ and $N_2(g)$, with the property that $\eta_i(g) = 0$, for $i \in N_1(g)$, while $\eta_j(g) = |N_2(g)| - 1$, for $j \in N_2(g)$.

2.2 Payoffs, equilibrium and stability

Let $\pi_i(g)$ denote the reduced form gross payoffs of player i. We will impose the following restrictions on gross payoffs:

(**A.1**) *Anonymity*: For any network $g = \{(g_{i,j})\}$, and any permutation function on the set of players $\rho : N \to N$, let $g^\rho = \{(g_{\rho(i),\rho(j)})\}$. Then:

$$\pi_i(g) \; = \; \pi_{\rho(i)}(g^\rho), \quad i \in N$$

(**A.2**) *Gross Payoff Monotonicity*: In any network g:

$$\pi_i(g + g_{i,j}) \geq \pi_i(g), \quad \pi_j(g + g_{i,j}) \geq \pi_j(g)$$

The anonymity condition formalizes the idea that the identity of the players in a network does not matter. The networks g and g^ρ have the same architecture but with different players at the nodes. Anonymity states that the gross payoff of player i in g is equal to that of player $\rho(i)$ who occupies the same node in g^ρ as i did in g. The payoff monotonicity condition says that given a network, g, a player always gains in terms of gross payoffs from forming an additional link.

Given a strategy profile $s = \{s_1, s_2, ..., s_n\}$, the (net) payoffs to a player are given by

$$\Pi_i(s_i, s_{-i}) = \pi_i(g(s)) - \eta_i(g(s))f \tag{1}$$

where f is the cost incurred by each player when a link is formed.

We now present two classes of games that are covered by our analysis. Given a network, g, let g_{-i} be the network obtained by deleting all the links of player i. In the first class of games, the payoffs to a player, i, from a network, g, can be written as follows:

$$\Pi_i(g) = \phi_i(g_{-i}, \eta_i(g)). \tag{2}$$

Thus the payoffs of a player i from a network g depend on the complementary network g_{-i} and the number of links he maintains in the network g, $\eta_i(g)$. The crucial feature here is that the identity of the partners – *viz.* the number of their links and more generally their location in the network – does not affect the payoffs of player i. Given a network g, the marginal payoffs of player i from an additional link are given by

$$\Pi_i(g + g_{i,j}) - \Pi_i(g) = \phi_i(g_{-i}, \eta_i(g) + 1) - \phi_i(g_{-i}, \eta_i(g)). \tag{3}$$

More generally, let k be the number of links of player i. It is useful to write the marginal returns from an additional link in terms of the complementary network g_{-i} and the number of links of player i as follows:

$$\Delta\phi_i(g_{-i}, k) = \phi_i(g_{-i}, k + 1) - \phi_i(g_{-i}, k) \tag{4}$$

The first part of our analysis deals with the implications of increasing and decreasing marginal returns and covers this class of games as a special case.

We also consider a class of games in which the payoffs of a player can be expressed as a function of his own links and the links of his partners:

$$\Pi_i(g) = \Phi_i(\eta_i(g), \{\eta_j(g)\}_{j \in \mathcal{N}_i(g)}). \tag{5}$$

Given a network g, the marginal payoffs to player i, from forming a link with player j are given as follows:

$$\Pi_i(g + g_{i,j}) - \Pi_i(g) = \Phi_i(\eta_i(g + g_{i,j}), \{\eta_k(g)\}_{k \in \mathcal{N}_i(g+g_{i,j})}) - \Phi_i(\eta_i(g)), \{\eta_k(g)\}_{k \in \mathcal{N}_i(g)}) \quad (6)$$

We develop conditions on marginal returns which relate to number of links of a player and his partners. This analysis covers the second class of games as a special case.

We study the architecture of networks that are strategically stable. Our notion of strategic stability is a refinement of Nash equilibrium. A strategy profile $s^* = \{s_1^*, s_2^*, ..., s_n^*\}$ is said to be a Nash equilibrium if $\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*)$, for all $s_i \in S_i$, and for all $i \in N$. In our model, a link requires that both players acquiesce in the formation of the link. It is then easy to see that an empty network is always a Nash equilibrium. More generally, for any pair i and j, it is always a mutual best response for the players to offer to form no link. To avoid this potential coordination problem we supplement the idea of Nash with the requirement of pair-wise stability. An equilibrium network is said to be pair-wise stable if players have no incentive to form a link that does not exist in the network. We borrow this idea from Jackson and Wolinsky (1996). We will focus on pair-wise stable equilibrium networks.

Definition 2.1 *A network g is a pair-wise stable equilibrium network if the following conditions hold.*

1. *There is a Nash equilibrium strategy profile which supports g.*
2. *For $g_{i,j} = 0$, $\pi_i(g + g_{i,j}) - \pi_i(g) > f \implies \pi_j(g + g_{i,j}) - \pi_j(g) < f$*

In what follows, for expositional simplicity we shall use the short form – pws-equilibrium – while referring to pair-wise stable equilibrium networks. It is arguable that the notion of pair-wise stability is very mild; for instance it does not consider deviations by more than two players. A more general stability requirement would allow for coordinated deviations by several players. In some of the settings we analyze, it is possible to see that this will alter the outcome (see example 2 in section 5 below). We choose to work with the notion of a pws-equilibrium on grounds of tractability: indeed, the main point of our paper is that this mild requirement by itself sharply restricts the set of equilibrium networks and gives us several insights into the relation between individual incentives and network architectures.

3 Own links

In this section we consider a class of network formation games in which the marginal returns from links for a player are increasing or decreasing in the number of links of the player.[8] Our analysis develops a characterization of equilibrium network architectures.

3.1 Increasing Returns

We start by examining the case where each player's marginal payoffs are strictly increasing in his own links. Formally:

Definition 3.1 *Fix some network g. The marginal payoff function of player i is strictly increasing in own links if for any distinct $j, k \in N$:*

$$\pi_i(g + g_{i,k} + g_{i,j}) - \pi_i(g + g_{i,j}) \ > \ \pi_i(g + g_{i,j}) - \pi_i(g) \tag{7}$$

We say that the payoff function of player i exhibits increasing returns (IR) if this property is true for any network $g \in \mathcal{G}$.

To facilitate interpretation of this definition, it is useful to recapitulate the first class of payoff functions we mentioned in section 2. The payoff function is given by $\Pi_i(g) = \phi_i(g_{-i}, \eta_i(g))$ and the marginal payoffs are given by $\Delta\phi_i(g_{-i}, k) = \phi_i(g_{-i}, k+1) - \phi_i(g_{-i}, k)$. Increasing

[8]In section 5 below we shall present a class of economic and social problems which satisfy this property. In particular, we shall show that R&D collaboration between firms in different market settings (Examples 1 and 2) satisfy the increasing returns property, while the patent race model (Example 3) satisfies the decreasing returns property.

returns imply that for every g_{-i}, $\Delta\phi_i(g_{-i}, k)$ is strictly increasing with respect to k. We note that we make no assumptions regarding how marginal payoffs are affected by links formed by other players.

Increasing returns imply the following "transitivity" property of link formation:

Lemma 3.1 *Consider a pws-equilibrium network g. Consider two players $i, j \in N$. If $g_{i,k} = g_{j,l} = 1$ for some $k, l \in N \backslash \{i, j\}$, then $g_{i,j} = 1$.*

Proof: Assume to the contrary that for some $i, j, k, l \in N$, $g_{i,k} = g_{j,l} = 1$ in a pws-equilibrium network g but $g_{i,j} = 0$. By the definition of pws-equilibrium:

$$\begin{aligned} \pi_i(g) - \pi_i(g - g_{i,k}) &\geq f \\ \pi_j(g) - \pi_j(g - g_{j,l}) &\geq f \end{aligned} \tag{8}$$

Since each player's marginal payoffs are increasing in own links:

$$\begin{aligned} \pi_i(g + g_{i,j}) - \pi_i(g) > \pi_i(g) - \pi_i(g - g_{i,k}) &\geq f \\ \pi_j(g + g_{i,j}) - \pi_j(g) > \pi_j(g) - \pi_j(g - g_{j,l}) &\geq f \end{aligned} \tag{9}$$

This implies that players i and j have an incentive to form a link and, therefore, g is not a pws-equilibrium, a contradiction that completes the proof. △

The above lemma helps us to develop the following characterization of pws-equilibrium networks:

Proposition 3.1 *Suppose the payoffs of each player satisfy (IR). Then, in the class of symmetric networks, the only pws-equilibrium networks are the empty and complete networks, while in the class of asymmetric networks, the only pws-equilibrium networks are those with the dominant group architecture.*

Proof: Consider the class of symmetric networks. For high enough f, the empty network is trivially stable. Now consider a non-empty pws-equilibrium symmetric network $g \neq g^c$ with degree $0 < \eta < n-1$. Since g is symmetric, we can find two players i and j such that $g_{i,l} = g_{j,m} = 1$ for some $l, m \in N$ but $g_{i,j} = 0$. However, from Lemma 3.1, this contradicts the hypothesis that g is a pws-equilibrium.

Next consider the class of asymmetric networks. We first show that any asymmetric network can have at most one non-singleton component. Let $C_1(g)$ and $C_2(g)$ be two non-singleton components in g. Suppose that $i, j \in C_1$ with $g_{i,j} = 1$ and $l, m \in C_2$, with $g_{l,m} = 1$. From the definition of a component, $g_{i,l} = 0$. However, from Lemma 3.1, this contradicts the hypothesis that g is a pws-equilibrium.

Next we show that the non-singleton component must be complete. If it is incomplete, then there are players i and j in the component such that $\eta_i(g) \geq 1$, $\eta_j(g) \geq 1$ and $g_{i,j} = 0$. However, from Lemma 3.1, this contradicts the pws-equilibrium hypothesis concerning g. $\triangle$

This result suggests that symmetric networks are generally difficult to sustain in the presence of increasing returns. Moreover, increasing returns also sharply restrict the nature of asymmetric networks that can arise: the dominant group architecture is the only candidate for equilibrium. An interesting question is how the size of the dominant group, k, varies with the cost of link formation, f. This requires an explicit comparison between the marginal returns and costs of forming links. We explore this issue in the context of *R&D* collaboration between firms in section 5. Figure 1 illustrates dominant group networks.

— Insert Figure 1 somewhere here —

3.2 Decreasing returns

We now take up the case where marginal returns to an additional link are decreasing.

Definition 3.2 *Fix some network g. The marginal payoff function of player i is strictly decreasing in own links if for any distinct $j, k \in N$:*

$$\pi_i(g + g_{i,k} + g_{i,j}) - \pi_i(g + g_{i,j}) \ < \ \pi_i(g + g_{i,j}) - \pi_i(g) \tag{10}$$

We say that the payoff function of player i exhibits decreasing returns if this property is true for any network $g \in \mathcal{G}$.

The analysis in this subsection will focus on the payoff functions of the form, $\Pi_i(g) = \phi(g_{-i}, \eta_i(g))$. In this context, the property of decreasing returns is stated as follows.

$$\textbf{(DR)} \quad \text{For all } \ g_{-i}, \ \ \phi_i(k+1, g_{-i}) - \phi_i(k, g_{-i}) < \phi_i(k, g_{-i}) - \phi_i(k-1, g_{-i}), \quad \forall\ k \in \{1, .., n-2\}.$$

The general analysis of equilibrium networks in the presence of decreasing returns is complicated. The main factor complicating the analysis is the presence of effects across links. To simplify matters we shall therefore take up different cases of effects – zero, positive and negative – separately. We start with the simplest case: zero effects.

Zero-effects We model zero-effects as follows:

$$\textbf{(ZE)} \qquad \pi_i(g) = \phi(\eta_i(g)), \forall\ g \in \mathcal{G}.$$

The following result helps in the delineation of sustainable symmetric networks.

Lemma 3.2 *Assume that the payoff function of each player satisfies (DR) and (ZE). For generic values of cost of forming links, f, there is at most one symmetric equilibrium.*

Proof: Suppose there are two symmetric networks with degree k and k', with $k < k'$ that are pws-equilibrium networks. Since the k-degree symmetric network is a pws-equilibrium, it follows from the definition of pair-wise stability and symmetry of the network that $\phi_i(k+1) - \phi_i(k) \leq f$. Similarly, since the k'-degree network is a pws-equilibrium network, it follows that $\phi_i(k') - \phi_i(k'-1) \geq f$. Since $k' > k$, the two inequalities cannot be simultaneously satisfied for generic values of f. △

We now turn to asymmetric networks.

Lemma 3.3 *Assume that the payoff function of each player satisfies (DR) and (ZE). Consider a pws-equilibrium network g. For generic values of f, the following property obtains: if $i \in N_q(g)$ and $j \in N_r(g)$, $1 \leq q, r < m$, then $g_{i,j} = 1$.*

Proof: We start by proving the following property: for any $i, j \in N$, if there exists a player $k \in N\backslash\{i, j\}$ such that $\eta_i(g) \leq \eta_j(g) < \eta_k(g)$, then $g_{i,j} = 1$. Suppose to the contrary that in a pws-equilibrium network g there exist players $i, j, k \in N$ such that $\eta_i(g) \leq \eta_j(g) < \eta_k(g)$ but $g_{i,j} = 0$. Then it is true that

$$\begin{aligned} f &\leq \pi_k(g) - \pi_k(g - g_{k_1,k}) \\ &= \phi(\eta_k(g)) - \phi(\eta_k(g) - 1) \\ &\leq \phi(\eta_j(g) + 1) - \phi(\eta_j(g)) \\ &= \pi_j(g + g_{i,j}) - \pi_j(g)) \end{aligned} \tag{11}$$

where the first weak inequality follows from pws-equilibrium hypothesis concerning g, the equality follows from zero-effects, and the weak-inequality follows from the fact that $\eta_j(g) < \eta_k(g)$, while the final inequality follows from the hypothesis that g is an equilibrium. For generic values of f, the first inequality is strict. An identical argument can be made for player i. Hence players i and j have an incentive to form a link. The proof follows. △

We summarize our analysis of the zero effects case in the following result.

Proposition 3.2 *Assume that the payoff function of each player satisfies (DR) and (ZE). (a). In the class of symmetric networks, a variety of networks can be supported depending on the costs of forming links. However, for generic values of f, there is at most one degree that can be supported in a pws-equilibrium. (b). In the class of asymmetric networks, a* *pws-equilibrium network has the property that all non-maximally connected players have a direct link.*

The above Proposition 3.2 has some interesting implications for the structure of asymmetric pws-equilibrium networks. First, if the players in $N_m(g)$, the element of the partition of N with the maximum number of links, form their own component, then the pws-equilibrium network will have only two components. Second, architectures such as the dominant group (with at least two singleton players), stars and interlinked stars are not sustainable in equilibrium.

Positive effects: We now take up the case of positive effects with respect to links of others. This is modeled as follows:

(PE) If $g_{-i} < g'_{-i}$, then $\phi_i(g_{-i}, k+1) - \phi_i(g_{-i}, k) < \phi_i(g'_{-i}, k+1) - \phi_i(g'_{-i}, k)$,
$\forall\, k \in \{1, 2, , .., n-2\}$.

We start with a consideration of symmetric networks. We expect that multiple symmetric networks can be sustained in equilibrium for a given value of f. The intuition is as follows. Suppose that a certain degree network is sustainable in a pws-equilibrium. Then it is possible that a higher degree network can also be sustained, since a higher degree can lead to higher marginal payoffs due to positive effects and this may offset the decreasing returns effect in some cases. We present a simple example to illustrate this intuition.

Example 3.1: Suppose $n = 4$ and individual payoffs are given as follows:

$$\pi_i(g) = \alpha\sqrt{\eta_i(g)} + \beta \sum_{j \neq i} \eta_j(g)\eta_i(g), \tag{12}$$

where $\alpha > \beta > 0$.[9] It follows that:

$$\phi_i(g_{-i}, \eta_i + 1) - \phi_i(g_{-i}, \eta_i) = \alpha(\sqrt{\eta_i + 1} - \sqrt{\eta_i}) + \beta \sum_{k \neq i} \eta_k + \beta + \beta\eta_i \tag{13}$$

[9]Note that if $\beta = 0$, then we are back to the case where the payoff function satisfies (DR) and (ZS).

It can be verified that the payoff function satisfies (DR) for α sufficiently greater than β. Further, it satisfies (PE) since an increase in g_{-i} leads to an increase in $\sum_{k\neq i} \eta_k$.

In this example, the symmetric pws-equilibrium networks are characterized as follows: *(a). If $\alpha + \beta < f$, then the empty network is a pws-equilibrium network. (b). If $0.41\alpha + 5\beta < f \leq \alpha + 3\beta$, then the symmetric network of degree 1, g^1, is a pws-equilibrium. (c). If $0.32\alpha+9\beta < f \leq 0.41\alpha+7\beta$ then the symmetric network of degree 2, g^2, is a pws-equilibrium. (d). If $f \leq 0.32\alpha + 11\beta$ then the complete network is a pws-equilibrium.*

This characterization can be derived by computing the payoffs to different players in the different networks. The payoffs in the complete network are given by $\Pi_i(g^c) = 1.72\alpha + 27\beta - 3f$. The payoffs in the empty network are given by $\Pi_i(g^e) = 0$. In the symmetric network of degree 1, $\Pi_i(g^1) = \alpha + 3\beta - f$, while in the symmetric network of degree 2, $\Pi_i(g^2) = 1.41\alpha + 12\beta - 2f$. The characterization stated above can be derived using these payoffs.

If we set $\alpha = 40$ and $\beta = 1$ then g^1, g^2 and g^c are pws-equilibrium networks over the range $21.8 < f \leq 23.4$. $\triangle$

What is the nature of asymmetric networks that can be sustained in equilibrium? In the above example it can be verified that *the only asymmetric pws-equilibrium network is the dominant group network with three players in a complete component and one isolated player, and this occurs over the range* $0.32\alpha + 7\beta < f \leq 0.41\alpha + 5\beta$.[10] This observation is in fact a special case of a more general result characterizing asymmetric equilibrium networks.

Lemma 3.4 *Suppose g is an asymmetric pws-equilibrium network. If $i, j \notin N_m(g)$, then $g_{i,j} = 1$.*

Proof: Take some player $k \in N_m(g)$. It follows from the pws-equilibrium hypothesis that

$$\phi_k(g_{-k}, \eta_k(g)) - \phi_k(g_{-k}, \eta_k(g) - 1) \geq f. \tag{14}$$

[10] This follows by noting that the payoff to a player in the complete component is $\Pi_i(g) = 1.41\alpha + 8\beta - 2f$ while the payoff to the isolated player is zero and then verifying the equilibrium conditions.

Suppose that $i, j \notin N_m(g)$, and that $g_{i,j} = 0$. Then it follows that either $\phi_i(g_{-i}, \eta_i(g) + 1) - \phi_i(g_{-i}, \eta_i(g)) \leq f$, or $\phi_j(g_{-j}, \eta_j(g) + 1) - \phi_j(g_{-j}, \eta_j(g)) \leq f$, or both. However, note that

$$g_{-i} = g_{-i-k} + \{g_{l,k}\}_{l \in N_k(g)}, l \neq i, k.$$
$$g_{-k} = g_{-i-k} + \{g_{i,m}\}_{m \in N_i(g)}, m \neq i, k. \tag{15}$$

Since $\eta_i(g) < \eta_k(g)$, it follows that $g_{-i} > g_{-k}$ (with a slight abuse of notation). Hence it follows that

$$\phi_i(g_{-i}, \eta_i(g) + 1) - \phi_i(g_{-i}, \eta_i(g)) \tag{16}$$
$$> \phi_i(g_{-k}, \eta_i(g) + 1) - \phi_i(g_{-k}, \eta_i(g)) \tag{17}$$
$$\geq \phi_k(g_{-k}, \eta_k(g)) - \phi_k(g_{-k}, \eta_i(g) - 1) \geq f. \tag{18}$$

In the above expression, the first inequality follows from (PE), while the second inequality follows from noting that $\eta_k(g) > \eta_i(g)$ and applying (DR). Note that we use the anonymity assumption in deriving this inequality as well. The final inequality follows from the equilibrium hypothesis. Similar reasoning establishes that

$$\phi_j(g_{-j}, \eta_j(g) + 1) - \phi_j(g_{-j}, \eta_j(g)) > f \tag{19}$$

Hence, i and j have a strict incentive to form a link, which contradicts our starting hypothesis that g is a pws-equilibrium. △

We are now in a position to summarize our analysis of network formation under decreasing returns and positive effects.

Proposition 3.3 *Suppose the payoff function of every player satisfies (DR) and (PE). For a given value of f, symmetric networks of different degrees can arise in equilibrium. Asymmetric pws-equilibrium networks have the property that all non-maximally linked players have a direct link.*

The first part of the above result shows that symmetric networks are easily sustained in equilibrium, and that for the same value of f, different degree networks can arise. This is in marked contrast to the situation under increasing returns where symmetric networks are in general not sustainable in equilibrium. The second part of the result rules out networks such as stars and dominant group architectures with two or more isolated players. It also rules out inter-linked stars. However, positive effects do allow for asymmetrically-sized complete components. One may interpret this result as saying that (DR) and (PE) together imply that networks can be at most moderately asymmetric. Figure 2 gives some examples of equilibrium networks.

— Insert Figure 2 somewhere here —

Negative effects: We finally take up the case where links of third parties have a negative spillover on a players' marginal returns. The idea of negative effects is modeled as follows:

$$\textbf{(NE)} \text{ If } g_{-i} < g'_{-i}, \text{then } \phi_i(g_{-i}, k+1) - \phi_i(g_{-i}, k) > \phi_i(g'_{-i}, k+1) - \phi_i(g'_{-i}, k), \quad \forall\, k \in \{1, 2, , .., n-2\}.$$

We first take up the case of symmetric networks.

Lemma 3.5 *Consider the class of symmetric networks. For a given value of f, symmetric networks of at most one degree k can be sustained in a pws-equilibrium.*

Proof: Suppose k^* and $\hat{k}$ are pws-equilibrium and $k^* < \hat{k}$. Then it follows that

$$\begin{aligned} &\phi_i(g^{k^*}_{-i}, k^*+1) - \phi_i(g^{k^*}_{-i}, k^*) \leq f. \\ &\phi_i(g^{k^*}_{-i}, k^*) - \phi_i(g^{k^*}_{-i}, k^*-1) \geq f. \end{aligned} \tag{20}$$

and moreover

$$\phi_i(g^{\hat{k}}_{-i}, \hat{k}) - \phi_i(g^{\hat{k}}_{-i}, \hat{k}-1) \geq f. \tag{21}$$

Combining these equations yields us the following:

$$
\begin{aligned}
f &\le \phi_i(g^{\hat{k}}_{-i}, \hat{k}) - \phi_i(g^{\hat{k}}_{-i}, \hat{k}-1) \\
&< \phi_i(g^{k^*}_{-i}, \hat{k}) - \phi_i(g^{k^*}_{-i}, \hat{k}-1) \\
&\le \phi_i(g^{k^*}_{-i}, k^*+1) - \phi_i(g^{k^*}_{-i}, k^*) \le f.
\end{aligned} \tag{22}
$$

The first inequality follows from the hypothesis that $g^{\hat{k}}$ is an equilibrium network, the second inequality follows from (NE), while the third inequality follows from (DR). The final inequality follows from the hypothesis that g^{k^*} is an equilibrium. This generates a contradiction which completes the proof. △

We now explore the nature of asymmetric equilibrium networks. We have been unable to provide a general characterization here. The following example illustrates some of the issues that arise.

Example 3.2: Suppose $n = 3$ and individual payoffs are given as follows:

$$\pi_i(g) = \alpha\sqrt{\eta_i(g)} - \beta \sum_{j \neq i} \eta_j(g)\eta_i(g), \tag{23}$$

where $\alpha > \beta > 0$. It is possible to verify that this payoff function satisfies (DR) and (NE). This is done as follows:

$$\phi_i(g_{-i}, \eta_i + 1) - \phi_i(g_{-i}, \eta_i) = \alpha(\sqrt{\eta_i + 1} - \sqrt{\eta_i}) - \beta \sum_{k \neq i} \eta_k - \beta - \beta\eta_i. \tag{24}$$

Clearly, this is decreasing in η_i and also decreasing in g_{-i}, since an increase in g_{-i} leads to an increase in $\sum_{k \neq i} \eta_k$.

In this example, the pws-equilibrium networks are characterized as follows: *(a). If* $f < 0.41\alpha - 5\beta$ *then the unique pws-equilibrium is the complete network. (b). If* $0.41\alpha - 5\beta < f < 0.41\alpha - 3\beta$ *then the unique pws-equilibrium is the star network. (c). If* $0.41\alpha - 3\beta < f < \alpha - \beta$ *then the unique pws-equilibrium is the partially connected network. (d). If* $\alpha - \beta < f$ *then the unique pws-equilibrium is the empty network.*

This characterization can be derived by computing the payoffs to different players in the different networks. The payoffs in the complete network are given by $\Pi_i(g^c) = 1.41\alpha - 8\beta - 2f$. The payoffs in the empty network are given by $\Pi_i(g^e) = 0$. In the partially connected network there is one pair of players linked and one isolated player. The payoffs of the linked players are as follows: $\Pi_i(g^{pc}) = \alpha - \beta - f$, while the payoffs of the isolated player are given by $\Pi_j(g^{pc}) = 0$. Finally, in the star there are two peripheral players with one link each while there is a central player with two links. The payoffs of the central player are given by $\Pi_i(g^s) = 1.41\alpha - 4\beta - 2f$, while the payoffs of the peripheral players are given by $\Pi_j(g^s) = \alpha - 3\beta - f$. The proof now follows from a straightforward comparison of these payoffs. △

This example illustrates that a variety of architectures, including some with sharp asymmetries such as the star, can be sustained in equilibrium if payoffs display decreasing returns and negative effects.

Several remarks can be made on the basis of the results we have obtained in this section. First, we note the effect of marginal returns on sustainability of symmetric networks in equilibrium. We found that it is difficult to sustain symmetric networks under increasing returns with respect to own links. By contrast, symmetric networks can be sustained in equilibrium under decreasing returns. Second, we note the effects on asymmetric networks. Here we found that increasing returns lead to a very specific form of asymmetric networks: the dominant group architecture. By contrast, the nature of asymmetric networks under decreasing differences depends on the spillover effects of third party links. If effects are absent or positive, then we found that dominant group and star like structures are not sustainable in equilibrium. In general, asymmetries across links of players are limited in equilibrium. Finally, if effects across links are negative, then stars and other asymmetric networks can arise in equilibrium.

4 Links of the Partner

In this section we consider a class of games in which the marginal returns of a player from a link are a function both of the number of links of the player and that of the potential partner. We distinguish between positive and negative spillovers based respectively on whether marginal payoffs from a link are influenced favorably or adversely by the number of links of the potential partner and the level of connectedness of the network.

4.1 Positive spillovers

Our notion of positive spillovers encompasses several related ideas: one, that marginal payoffs from a link increase as the network gets more connected. Two, that marginal payoffs are greater the more connected is the player with whom the link is being formed. Finally, that marginal returns from a link are greater the greater the number of links the player forming the link has. These ideas are formally defined below.

Definition 4.1 *Fix some network g. The payoff function of player i satisfies the positive spillovers (PS) property if:*

1. $\pi_i(g + g_{l,k} + g_{i,j}) - \pi_i(g + g_{l,k}) \;\geq\; \pi_i(g + g_{i,j}) - \pi_i(g)$
2. $\pi_i(g + g_{i,j}) - \pi_i(g) > \pi_i(g + g_{i,l}) - \pi_i(g)$ *if* $\eta_j(g) > \eta_l(g)$
3. $\pi_i(g + g_{i,k}) - \pi_i(g) > \pi_j(g + g_{j,l}) - \pi_j(g)$ *if* $\eta_i(g) > \eta_j(g)$ *and* $\eta_k(g) \geq \eta_l(g)$

It is worth noting that increasing returns and positive spillovers are distinct conditions on payoffs. This is easiest to illustrate by looking at examples in section 5. Example 1 which deals with collaboration among Cournot firms satisfies increasing returns, but violates positive spillovers (cf. conditions (1) and (2) in positive spillovers). On the other hand, example 3 about information sharing of a public good satisfies positive spillovers but violates increasing returns. (An implication of this is the finding that the star network can be an equilibrium under positive spillovers but is ruled out under increasing returns.) We now characterize pws-equilibrium networks for payoff functions which satisfy the positive spillovers property.[11] We start with a consideration of symmetric networks.

Lemma 4.1 *Suppose payoffs of every player satisfy (PS) and let g be a symmetric network. If g is a pws-equilibrium network then it is either empty or complete.*

Proof: Let g be a symmetric network which is neither empty nor complete. Let every firm have k links, with $0 < k < n-1$. Suppose g is an pws-equilibrium. Then it follows that there exist firms i and j with $g_{i,j} = 0$, whose payoffs satisfy the following conditions:

$$\begin{aligned} \pi_i(g) - \pi_i(g - g_{i,l}) &\geq f \\ \pi_j(g) - \pi_j(g - g_{j,p}) &\geq f \end{aligned} \tag{25}$$

with $l, p \neq i, j$. Since g is symmetric $\eta_j(g - g_{i,l}) > \eta_l(g - g_{i,l})$, and $\eta_i(g - g_{j,p}) > \eta_p(g - g_{j,p})$. From part (1) of positive spillovers it follows that $\pi_i(g + g_{i,j}) - \pi_i(g) \geq \pi_i(g - g_{i,l} + g_{i,j}) - \pi_i(g - g_{i,l})$. From part (2) of positive spillovers it follows that $\pi_i(g - g_{i,l} + g_{i,j}) - \pi_i(g - g_{i,l}) > \pi_i(g) - \pi_i(g - g_{i,l})$. Thus we have the following implication:

$$\begin{aligned} \pi_i(g + g_{i,j}) - \pi_i(g) > \pi_i(g) - \pi_i(g - g_{i,l}) &\geq f \\ \pi_j(g + g_{i,j}) - \pi_j(g) > \pi_j(g) - \pi_j(g - g_{j,p}) &\geq f \end{aligned} \tag{26}$$

This implies that players i and j have an incentive to form a link and therefore g is not a pws-equilibrium, a contradiction that completes the proof. $\triangle$

Our next result looks at asymmetries that may arise in pws-equilibrium networks.

Lemma 4.2 *Suppose payoffs of every player satisfy (PS) and g is a pws-equilibrium network. Then g has at most one non-singleton component.*

[11] We note that in the pure public good case (Example 4 in section 5, below), the marginal returns to player i from a link $g_{i,j}$ are given by $3/2 + n_i(g) + 2n_j(g)$. It can be checked that these marginal returns satisfy the three requirements of positive spillovers stated above.

Proof: Let C_1 and C_2 be two non-singleton components in g. Suppose that $i, j \in C_1$ with $g_{i,j} = 1$ and $l, m \in C_2$, with $g_{l,m} = 1$. Moreover, let $\eta_i(g) \geq \eta_p(g), \forall p \in N$.

Since g is pws-equilibrium it follows that:

$$\begin{aligned} \pi_l(g) - \pi_l(g - g_{l,m}) &\geq f \\ \pi_m(g) - \pi_m(g - g_{l,m}) &\geq f. \end{aligned} \tag{27}$$

Since $\eta_i(g) \geq \eta_p(g)$, for all $p \in N$, it follows from parts (1) and (3) of the positive spillovers condition that $\pi_i(g+g_{i,l})-\pi_i(g) \geq \pi_i(g-g_{l,m}+g_{i,l})-\pi_i(g-g_{l,m}) > \pi_m(g)-\pi_m(g-g_{l,m}) \geq f$. We next note that from part (1) and (2) of the positive spillovers condition it follows that $\pi_l(g + g_{i,l}) - \pi_l(g) \geq \pi_l(g - g_{l,m} + g_{i,l}) - \pi_l(g - g_{l,m}) \geq \pi_l(g) - \pi_l(g - g_{l,m}) \geq f$. Thus players i and l have an incentive to form an additional link, contradicting the hypothesis that the network g is a pws-equilibrium. This completes the proof. △

We are now ready to provide a characterization of pws-equilibrium networks in the positive spillovers case.

Proposition 4.1 *Suppose payoffs of every player satisfy (PS) and g is a pws-equilibrium network. Then either g is empty or it has a unique non-singleton component. In the latter case, the non-singleton component is either complete or has the interlinked stars architecture.*

Proof: Consider a non-empty network g and let $C(g)$ be a non-singleton component in this network. Suppose that $C(g)$ is not complete. Let $i \in N_m(g)$; it is easy to see that $i \in C(g)$. We wish to show that $g_{i,j} = 1$, for all $j \in C(g)\backslash\{i\}$. Suppose that $j \in C(g)$ and $g_{i,j} = 0$. Since $j \in C(g)$, there is some $l \in C(g)$ such that $g_{j,l} = 1$. Since g is a pws-equilibrium network, it follows that

$$\begin{aligned} \pi_j(g) - \pi_j(g - g_{j,l}) &\geq f \\ \pi_l(g) - \pi_l(g - g_{j,l}) &\geq f \end{aligned} \tag{28}$$

It now follows from the definition of positive spillovers that:

$$\begin{aligned}\pi_i(g+g_{i,j})-\pi_i(g) &\geq \pi_i(g-g_{j,l}+g_{i,j})-\pi_i(g-g_{j,l}) > \pi_l(g)-\pi_l(g-g_{j,l}) \geq f \\ \pi_j(g+g_{i,j})-\pi_j(g) &\geq \pi_j(g-g_{j,l}+g_{i,j})-\pi_j(g-g_{j,l}) > \pi_j(g)-\pi_j(g-g_{j,l}) \geq f\end{aligned} \tag{29}$$

Here we have used part (1) of the (PS) definition to derive the weak inequality in both equations. Part (3) of (PS) implies the strict inequality in the first equation, while part (2) implies the strict inequality in the second equation. Thus players i and j have an incentive to form a link. This contradicts the pws-equilibrium hypothesis of the network g. The proof now follows from Lemma 4.1 and Lemma 4.2.

Let $N_1(g)$ be the set of minimally linked players in the component $C(g)$. We finally show that $\eta_i(g) = |N_m(g)|$ for all $i \in N_1(g)$. Suppose not. Then there is some $i \in N_1(g)$ such that $g_{i,j} = 1$, and $j \notin N_m(g)$. But then it follows from parts (1) and (2) of positive spillovers that player j has a strict incentive to form a link with every other player since $\eta_i(g-g_{i,j}) < \eta_k(g)$, for all $k \in N$; moreover, parts (1) and (3) of (PS) imply that every other player k has an incentive to reciprocate the link with j. Thus player j must be a member of $N_m(g)$, a contradiction that completes the proof. $\triangle$

— Insert Figure 3 somewhere here —

4.2 Negative spillovers

In some interesting economic applications, congestion effects may dominate and links consequently exhibit negative spillovers. Our definition of negative spillovers incorporates the following ideas: one, that marginal payoffs from a link decrease as the network gets more connected. Two, that marginal payoffs are smaller the more connected is the player with whom the link is being formed. Finally, that marginal returns from a link are smaller the greater the number of links of the player forming the link. These are defined below:

Definition 4.2 *Fix some network g. The payoff function of player i satisfies the negative spillovers (NS) property if:*

1. $\pi_i(g + g_{l,k} + g_{i,j}) - \pi_i(g + g_{l,k}) \ \leq \ \pi_i(g + g_{i,j}) - \pi_i(g)$

2. $\pi_i(g + g_{i,j}) - \pi_i(g) \leq \pi_i(g + g_{i,l}) - \pi_i(g)$ *if* $\eta_j(g) \geq \eta_l(g)$

3. $\pi_i(g + g_{i,l}) - \pi_i(g) \leq \pi_j(g + g_{j,k}) - \pi_j(g)$ *if* $\eta_i(g) \geq \eta_j(g)$ *and* $\eta_l(g) \geq \eta_k(g)$

The negative spillovers case is considerably more difficult to characterize than the positive. In order to gain some insight into the nature of equilibrium networks, we start with an example satisfying the negative spillovers property. The example demonstrates that both symmetric networks as well as very asymmetric networks can be sustained in an equilibrium under negative spillovers.

Example 4.1: Let $n = 4$. Suppose the payoff function of the players in the network g is given by:

$$\pi_i(g) \ = \ \alpha\sqrt{\eta_i(g)} - \beta \sum_{l \in N} \eta_l^2(g)\eta_i(g), \quad i \in N \tag{30}$$

The marginal payoffs are given by:

$$\begin{aligned} \pi_i(g + g_{i,j}) - \pi_i(g) \ &= \ \alpha\left\{\sqrt{\eta_i(g)+1} - \sqrt{\eta_i(g)}\right\} \\ &\text{-}\beta\sum_{l\in N}\eta_l^2(g) - 2\beta\left\{\eta_i(g)+\eta_j(g)+1\right\}\left\{\eta_i(g)+1\right\} \end{aligned} \tag{31}$$

It can be verified that payoffs satisfy the negative spillovers property. In this example, the symmetric pws-equilibria are characterized as follows: *(a). The empty network is a pws-equilibrium for* $\alpha - 2\beta < f$*. (b). The symmetric network of degree 1,* g^1*, is a pws-equilibrium for* $0.41\alpha - 16\beta < f \leq \alpha - 4\beta$*. (c). The symmetric network of degree 2,* g^2*, is a pws-equilibrium for* $0.32\alpha - 46\beta < f \leq 0.41\alpha - 22\beta$*. (d). The complete network is a pws-equilibrium for* $0.32\alpha - 56\beta \geq f$*.*

This characterization of symmetric networks can be derived by computing the payoffs for each player in the different networks. For example, $\Pi_i(g^e) = 0$ while $\Pi_i(g^c) = 1.72\alpha - 108\beta - 3f$.

Similarly, $\Pi_i(g^1) = \alpha - 4\beta - f$ and $\Pi_i(g^2) = 1.41\alpha - 32\beta - 2f$. The characterization follows by verifying the equilibrium conditions using these payoffs.

We turn next to a complete characterization of asymmetric networks. *(a). The dominant group network with one isolated player is a pws-equilibrium for* $0.32\alpha - 30\beta < f \leq 0.41\alpha - 18\beta$ *while the dominant group with two isolated players is a pws-equilibrium for* $\alpha - 4\beta < f \leq \alpha - 2\beta$. *(b). The interlinked star network in which* $N_1(g) = \{i\}$, $N_2(g) = \{j, k\}$ *and* $N_3(g) = \{l\}$ *is a pws-equilibrium for* $0.32\alpha - 42\beta < f \leq 0.32\alpha - 34\beta$*; the interlinked star network in which* $N_1(g) = \{i, j\}$ *and* $N_2(g) = \{k, l\}$ *is a pws-equilibrium for* $0.32\alpha - 56\beta < f \leq 0.32\alpha - 42\beta$. *The star network, however, is not a pws-equilibrium. (c). The "line" network, in which two players have two links and two players have one link, is a pws-equilibrium for* $0.41\alpha - 22\beta < f \leq 0.41\alpha - 16\beta$. *(d) The non-singleton component may be incomplete: the network with one isolated player and the other three players in a star component is a pws-equilibrium for* $0.41\alpha - 14\beta < f \leq 0.41\alpha - 10\beta$. △

Two observations follow from the above example. *First*, we see that symmetric networks and highly asymmetric networks can coexist as equilibria under negative spillovers. This is mainly due to condition (1) of negative spillovers: the marginal payoffs to players i and j from a link are adversely affected as the network becomes more connected; therefore, if some players have established a large number of links, then they can deter other players from forming links. Of course, if all firms are symmetrical with respect to links, then over some cost range they will find it unprofitable to establish or delete a link. *Second*, we note that a wide range of possible architectures can be supported as pws-equilibria. In fact, for the case $n = 4$, all network architectures except the star can be sustained as pws-equilibria over some cost range. In particular, the dominant group and the interlinked stars architecture, which were equilibria under positive spillovers, continue to be equilibria under negative spillovers as well.

We would like to gain a better understanding of the role played by the number of links of the potential partner. In the process, we would like to differentiate more critically between positive and negative spillovers by restricting the architectures that can be sustained as equilibria under the latter. To accomplish this, we now replace condition (1) of negative spillovers with the following stronger condition:

$$1^*a. \quad \pi_i(g+g_{l,k}+g_{i,j})-\pi_i(g+g_{l,k}) \;<\; \pi_i(g+g_{i,j})-\pi_i(g),\ l=i,j \text{ and } k\neq i,j$$

$$1^*b. \quad \pi_i(g+g_{l,k}+g_{i,j})-\pi_i(g+g_{l,k}) \;=\; \pi_i(g+g_{i,j})-\pi_i(g),\ l,k\neq i,j$$

Condition (1*) states that the marginal payoffs of players i and j from establishing a link are adversely affected if either i or j gets more connected; however, the marginal payoffs are unafffected if players other than i and j form or delete links. Therefore, in contrast to condition (1), link formation by other players does not create negative externalities for players i and j. We shall denote the combination of (1*), (2) and (3) as (NS*).[12] We start with an example to show that condition (1*a) restricts the possible equilibria that can emerge.

Example 4.2: Let the payoff function be given by:

$$\pi_i(g) \;=\; \alpha\sqrt{\eta_i(g)}-\beta\sum_{k\neq i}\eta_k^2(g) \tag{32}$$

The marginal payoffs are given by:

$$\pi_i(g+g_{i,j})-\pi_i(g) \;=\; \alpha\left\{\sqrt{\eta_i(g)+1}-\sqrt{\eta_i(g)}\right\}-2\beta\eta_j(g)-\beta \tag{33}$$

It can be verified that the above payoff function satisfies conditions (NS*). We now offer a complete characterization all pws-equilibrium networks for the case $n=4$.*In the class of symmetric networks: (a). The empty network is a pws-equilibrium for* $f>\alpha-\beta$. *(b). The symmetric network of degree 1,* g^1*, is a pws-equilibrium for* $0.41\alpha-3\beta<f\leq\alpha-\beta$. *(c). The symmetric network of degree 2,* g^2*, is a symmetric equilibrium for* $0.32\alpha-5\beta<f\leq 0.41\alpha-$

[12]We note that in the example with free-trade agreements (example 5 in section 5, below) the marginal returns satisfy conditions (NS*) for neighborhood sizes, $n_i(g)=\eta_i(g)+1\geq 3$ and $n_j(g)=\eta_j(g)+1\geq 1$.

3β. *(d). The complete network is a pws-equilibrium for* $f \leq 0.32\alpha - 5\beta$. This characterization result follows by noting that $\Pi_i(g^e) = 0$, $\Pi_i(g^1) = \alpha - 3\beta - f$, $\Pi_i(g^2) = 1.41\alpha - 12\beta - 2f$ and $\Pi_i(g^c) = 1.72\alpha - 27\beta - 3f$ and then verifying the equilibrium conditions.

In the class of asymmetric networks, there are only two equilibrium architectures: (a). The dominant group network with one isolated player is a pws-equilibrium for $0.32\alpha - \beta < f \leq 0.41\alpha - 3\beta$. *(b). The star network is a pws-equilibrium for* $0.41\alpha - 3\beta < f \leq min\{0.32\alpha - \beta, \alpha - 5\beta\}$. In the dominant group network, the payoff to the player in the non-singleton component is $\Pi_i(g) = 1.41\alpha - 8\beta - 2f$ while that to the isolated player is $\Pi_j(g) = -12\beta$; in the star network, the payoff of the central player is $\Pi_i(g) = 1.72\alpha - 3\beta - 3f$ while the peripheral players earn $\Pi_j(g) = \alpha - 11\beta - f$. The characterization result now follows from the equilibrium conditions. △

We observe that in contrast to Example 4.1, the star network is a pws-equilibrium; on the other hand, networks such as the interlinked star, or the dominant group with two or more isolated players, are no longer equilibria in Example 4.2. We now examine the generality of these findings.

Our first result establishes a general property of equilibrium networks under (NS*), which is specially relevant for asymmetric networks.

Lemma 4.3 *Consider any network* g *in which there exists distinct players* i, j, k, l *satisfying* $g_{k,l} = 1$ *and* $\eta_i(g) \leq \eta_j(g) \leq \eta_k(g) \leq \eta_l(g)$ *such that* $\eta_i(g - g_{k,l}) \leq \eta_k(g - g_{k,l})$ *and* $\eta_j(g - g_{k,l}) \leq \eta_l(g - g_{k,l})$. *If* g *is a pws-equilibrium network, then for generic values of cost of link formation,* $g_{i,j} = 1$ *in* g.

Proof Suppose not and let $g_{i,j} = 0$. It follows that:

$$\begin{aligned} \pi_i(g + g_{i,j}) - \pi_i(g) &= \pi_i(g - g_{k,l} + g_{i,j}) - \pi_i(g - g_{k,l}) \\ &\geq \pi_l(g) - \pi_l(g - g_{k,l}) \\ &\geq f \end{aligned} \tag{34}$$

where the equality follows from property (1*b) of negative spillovers and the first inequality from property (3) since $\eta_l(g - g_{k,l}) \leq \eta_j(g - g_{k,l})$ and $\eta_k(g - g_{k,l}) \leq \eta_i(g - g_{k,l})$; the final inequality follows from the hypothesis that g is a pws-equilibrium. This will be strict for generic values of f. A similar argument holds for player j. Therefore, i and j have an incentive to form a link, contradicting the hypothesis that g is a pws-equilibrium network.$\triangle$

Lemma 4.3 shows that if there are two or more unlinked players in $N_1(g)$, then players in $N_q(g)$, $q > 1$, cannot have links with each other. In particular, this implies that (localized) negative spillovers rule out the "line" architecture and the interlinked stars architecture (for $|N_m| \geq 2$) as pws-equilibria. In the "line" network (with $n \geq 4$), we can find peripheral players i and j, and non-peripheral players k and l such that $1 = \eta_i(g) = \eta_j(g) < \eta_k(g) = \eta_l(g) = 2$; however, $g_{i,j} = 0$ and therefore g is not a pws-equilibrium. In the interlinked star (for $|N_m| \geq 2$) network, there exists $i \in N_1(g)$, $l \in N_m(g)$, $j \in N_q(g)$, $q \in \{1, 2, ..., m-1\}$ and $k \in N_r(g)$, $r \in \{2, 3, .., m\}$; therefore, the conditions of Lemma 4.3 are satisfied but $g_{i,j} = 0$. However, if the central players have no link with each other but are connected only to the peripheral players, then we have an "interlinked star-like" structure which may be a pws-equilibrium. To see this, consider the payoff function of example 4.2 once again and let $n = 6$. Consider the network where players 1 and 2 are linked to players 3,4,5 and 6 but not with each other; the peripheral players 3,4,5 and 6 have no links with each other. This network is a pws-equilibrium for $0.32\alpha - 5\beta < f \leq \min\{0.28\alpha - 3\beta, 0.41\alpha - 7\beta\}$. It is worth noting that a star network cannot be ruled out on the basis of Lemma 4.3 because for any players i, j, k, l, with $g_{k,l} = 1$, $\eta_i(g - g_{k,l}) > \eta_k(g - g_{k,l})$.

One of the networks that has played a prominent role in the analysis so far is the dominant group architecture, in which there is one non-singleton component and a set of isolated players. More generally one can consider the class of networks in which there is a set of isolated players. The above result has a direct implication for the size of the set of singleton players: there is at most one singleton component in a (non-empty) pws-equilibrium network. Suppose we consider the class of networks with one singleton player i. What can we say about the architecture of the network g_{-i}? The next result derives an important property of this network.

Proposition 4.2 *Suppose payoffs of every player satisfy (NS*) and suppose g is a pws-equilibrium network. If g has a singleton component consisting of player i, then for generic values of cost of forming links, f, the network g_{-i} is symmetric. Moreover there is a unique degree, η^f, for the network g_{-i}, which can be sustained in a pws-equilibrium.*

Proof We first show that the network g_{-i} is symmetric. Suppose to the contrary that in a pws-equilibrium network g in which player i belongs to the unique singleton component, g_{-i} is asymmetric. In the network g_{-i}, there exists players j and l with the minimum and maximum number of links respectively such that $0 < \eta_j(g) < \eta_l(g)$. Since $n \geq 4$, there also exists a player k in g_{-i} such that $\eta_j(g) \leq \eta_k(g) \leq \eta_l(g)$ with $g_{j,k} = 0$ and $g_{k,l} = 1$. Since g is a pws-equilibrium network:

$$\pi_k(g) - \pi_k(g - g_{k,l}) \geq f \ , \ \pi_l(g) - \pi_l(g - g_{k,l}) \geq f \tag{35}$$

First consider player j and note that $\pi_j(g + g_{i,j}) - \pi_j(g) = \pi_j(g - g_{k,l} + g_{i,j}) - \pi_j(g - g_{k,l}) \geq \pi_l(g - g_{k,l} + g_{i,l}) - \pi_l(g - g_{k,l}) \geq \pi_l(g) - \pi_l(g - g_{k,l}) \geq f$. The equality follows from property (1*b), the first inequality from property (3) since $\eta_l(g - g_{k,l}) \geq \eta_j(g - g_{k,l})$, and the second inequality from property (2) since $\eta_k(g - g_{k,l}) \geq \eta_i(g - g_{k,l})$; the final inequality follows from (35) and will be strict for generic values of f. We can similarly establish for player i that $\pi_i(g + g_{i,j}) - \pi_i(g) = \pi_i(g - g_{k,l} + g_{i,j}) - \pi_i(g - g_{k,l}) \geq \pi_k(g - g_{k,l} + g_{j,k}) - \pi_k(g - g_{k,l}) \geq \pi_k(g) - \pi_k(g - g_{k,l}) \geq f$. The last inequality will be strict for generic values of f. Therefore, players i and j have an incentive to form a link. This contradicts the hypothesis that g is a pws-equilibrium network.

We next show the uniqueness property. Let g be a pws-equilibrium network with player l as the singleton component. If η is the degree of symmetry, then we will denote this network by g^{η}_{-l}. Since all players in g^{η}_{-l} have the same number of links, pick any representative player i. The marginal returns of this player from deleting a link with another player j in g^{η}_{-l} can be written as function of the degree of the network as $\delta\pi_i(\eta) = \pi_i(g^{\eta}_{-l}) - \pi_i(g^{\eta}_{-l} - g_{i,j})$. Note from property (2) of negative spillovers that $\pi_i(g^{\eta}_{-l} + g_{i,l}) - \pi_i(g^{\eta}_{-l}) \geq \pi_i(g^{\eta}_{-l} + g_{i,k}) - \pi_i(g^{\eta}_{-l})$ for any $k \neq l$ since k has more links than l. Therefore, if player i has no incentive to form a link with the isolated player l, then i will also not have an incentive to form a link with player $k \neq l$. Let $\Delta\pi_i(\eta) = \pi_i(g^{\eta}_{-l} + g_{i,l}) - \pi_i(g^{\eta}_{-l})$ denote the marginal returns to a player in g^{η}_{-l} from adding a link with the isolated player. We will show that both $\Delta\pi_i(\eta)$ and $\delta\pi_i(\eta)$ are (strictly) decreasing in η. Moreover, $\delta\pi_i(\eta) \leq \Delta\pi_i(\eta - 1)$.

First, we observe that in moving from g^{η}_{-l} to $g^{\eta+1}_{-l}$, the number of direct links of player i with players other than l increases by 1; therefore, by virtue of property (1*a) of negative spillovers:

$$\pi_i(g^{\eta+1}_{-l} + g_{i,l}) - \pi_i(g^{\eta+1}_{-l}) < \pi_i(g^{\eta}_{-l} + g_{i,l}) - \pi_i(g^{\eta}_{-l}). \tag{36}$$

Also by virtue of property (1*a) of negative spillovers:

$$\pi_i(g^{\eta+1}_{-l}) - \pi_i(g^{\eta+1}_{-l} - g_{i,j}) < \pi_i(g^{\eta}_{-l}) - \pi_i(g^{\eta}_{-l} - g_{i,j}). \tag{37}$$

Finally, we note that:

$$\begin{aligned} \pi_i(g^{\eta}_{-l}) - \pi_i(g^{\eta}_{-l} - g_{i,j}) &\leq \pi_i(g^{\eta}_{-l} - g_{i,j} + g_{i,l}) - \pi_i(g^{\eta}_{-l} - g_{i,j}) \\ &= \pi_i(g^{\eta-1}_{-l} + g_{i,l}) - \pi_i(g^{\eta-1}_{-l}) \end{aligned} \tag{38}$$

where the first inequality holds due to property (2) of negative spillovers (since j is better linked than l) and the equality holds due to property (1*b) of negative spillovers (since in moving from $g^{\eta}_{-l} - g_{i,j}$ to $g^{\eta-1}_{-l}$, the only links that are deleted are those of players other than i and l).

Let f be the cost of forming links and consider g^{η}_{-l}. Since g is a pws-equilibrium, we must have $\delta\pi_i(\eta) \geq f$ and $\Delta\pi_i(\eta) < f$. Now consider any $\eta' \neq \eta$. If $\eta' > \eta$, then $f > \Delta\pi_i(\eta) \geq \delta\pi_i(\eta')$, and players in $g^{\eta'}_{-l}$ have an incentive to delete their links. Similarly, if $\eta' < \eta$, then $\Delta\pi_i(\eta') \geq \delta\pi_i(\eta) \geq f$, and a player in $g^{\eta'}_{-l}$ has an incentive to form a link with the isolated player for generic values of f. This proves the result. △

We finally turn to the nature of symmetric networks that can be supported in a pws-equilibrium. The following result summarizes our analysis for this class of networks.

Proposition 4.3 *Suppose that payoffs of every player satisfy (NS*). Given any generic cost $f \geq 0$ of forming links, there exists a constant $\eta(f)$ such that the symmetric network $g^{\eta(f)}$ is the unique pws-equilibrium network in the class of symmetric networks.*

Proof: Since all players have the same number of links in a symmetric network, pick any representative player i. Define $\Delta\pi_i(\eta) = \pi_i(g^\eta + g_{i,j}) - \pi_i(g^\eta)$ as the marginal returns from adding a link and $\delta\pi_i(\eta) = \pi_i(g^\eta) - \pi_i(g^\eta - g_{i,j})$ as the loss in returns from a deleting a link. We will show that both $\Delta\pi_i(\eta)$ and $\delta\pi_i(\eta)$ are (strictly) decreasing in η. Moreover, $\Delta\pi_i(\eta) < \delta\pi_i(\eta)$ and $\Delta\pi_i(\eta) = \delta\pi_i(\eta+1)$ for all η.

First, we observe that in moving from g^η to $g^{\eta+1}$, the number of direct links of players i and j increase by 1; therefore, by virtue of property (1*a) of negative spillovers:

$$\Delta\pi_i(\eta+1) = \pi_i(g^{\eta+1} + g_{i,j}) - \pi_i(g^{\eta+1}) < \pi_i(g^\eta + g_{i,j}) - \pi_i(g^\eta) = \delta\pi_i(\eta). \tag{39}$$

Also, by virtue of property (1*a) of negative spillovers:

$$\delta\pi_i(\eta+1) = \pi_i(g^{\eta+1}) - \pi_i(g^{\eta+1} - g_{i,j}) < \pi_i(g^\eta) - \pi_i(g^\eta - g_{i,j}) = \delta\pi_i(\eta). \tag{40}$$

Further, we note that:

$$\begin{aligned} \Delta\pi_i(\eta) &= \pi_i(g^\eta) - \pi_i(g^\eta - g_{i,j}) \\ &\geq \pi_i(g^\eta - g_{i,j} + g_{i,k}) - \pi_i(g^\eta - g_{i,j}) \\ &> \pi_i(g^\eta + g_{i,k}) - \pi_i(g^\eta) = \delta\pi_i(\eta) \end{aligned} \tag{41}$$

where the first inequality holds due to property (2) of negative spillovers (since k is better linked than j) and the second inequality holds due to property (1*a). Finally, we note using property (1*b) of negative spillovers that:

$$\begin{aligned} \Delta\pi_i(\eta) = \pi_i(g^\eta + g_{i,j}) - \pi_i(g^\eta) &= \pi_i(g' + g_{i,j}) - \pi_i(g') \\ &= \pi_i(g^{\eta+1}) - \pi_i(g^{\eta+1} - g_{i,j}) = \delta\pi_i(\eta) \end{aligned} \tag{42}$$

where g' is obtained from g^η by adding for every player, other than i and j, an additional link. There are three possible cases to consider:

(i) $f < \delta\pi_i(n-1)$.

In this case, the complete network is a pws-equilibrium because the players cannot add any links, and have no incentive to delete any links. We now argue that no other degree can be sustained as an equilibrium over this range of costs. Consider any degree $\eta < (n-1)$. In g^η, there will exist players i and j such that $g_{i,j} = 0$. However:

$$f < \delta\pi_i(n-1) \leq \Delta\pi_i(\eta) = \pi_i(g^\eta + g_{i,j}) - \pi_i(g^\eta) \tag{43}$$

Therefore, players i (and by symmetry) j have an incentive to form a link; thus g^η is not a pws-equilibrium.

(ii) $f > \Delta\pi_i(0)$

In this case, the empty network is a pws-equilibrium because players have no incentive to add a link, and there are no links to delete. Consider any degree of symmetry $\eta > 0$. In g^η, there will exist players i and j such that $g_{i,j} = 1$. However:

$$f > \Delta\pi_i(0) \geq \delta\pi_i(\eta) = \pi_i(g^\eta) - \pi_i(g^\eta - g_{i,j}) \tag{44}$$

Therefore, players i (and by symmetry) j have an incentive to delete their link; thus g^η is not a pws-equilibrium.

(iii) $\delta\pi_i(n-1) < f < \Delta\pi_i(0)$

In this case, we can find a degree η' such that $\Delta\pi_i(\eta') \leq f \leq \delta\pi_i(\eta')$, where at least one inequality is strict. Then $g^{\eta'}$ is a pws-equilibrium because players have no incentive to delete their links, and, for values of f in the interval $(\Delta\pi_i(\eta'), \delta\pi_i(\eta'))$, no incentive to form additional links. Now consider any $\eta \neq \eta'$. If $\eta > \eta'$, then $f \geq \Delta\pi_i(\eta') \geq \delta\pi_i(\eta)$, and players in g^η have a strict incentive to delete their links for generic values of f in the interval

$(\Delta\pi_i(\eta'), \delta\pi_i(\eta))$. Similarly, if $\eta < \eta'$, then $\Delta\pi_i(\eta) \geq \delta\pi_i(\eta') \geq f$, and unlinked players in g^η have an incentive to form a link for generic values of f in the interval $(\Delta\pi_i(\eta), \delta\pi_i(\eta'))$. This proves the result. $\triangle$

5 Applications

In this section we present some social and economic examples to illustrate the scope of the analysis.

Example 1: *R&D collaboration between Cournot competitors*[13]

Consider a homogeneous product Cournot duopoly consisting of n ex-ante symmetric firms who face the linear inverse demand: $p = \alpha - \sum_{i \in N} q_i$, $\alpha > 0$. Before engaging in quantity competition, the firms can form collaboration links with other firms. A collaboration link is an agreement to jointly invest in cost-reducing R&D activity. The firms are initially symmetric with zero fixed costs and identical constant returns-to-scale cost functions. Collaborations lower marginal costs of production along the following lines: $c_i(g) = \gamma_0 - \gamma\eta_i(g)$, $i \in N$, where γ_0 is a positive parameter representing a firm's marginal cost if it has no links. In this case, firm i's marginal costs are *linearly* declining in the number of links it has with other firms.

Given any network g, the Cournot equilibrium output can be written as:

$$q_i(g) \;=\; \frac{(\alpha - \gamma_0) + n\gamma\eta_i(g) - \gamma \sum_{j \neq i} \eta_j(g)}{(n+1)}\,, \quad i \in N \tag{45}$$

In order to ensure that each firm produces a strictly positive quantity in equilibrium, we will assume that $(\alpha - \gamma_0) + (n-1)^2\gamma > 0$. The Cournot profits for firm $i \in N$ are given by $\pi_i(g) = q_i^2(g)$.

[13]This example is taken from Goyal and Joshi (1999a).

Given a network g, the marginal (gross) returns from forming an additional link are as follows: $\pi_i(g+g_{i,j}) - \pi_i(g) = q_i^2(g+g_{i,j}) - q_i^2(g)$. It can be verified that the Cournot output of firm i is strictly increasing with each additional link: $q_i(g+g_{i,j}) - q_i(g) \; = \; \gamma(n-1)/(n+1) \; > \; 0$. Using this fact, it is easy to verify that $\pi_i(g+g_{i,j}+g_{i,k}) - \pi_i(g+g_{i,j})\pi_i(g+g_{i,j}) - \pi_i(g)$. Thus, given a network g, the marginal returns to player i from additional links are increasing in the number of links. We now use our general results from section 3 to provide a complete characterization of pws-equilibrium networks in this example.

First we note from Proposition 3.1 that a pws-equilibrium network has the dominant group architecture. Recall that a dominant group architecture has at most one non-singleton component which is complete and all the other players are isolated. Thus a dominant group architecture is fully defined in terms of the size of the dominant group, k, where we will allow $k \in \{1, ..., n\}$. In the non-singleton component of size k every firm should should have no incentive to delete any subset of its links. In the present example, marginal payoffs from links are increasing. So a strategy of retaining a strict subset of links is always dominated either by the strategy of no links or the strategy of maintaining all the existing links. Thus we only have to check if the payoff from the latter strategy is higher than the payoff from the strategy of having no links.[14] Therefore, the firm should have no incentive to delete all its links. After some rearrangement, this requirement can be written as follows:

$$Y(k) \;\equiv\; \frac{(n-1)\gamma}{(n+1)^2}\left[2(\alpha-\gamma_0) + (k-1)(n+3-2k)\gamma\right] \;\geq\; f \tag{46}$$

If the above condition is satisfied, a firm in the non-singleton component would always want to form a link with an isolated firm. This is because the firm's marginal payoffs are increasing in its own links. Therefore, if g is stable, then the isolated firm should have no incentive to form a link with a firm in the non-singleton component. This requires:

$$X(k) \;\equiv\; \frac{(n-1)\gamma}{(n+1)^2}\left[2(\alpha-\gamma_0) + (n-1)\gamma - 2k(k-1)\gamma\right] \;<\; f \tag{47}$$

[14]This analysis is drawn from our earlier paper, Goyal and Joshi (1999a).

A network g^k is an pws-equilibrium if and only if it satisfies (46) and (47). By inspection, we see that $X(k)$ is declining in k. Let $X(n-1) = F_0$. Regarding $Y(k)$, it is initially increasing and then decreasing in k. Now let $F_1 = Y(n)$, $F_2 = Y(2) = X(1)$, and $F_3 = Y(k^*)$. Note that $F_0 < F_1 < F_2 < F_3$. The functions $X(k)$ and $Y(k)$ are shown in Figure 4.

— Insert Figure 4 somewhere here —

Figure 4 illustrates the nature of pws-equilibrium architectures as the cost of forming links f varies. When costs are low, i.e. $f < F_0$, the complete network is uniquely stable. When costs are moderate, i.e. $F_0 \leq f < F_1$, only networks with relatively large dominant groups are stable. When costs are high, i.e. $F_1 \leq f < F_3$, only medium size dominant groups are pws-equilibrium (small and large groups are not sustainable). The empty network is pws-equilibrium when $f > F_2$ and is uniquely pws-equilibrium for very high costs, $f > F_3$. Hence, the effect of R&D costs on the size of the dominant group is *non-monotonic*.

Example 2: *R&D collaboration between local monopolies* [15]

Consider n firms but now suppose that each of them is a monopolist and operates in a market with linear inverse demand: $p = \alpha - q_i$, $\alpha > 0$. As in example 1, firms can form collaboration links with other firms. The firms are initially symmetric with zero fixed costs and identical constant returns-to-scale cost functions. Collaborations lower marginal costs of production in the following way: $c_i(g) = \gamma_0 - \gamma\eta_i(g)$, $i \in N$ where γ_0 is a positive parameter representing a firm's marginal cost when it has no links. In this case, firm i's marginal costs are *linearly* declining in the number of links it has with other firms. Given any network g, the monopoly output is $q_i(g) = [(\alpha - \gamma_0) + \gamma\eta_i(g)]/2$, $i \in N$. The profits for a firm are given by $\pi_i(g) = q_i^2(g)$.[16] For a given network g, the marginal returns to player i from a link $g_{i,j}$ are given by:

[15] This example is adapted from Goyal and Moraga (2001). In that paper, there are no costs to forming links, but firms choose cost-reducing effort level after forming links. By contrast, in the version presented below every firm has some exogenously given useful information, but links are costly.

[16] We shall assume that γ and n are small relative to α, so that the costs are always positive and the optimum is well defined.

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \frac{1}{4}\{2(\alpha - \gamma_0) + 2\gamma\eta_i(g) + \gamma\}\{\gamma\}. \tag{48}$$

Clearly, marginal payoffs are increasing in $\eta_i(g)$, the number of links that firm i has. Proposition ** implies that pws-equilibrium networks are either empty or have the dominant group architecture. The precise range of archiectures depends on the value of f. We need to verify two types of incentive constraints. The first set of conditions pertain to the incentives of the players in the dominant group. Given the increasing returns property, we only need to check the optimality of zero links as against all existing links. A player i in the dominant group (of size k) has an incentive to retain all links if:

$$\left[\frac{\alpha - \gamma_0 + \gamma(k-1)}{2}\right]^2 - (k-1)f \geq \left[\frac{\alpha - \gamma_0}{2}\right]^2 \tag{49}$$

This is equivalent to the requirement that $f \leq \gamma[2(\alpha - \gamma_0) + \gamma(k-1)]/4$. Player i has no incentive to form an additional link with one of the isolated players if $f \geq \gamma[2(\alpha - \gamma_0) + 2\gamma(k-1) + \gamma]/4$. Similarly, an isolated player has no incentive to form a link if $f \geq \gamma[2(\alpha - \gamma_0) + \gamma]/4$. Thus it follows that a dominant group of size k is an pws-equilibrium if and only if $\gamma[2(\alpha - \gamma_0) + \gamma]/4 \leq f \leq \gamma[2(\alpha - \gamma_0) + \gamma(k-1)]/4$.

Let $F_0 = \gamma[2(\alpha - \gamma_0) + \gamma]/4$ and $F_1 = \gamma[2(\alpha - \gamma_0) + \gamma(n-1)]/4$. Figure 5 illustrates the nature of pws-equilibrium architectures as a function of the cost of forming links. The complete network is a pws-equilibrium when $f \leq F_1$ and is a unique pws-equilibrium for very low costs, $f < F_0$. The empty network is a pws-equilibrium when $f > F_0$ and is a unique pws-equilibrium for very high costs, $f > F_1$. The dominant group architecture for $k \in \{2, 3, ..., n-1\}$ is a pws-equilibrium over the intermediate range, $F_0 \leq f \leq F_1$. Over this range, the dominant group architecture exhibits the following monotonicity property: as the cost of forming links increases, only the larger-sized dominant groups are sustainable in equilibrium. The intuition for this result is as follows: over the intermediate range of costs, the incentive constraint for the firm in the dominant group to retain all its links is binding; an increase in cost f requires that the size of the dominant group be large in order to ensure that each member firm in the non-singleton component can profitably retain all its links.

— Insert Figure 5 somewhere here —

Example 3: ***Patent races***[17]

Consider n firms who are racing to innovate a new product or process. The race is conducted in continuous time. The firm which succeeds in innovating first wins a patent which prevents the innovation from imitation or duplication for perpetuity. Let the discounted value of the patent be V; without loss of generality, we will use the normalization $V = 1$. All other firms get a payoff of 0. All firms use the same discount rate ρ.

All firms are inelastically endowed with one unit of R&D capability (or technical know-how). Firms race to innovate by forming bilateral links with other firms; these links represent agreements to mutually share R&D capability or technical information. Let $\tau(\eta_i(g))$ denote the random time at which firm i innovates in a network g in which firm i has established $\eta_i(g)$ bilateral links. We assume that τ has an exponential distribution:

$$Pr\{\tau(\eta_i(g)) \leq t\} \ = \ 1 - e^{-\eta_i(g)t} \tag{50}$$

As firm i establishes more links, it increases the probability of innovating successfully before time t. In addition to this technological uncertainty, there is also market uncertainty: any of the rival $n-1$ firms may successfully innovate before firm i. Assuming that the distribution of the time of innovation is stochastically independent for the firms, the probability that firm i is the first to successfully innovates by time t is:

$$Pr\{\tau(\eta_i(g)) \in [t, t+dt],\ \tau(\eta_j(g)) > t\ \forall j \neq i\} \ = \ \eta_i(g)e^{-t\sum_{j=1}^{n}\eta_j(g)}dt \tag{51}$$

There are both benefits and costs from establishing bilateral links. The benefit to i from linking with j is that it increases the probability of i innovating successfully before time t; the cost is that the probability that j will innovate successfully before time t also increases.

[17] The formulation of patent races in terms of a memoryless dynamic process is due to Dasgupta and Stiglitz (1980) and Loury (1979), among others.

This tension between benefits and costs will determine the equilibrium architecture of the network.

In the network g, the (expected) payoff to firm i is given by:

$$\begin{aligned}\pi_i(g) &= \int_0^\infty e^{-\rho t}\eta_i(g)e^{-t\sum_{j=1}^n \eta_j(g)}dt \\ &= \frac{\eta_i(g)}{\rho + \sum_{j=1}^n \eta_j(g)} \qquad (52)\end{aligned}$$

Let $H(g) \equiv \sum_{j=1}^n \eta_j(g)$. It is easily verified that:

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \frac{\rho + H(g) - 2\eta_i(g)}{[\rho + H(g)][\rho + H(g) + 2]} \qquad (53)$$

Marginal profits are always positive, i.e. link monotonicity is satisfied. To see this, rewrite the numerator of (53) as $\rho + \sum_{k \neq i} \eta_k(g) - \eta_i(g)$. It is implicit in (53) that $g_{i,j} = 0$. The highest value that $\eta_i(g)$ can assume in any network g with $g_{i,j} = 0$, and the lowest value $\sum_{k \neq i} \eta_k(g)$ can assume, is when i is the center of a star component with $n - 1$ firms (so that $\eta_i(g) = n - 2$ and $\eta_k(g) = 1$ for $k \neq i, j$) and j belongs to a singleton component. But, in such a network:

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \frac{\rho}{[\rho + H(g)][\rho + H(g) + 2]} > 0 \qquad (54)$$

Noting that $H(g + g_{i,j}) = H(g) + 2$ for any g, it follows that marginal returns in the racing model are decreasing in the number of links of player i:

$$\begin{aligned}\pi_i(g + g_{i,l} + g_{i,j}) - \pi_i(g + g_{i,j}) &= \frac{\rho + H(g + g_{i,j}) - 2\eta_i(g + g_{i,j})}{[\rho + H(g + g_{i,j})][\rho + H(g + g_{i,j}) + 2]} \\ &= \frac{\rho + H(g) - 2\eta_i(g)}{[\rho + H(g) + 2][\rho + H(g) + 4]} \\ &< \pi_i(g + g_{i,j}) - \pi_i(g) \qquad (55)\end{aligned}$$

Finally, we note that for large ρ, the marginal returns display negative effects, i.e., $\pi_i(\eta_i(g)+1, g_{-i})-\pi_i(\eta_i(g), g_{-i}) > \pi_i(\eta_i(g)+1, g'_{-i})-\pi_i(\eta_i(g), g'_{-i})$ for $g'_{-i} > g_{-i}$. Recall that the general results obtained in section 3 tell us that in the presence of decreasing returns and negative spillovers, equilibrium networks can have a variety of architectures. We illustrate this with the help of an example with 3 firms. In this case there are four possible network architectures, the empty network (with zero links), the complete network (with three links), the partially connected network (with one link) and the line network (with two links). We suppose that $\rho > 2$ to ensure that negative spillovers obtain. The payoffs under the different networks are given as follows: $\pi_i(g^e) = 0$, for all $i \in N$; $\pi_i(g^c) = 2/(\rho+6)$, for all $i \in N$; $\pi_i(g^1) = 1/(\rho+2)$, for the firms with a link and $\pi_i(g^1) = 0$ for the isolated firm; $\pi_i(g^2) = 1/(\rho+4)$, for the firms with one link and $\pi_i(g^2) = 2/(\rho+4)$ for the firm with two links. It is now straightforward to obtain the following characterization: *If $f > 1/(\rho+2)$ then the empty network is a pws-equilibrium; if $\rho/(\rho+2)(\rho+4) < f < 1/(\rho+2)$ then the partially connected network is a pws-equilibrium; if $(\rho+2)/(\rho+4)(\rho+6) < f < \rho/(\rho+2)(\rho+4)$ then the line network is a pws-equilibrium, while if $f < (\rho+2)/(\rho+4)(\rho+6)$ then the complete network is a pws-equilibrium. Thus each of the four possible architectures is a pws-equilibrium under suitable parameters.*

While we know that symmetric equilibrium networks do not overlap, an interesting question is whether symmetric and asymmetric networks can be sustained as equilibria over the same range of costs? The answer is in the affirmative, as can be seen with an example with 4 players. The symmetric network of degree 2 is a pws-equilibrium for $n = 4$ over the range $(\rho+4)/(\rho+8)(\rho+10) < f < (\rho+4)/(\rho+6)(\rho+8)$. Assuming that $\rho > 2$, three asymmetric networks can be sustained as equilibria over subsets of this range of costs. For example, the dominant group network with one isolated player and the line network are pws-equilibria for $(\rho+2)/(\rho+6)(\rho+8) < f < (\rho+4)/(\rho+6)(\rho+8)$. The interlinked star, with $N_1(g) = \{1\}$, $N_2(g) = \{2,3\}$ and $N_3(g) = \{4\}$, is a pws-equilibrium for $(\rho+4)/(\rho+8)(\rho+10) < f < (\rho+2)/(\rho+6)(\rho+8)$. Therefore, symmetric networks can coexist as an equilibrium with a variety of asymmetric architectures.

Example 4: ***Provision of a pure public good***[18]

There are n persons, each of whom is deciding on what output share, x_i to produce of a pure public good. Given each person's output, the utility of person i is: $u_i(x) \; = \; x_i + \sum_{j \neq i} x_j$. A collaboration link between two persons can be interpreted as an agreement to share knowledge about the production of a public good. Let $f > 0$ be the fixed investment required from each person in such a link. In any network g in which person i has a neighborhood of size $n_i(g) = \eta_i(g) + 1$, the cost of producing output x_i is given by:

$$C_i(x_i) \; = \; \frac{1}{2} \left(\frac{x_i}{n_i(g)} \right)^2 \tag{56}$$

Given any network g from the first stage, person i will choose output to maximize utility net of production costs. This yields an optimal output of $x_i(g) = \eta_i^2(g)$. Therefore, the reduced form gross payoff of person i is:

$$\pi_i(g) \; = \; \frac{1}{2} n_i^2(g) \; + \; \sum_{j \neq i} n_j^2(g) \tag{57}$$

The marginal returns to person i from an additional link $g_{i,j}$, can be written as follows:

$$\pi_i(g + g_{i,j}) - \pi(g) \; = \; \frac{3}{2} + n_i(g) + 2n_j(g). \tag{58}$$

We note that the marginal payoffs to person i from a link $g_{i,j}$ are increasing in the size of his own neighborhood, $n_i(g)$, and they are also increasing in the number of links person j has, $n_j(g)$. Finally, note that the marginal payoffs are insensitive to to the links between third parties. This example thus satisfies the requirements of positive spillovers. We now use our analysis of positive spillovers from section 4 to obtain a characterization of equilibrium networks in this case.

[18]This example is a networks version of the public good problem presented in Bloch (1997); that paper studies the formation of coalitions.

First, it follows from Proposition 4.1 that a pws-equilibrium network is either empty or has a unique non-singleton component. Moreover the non-singleton component has the inter-linked stars architecture or is complete. In the following result we characterize the conditions under which different types of interlinked stars and dominant groups of different sizes can arise in equilibrium. We need some additional notation to state this result. We shall consider interlinked stars with two types of players, those who are linked to everyone and a second group of players who are only linked to this former set. Recall that $N_m(g)$ ($N_1(g)$) is the collection of players which has the largest (smallest) number of links in a network g. For the class of inter-linked stars that we are looking at, we can set $m = |N_m(g)|$ and $n-m = |N_1(g)|$. Also define $x_m = (n+1)/2+(m-1)(2n-1)/(n-1)+(n-m)(2m+1)/(n-1)$, $y_m = m/2+2n$, and $z_m = 3m + 9/2$.

Proposition 5.1 *(a) An inter-linked star with* $|N_m(g)| = m$*, where* $m \in \{1, 2, .., n-2\}$*, is a pws-equilibrium if and only if* $z_m \le f \le \min\{x_m, y_m\}$*. (b) A dominant-group network of size* k*, where* $k \in \{2, ..., n-1\}$ *is a pws-equilibrium if and only if* $7/2 + k < f < 5k/2 - 1/2$*. The complete network is a pws-equilibrium if* $f < 5n/2 - 1/2$*, while the empty network is a pws-equilibrium if* $f > 9/2$*.*

Proof: (a) Denote an interlinked star with $|N_m(g)| = m$ and $|N_1(g)| = n - m$ by g^{ms}. For such a network to be a pws-equilibrium it must be the case that (i) each player $i \in N_m(g)$ has no incentive to delete any subset of his links (ii) each player $j \in N_1(g)$ has no incentive to delete any subset of his links and also has no incentive to form an additional link with any another peripheral player $k \in N_1(g)$. We note first that for a given network g, the marginal returns to a player i from a link $g_{i,j}$ are given by $3/2 + n_i(g) + 2n_j(g)$. It therefore follows that for any player it is optimal to have either zero links or maintain all the existing links. Let player n be a typical member of $N_m(g)$. Then we require that:

$$\begin{aligned} \Pi_n(g^{ms}) &= \frac{1}{2}n^2 + (m-1)n^2 + (n-m)(m+1)^2 - (n-1)f \\ &\ge \frac{1}{2} + (m-1)(n-1)^2 + (n-m)m^2 = \Pi_n(g^{ms}_{-n}) \end{aligned} \tag{59}$$

Rewriting, we find that that this is equivalent to the requirement that:

$$f \leq \frac{1}{2}(n+1) + \frac{(m-1)(2n-1)}{n-1} + \frac{(n-m)(2m+1)}{n-1} = x_m \tag{60}$$

Similarly, we can verify that for a peripheral player the condition that he has no incentive to delete any links is given by

$$f \leq \frac{1}{2}m + 2n = y_m \tag{61}$$

Finally, the requirement that the peripheral player has no incentive to form an additional link with another peripheral player is given by

$$f \geq 3m + \frac{9}{2} = z_m \tag{62}$$

It then follows that an inter-linked star g^{ms} with $m \in \{1, 2, .., n-2\}$ is a pws-equilibrium if and only if $z_m \leq f \leq \min\{x_m, y_m\}$. We note that if $m = 1$, i.e., for the star, these conditions are simultaneously satisfied if $15/2 < f < (n+1)/2 + 3$; this interval is non-empty so long as $n \geq 9$. Figure 6 illustrates this pattern.

(b) Suppose that $1 < k < n$. In that case, we need to examine the incentives of players to both form additional links as well as delete existing ones. We first check the incentives of a player i in the dominant group. Again, from positive spillovers, it follows that we only need to compare the payoffs from retaining all links and not forming any links. This constraint is:

$$\frac{1}{2}k^2 + (k-1)k^2 + (n-k) - (k-1)f \geq \frac{1}{2} + (k-1)(k-1)^2 + (n-k) \tag{63}$$

Simplifying, we arrive at the following inequality: $f < 5k/2 - 1/2$. The marginal returns to player i from an additional link with player j who is a singleton are given $3/2+k+2$. Similarly, the marginal return to player j from forming such a link with player i are given by $3/2+1+2k$. Clearly, we require that $f > \min\{7/2+k, 5/2+2k\}$. We note that $7/2+k \leq 5/2+2k$ for all

$k \geq 1$. Finally, we require that two singleton players should not have an incentive to form a link. This is equivalent to the requirement that $f > 9/2$. Clearly the incentives to form a link for players i and j dominate the incentives to form a link between two singleton players. This is a consequence of the positive spillovers property. Thus for a fixed f, a dominant group architecture g^k is pws-equilibrium if and only if $7/2 + k < f < 5k/2 - 1/2$. We note that the interval is non-empty for $k \geq 3$.

We next examine the complete network, where $k = n$. In this case, we only need to check if a player has incentives to maintain all links. This is equivalent to the requirement that $f < 5n/2 - 1/2$. Finally, we check the empty network, where $k = 1$. In this case, we require that no player has an incentive to form a link. This is equivalent to the requirement that $f > 9/2$. Figure 7 illustrates this pattern. $\triangle$

— Insert Figures 6 and 7 somewhere here —

In a recent paper Yi (1997) studies coalition formation under different rules for a similar public good setting. Yi finds that coalitions of unequal size will be stable under different rules. Our finding is that equilibrium networks are either dominant groups or inter-linked stars.The results from the coalition approach and our findings are similar in that asymmetric structures emerge. However, the precise structures we obtain are very different. For instance we find that the star network is an equilibrium in a large class of cases, while this network is ruled out in the coalition framework.

Example 5: *Free trade agreements among countries* [19]

Suppose there are n countries. In each country there is one firm producing a homogeneous good and competing as a Cournot oligopolist in all countries. We let the output of firm j in country i be denoted by Q_i^j. The total output in country i is given by $Q_i = \sum_{j \in N} Q_i^j$. In each country $i \in N$, a firm faces an identical inverse linear demand given by:

[19] In an earlier paper, Goyal and Joshi (1999b) we considered a setting where links between countries are interpreted as free trade agreements. In that paper the links were costless; by contrast, in the present paper, we assume that links are costly and the focus is on the relation between returns and these costs of forming links.

$$P_i = \alpha - Q_i \,, \quad \alpha > 0 \tag{64}$$

All firms have a constant and identical marginal cost of production, $\gamma > 0$. We assume that $\alpha > \gamma$. Let the initial pre-agreement import tariff in each country be $T > \alpha$. Countries can form agreements which lower the tariff to 0. The natural interpretation of such an agreement is as a *bilateral free trade agreement.* We suppose that tariffs remain prohibitively high between countries that do not have a bilateral free-trade agreement. The assumption that $T > \alpha$ ensures that a firm i sells in country j if and only if there is a trade agreement between the two countries. Therefore, $n_i(g) = \eta_i(g) + 1$ is the number of firms active in country i given the network g. If firm i is active in market j, then its output is given by $Q^i_j = (\alpha - \gamma)/(n_j(g) + 1)$. The social welfare of country i is given by:[20]

$$S_i(g) = \frac{1}{2}\left[\frac{(\alpha-\gamma)n_i(g)}{n_i(g)+1}\right]^2 + \sum_{j\in N_i(g)} \left[\frac{\alpha-\gamma}{n_j(g)+1}\right]^2 \tag{65}$$

The marginal return from an additional free trade agreement is given by:

$$\begin{aligned} S_i(g+g_{ij}) - S_i(g) &= \frac{1}{2}\left[\frac{(\alpha-\gamma)(n_i(g)+1)}{n_i(g)+2}\right]^2 - \frac{1}{2}\left[\frac{(\alpha-\gamma)n_i(g)}{n_i(g)+1}\right]^2 + \left[\frac{\alpha-\gamma}{n_i(g)+2}\right]^2 \\ &\quad - \left[\frac{\alpha-\gamma}{n_i(g)+1}\right]^2 + \left[\frac{\alpha-\gamma}{n_j(g)+2}\right]^2 \end{aligned} \tag{66}$$

[20] An important concern in the literature has been the negative effects of (regional and bilateral) free-trade agreements on third parties. One aspect of this effect is "concession diversion". The above expression allows us to examine the nature of concession diversion explicitly. Fix a network g and a country i. Consider a country $j \in N_i(g)$. The firm from country j earns profits $(\alpha-\gamma)^2/(n_i(g)+1)^2$ from its operations in country i. Now consider what happens when country i forms an additional bilateral trade agreement with, say, country k. This allows the firm of country k to enter the market of country i, thus raising the level of competition. In this new network $g + g_{ik}$, the profits of country j firm from its operations in country i are given by $(\alpha-\gamma)^2/(n_i(g)+2)^2$. Suppose that $j \notin N_k(g)$. It follows that profits from all other operations remain the same. Thus the effect of this additional free trade agreement between country i and country k on the profits of firm j is given by $(\alpha-\gamma)^2/(n_i(g)+2)^2 - (\alpha-\gamma)^2/(n_i(g)+1)^2$. This term is negative: this is the measure of concession diversion created by the new bilateral free-trade agreement and therefore the extent of negative spillovers that a link creates on a countries other trading partners.

Simplifying the above expression, we get:

$$S_i(g + g_{i,j}) - S_i(g) = (\alpha - \gamma)^2 \left[\frac{1}{2(n_i(g) + 2)^2 (n_i(g) + 1)^2} \{2n_i^2(g) - 5\} + \frac{1}{(n_j(g) + 2)^2} \right] \quad (67)$$

The marginal payoff of country i from link $g_{i,j}$ is sensitive to the number of links that country i has as well as the number of links that country j has. In particular, the marginal payoff is declining with respect to $n_i(g)$, for $n_i(g) \geq 3$, and declining with respect to $\eta_j(g)$, for $n_j(g) \geq 1$.

We now offer a characterization of all pws-equilibrium networks for the case where $n = 4$. Without loss of generality, we will let $(\alpha - \gamma)^2 = 1$. *(a). The complete network is a pws-equilibrium for* $f \leq 0.056$*. (b). The symmetric network of degree 2 is a pws-equilibrium for* $0.056 < f \leq 0.073$*. (c). The empty network is a pws-equilibrium for* $f > 0.069$*. (d). The dominant group network with one isolated player is a pws-equilibrium for* $0 < f \leq 0.073$.

6 Conclusion

It is widely felt that connections matter both at the individual level as well as in the aggregate. In particular, well-connected individual entities are seen to be at an advantage as compared to their less connected cohorts. In this paper, we present a strategic model of link formation to explore the circumstances which give rise and may support unequal connections across individual players.

In particular, we consider a class of network formation games in which payoffs from links are related to the 'local' structure of the network – the number of links of the individual player in question and the number of links of his partners. Our analysis derives relations between the shape of the marginal payoffs function and the architecture of equilibrium networks. We find that relatively simple and general restrictions on the marginal payoffs lead to sharp predictions with regard to the architecture of equilibrium networks. In particular, we find that if marginal returns are increasing with respect to the number of links of a player then equilibrium networks will typically be asymmetric, with players having different number of links and correspondingly different payoffs. We are able to characterize the architecture of these asymmetric networks fairly tightly: they have the dominant group structure, with a group of players which is fully connected and a complementary group which consists of individually isolated players. By contrast, if marginal returns from links are decreasing, then only moderate levels of inequality in connections can be sustained in equilibrium. These differences are accentuated once we allow for spillovers across links and make individual payoffs contingent on the number of links of the partners. In case of positive spillovers, in addition to the dominant group, we now see the emergence of star-like structures (where some players are fully connected while others are peripheral and have links with these fully connected players only). By contrast, in the case of negative spillovers, symmetric structures figure prominently.

References

[1] V. Bala and S. Goyal (2000), A non-cooperative model of network formation, *Econometrica,* 68, 5, 1181-1231.

[2] V. Bala and S. Goyal (1998), Learning from neighbours, *Review of Economic Studies,* 65, 595-621.

[3] R. Benabou (1993), The workings of a city: location, education and production, *Quarterly Journal of Economics*, 108, 619-652.

[4] F. Bloch (1995), Endogenous structures of association in oligopolies, *Rand Journal of Economics*, 26, 537-556.

[5] F. Bloch (1997), Non-cooperative models of coalition formation with spillovers, in C. Carraro and D. Siniscalco (eds) *The Economic Theory of the Environment*, Cambridge University Press.

[6] R. Burt (1992), Structural Holes: The Social Structure of Competition, Harvard University Press. Cambridge, MA.

[7] S. Boorman (1975), A combinatorial optimization model for transmission of job information through contact networks, *Bell Journal of Economics*, 6, 1, 216-49.

[8] A. Calvo (2000), Job Contact Networks, *mimeo*, University of Carlos III, Madrid.

[9] B. Dutta, A. van den Nouweland and S. Tijs (1995), Link formation in cooperative situations, *International Journal of Game Theory*, 27, 245-256.

[10] P. Dasgupta and J. Stiglitz (1980), Uncertainty, Industrial Structure and the speed of R&D, *Bell Journal of Economics* , 11, 1-28.

[11] M. Delapierre and L. Mytelka (1998). "Blurring Boundaries: New inter-firm relationships and the emergence of networked, knowledge-based oligopolies", in M. G. Colombo (eds), (1998), *The changing boundaries of the firm.* London: Routledge Press.

[12] Ellison, G. and D. Fudenberg (1993), Rules of Thumb for Social Learning, *Journal of Political Economy* 101, 612-644.

[13] Glaeser, E., B. Sacredote and J. Scheinkman (1996), Crime and Social Interactions, *Quarterly Journal of Economics*, 111, 507-548.

[14] Goyal, S. (1996), Interaction Structure and Social Change, *Journal of Institutional and Theoretical Economics*, 152, 3, 472-495.

[15] S. Goyal and S. Joshi, (1999a), Networks of collaboration in oligopoly, *Game and Economic Behavior*, forthcoming.

[16] S. Goyal and S. Joshi, (1999b), Bilateralism and Free-trade, *Econometric Institute Report 9953/A*, Erasmus University Rotterdam.

[17] S. Goyal and J.L. Moraga, (2001), R&D Networks, *Rand Journal of Economics*, forthcoming.

[18] J. Hagedoorn and J. Schakenraad (1990), *Alliances and Partnerships in Biotechnology and Information Technologies*, MERIT, University of Maastricht, Netherlands.

[19] Granovetter, M (1985), Economic Action and Social Structure: The Problem of Embeddedness, *American Journal of Sociology* 3, 481-510.

[20] Jackson, M. and A. Wolinsky (1996), A Strategic Model of Economic and Social Networks, *Journal of Economic Theory* 71, 1, 44-74.

[21] R. Kranton and D. Minehart (2001), A Theory of Buyer-Seller networks, *American Economic Review,* 91, 485-508.

[22] G. Loury (1979), Market Structure and Innovation, *Quarterly Journal of Economics*, 93, 395-410.

[23] Morris, S. (2000), Contagion, *Review of Economic Studies*, 67, 1, 57-79.

[24] D. Ray and R. Vohra (1997), Equilibrium binding agreements, *Journal of Economic Theory*, 73, 30-78.

[25] D. Siedman (2001), Preferential Trading Arrangements as Strategic Positioning, *mimeo*, University of Newcaste.

[26] T. Schelling (1975), *Micromotives and macrobehaviour*, Norton. New York.

[27] B. Wellman and S. Berkowitz (1988), *Social Structure: A Network Approach*, Cambridge University Press, Cambridge.

[28] S. Yi (1998), Endogenous formation of joint ventures with efficiency gains, *Rand Journal of Economics*, 29(3), 610-631.