Unforeseen Contingencies*

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Abstract. We develop a model of unforeseen contingencies. These are contingencies that are understood by economic agents — their consequences and probabilities are known — but are such that every description of such events necessarily leaves out relevant features that have a non-negligible impact on the parties' expected utilities. Using a simple co-insurance problem as backdrop, we introduce a model where states are described in terms of objective features, and the description of an event specifies a finite number of such features. In this setting, unforeseen contingencies are present in the co-insurance problem when the first-best risk-sharing contract varies with the states of nature in a complex way that makes it highly sensitive to the component features of the states. In this environment, although agents can compute expected payoffs, they are unable to include in any ex-ante agreement a description of the relevant contingencies that captures (even approximately) the relevant complexity of the risky environment.

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1. Introduction

1.1. Motivation

In very many circumstances, economic agents, even though they may understand the consequences and probabilities associated with the environment in which they operate, are unable to describe adequately certain complex contingencies that may arise. Often, the gap between the complexity of the environment and what the agents are able to describe ex-ante is non-negligible in the sense that it has a significant impact on the agents' expected utilities.

These contingencies have often been called "unforeseen" by economists (Tirole 1999, p. 743). It is clear that contingencies that are unforeseen in the sense intuitively sketched out above cannot possibly be included in any ex-ante contractual agreement that the agents may contemplate. The resulting contracts are often termed incomplete by economists (Grossman and Hart 1986).[1]

The goal of this paper is to provide a formal model of unforeseen contingencies. We set forth a contractual environment that displays contingencies that are understood by the contracting agents in the sense that their consequences and probabilities are known to them, but where every *feasible* ex-ante description of such events necessarily leaves out relevant features that have a non-negligible impact on the parties' expected utilities. Although the contracting parties are able to carry out expected utility computations to evaluate their decisions in reaching a contractual agreement, they are not able to describe ex-ante some relevant future contingencies. Any attempt to "fill the gap" by describing in more and more detail the relevant set of states will not even approximate a viable description of the unforeseen contingencies that the agents face.

[1] Of course, the fact that a contingency cannot be included in an ex-ante agreement, does not, in general, imply that the *outcome* of the contractual situation cannot depend on such a contingency. This is because of the possible role of ex-post implementation mechanisms (Maskin and Tirole 1999). We return to this issue at some length in Section 2 below.

1.2. The Contractual Environment

The first paragraph of this paper is a statement of fact. However, there is a sense in which it is also not far from an explanation of what precisely an *unforeseen contingency* is. A model in which ex-ante descriptions of certain contingencies are necessarily significantly less complex than the actual contractual environment, can provide a possible formal model of unforeseen contingencies. Of course the value of such model is commensurate to the appeal of the complex environment that it captures, and to the appeal of the class of feasible descriptions that is considered.

In this subsection, we provide an informal description of the state space that embodies the uncertainty that the agents face. We then briefly describe the class of feasible descriptions of a contingency (event) that we consider formally below. We postpone a discussion of our modelling choices until Subsection 1.3 that follows.

Purely for the sake of simplicity, we focus on two agents who enter a contractual relationship whose outcome is affected by the realization of a state of nature. To keep matters simple, we consider a co-insurance problem in which two risk-averse agents face a random environment that makes it mutually beneficial for them to draw up a contract to smooth their consumption across states.

We consider a countable infinity of physical states of nature. These states can be described by means of a language in which a countable infinity of elementary statements are possible. Each elementary statement represents a particular feature that can be either present or not in a given state of nature (the sky can be either "blue" or "not blue").

Slightly more formally, we work with a model in which each state of nature s_n ($n = 1, 2, \ldots$) is characterized by an infinite list of elementary statements $\{s_n^1, \ldots, s_n^i, \ldots\}$ that determine which features are present in the state. Each feature s_n^i can either be present ($s_n^i = 1$) or not ($s_n^i = 0$) in each state.

Real world situations in which the description of each state is by itself potentially highly complex abound. Just as an example consider a situation in which the object of economic interest is the overall configuration of active connections in the US telephone system at any one point in time.[2] In principle, of course, a finite set of features will suffice to describe the state of nature in this case. However, for all intents and purposes such description is not feasible.[3]

An example involving more familiar ingredients is the description of the "output" of an academic (on which, for instance, a promotion decision might be based). Of course, the full set of papers that the academic might write will again in principle be finite. However, for all intents and purposes the features that fully identify the academic's output might be taken again as belonging to an infinite set.

For reasons that we will discuss in detail in Subsection 1.3 below, we work with an "atomless" measure over our countable state space. Roughly speaking, the probability of a set of states of nature will be set equal to its "limit frequency" within the state space. Thus, while any finite set of states will be assigned zero probability by our measure, an infinite set consisting of say "every third state" will receive a probability of 1/3.

The set of events that we consider "describable" — or equivalently the set of contingencies that are *not* unforeseen (for want of a better term *foreseen contingencies* from now on) — is not hard to outline intuitively. A feasible description of an event in our model is an object that must be *finite*. In other words a foreseen contingency is an event that can be fully described with reference to a *finite* set of the constituent features of each state. A describable event must be entirely pinned down by a finite set of statements in the language used to describe the states.

The main result of this paper can be paraphrased as follows. In the set up we have just briefly outlined, it is possible to envisage events that have a well defined probability (frequency), but that are not describable (are unforeseen) in the sense above. Any attempt to capture these events using a finite set of statements in the language used to describe the states will result in the definition of a set of states that

[2]This might be relevant for instance to determine the fully contingent pricing of a further connection to be activated on demand.

[3]For instance, routing of trunk calls is determined by solving a finite set of "local" optimization problems that take into account a particular subset of the system and proceed "as if" the entire system is in fact an object of infinite size.

will *significantly differ* (with positive measure) from the unforeseen contingency that we were trying to capture in the first place.

To go back to the familiar turf concerning the output of an academic (an assistant professor this time), consider the event that she gets tenure. That is consider the set of states of nature that correspond to an academic output sufficient to obtain tenure in the candidate's school. It seems reasonable to assert that any finite description of the assistant professor's output will not capture exactly the set of states of nature in which tenure is given. The probability that the candidate gets tenure if she publishes one paper in a particular journal is neither 1 nor 0. For m large, the probability that the candidate gets tenure with m published papers may even be 1. However, surely the probability of getting tenure with $m-1$ published papers is in fact not 0.

1.3. Discussion

Our choice of what constitutes a feasible description of an event (a foreseen contingency) is not hard to justify. Once a language to describe the states is given, it seems natural and compelling to restrict attention to *finite* "sentences" in the language.

At this point it is useful to notice two features of our definition of foreseen contingencies. The first is that since we are only restricting our descriptions of events to be *finite*, our results below are immune to changes in the elementary statements in the language that, for instance, re-code feature "1" and feature "14" into a single one. A finite statement in one language will correspond to a finite statement in the new one and vice-versa. This immunity to re-coding is a relevant feature in a world in which languages obviously evolve to capture more efficiently concepts that may once have been considered complex or difficult. A contract concerning the content of an e-mail message would have required a much larger number of words 10 years ago than it does now.

The second feature of our definition of a foreseen contingency is that clearly it yields results that must hold in a world in which each statement in the language is associated with a cost. In fact any cost function of the number of statements that guarantees that an infinite number of statements is infinitely costly must yield results that are at least as strong as the ones that we present below. Of course restricting attention only to finite statements also affords us the luxury of not having to specify what the (inevitably arbitrary, and possibly sensitive to re-coding) form of the cost function of longer statements in the language might be.

We are now ready to turn to a discussion of our modelling choices concerning the state space that we have described above. There are two issues of concern. The *cardinality* of the state space, and the *atomless* probability measure that we place on it.

The set of possible states of nature is countably infinite in our model. The reason why an infinity of states is needed to model a complex world is an obvious one. If the set of states is finite, then only finitely many features of each state can possibly matter. Any two states can be "separated" by identifying finitely many of their constituent features. Thus the restriction that a foreseen contingency must use only a finite number of features would have no bite in a model with finitely many states.[4]

Intuitively, the reason we work with a *countable* infinity of states rather than a continuum is as follows. It turns out to be the case that if we consider a continuum of states, those contingencies for which expected utility can be computed are also those that can be handled (at least approximately) by agents who are restricted to condition on foreseen contingencies alone. Roughly speaking this is because, with a continuum of states, those ex-ante agreements that can be "integrated" to yield well defined expected utility values are also those agreements that can be approximated by a sequence of (step functions) agreements that specify foreseen contingencies alone (Anderlini and Felli 1994).[5]

Any "standard" (countably additive) probability measure over a countable set also poses a problem to model the complex world that we try to capture here. Suppose that we were to place a countably additive probability measure over our state space.

[4]Of course this would no longer be true in a model in which "writing costs" are explicitly modelled (Anderlini and Felli 1999, Battigalli and Maggi 2002). As we mentioned above, modelling such costs is something we specifically want to avoid here.

[5]We return to this point in Section 2 below.

Then we could be sure to find a finite subset of the state space that "approximates" the entire set of possible states in the sense that whatever is left out has arbitrarily small probability. Therefore, in this world, the agents could approximate whatever is best for them by effectively considering a finite problem. But, as we noted above, once we are dealing with a finite problem, only finitely many features of each state can possibly matter. Hence, the restriction to conditioning only on foreseen contingencies would have no bite in this case.

To avoid the "approximation" problem that we have just outlined, we choose to work with an atomless (finitely additive) probability measure over our countable state space.

1.4. Overview

The plan of the rest of the paper is as follows. We begin by reviewing some related literature in Section 2. In Section 3 we set up the co-insurance problem we use as a backdrop and derive the benchmark efficient allocation that the parties can achieve in the absence of any constraint. We then define the state space that we described intuitively above, and the associated probability measure in Section 4. In Section 5 we proceed to give a formal definition of the notion of a finite contract. In Section 6 we piece together all these elements and proceed to evaluate the parties' expected utilities associated with any finite contract. Section 7 presents our first batch of results: we show that for some instances of our basic co-insurance problem the only transfers that the parties would like to specify are contingent on unforeseen contingencies. As a consequence, the optimal finite contract is to specify no transfers at all: the no-contract outcome obtains. Sections 8 and 9 generalize the results of Section 7 to the case in which *some* of the variability of the environment can be captured by a finite contract, but a non-negligible amount of uncertainty cannot be captured in this way. In Section 8 we consider a continuous "smoothed" contracting problem that can be associated with every instance of our basic co-insurance problem with a countable state space, and we establish some of its basic properties. In Section 9 we use the tools developed in Section 8 to characterize the shape of optimal finite contracts in the general case for our basic co-insurance problem. Section 9 also includes two leading examples of how our tools can be used to characterize optimal finite contracts in specific cases. Section 10 concludes the paper. For ease of exposition, all proofs have been relegated to the Appendix.[6]

2. Related Literature

The intuitive notion of a contingency that is "[...]prohibitively difficult to think about and describe unambiguously in advance" (Grossman and Hart 1986, p. 696) has been extensively used in the contracting literature. In short, if we take *as given* that some contingencies cannot be included in an ex-ante agreement (although their consequences and probabilities are understood by the agents), and therefore that contracts are incomplete, we can then focus on the institutional arrangements that may reduce the inevitable inefficiencies that are associated with this lack of detail of the ex-ante contracts that the parties draw up.

This line of research has proved extremely fertile. Among other things, it has afforded important insights concerning the boundaries of a firm (Grossman and Hart 1986), the allocation of ownership rights over physical assets (Hart and Moore 1990), the allocation of authority (Aghion and Tirole 1997) and power (Rajan and Zingales 1998) in organizations and the judicial role of the courts of law (Anderlini, Felli, and Postlewaite 2001).

Perhaps precisely because of its prominence and usefulness in modelling a wide range of economic phenomena, the plain assumption that contracting agents may face some contingencies that are unforeseen has itself been the subject of intense scrutiny in a number of recent papers. It seems useful to distinguish between two literature strands here. One that investigates the *foundations* of the notion of unforeseen contingencies in a contractual set-up, and one that addresses the necessary *effects* on the contractual outcomes that can be achieved when unforeseen contingencies are present.[7]

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[6]In the numbering of equations, definitions, remarks and so on, a prefix of "A" indicates that the relevant item is to be found in the Appendix.

This paper is a contribution to the literature that concerns the *foundations* of the notion of an event that has known consequences and probabilities but which is impossible to include in an ex-ante agreement.[8]

Anderlini and Felli (1994) and Al-Najjar (1999) are two existing contributions that are closely related to the results presented here.

In Anderlini and Felli (1994), the contracting parties are restricted to ex-ante agreements that are finite in a sense that is analogous to the one we postulate in this paper. However, crucially, in Anderlini and Felli (1994), there is a *continuum* of states of nature. One of the results reported there is the so-called *approximation result*: in a model with a continuum of states, under general conditions of continuity, the restriction that only finitely many of the constituent features of a state of nature can be included in any ex-ante agreement has a *negligible impact* on the parties' expected utilities.

The restriction to finite agreements clearly precludes the agents from writing some possible ex-ante contracts.[9] Intuitively, the reason why the impact of this restriction is in fact negligible lies in the requirement that the parties must be able to compute the *expected utilities* that an ex-ante agreement generates. In short, if an ex-ante agreement yields well defined expected utilities to the contracting parties, then it must yield them utility levels that are "integrable" as a function of the state of nature. Since a function that is integrable can always be approximated by a sequence of step functions, it is now enough to notice that (a "sufficiently rich" set of) step

[7]It should be noted at this point that the term "unforeseen contingencies" has also been used in a number of decision-theoretic and epistemic models (see for instance Kreps (1992), and more recently Dekel, Lipman, and Rustichini (2001) and the survey in Dekel, Lipman, and Rustichini (1998)). Once again, here we are using the term unforeseen contingency in a restricted sense. Our contracting parties understand (have common knowledge of) the consequences and probabilities of unforeseen contingencies. They are simply unable to describe them in advance and hence to incorporate them in any ex-ante agreement.

[8]Anderlini and Felli (2000) contains a partial review of the literature that links the notion of complexity costs to that of contractual incompleteness as generated by unforeseen contingencies.

[9]A simple counting argument suffices to prove this point. It is easy to see that in the world of Anderlini and Felli (1994) there are countably many possible finite ex-ante contracts, while there are uncountably many possible ex-ante agreements.

functions can be viewed as *finite* ex-ante agreements.

The key differences between the analysis in Anderlini and Felli (1994) and the results that we report below is that the approximation result *does not hold* in the model we analyze here. Intuitively the stark difference between the two environments can be traced to the cardinality of the state space (countable versus continuous) and the nature of the associated probability measure (finitely additive "frequencies" in this paper, "standard" probability measures over the interval $[0, 1]$ in Anderlini and Felli (1994)).

In Al-Najjar (1999) the state space is akin to the one used here: it is discrete and is equipped with finitely additive "frequencies," as in the analysis below. Using this apparatus, in a very different set-up from the one analyzed below, Al-Najjar (1999) addresses the question of whether competitive differences between agents get washed out by imitation. Roughly speaking, imitation is limited to those features that can be finitely defined. In a model with a continuum of states it is possible to show that the performance of a successful agent can be replicated asymptotically as more and more data become available: a version of the approximation result described above holds in this case. However, in a complex environment (embodied in a state space similar to the one used in the present paper) imitation does not eliminate all competitive advantages, even in the limit when an arbitrarily large amount of data becomes available.

Two further papers have investigated contractual environments in which the approximation result described above fails. The analysis in both Anderlini and Felli (1998) and Krasa and Williams (1999) centers on the observation that the approximation result in Anderlini and Felli (1994) requires the parties utilities to be *continuous* in an appropriate way. The focus of Anderlini and Felli (1998) is to characterize the effects of discontinuities in the parties' utilities in a principal-agent model in which only finite agreements are allowed. Krasa and Williams (1999) focus on a condition that they label "asymptotic decreasing importance" which, in their model, is necessary and sufficient for the required continuity conditions, and hence for the approximation result, to hold. By contrast, in this paper the parties' utilities *are assumed* to be continuous in outcomes. The fact that the approximation result fails in our model below is not due to any form of discontinuity in the agents' preferences.

Using the contribution by Hart and Moore (1988), as their point of departure Maskin and Tirole (1999) have highlighted a tension between the restrictions that the agents face in drawing up an ex-ante agreement and their impact on the possible contractual outcomes *ex-post*.[10] Hart and Moore (1988) analyze an environment in which there are unforeseen contingencies and all payoffs and their probabilities are *common knowledge* between the contracting parties. Maskin and Tirole (1999) argue that this does not necessarily impose any restrictions on the outcomes (payoffs) that the contracting parties can achieve *ex-post*.

In short, Maskin and Tirole (1999) argue that, instead of relying on an infeasible ex-ante description of all relevant contingencies, the parties can write an ex-ante contract that commits them to playing an ex-post revelation game. In this revelation game the players are required to report the payoff relevant information associated to the realized states or any uniquely defined coding of this information. Provided that the coding is common knowledge, the game can be designed so that the parties in equilibrium report the truth, and the allocation implemented by such a mechanism coincides with the allocation implemented by the contract that is optimal in the absence of any unforeseen contingencies. In other words, an ex-post implementation mechanism allows the parties to render the realized outcome contingent on the unforeseen contingencies that at an ex-ante stage the parties could not describe.

This paper provides a formal foundation for the notion of unforeseen contingencies that fits the environment considered by Hart and Moore (1988) and Maskin and Tirole (1999). If we are in a world in which the Maskin and Tirole (1999) "critique" applies, then our contribution can be viewed as providing a rigorous model in which *any efficient outcome must necessarily be implemented ex-post using a message game* (Moore and Repullo 1988, Maskin and Tirole 1999). If, on the other hand, the relevant environment is one to which the Maskin and Tirole (1999) critique does not apply,

[10]Tirole (1999) provides an authoritative account of the state of this literature and the debate that it has generated.

then our results below can be viewed as a foundation for contractual agreements that are genuinely incomplete.

We conclude our discussion of related literature by recalling that the possibility of renegotiation tempers the benefits to the contracting parties of an ex-post implementation mechanism. If the contracting parties are allowed to write message-contingent mechanisms but they cannot commit ex-ante *not* to renegotiate the agreed mechanism if an ex-post mutually beneficial opportunity arises, the gain from these mechanisms may be greatly reduced as the complexity (Segal 1999, Hart and Moore 1999) or symmetry (Reiche 2001) of the environment increases. In particular Segal (1999) analyzes an environment in which all "states of nature" have an equal probability and an equal impact on the complexity of the message game the parties optimally commit to. Hence as the number of states of nature (the number of "widgets" in his case) increases without bound, the welfare benefits of the message game decrease asymptotically to zero. In the limit the parties' welfare coincides with their welfare in the absence of any ex-ante contract. The state space that we consider in this paper could be embedded in a different contracting model to formalize the limit world of Segal (1999). Although our analysis abstracts from the message contingent mechanism, we propose an environment with a countable infinity of states of nature in which the parties cannot approximate the first best by focusing on any finite subset of states.

3. The Contracting problem

For the sake of concreteness, throughout the paper we work using a standard co-insurance problem as backdrop. Two risk-averse agents, labelled $i = 1, 2$ face a risk-sharing problem. The uncertainty in the environment is captured by the realization of a state of nature, denoted by s; the set of all possible states of nature is denoted by $\mathcal{S}$. The preferences of agent i are represented by the state contingent utility function $U_i : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$. For simplicity only, we assume that the agents' utilities depend on s only according to whether or not s falls in a subset $\mathcal{Z}$ of the state space $\mathcal{S}$.

The two agents can agree to a state-contingent monetary transfer $t \in \mathbb{R}$, which by convention represents a payment from 2 to 1. We write the utility of 1 in state s, if the transfer is t as

$$U_1(t,s) = \begin{cases} V(1+t) & \text{if } s \in \mathcal{Z} \\ V(t) & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{1}$$

where $\overline{\mathcal{Z}}$ denotes the complement of $\mathcal{Z}$ in $\mathcal{S}$. Party 2's utility in state s is instead written as

$$U_2(t,s) = \begin{cases} V(-t) & \text{if } s \in \mathcal{Z} \\ V(1-t) & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{2}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a twice differentiable, increasing and strictly concave function satisfying the Inada conditions

$$\lim_{y \to -1} V'(y) = +\infty, \quad \lim_{y \to +1} V'(y) = 0.$$

Ex-ante, 1 makes a take-it-or-leave-it offer of a contract $t : \mathcal{S} \to \mathbb{R}$ to 2, where $t(s)$ is the monetary transfer from 2 to 1 if state s is realized. Of course, 1's take-it-or-leave-it offer to 2 will have to satisfy a participation constraint for 2 which will be specified shortly.

The co-insurance problem we have just described is a completely standard one. Since in (1) and (2) we have specified the agents utilities so that complete insurance is in fact feasible, in the absence of any additional restrictions, the optimal contract $t^*(\cdot)$ will involve only two levels of transfers $t_{\mathcal{Z}}$ and $t_{\overline{\mathcal{Z}}}$ with

$$t^*(s) = \begin{cases} t_{\mathcal{Z}} & \text{if } s \in \mathcal{Z} \\ t_{\overline{\mathcal{Z}}} & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{3}$$

and $1 + t_{\mathcal{Z}} = t_{\overline{\mathcal{Z}}}$ so that

$$U_1(t(s), s) = V(1 + t_{\mathcal{Z}}) = V(t_{\overline{\mathcal{Z}}}) \quad \forall\, s \in \mathcal{S} \tag{4}$$

and

$$U_2(t(s), s) = V(-t_{\mathcal{Z}}) = V(1 - t_{\overline{\mathcal{Z}}}) \quad \forall\, s \in \mathcal{S} \tag{5}$$

Agent 2's participation constraint can be easily specified if we define the probability $p = \Pr\{s \in \mathcal{Z}\}$ that s falls in $\mathcal{Z}$. In the absence of any agreed transfers 2's expected utility is $pV(0) + (1-p)V(1)$. Since 2 is the recipient of a take-it-or-leave-it offer, his participation constraint will bind. Therefore, in addition to (4) and (5) the optimal contract $t^*(\cdot)$ is characterized by

$$pV(-t_{\mathcal{Z}}) + (1-p)V(1 - t_{\overline{\mathcal{Z}}}) = pV(0) + (1-p)V(1) \tag{6}$$

Clearly, equations (4), (5) and (6) uniquely pin down the values of $t_{\mathcal{Z}}$ and $t_{\overline{\mathcal{Z}}}$, so that the characterization of the solution to our co-insurance problem in the standard case is complete.

4. States and Probabilities

We are now ready to proceed with a formal description of our state space $\mathcal{S}$ and the associated probability measure μ.

As we mentioned above, both of these ingredients of our model are not of a standard form. They are building blocks of a world in which *details*, no matter how small, can matter a lot. The inability to capture these details in any finite ex-ante agreement is at the center of our model of unforeseen contingencies.

4.1. The State Space

We think of there being a countable infinity of *physical states of the world* $\mathcal{S} = \{s_1, \ldots, s_n, \ldots\}$.

The parties have a common language to *describe* each state s_n. The language consists of a countable infinity of *elementary statements* (characteristics) that can be true or false about each state of nature s_n. Hence the *complete* description of a state of nature s_n can be thought of as an infinite sequence $\{s_n^1, \ldots, s_n^i, \ldots\}$ of 0's and 1's. Each element of the sequence is simply interpreted as reporting whether the i-th elementary statement is true ($s_n^i = 1$) or false ($s_n^i = 0$) about state s_n.

The formal definition of our state space simply encapsulates what we have stated so far about $\mathcal{S}$.

Definition 1. *<u>State Space</u>: The state space $\mathcal{S}$ is a countably infinite set $\{s_1,\ s_2,\ \ldots s_n, \ldots\}$. Each s_n is in turn an infinite sequence of the type $\{s_n^1, \ldots, s_n^i, \ldots\}$ with $s_n^i \in \{0,1\}$ for every i and n.*

4.2. Probabilities

The second ingredient that is crucial to our model of unforeseen contingencies is the probability measure over the set of possible states of nature $\mathcal{S}$. Again, our goal is to model a world in which small details can have a non-negligible impact on the contract that the parties would like to draw-up and hence on their expected utilities.

Any countably additive probability measure $p(\cdot)$ over a countable set like $\mathcal{S}$ cannot be "atomless" in the following obvious sense. For every $\xi > 0$ we can find $\overline{n}$ such that $\sum_{n=\overline{n}}^{\infty} p(s_n) < \xi$. It follows that (provided utilities are bounded below) the risk-sharing problem we described in Section 3 above can be approximated arbitrarily closely by considering a finite problem that ignores all states of nature s_n with $n \geq \overline{n}$. The expected utility loss from a contract that prescribes an arbitrary sharing of surplus for all but finitely many states is proportional to ξ.[11] Since only finitely many states matter, it is now clear that only finitely many features of each state can possibly matter in the contracting problem. A finite set of features will be sufficient to "distinguish" between any two states in the relevant finite set. Hence, with a countably additive probability measure over $\mathcal{S}$, the features of each state beyond a certain level have a negligible impact on the contracting problem. Small details are negligible in some well defined sense.

Therefore, to proceed in our modelling of unforeseen contingencies in which details matter we have to abandon the requirement that the probability measure we place over $\mathcal{S}$ is countably additive, and consider a genuinely atomless, finitely additive, probability measure. Our first step is to define the *density* of a set of states.

[11] This is the basic intuition behind the version of the Approximation Result for the case of a countable state space with a countably additive probability measure reported in Anderlini and Felli (2000).

Definition 2. *Density: Given any $Q \subseteq \mathcal{S}$, let χ_Q denote the indicator function of Q so that $\chi_Q(s_n) = 1$ if $s_n \in Q$ and $\chi_Q(s_n) = 0$ if $s_n \notin Q$. We define the* density *of Q to be*

$$\mu(Q) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_Q(s_n) \tag{7}$$

when the limit in (7) exists. The density is otherwise left undefined. We denote by $\mathcal{D}$ the collection of subsets of $\mathcal{S}$ that have a well defined density.

Two points should be noted. First of all the density of a set $\mu(Q)$ is its "frequency" in the standard sense of the word. Thus, for instance, every finite set of states has a density of zero and the set of all "even numbered" states $\{s_2, s_4, s_6, \ldots\}$ has a density of 1/2. Secondly, the definition of density we have given (both whether the density of a given set is defined and the value that it takes if it is defined) depends on the *ordering* of the states $\{s_1, \ldots, s_n, \ldots\}$. This ordering is taken as given and fixed throughout the paper.[12]

We conclude this subsection with two observations that will become useful below.

First, given two sets Q' and Q'' that have well defined densities and such that $\mu(Q') > 0$ and $\mu(Q' \cap Q'')$ is also well defined, then we can define the *conditional density* $\mu(Q'' \mid Q')$ as $\mu(Q' \cap Q'')/\mu(Q')$.

Secondly, if we let Σ be the set of all subsets of $\mathcal{S}$. Then there exists an extension to Σ of the density μ in Definition 2 above which is a finitely additive probability measure. In other words

Remark 1. *Finitely Additive Probability Measure: There exists a finitely additive probability measure $\tilde{\mu}$ over $(\mathcal{S}, \Sigma)$ that for every set of states $B \subset \mathcal{S}$ satisfies $\tilde{\mu}(B) = \mu(B)$, whenever $\mu(B)$ is defined.*[13]

[12]The class of permutations of the states of nature that leave our results unaffected includes all finite permutations. We do not attempt a general characterization of such permutations in this paper.

[13]See, for example, Rao and Rao (1983), p. 41) for a proof.

5. Finitely Definable Sets and Finite Contracts

The set of ex-ante contracts that our agents can draw up intuitively coincides with those agreements that are *finite* in a sense to be defined shortly in a formal way.

There is a further important issue to clarify at this point. We interpret the words "ex-ante" to mean that a contract prescribes outcomes (transfers) as a function of the actual realized state. Any other variables cannot be included in an ex-ante agreement because they cannot be "verified" by an enforcing authority (a court).[14]

It is convenient to start our description of what a finite contract is by introducing the notion of a *finitely definable set*. Intuitively, these are subsets of $\mathcal{S}$ that can be defined referring only to a *finite* set of their constituent features.

For each state of nature s_n, let $s_n^i \in \{0,1\}$ indicate the value of the i-th feature of s_n. Define also

$$A(i,j) \quad = \quad \{s_n \in \mathcal{S} \quad \text{such that} \quad s_n^i = j\} \tag{8}$$

so that $A(i,j)$ is the set of those states of nature that have the i-th feature equal to $j \in \{0,1\}$.

We are now ready to define the set of finitely definable subsets of $\mathcal{S}$.

Definition 3. *Finitely Definable Sets: Consider the* algebra *of subsets of $\mathcal{S}$ generated by the collection of sets of the type $A(i,j)$ defined in (8). Let this algebra be denoted by $\mathcal{A}$. We refer to any $A \in \mathcal{A}$ as a finitely definable set.*

Elements of $\mathcal{A}$ can be obtained by complements and/or finite intersections and/or finite unions of the sets $A(i,j)$. Hence every element of $\mathcal{A}$ can be defined by finitely many elementary statements about the features of the states of nature that it contains.

[14] As we remarked above (see our discussion of Maskin and Tirole (1999) in Section 2) the parties to a contract can attempt to implement the desired outcomes as a function of other (non-verifiable) variables (e.g. utility levels) committing to an *ex-post* game. Our focus here is what can be achieved by means of ex-ante agreements. In a world with unforeseen contingencies as we model here, the agents may indeed be forced to rely on such ex-post mechanisms to implement certain outcomes. We come back to this point once again in the concluding section of the paper.

A suitable definition of a finite contract is now easy to get. The key feature of a finite contract is that it should specify a set of transfers that is conditional only on finitely definable sets. For simplicity we also restrict attention to contracts that specify a finite set of values for the actual transfer t. This is clearly without loss of generality in our simple co-insurance problem described in Section 3 above.

Definition 4. *Finite Contracts: A contract is* finite *if and only if the transfer rule $t(\cdot)$ that it prescribes is measurable with respect to $\mathcal{A}$, and takes finitely many values $\{t_1, \ldots, t_M\}$. The set of finite contracts is denoted by $\mathcal{F}$.*

Above, we have justified informally the fact that one might want to restrict attention to finite contracts using the idea that contracts must be finite objects in some sense. While it is possible to take Definition 4 as a primitive that embodies the notion of a contract as a finite object, it is important to point out that this requirement can be supported in a different way (than just taking Definition 4 at face value).

Anderlini and Felli (1994) put forward the idea that it is natural to consider contracts that yield a value for a sharing rule that is *computable* by a Turing machine as a function of the state of nature. The justification for this requirement is a claim that if a function is computable in a finite number of steps by any imaginable finite device then it must be computable by a Turing machine.[15] Obviously, any finite contract must be computable. It is also possible to show that the converse holds: requiring that contracts be finite exhausts the set of all computable contracts. For reasons of space, we omit any formal analysis of this topic.

6. Computing Expected Utilities

We now have set out all the ingredients of our model. In essence we want to characterize what the agents can achieve using finite contracts when the state space and associated probability measure are as in Section 4.

[15]This claim is known in the literature on computable functions as *Church's thesis*. See for instance Cutland (1980), or Rogers (1967).

As we mentioned already, we want to restrict attention to those cases in which the agents can base their choices on the *expected utility* that an ex-ante contract yields. Since we want the agents to be able to contemplate *all* possible finite contracts, we need to ensure that all such contracts can be evaluated in this way. So far, there is nothing in our framework that guarantees that this is the case. This is because our Definition 2 above does not, by itself, guarantee that all finitely definable sets have a well defined density. Our next assumption guarantees that this property holds.

Assumption 1. *Densities of Finitely Definable Sets: The state space $\mathcal{S}$ is such that every $A \in \mathcal{A}$ has a well defined density $\mu(A)$. In other words, $\mathcal{S}$ is such that $\mathcal{A} \subseteq \mathcal{D}$.*

Of course, at this point we need to show that a state space $\mathcal{S}$ as in Definition 1 that also satisfies Assumption 1 does indeed exist.

Proposition 1. *Existence: There exists a state space $\mathcal{S}$ as in Definition 1 that satisfies Assumption 1.*

The proof of Proposition 1 is a simple consequence of the law of large numbers. Think of $\mathcal{S}$ as a realization of countably many i.i.d. draws from, say, a (countably additive) density $\hat{\mu}$ over $\{0, 1\}^{\mathbb{N}}$. It is then sufficient to observe that the law of large numbers guarantees that, with probability one, the fraction of draws that falls into any finitely definable set A is in fact well defined and equal to its density $\hat{\mu}(A)$. The set of realizations of these i.i.d. draws that have the properties required to satisfy Assumption 1 has probability one in the space of realizations of this process. It then follows that it must be not empty. Hence, setting $\mathcal{S}$ to be equal to a "typical" realization of these i.i.d. draws as described is sufficient to prove the claim.

To evaluate the expected utility accruing to each party from any finite contract we will also need to refer to the *conditional* densities of certain events. This is an easy task if we restrict attention to finitely definable sets. The following remark is stated without proof since it is a direct consequence of the fact that, by assumption, since $\mathcal{A}$ is an algebra, the intersection of two finitely definable sets is itself finitely definable.

Remark 2. *Well Defined Conditional Densities: Let Assumption 1 hold and let A' and A'' be two finitely definable sets with $\mu(A') > 0$. Then the conditional density $\mu(A'' \mid A')$ is well defined.*

In any simple co-insurance problem of the type described in Section 3, the expected utility that a finite contract yields to the agents is, of course, the result of two maps: the contract itself, and the way in which the agents' state-dependent utilities vary with the state of nature. Therefore, to ensure that the agents' expected utilities from any finite contract is well defined we need to further impose a restriction on the second of these two maps. Clearly, without doing so, it is possible that the agents' utilities vary with the state of nature in a way that makes it impossible to assign a frequency to the contract prescribing a certain transfer t *conditional* on the state of nature belonging to a particular set on which the agents' utilities depend.

Notice that, a *stronger* restriction on the way that the agents' utilities depend on the state of nature will make our results below *stronger* rather than weaker. This is because our results will show that the agents' utilities can be made to depend on the state in such a way that any finite contract is unable to capture (all or a significant part) of such variability. Clearly, the smaller the class of state-dependent utilities to which we refer, the stronger the result.

It is useful to start with an abstract definition of what it means for a function to vary with the state of nature so that its frequencies can be computed, conditional on any finitely definable set. This, roughly speaking, is the maximum rate of state-dependence that we will allow for the agents' utilities.

Consider a function $f : \mathcal{S} \to \{f_1, \ldots, f_M\}$ and denote $F_i = f^{-1}(f_i)$, for $i \in \{1, \ldots, M\}$ the inverse images of each of the values f_i. Then we say that the function $f(\cdot)$ has well-defined frequencies if it is possible to compute the density of F_i, conditional on any finitely definable set.

Definition 5. *Well-Defined Frequencies: The function $f : \mathcal{S} \to \{f_1, \ldots, f_M\}$ has* well defined frequencies *if*

$$F_i \cap A \in \mathcal{D} \quad \forall\, A \in \mathcal{A}, \quad \forall i \in \{1, \ldots, M\}$$

in other words the inverse image sets of f have densities, conditional on any finitely definable set A (provided of course that $\mu(A) > 0$).

We have now introduced all the elements that will allow us to study a class of co-insurance problems in which unforeseen contingencies can arise, and in which the expected utilities for both agents from any finite contract are well defined and can be computed in a simple way.

The fact that unforeseen contingencies can arise in this model is the subject of our next section. For the time being, we remark that the expected utilities from any finite contract are well defined.

Our next statement takes the shape of a *definition* (rather than a proposition) since we are in fact defining what the natural meaning of expected utilities is in a world in which probabilities are equated with the densities of Definition 2 above.

Definition 6. *Expected Utilities: Consider the co-insurance problem described in Section 3. Let a density μ as in Definition 2 be given and let Assumption 1 hold. Assume further that the function $f : \mathcal{S} \to \{0, 1\}$ defined as $f(s) = 1$ if $s \in \mathcal{Z}$ and $f(s) = 0$ if $s \in \overline{\mathcal{Z}}$ has well defined frequencies in the sense of Definition 5. Let also any finite contract $t : \mathcal{S} \to \{t_1, \ldots, t_M\}$ be given. Then the expected utility to agent 1 from contract t is defined as*

$$EU_1(t) \;=\; \sum_{i=1}^{M} V(1 + t_i)\, \mu[t^{-1}(t_i) \cap \mathcal{Z}] + \sum_{i=1}^{M} V(t_i)\, \mu[t^{-1}(t_i) \cap \overline{\mathcal{Z}}\,] \tag{9}$$

while 2's expected utility is

$$EU_2(t) \;=\; \sum_{i=1}^{M} V(-t_i)\, \mu[t^{-1}(t_i) \cap \mathcal{Z}] + \sum_{i=1}^{M} V(1 - t_i)\, \mu[t^{-1}(t_i) \cap \overline{\mathcal{Z}}\,] \tag{10}$$

We conclude this section with an observation. Using the finitely additive probability measure $\tilde{\mu}$ of Remark 1 that extends μ to all subsets of $\mathcal{S}$ it is possible to compute the density of every set $D \in \Sigma$. This in turn would allow us to compute the expected utility of a much broader class of contracts that are not necessarily finite, allowing also for a much broader class of state-dependent utilities. Of course, to do this we would need a way to integrate a much broader class of functions $f : \mathcal{S} \to \mathbb{R}$ with respect to $\tilde{\mu}$. Fortunately, there is an elaborate theory of integration with respect to finitely additive probabilities, which for the most part is analogous to the usual theory of integration.[16]

In this paper, we restrict attention to contracts that are measurable with respect to $\mathcal{A}$. Of course, when we restrict attention to this case, the more general type of integration that we are referring to gives exactly the expected utilities that we have defined above.

To simplify matters further, we also restrict attention (without any loss in generality in our co-insurance setup) to contracts that take a finite number of values. It should be noted, however, that the restriction to finitely-valued functions, is introduced only for expository simplicity; our analysis is applicable more generally (although this would require some additional machinery).

7. Unforeseen Contingencies

7.1. *Finite Invariance and Fine Variability*

In contrast to the cases of a continuous state space and of a countable state space with a countably additive probability measure, finite contracts *cannot* always *approximate* the first best in the model we have set-up here. The idea is the allocation t^* that the agents may be trying to attain could exhibit "fine" variability as a function of the state of nature. Any finite contract is bound not to capture part (or all) of this

[16]Dunford and Schwartz (1958) is a classic textbook which provides a unified treatment of integration for both finite and countably additive measures. A more specialized treatment can be found in Rao and Rao (1983).

variability. It is important to stress again that this is in fact possible when the state-dependence of the agents' preferences is such that the expected utility of *any* finite contract (Definition 6) is well defined.

We begin with two abstract definitions that capture the idea that in the model we have set up it is possible that a function $f : \mathcal{S} \to \{f_1, \ldots, f_M\}$ may "look the same" if we look at its restriction over any finitely definable set, but at the same time may vary "finely" with the state of nature. It will be precisely this type of fine variability that finite contracts cannot capture and hence give rise to unforeseen contingencies below.

Definition 7. *Finite Invariance: Let $f : \mathcal{S} \to \mathbb{R}$ be a function that takes finitely many distinct values $\{f_1, \ldots, f_M\} \in \mathbb{R}^M$ and let $F_i = f^{-1}(f_i)$ for every $i \in \{1, \ldots, M\}$. We say that f displays* finite invariance *over A (with $\mu(A) > 0$) if for every subset $A' \subseteq A$ such that $A' \in \mathcal{A}$ and $\mu(A') > 0$,*

$$\mu(F_i|A') = \mu(F_i|A) \qquad \forall\, i \in \{1, \ldots, M\} \tag{11}$$

So, f displays finite invariance over A, if the densities of the sets F_i are the same, conditional on all finitely definable sets that are subsets of A.

In other words, if f displays finite invariance over, say, the whole of $\mathcal{S}$, knowing that s belongs to any finitely definable subset of $\mathcal{S}$ does not help us to "predict" better the values that f will take. It should be noted at this point that the possibility that Definition 7 may have a non-trivial content is a feature of the model we have set up, which *does not* hold in say a standard model with a continuum of states when f is a measurable function of s. In fact, it is clear that in this case if f is finitely invariant over A then it must be (essentially) constant over A. This is not the case in our model, as we will demonstrate shortly in Proposition 2 below.

The second abstract definition that we state is a property that we label fine variability: roughly speaking this is a measure of the degree of variability of a finitely-valued function that displays finite invariance.

Definition 8. *Fine Variability: Let $f : \mathcal{S} \to \{f_1, \ldots, f_M\}$ finitely invariant over $A \in \mathcal{A}$ with $\mu(A) > 0$ be given.*

We say that $f(\cdot)$ displays fine variability *of degree $v \in \{1, \ldots, M-1\}$ over A if and only if there are $v+1$ elements of the range of f that have positive density conditional on A.*

In other words $\mu(F_i|A) > 0$ for $v+1$ distinct F_i's.

The properties that we have just defined may *simultaneously* hold for a function that is also well defined in terms of frequencies.

Our next proposition asserts that, for some state spaces $\mathcal{S}$ satisfying Assumption 1 even though a function may be well defined in terms of frequencies and display finite invariance over a finitely definable set A, it may be far from being constant over A. In other words, f may be well defined in terms of frequencies and display finite invariance over A, but at the same time exhibit an arbitrary amount of fine variability over the same set A.

Proposition 2. *Finite Invariance and Fine Variability: There exists an $\mathcal{S}$ such that the following is true.*

Let $v \in \{1, \ldots, M-1\}$ and any $(p_1, \ldots, p_M) \in \Delta^{M-1}$ be given.[17] *Let also $A \in \mathcal{A}$ be given, with $\mu(A) > 0$.*

Then there exists a function $f : \mathcal{S} \to \{f_1, \ldots, f_M\}$ with well defined frequencies that displays both finite invariance and *fine variability of degree v over A. Moreover the density of $F_i = f^{-1}(f_i)$ conditional on A is equal to p_i for every $i = 1, \ldots, M$.*

The formal proof of Proposition 2 is in the Appendix. Here we sketch the argument in the simple case in which f takes only two values $\{f_1, f_2\}$, $A = \mathcal{S}$ and $p_1 = p_2 = 1/2$.

Let $\mathcal{S}$ be as in Proposition 1. We can then construct the function f in the following way. For each given state of nature $s_n \in \mathcal{S}$ we set $f(s_n)$ equal to f_1 or f_2 with equal

[17]Here, and throughout the rest of the paper the notation Δ^{M-1} denotes the $(M-1)$-dimensional simplex in $\mathbb{R}^M$.

probability, and with i.i.d draws across all the states s_n. The law of large numbers again guarantees that we can take f to be a "typical" realization of this process to prove the claim.

In fact, in any such typical realization, the law of large numbers ensures that the event "the function f takes value f_i" has a density that is well defined and is equal to $1/2$ conditional on any finitely definable subset of states. This clearly guarantees that f exhibits both finite invariance and fine variability of degree 1, as well as displaying well defined frequencies, as required.

As we mentioned above, the type of fine variability that is found in Proposition 2 is at the root of our model of unforeseen contingencies. Our next task is to examine its impact on the simple co-insurance model described in Section 3 above.

7.2. Unforeseen Contingencies and Fine Variability

The possibility that the contract t^* in the co-insurance problem described in Section 3 above may have the fine variability described in Proposition 2 has far reaching consequences on what the contracting parties can achieve by means of a finite contract.

In this section, we characterize the impact of fine variability in its simplest form — namely when it is associated with finite invariance over the entire state space. In this case, any finite contract will be unable to capture any of the fine variability of t^*. As a consequence the agents will choose a trivial contract that prescribes a transfer of $t = 0$ in every possible state. This is of course the same as saying that *no contract* will be drawn up.

Consider the co-insurance problem described in Section 3. For a given $\mathcal{S}$, μ and $\mathcal{Z}$, let t^{**} be the optimal *finite* co-insurance contract, if it exists. In other words, if it is well defined let t^{**} be the solution to

$$\begin{aligned} \max_t \quad & EU_1(t) \\ \text{s.t.} \quad & EU_2(t) \ \geq \ \mu(\mathcal{Z})V(0) + \mu(\overline{\mathcal{Z}})V(1) \\ & t \in \mathcal{F} \end{aligned} \tag{12}$$

where $EU_i(t)$ are the parties' expected utilities as in Definition 6 above.

Proposition 3. *Optimal Finite Contract I: Consider the co-insurance problem described in Section 3. Then there exist an $\mathcal{S}$, μ and $\mathcal{Z}$ with $\mu(\mathcal{Z}) \in (0,1)$ with the following properties.*

1. The characteristic function of $\mathcal{Z}$ is well defined in terms of frequencies.

2. The optimal finite contract t^{**} that solves problem (12) exists unique, up to a set of states of μ-measure zero.

3. The optimal finite contract t^{**} prescribes no transfer between the agents in every state of nature. In other words $t^{**}(s) = 0$ for every $s \in \mathcal{S}$, up to a set of states of μ-measure zero.

Once again the formal proof of Proposition 3 is presented in the Appendix. Intuitively, Proposition 3 is a fairly direct consequence of Propositions 1 and 2 coupled with the strict concavity (in t) of the agents' preferences.

Again, we start with an $\mathcal{S}$ as in Proposition 1. Recall now that in the co-insurance problem described in Section 3 above the parties are able to achieve full insurance by agreeing on a transfer contingent on the event $\mathcal{Z}$. We now choose the event $\mathcal{Z}$ so that its characteristic function exhibits finite invariance and fine variability as in Proposition 2 over the whole of $\mathcal{S}$. Let $p_{\mathcal{Z}}$ and $p_{\overline{\mathcal{Z}}}$ be the densities of $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ respectively, conditional on any $A \in \mathcal{A}$.

Notice that by definition of finite invariance the event $\mathcal{Z}$ has been defined so that any attempt by the parties to condition on a finite set of characteristics (the only feasible ex-ante description available to them) will leave them with a set of states of which only a fraction $p_{\mathcal{Z}}$ actually belongs to $\mathcal{Z}$. This is true whatever finitely definable subset of $\mathcal{S}$ the parties decide to condition their contract on. The fact that the parties are risk averse now implies that the optimal finite contract should specify the same transfer from 2 to 1 contingent on *any* finitely definable subset of $\mathcal{S}$. Any transfer function that varies across two finitely definable sets of states will be strictly dominated (in terms of the parties expected utility) by a constant transfer that coincides with the average of the transfer function we started from.

The optimal contract t^{**} is now immediately obtained from the observation that the only constant (across all states) transfers from 2 to 1 that are compatible with 2's participation constraint are non-positive. Since 1's expected utility is monotonically increasing in the constant transfer from 2, the optimal finite contract must clearly prescribe a transfer of 0 in all states.

The allocation entailed by the optimal finite contract coincides with the no-contract outcome. Clearly the fact that the two parties to the contract are strictly risk averse implies that party 1's expected utility associated with the no-contract outcome is bounded away from the full-insurance contract t^* described in Section 3.

In our terminology, the event $\mathcal{Z}$ is an unforeseen contingency. The agents understand its probability $p_{\mathcal{Z}}$ and use it in their expected utility computations. However, no matter how finely they attempt to describe it in a finite ex-ante agreement, they will only be correct a fraction $p_{\mathcal{Z}}$ of the time. The extreme prediction that the parties will choose an allocation equivalent to no-contract of course derives from the particular event $\mathcal{Z}$ we constructed above.

8. The Discrete and the Continuous State Spaces

Our next task in this paper is to characterize tightly optimal finite contracts in a more general case that we have done so far. In particular we would like to "solve" our basic contracting problem in the more general case in which f — the characteristic function of $\mathcal{Z}$ — exhibits fine variability but not necessarily finite invariance.

We attack this problem directly in Section 9 below. In this section we develop some of the tools that are needed for this task. For reasons that will be apparent below, we need to investigate the relationship between a given contracting problem in our discrete state space $\mathcal{S}$ and a closely related problem formulated in a continuous state space.

Recall that our state space $\mathcal{S}$ is a countable set of sequences of 0s and 1s. It is clearly a natural question to ask what happens if we consider instead the set $\hat{\mathcal{S}} \equiv \{0,1\}^{\mathbb{N}}$ of *all* binary sequences.[18]

To proceed with the translation of our basic contracting problem into one on the continuous state space $\hat{\mathcal{S}}$, we begin with a basic observation: every elementary set of the form $A(i,j)$, has a natural image in $\hat{\mathcal{S}} \equiv \{0,1\}^{\mathbb{N}}$. Formally, define

$$\hat{A}(i,j) = \{s \in \hat{\mathcal{S}} \qquad \text{such that} \;\; s^i = j\} \tag{13}$$

Clearly, $A(i,j) = \hat{A}(i,j) \cap \mathcal{S}$, so there is a natural one-to-one correspondence between elementary sets in the two models. Intuitively, one may think of this correspondence as follows: $A(i,j)$ and $\hat{A}(i,j)$ are representations in two different state spaces of the same statement, namely "the set of states where feature i takes value j." This statement makes sense independently of the set of physical states. Note also that the identification of elementary sets in $\mathcal{S}$ and $\hat{\mathcal{S}}$ extends to the algebras they generate. Thus, the algebra $\hat{\mathcal{A}}$ generated by all sets of the form $\hat{A}(i,j)$, corresponds in a natural way to the algebra $\mathcal{A}$ through the relationship: $\hat{A} \in \hat{\mathcal{A}}$ if and only if $\hat{A} \cap \mathcal{S} \in \mathcal{A}$.

We now introduce the main idea of this section: every contracting problem in the discrete model has a unique, natural extension to a contracting problem on $\hat{\mathcal{S}}$.

Recall that our basic co-insurance problem (12) is defined by three elements: the state space $\mathcal{S}$, the measure μ over $\mathcal{S}$ and the set $\mathcal{Z}$, or equivalently its characteristic function f.

Our first result is that we can find a measure $\hat{\mu}$ and a measurable[19] function $\hat{f}$ on

[18]The space $\hat{\mathcal{S}}$ is closely related to [0,1], as can be seen by viewing each element of $\hat{\mathcal{S}}$ as a binary expansion of a real number. There are two formal differences, however. First, some real numbers have more than one binary expansion, so two points in $\hat{\mathcal{S}}$ may correspond to the same real number in [0,1]. However, this happens for only a countable number of points in [0,1], which is negligible if one takes a diffuse measure, such as the uniform distribution. Second, the usual metric on [0,1] is generated by a specific order of the features, while no such order is implied in the definition of $\hat{\mathcal{S}}$.

[19]Here and in the rest of the paper, all measurability statements about functions defined on $\hat{\mathcal{S}}$ are with respect to the σ-algebra generated by $\hat{\mathcal{A}}$, which is in fact the Borel σ-algebra generated by the product topology on $\hat{\mathcal{S}}$. This is easily seen by noting that $\hat{\mathcal{A}}$ is a base for the product topology on

$\hat{\mathcal{S}}$ which "replicate" μ and f in the following sense.

Proposition 4. *Continuum Extension: Let $\mathcal{S}$ be a (discrete) state space as in Definition 1, and assume that $\mathcal{S}$ satisfies Assumption 1. Let μ be the (finitely additive) density (measure) of Definition 2. Let $\mathcal{Z} \subseteq \mathcal{S}$ be a subset of states in $\mathcal{S}$ and assume that the characteristic function f of $\mathcal{Z}$ has well defined frequencies in the sense of Definition 5.*

For every pair (f, μ) as above, there is a unique countably additive measure $\hat{\mu}$ on $\hat{\mathcal{S}} \equiv \{0,1\}^{\mathbb{N}}$ and a measurable function $\hat{f} : \hat{\mathcal{S}} \to I\!R$, unique (up to equivalence), such that

$$\mu(A) = \hat{\mu}(\hat{A}) \qquad \text{for every } \ A \in \mathcal{A} \tag{14}$$

and

$$\int_A f d\mu = \int_{\hat{A}} \hat{f} d\hat{\mu} \qquad \text{for every } \ A \in \mathcal{A} \tag{15}$$

where for any $A \in \mathcal{A}$, $\hat{A}$ denotes the corresponding set in the algebra $\hat{\mathcal{A}}$ as given by (13) above.

Given any triple $(\mathcal{S}, f, \mu)$, we call the pair $(\hat{f}, \hat{\mu})$ its extension to the continuum.[20]

The proof of Proposition 4 is in the Appendix. Besides being useful to characterize optimal finite contracts in the more general case considered in the next section, Proposition 4 sheds light on the nature of our original model with the discrete state space $\mathcal{S}$.

On the one hand, equation (14) reassures us that the (finitely additive) measure μ on the state space $\mathcal{S}$ must treat all sets in the algebra $\mathcal{A}$ in a way that is "compatible"

$\hat{\mathcal{S}}$.

[20]Notice that we map a triple into a pair simply because the state space in the continuous extension is always $\hat{\mathcal{S}}$.

with a standard countably additive measure $\hat{\mu}$ on the continuous state space $\hat{\mathcal{S}}$. This part of Proposition 4 is in fact a consequence of Kolmogorov's existence theorem.

On the other hand, equation (15)tells us that for some *measurable* function $\hat{f}$, for any finitely definable set A we can think of the density of $\mathcal{Z} \cap A$ as the integral of f on the corresponding $\hat{\mathcal{A}}$. In some sense, $\hat{f}$ contains all the information about the behavior of f, that we can ever hope to capture if we can only condition on finitely definable sets. Not surprisingly then, if f exhibits finite invariance with $\mu(\mathcal{Z}|A) = p_{\mathcal{Z}}$ for every $A \in \mathcal{A}$, it is immediate from (14) and (15) that we would get $\hat{f}(s) = p_{\mathcal{Z}}$ for every $s \in \hat{\mathcal{S}}$.

This last observation highlights a key insight about the nature of fine variability in our model with discrete state space $\mathcal{S}$. If f does *not* display finite invariance it is again immediate from (15) that the corresponding $\hat{f}$ would *not* be constant over $\hat{\mathcal{S}}$. So, *some* of the variability of f may be meaningfully captured preserving measurability with respect to the algebra $\mathcal{A}$. This part of the variability of f is what can be incorporated in the corresponding measurable function $\hat{f}$. Working with the discrete state space $\mathcal{S}$ allows for (fine) variability that cannot possibly be captured in this way in a model with the continuous state space $\hat{\mathcal{S}}$.

9. Fine Variability Without Finite Invariance

So far we have considered contracting problems that exhibit both fine variability and finite invariance. This is clearly a canonical extreme case. Proposition 3 tells us that in this case the optimal finite contract will simply ignore *all* the variability embodied in the contracting problem. Finite invariance guarantees that the contracting problem exhibits *no* variability that can be usefully captured by any finite contract.

We now ask the question of what optimal finite contracts look like in a contracting problem that does not necessarily exhibit finite invariance. Clearly, it can still be the case that some of the variability of the contracting problem is just "too fine" to be usefully captured by any finite contract. The question that remains, however, is whether there is any part of the variability of a contracting problem that will be reflected in optimal finite contracts. If this is the case, can we characterize what part of the variability embodied in a contracting problem will in fact be reflected in an optimal finite contract?

The answer to the above question stems from Proposition 4 above. To know what part of the variability of the environment will be reflected in the optimal finite contract in the general case it is sufficient to consider the continuous extension of the contracting problem to which Proposition 4 refers. In a contracting problem that does not exhibit finite invariance it will indeed be the case that the optimal finite contract will not be "flat" as in Proposition 3. Moreover, the optimal finite contract in the absence of finite invariance can be tightly characterized. As it turns out, using the continuous extension of the given contracting problem we will able to state basic first order conditions which completely characterize the optimal finite contract.

9.1. The Contracting Problem and Optimizing Sequences

The contracting problem that we consider here is still the one described in Section 3, and defined formally in (12). The novelty now is that, because we are not restricting attention to the case of finite invariance, we are unable to use the concavity arguments in the proof of Proposition 3 to characterize the optimal finite contract between the agents.

Indeed, as will be apparent below, once we allow contracting problems without finite invariance, an optimal finite contract may or may *not* exist. The possibility that an optimal finite contract may not exist is due to a simple "closure" problem. In other words, the contracting problem may be associated with a sequence of finite contracts that yield higher and higher expected utility to agent 1 (approaching a finite supremum of course), while still meeting the participation constraint of agent 2. In these cases our results below characterize tightly the "shape" of any sequence of feasible finite contracts that approaches agent 1's supremum of expected utility. Some extra notation is needed to handle this point formally.

Consider again problem (12) above. If a solution to this problem does not exist, let V_1^{**} be the *supremum* of agent 1's expected utility over the set of contracts that satisfy both constraints in problem (12).[21] We can now proceed with our next formal definition.

Definition 9. *Optimizing Sequence: Consider the contracting problem (12) above.*

Consider now a sequence of finite contracts $t_n \in \mathcal{F}$ that satisfies the first constraint in problem (12) and such that

$$\lim_{n\to\infty} E\, U_1(t_n) \quad = \quad V_1^{**} \tag{16}$$

then we say that t_n is an optimizing sequence *of finite contracts. Below, an optimizing sequence of finite contracts will typically be denoted by $\{t_n^{**}\}_{n=1}^{\infty}$.*

9.2. Characterization of Optimal Finite Contracts

The characterization of optimal finite contracts (or optimizing sequences) that we provide in Proposition 5 below is obtained via the solution to the continuous extension to our original co-insurance problem that we defined in Proposition 4 above.

Before we state our proposition and begin providing some intuition for it, it is useful to define the class of maximization problem that yields the solution to the continuous extension that we will use below.

Definition 10. *Auxiliary Problem: Consider a triple $(\mathcal{S}, \mu, f)$ defining a co-insurance problem as in (12). Assume that $(\mathcal{S}, \mu, f)$ satisfies the hypotheses of Proposition 4 and, as before, denote by $(\hat{\mu}, \hat{f})$ it continuous extension.*

Let $\mathcal{M}$ be the set of all bounded measurable functions on $\hat{\mathcal{S}}$.

The auxiliary problem *for the original contracting problem is*

[21] Notice that it is trivial that V_1^{**} is finite.

$$\max_{\hat{t} \in \mathcal{M}} \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1 + \hat{t}(s)] \;\; + \;\; [1 - \hat{f}(s)]\, V[\hat{t}(s)]\, \mathrm{d}\hat{\mu}$$

$$\text{s.t.} \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[-\hat{t}(s)] \;\; + \;\; [1 - \hat{f}(s)]\, V[1 - \hat{t}(s)]\, \mathrm{d}\hat{\mu} \;\geq\; \hat{\mu}(\mathcal{Z})V(0) + \hat{\mu}(\overline{\mathcal{Z}})V(1) \tag{17}$$

Notice that, given the one-to-one correspondence between the sets in the algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ that we discussed in Section 8, given a finite contract $t : \mathcal{S} \to \mathbb{R}$, its "translation" in the obvious way to the state space $\hat{\mathcal{S}}$ — typically denoted by $\hat{t} : \hat{\mathcal{S}} \to \mathbb{R}$ — is unambiguously well defined. In the sequel, when we refer to a contract $\hat{t} : \hat{\mathcal{S}} \to \mathbb{R}$ as *finite*, we mean a contract $\hat{t}$ with a finite range $\{t_1, \ldots, t_M\}$ and such that $\hat{t}^{-1}(t_j) \in \hat{\mathcal{A}}$ for every $j = 1, \ldots, M$. Clearly, its "translation" to the state space $\mathcal{S}$ — typically denoted by $t : \mathcal{S} \to \mathbb{R}$ — is also unambiguously well defined.

We are now ready to state our main characterization result.

Proposition 5. *Optimal Finite Contract II: Consider any contracting problem as in (12). Assume that $\mathcal{S}$ satisfies Assumption 1 and that the characteristic function of $\mathcal{Z}$ has well defined frequencies as in Definition 5.*

Then the following statements hold for the original problem $(\mathcal{S}, \mu, f)$ and the associated auxiliary problem $(\hat{\mu}, \hat{f})$.

1. The solution $\hat{t}^{**}$ to the auxiliary problem exists and is unique up to a set of $\hat{\mu}$-measure zero of points $s \in \hat{\mathcal{S}}$.

2. The optimal finite contract t^{**} for the original problem $(\mathcal{S}, \mu, f)$ exists, if and only if there exists a finite contract $\hat{t}^{**} : \hat{\mathcal{S}} \to \mathbb{R}$ that solves the auxiliary problem $(\hat{\mu}, \hat{f})$.

3. If there exists a finite contract $\hat{t}^{**} : \hat{\mathcal{S}} \to \mathbb{R}$ that solves the auxiliary problem $(\hat{\mu}, \hat{f})$, then its translation $t^{**} : \mathcal{S} \to \mathbb{R}$ to the state space $\mathcal{S}$ is an optimal finite contract for the original problem $(\mathcal{S}, \mu, f)$.

4. Let $\{t_n^{**}\}_{n=1}^{\infty}$ *be any optimizing sequence of contracts for the original problem, and let* $\hat{t}^{**}$ *denote the (unique up to equivalence) solution to the auxiliary problem. Then for every* $A \in \mathcal{A}$*, with* $\mu(A) > 0$*, and corresponding* $\hat{A} \in \hat{\mathcal{A}}$

$$\lim_{n\to\infty} E[U_1(t_n^{**})|A] \;=\; \frac{1}{\hat{\mu}(\hat{A})}\int_{\hat{A}} \hat{f}(s)\, V[1+\hat{t}^{**}(s)] \;+\; [1-\hat{f}(s)]\, V[\hat{t}^{**}(s)]\, \mathrm{d}\hat{\mu} \quad (18)$$

That is, whether an optimal finite contract exists or not, the expected payoff along the sequence $\{t_n^{**}\}_{n=1}^{\infty}$ *converges to the expected payoff under* $\hat{t}^{**}$*, conditional on any* $A \in \mathcal{A}$ *of positive* μ*-measure.*

As usual, the proof of the proposition is in the Appendix. Intuitively, this result is a fairly direct consequence of Proposition 4 and of the way we have set up the auxiliary problem in (17) above. Once we find a solution to the auxiliary problem $\hat{t}^{**}$, if the solution to the original problem were not (at least approximately) equal to it, then we could easily contradict the fact that $\hat{t}^{**}$ is optimal in the first place.

Our last result of this subsection states that the solution to the auxiliary problem invoked in Proposition 5 is in turn easy to characterize via the associated first order conditions.

Proposition 6. *First Order Conditions: Let any contracting problem as in (12), identified by the triple* $(\mathcal{S}, \mu, f)$*, be given. Assume that* $\mathcal{S}$ *satisfies Assumption 1 and that the characteristic function of* $\mathcal{Z}$ *has well defined frequencies as in Definition 5.*

Then the solution to the associated auxiliary problem $(\hat{\mu}, \hat{f})$ *satisfies the following first order conditions, up to a set of* $\hat{\mu}$*-measure zero of points* $s \in \hat{\mathcal{S}}$*.*

$$\begin{aligned} &\hat{f}(s)\, V'[1+\hat{t}^{**}(s)] \;+\; [1-\hat{f}(s)]\, V'[\hat{t}^{**}(s)] \;= \\ &= \;\gamma \left\{ \hat{f}(s)\, V'[-\hat{t}^{**}(s)] \;+\; [1-\hat{f}(s)]\, V'[1-\hat{t}^{**}(s)] \right\} \end{aligned} \quad (19)$$

where γ *is a positive constant.*

Together with the obvious observation that the constraint in (17) must hold with equality, the first order conditions (19) completely characterize the solution to the auxiliary problem (17).

The proof of Proposition 6 is a matter of routine, and appears below in the Appendix to the paper.

9.3. *Two Leading Examples*

We conclude this section with two examples of how the characterization of optimal finite contracts (or optimizing sequences) we have just given may be used to solve a contracting problem with fine variability but without finite invariance.

The first example is easy to describe. We partition the state space $\mathcal{S}$ into two subsets: $A_0 = A(1,0)$ in which $\mathcal{Z}$ is more "likely," and $A_1 = A(1,1)$ in which it is less likely.[22]

Therefore, we consider an $\mathcal{S}$ that satisfies Assumption 1 and that also satisfies $\mu(A_0) = q \in (0,1)$ and $\mu(A_1) = 1 - q$, together with a $\mathcal{Z}$, and associated f as follows. For any $A \in \mathcal{A}$ with $A \subseteq A_0$ we have that $\mu(\mathcal{Z}|A_0) = \underline{p}$, while for any $A \in \mathcal{A}$ with $A \subseteq A_1$ we have that $\mu(\mathcal{Z}|A_1) = \overline{p}$, with $0 < \underline{p} < \overline{p} < 1$. So, the problem exhibits finite invariance over A_0 and A_1 considered separately, but not over the entire state space $\mathcal{S}$.

Notice that it is immediate that in this case $\hat{f}(s) = \underline{p}$ for every $s \in A_0$, while $\hat{f}(s) = \overline{p}$ for every $s \in A_1$. Therefore, the first order conditions (19) in this case read

$$\begin{aligned} &\underline{p}\, V'[1 + \hat{t}^{**}(s)] \;+\; [1 - \underline{p}]\, V'[\hat{t}^{**}(s)] \;= \\ &= \;\gamma \left\{ \underline{p}\, V'[-\hat{t}^{**}(s)] \;+\; [1 - \underline{p}]\, V'[1 - \hat{t}^{**}(s)] \right\} \quad \forall\, s \in A_0 \end{aligned} \tag{20}$$

[22]Therefore, A_0 can be thought of as the set of states in which the first feature is equal to 0, while A_1 is the set of states in which it is equal to one. Notice also that, the images $\hat{A}_0$ and $\hat{A}_1$ in the continuous state space $\hat{\mathcal{S}}$ can be, loosely speaking, be thought of as the real intervals $[0, 1/2]$ and $(1/2, 1]$ respectively.

and

$$\overline{p}\,V'[1+\hat{t}^{**}(s)] \;\;+\;\; [1-\overline{p}\,]\,V'[\hat{t}^{**}(s)] \;=$$
$$=\;\gamma\left\{\overline{p}\,V'[-\hat{t}^{**}(s)] \;\;+\;\; [1-\overline{p}\,]\,V'[1-\hat{t}^{**}(s)]\right\} \quad \forall\, s\in A_1 \tag{21}$$

Using (20), (21) and the concavity of V it is a matter of routine to check that $\hat{t}^{**}$ must be as follows.

$$t^{**}(s) \;=\; \begin{cases} t_0 & \text{if } \; s\in A_0 \\ t_1 & \text{if } \; s\in A_1 \end{cases} \tag{22}$$

with $t_1 < t_0$.

Intuitively, the two agents face two sources of uncertainty. These are the two events $s\in A_0$ as opposed to $s\in A_1$, and $s\in\mathcal{Z}$ as opposed to $s\in\overline{\mathcal{Z}}$. The first event is insurable since A_0 and A_1 are finitely definable sets. Because of the fine variability that underlies $\mathcal{Z}$, the second cannot be usefully captured by a finite contract, except for the fact that it is *correlated* with the first. The optimal co-insurance contract between the agents then exhibits partial insurance against the event $s\in A_0$ as opposed to $s\in A_1$.

Our second leading example involves a co-insurance problem that cannot be partitioned (as was the case with our first example above) into subsets within which finite invariance holds. Typically in this type of problem an optimal finite contract does not exist. In this case, in the light of Proposition 5, we proceed to characterize the solution to the auxiliary problem, which we know completely pins down the limit behavior of any optimizing sequence of finite contracts.

To describe the density μ and the characteristic function f of $\mathcal{Z}$ for our second example, we need to introduce some extra notation first. Let $\mathbf{A}(s^1,\ldots,s^m)$ be the set of states of nature $s\in\mathcal{S}$ such that the first m digits of each state s are equal to the finite sequence $(s^1,\ldots,s^m)$, $s^i\in\{0,1\}$. Clearly this set $\mathbf{A}(s^1,\ldots,s^m)=\bigcap_{i=1}^m A(i,s^i)$ is finitely definable.

We consider an $\mathcal{S}$ that satisfies Assumption 1 and such that

$$\mu(\mathbf{A}(s^1,\ldots,s^m)) = \frac{1}{2^m} \tag{23}$$

so that in a well defined sense the distribution of states is "uniform" across the state space $\mathcal{S}$.[23]

Intuitively, in this example the likelihood of $\mathcal{Z}$ is an increasing function of the state s. Formally, we let

$$\mu(\mathcal{Z}|\mathbf{A}(s^1,\ldots,s^m)) = \left(\sum_{i=1}^{m} \frac{s^i}{2^i} + \frac{1}{2^{m+1}}\right) \tag{24}$$

for every $\mathbf{A}(s^1,\ldots,s^m)$, every sequence $(s^1,\ldots,s^m)$ and every m.

It is not hard to see that, given (23) and (24), the auxiliary problem in this case entails a distribution $\hat{\mu}$ that is uniform on $\{0,1\}^{\mathbb{N}}$ and an $\hat{f}(s) = s$.[24] Therefore the first order conditions (19) for the auxiliary problem in this case read

$$\begin{aligned} s\,V'[1+\hat{t}^{**}(s)] \;\;+\;\; (1-s)\,V'[\hat{t}^{**}(s)] \;\;= \\ =\;\gamma\left\{s\,V'[-\hat{t}^{**}(s)] \;\;+\;\; (1-s)\,V'[1-\hat{t}^{**}(s)]\right\} \quad \forall\, s\in\hat{\mathcal{S}} \end{aligned} \tag{25}$$

Using the implicit function theorem and the concavity of $V(\cdot)$, the solution to the auxiliary problem $\hat{t}^{**}(s)$ must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\,\hat{t}^{**}(s) < 0$$

In the special case $V(x) = x - x^2/2$ some straightforward calculations yield a

[23]It is implicit in the way we set things up in this example that we add to each s viewed as a sequence of 0's and 1's its interpretation as a real number in $[0,1]$ by setting the "value" of each state as a real number equal to $\sum_{i=1}^{m} s^i/2^i$. See also footnote 24 below.

[24] Without grinding through the tedious details notice that if we let $\underline{s} = \sum_{i=1}^{m}(s^i/2^i)$ and $\overline{s} = \sum_{i=1}^{m}(s^i/2^i) + (1/2^m)$ then $\mu(\mathbf{A}(s^1,\ldots,s^m)) = \int_{\underline{s}}^{\overline{s}} \mathrm{d}s$ and $\mu(\mathcal{Z}|\mathbf{A}(s^1,\ldots,s^m)) = \int_{\underline{s}}^{\overline{s}} s\,\mathrm{d}s$.

closed-form solution for $\hat{t}^{**}(s)$. In this case $\hat{t}^{**}(s)$ is linear and takes the form.[25]

$$\hat{t}^{**}(s) = \frac{1}{1+\gamma} - s$$

As in our first example, the parties to our coinsurance problem face an environment in which risk can be decomposed into two parts. An uncertain state, and a likelihood of $\mathcal{Z}$ that is a function of the state itself. While it is impossible for the parties to capture directly the event $s \in \mathcal{Z}$ with any finite contract, they find it mutually advantageous to make their co-insurance contract reflect the fact that as s increases so does the likelihood that s does in fact belong to $\mathcal{Z}$.

Using Proposition 5 we know that any sequence of finite contracts that approximates a risk-sharing arrangement that is both feasible and optimal, in the limit will behave like the increasing function $\hat{t}^{**}$. Even in the limit, the parties will only achieve partial co-insurance against the risk they are faced with.

10. Conclusions

We have shown that it is possible to construct a contracting environment in which some contingencies have the following properties. Their probabilities and consequences are understood by all concerned, and all agents involved use this information to compute expected utilities arising from any possible finite ex-ante contract. Yet these contingencies are unforeseen in the sense that any attempt to describe them in a finite ex-ante agreement must fail. The contracting parties cannot describe ex-ante these contingencies to any degree that will improve their expected utilities relative to an agreement that ignores them altogether.

Given that these unforeseen contingencies cannot be taken into account ex-ante, ours may be taken as a model that formalizes the underpinnings of many contributions in the incomplete contracting literature. On the other hand, following the critique put forward by Maskin and Tirole (1999) our model may be one of a world in which the agents are *forced* to resort to mechanisms that implement the desired outcome

[25]Notice that clearly $0 < \hat{t}^{**}(0) = 1/(1+\gamma) < 1$ and $-1 < \hat{t}^{**}(1) = -\gamma/(1+\gamma) < 0$.

ex-post. In our view, whether the former or the latter is the case depends on many other details of the situation at hand.

Appendix

Proof of Proposition 1: Consider the set $\hat{\mathcal{S}}$ of infinite sequences of 0s and 1s, $\hat{\mathcal{S}} = \{0,1\}^{\mathbb{N}}$, with typical element $\hat{s}$ and let $\hat{s}^i$ be the i-th digit of the sequence $\hat{s}$. Let also

$$\hat{A}(i,j) \;=\; \{\hat{s} \in \hat{\mathcal{S}} \;\text{ such that }\; \hat{s}^i \;=\; j\} \tag{A.1}$$

Let H denote the set of all infinite sequences $\{\hat{s}_1, \ldots, \hat{s}_n, \ldots\}$ with $\hat{s}_n \in \hat{\mathcal{S}}$ for every n. Let $\{\tilde{s}_n\}_{n=1}^{\infty}$ be an infinite sequence of i.i.d. random variables with (countably additive) distribution $\hat{\mu}$ over $\hat{\mathcal{S}}$, and let P be the (product) probability distribution that this yields for H.

For any i and j now consider the event $M(i,j) \subset H$ such that $\lim\limits_{N\to\infty}(1/N)\sum_{n=1}^{N}\chi_{\hat{A}(i,j)}(\hat{s}_n) = \hat{\mu}(\hat{A}(i,j))$. By the law of large numbers, $P(M(i,j)) = 1$ for every i and j.

Now define,

$$M \;=\; \bigcap_{\substack{i\in\mathbb{N}\\ j\in\{0,1\}}} M(i,j) \tag{A.2}$$

Clearly, since $P(M(i,j)) = 1$ for every i and j, and of course P is countably additive, we must also have $P(M) = 1$, and therefore $M \neq \emptyset$.

It is now sufficient to choose $\mathcal{S}$ to be equal to any element of M to prove the claim. ■

Proof of Proposition 2: Fix $(p_1, \ldots, p_M)$ as in the statement of the proposition, and assume for the moment that $A = \mathcal{S}$. Assume that $\mathcal{S}$ is as in Proposition 1, and that it has the property that any finitely definable set B contains a countable infinity of elements. This is clearly possible from the construction in the proof of Proposition 1.

Define a stochastic process $\{\tilde{h}_1, \ldots, \tilde{h}_n, \ldots\}$ where each random variable $\tilde{h}_n$ takes values in the finite set $\{f_1, \ldots, f_M\}$. Let H denote the set of all realizations of this process, and let P be the probability distribution on H under which $\{\tilde{h}_1, \ldots, \tilde{h}_n, \ldots\}$ are i.i.d. random variables with distribution $(p_1, \ldots, p_M)$. Notice that a realization $h \in H$ of this process can be taken to be a candidate for our $f : \mathcal{S} \to \{f_1, \ldots, f_M\}$, so that the realized value h_n of $\tilde{h}_n$ is the value assigned to $f(s_n)$. We now proceed to show that the claim can be proved by setting f equal to any such realization of this process in a set of probability 1.

Let any $h \in H$ be given and let $B(m,h)$ be the set of states s_n such that $(s_n \in B)$ and $(f(s_n) = h_n = f_n)$. The law of large numbers holds for any $B \in \mathcal{A}$ in the following sense. There is a set $H_B \subset H$ with $P(H_B) = 1$ such that $h \in H_B$ implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{B(m,h)}(s_n) = p_m\, \mu(B). \tag{A.3}$$

Since $P(H_B) = 1$, clearly $Q = \bigcap_{B\in\mathcal{A}} H_B$ also has probability 1. Therefore $Q \neq \emptyset$. Now select any element $h = \{h_1, \ldots, h_n, \ldots\}$ of Q, and set $f(s_n) = h_n$ for every n. This is our candidate $f(\cdot)$.

Since equation (A.3) holds for any $B \in \mathcal{A}$ it is obvious that $f(\cdot)$ satisfies finite invariance as in Definition 7 over the whole of $\mathcal{S}$. Again from the fact that equation (A.3) holds for any $B \in \mathcal{A}$, it is clear that $f(\cdot)$ has well defined frequencies as in Definition 5. Lastly, again from equation (A.3) it is immediate that for any $B \in \mathcal{A}$ with $\mu(B) > 0$ we must have that $\mu(F_i|B) = p_i$ for every $i \in \{1, \ldots, M\}$, as required.

The argument for $A \subset \mathcal{S}$ is completely analogous to the one we have just given. The details are therefore omitted. ■

Lemma A.1: *Consider problem (12). Let $\mathcal{Z}$ be such that its characteristic function*

$$f(s) = \begin{cases} 1 & \text{if } s \in \mathcal{Z} \\ 0 & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{A.4}$$

has well-defined frequencies, as in Definition 5. Let also f be finitely invariant over $A \in \mathcal{A}$ with $\mu(A) > 0$, as in Definition 7.

Let any finite contract $t(\cdot) \in \mathcal{F}$ that is feasible in problem (12) be given, and $\{t_1, \ldots, t_M\}$ be the range of $t(\cdot)$. Finally, for every $i = 1, \ldots, M$, let T_i be the inverse image of t_i under $t(\cdot)$.

Assume now that $t(\cdot)$ has the following property. There exist an $i \in \{1, \ldots, M\}$ and a $j \in \{1, \ldots, M\}$ such that $\mu(A \cap T_i) > 0$ and $\mu(A \cap T_j) > 0$. Then there exists another finite contract $t'(\cdot) \in \mathcal{F}$ that is constant over *$(A \cap T_i) \cup (A \cap T_j)$, which is also feasible in problem (12) and which yields a higher expected utility for agent* 1.

PROOF: Let $t'(\cdot)$ be the same as $t(\cdot)$ for every $s_n \notin (A \cap T_i) \cup (A \cap T_j)$, and set

$$t'(s_n) \;=\; \frac{\mu(T_i)t_i \;+\; \mu(T_j)t_j}{\mu(T_i) \;+\; \mu(T_j)} \qquad \forall s_n \in (A \cap T_i) \cup (A \cap T_j) \tag{A.5}$$

The claim now follows directly by concavity of V, defining U_1 and U_2 as in (1) and (2). The rest of the details are omitted. ■

Lemma A.2: *Let $\mathcal{Z}$ be such that its characteristic function, as in (A.4), has well-defined frequencies (as in Definition 5) and displays finite invariance (as in Definition 7) over the entire state space $\mathcal{S}$. Then an optimal finite contract t^{**} that solves problem (12) exists unique, up to a set of states of μ-measure zero. Moreover, $t^{**}(s_n) = 0$ for all $s_n \in \mathcal{S}$, up to a set of states of μ-measure zero.*

Proof: Let $\mathcal{Z}$ as in the statement of the Lemma be given. Consider now the following maximization problem.

$$\begin{aligned} \max_{x} \quad & V(1+x)\,\mu(\mathcal{Z}) + V(x)\,\mu(\overline{\mathcal{Z}}) \\ \text{s.t.} \quad & V(-x)\,\mu(\mathcal{Z}) + V(1-x)\,\mu(\overline{\mathcal{Z}}) \geq V(0)\,\mu(\mathcal{Z}) + V(1)\,\mu(\overline{\mathcal{Z}}) \\ & x \in \mathbb{R} \end{aligned} \tag{A.6}$$

The strict concavity of $V(\cdot)$ implies that problem (A.6) has a unique solution by completely standard arguments. Let this solution be denoted by $\tilde{x}$.

The expected utility $V(-x)\,\mu(\mathcal{Z}) + V(1-x)\,\mu(\overline{\mathcal{Z}})$ is monotonically decreasing in x. Therefore the constraint in problem (A.6) is satisfied only when $x \leq 0$. Since the objective function in problem (A.6), $V(1+x)\,\mu(\mathcal{Z}) + V(x)\,\mu(\overline{\mathcal{Z}})$, is monotonically increasing in x we conclude that the unique solution of problem (A.6) is $\tilde{x} = 0$.

From Lemma A.1 above it is immediate that a solution to problem (A.6) must yield a solution to problem (12). Therefore setting $t^{**}(s_n) = 0$ for every $s_n \in \mathcal{S}$ yields the unique (up to a set of μ-measure zero) solution to problem (12). ■

Proof of Proposition 3: Let $\mathcal{S}$ be as in Proposition 1. Using Proposition 2 we can now choose $\mathcal{Z}$ such that its characteristic function has well defined frequencies, displays finite invariance over the whole of $\mathcal{S}$ and exhibits fine variability of degree 1 over $\mathcal{S}$ with $\mu(\mathcal{Z}) \in (0,1)$. The claim now follows directly from Lemma A.2. ■

Lemma A.3: *Let $\hat{\mathcal{A}}_\sigma$ be the σ-algebra generated by the algebra $\hat{\mathcal{A}}$. For any finitely additive measure λ on $\mathcal{A}$ there is a unique countably additive measure $\hat{\lambda}$ on $\hat{\mathcal{A}}_\sigma$ such that for every $\hat{A} \in \hat{\mathcal{A}}$ we have that $\hat{\lambda}(\hat{A}) = \lambda(\hat{A} \cap \mathcal{S})$.*

Proof: Let $\hat{\lambda}$ be the measure on $\hat{\mathcal{A}}$ defined by $\hat{\lambda}(\hat{A}) = \lambda(\hat{A} \cap \mathcal{S})$. Obviously, $\hat{\lambda}$ is a finitely additive probability measure on $\hat{\mathcal{A}}$. The lemma claims that $\hat{\lambda}$ can be uniquely extended to a countably additive measure on $\hat{\mathcal{A}}_\sigma$. The result follows from an application of the Kolmogorov Existence Theorem (Billingsley 1979, p. 433). This result guarantees that the extension we seek exists as required, provided that a consistency condition is satisfied.

To define consistency formally, for any finite set of indices $Y \equiv \{y_1, \ldots, y_n\} \subset \mathbb{N}$, define $\hat{\mathcal{A}}^Y \subset \hat{\mathcal{A}}$ to be the algebra of subsets of $\hat{\mathcal{S}}$ generated by the features of a state in positions $\{y_1, \ldots, y_n\}$. Let $\hat{\lambda}^Y$ denote the restriction of $\hat{\lambda}$ to $\hat{\mathcal{A}}^Y$. Consistency requires that for any two sets of indices Y and Z such that $Y \subset Z$, $\hat{\lambda}^Y$ is the marginal of $\hat{\lambda}^Z$ on $\hat{\mathcal{A}}^Y$. This is obviously satisfied because the $\hat{\lambda}^Y$'s are obtained as the restrictions of the finitely additive measure $\hat{\lambda}$. ■

Proof of Proposition 4: Obviously, μ satisfies the assumption of Lemma A.3 and so it has a unique countably additive extension, $\hat{\mu}$ on $\hat{\mathcal{A}}_\sigma$.

To prove the second claim, for every $A \in \mathcal{A}$ let $\phi(A) \equiv \mu(A \cap \mathcal{Z})$. Note that ϕ is well defined since we are assuming that $\mathcal{Z}$ has well defined frequencies. Clearly, ϕ is a finitely additive measure on $\mathcal{S}$ and so Lemma A.3 applies again, yielding a unique countably additive measure $\hat{\phi}$ on $\hat{\mathcal{A}}_\sigma$.

The desired function $\hat{f}$ in the statement of the proposition (if it exists) must be the Radon-Nikodym derivative of $\hat{\phi}$ with respect to $\hat{\mu}$. For such derivative to exist, we must show that $\hat{\phi}$ is absolutely continuous with respect to $\hat{\mu}$. To prove the latter, we use a characterization of absolute continuity in Shiryayev (1984). Define $\hat{\mathcal{A}}_m$ to be the finite algebra of subsets of $\hat{\mathcal{S}}$ generated by the first m features of each state. Clearly $\hat{\mathcal{A}}_m \subset \hat{\mathcal{A}}_{m+1}$, and $\bigcup_m \hat{\mathcal{A}}_m$ generates $\hat{\mathcal{A}}_\sigma$. Define

$$z_m(s) \equiv \frac{\hat{\phi}(\hat{A}_m(s))}{\hat{\mu}(\hat{A}_m(s))}$$

where $\hat{A}_m(s)$ is the smallest set in $\hat{\mathcal{A}}_m$ containing s (this is well defined since $\hat{\mathcal{A}}_m$ is a finite algebra).

It is known that the family of functions $\{z_m\}_{m=1}^\infty$ is a martingale under $\hat{\mu}$ (Shiryayev 1984, p. 493), and thus converge $\hat{\mu}$-almost everywhere to a limiting function $z_\infty : \hat{\mathcal{S}} \to \mathbb{R}$, measurable with respect to $\hat{\mathcal{A}}_\sigma$. It is known (Shiryayev 1984, p. 493) that $\hat{\phi}$ is absolutely continuous with respect to $\hat{\mu}$ if and only if $\hat{\phi}(\{s : z_\infty(s) = \infty\}) = 0$. To see this, note that since $\hat{\phi}$ agrees with ϕ on every $A \in \hat{\mathcal{A}}_m$, using the definition of ϕ we immediately have that

$$z_m(s) \equiv \frac{\hat{\phi}(\hat{A}_m(s))}{\hat{\mu}(\hat{A}_m(s))} \leq 1$$

Thus, there is a uniform bound on the values of $z_m(s)$ for every s and m, so $\hat{\phi}(\{s : z_\infty(s) = \infty\}) = 0$ as required.

In summary, $\hat{\phi}$ is countably additive measure on $\hat{\mathcal{A}}_\sigma$ that is absolutely continuous with respect to $\hat{\mu}$. It follows, by the Radon-Nikodym theorem that there is a measurable function $\hat{f}$ (unique up to equivalence) such that for every $\hat{A} \in \hat{\mathcal{A}}_\sigma$ we have that $\hat{\phi}(\hat{A}) = \int_{\hat{A}} \hat{f}\, d\hat{\mu}$. This function clearly satisfies all the required properties and hence the proposition is proved. ■

Lemma A.4: *A solution to the auxiliary problem (17) exists.*

PROOF: Let $\hat{V}^{**}$ be the supremum (clearly finite) of the maximand of problem (17) over the feasible set.

Let $\{\hat{t}_n\}_{n=1}^{\infty}$ be a sequence of bounded measurable functions $\hat{S} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\hat{t}_n(s)] \quad + \quad [1-\hat{f}(s)]\, V[\hat{t}_n(s)]\; \mathrm{d}\hat{\mu} \quad = \quad \hat{V}^{**} \tag{A.7}$$

We first note that, passing to subsequences if necessary, if the sequence $\{\hat{t}_n\}_{n=1}^{\infty}$ converges for a set of $s \in \hat{\mathcal{S}}$ of $\hat{\mu}$-measure 1, then the limit function, denoted $\hat{t}$, is measurable. Since $\{\hat{t}_n\}_{n=1}^{\infty}$ is bounded, the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\hat{t}_n(s)] \quad + \quad [1-\hat{f}(s)]\, V[\hat{t}_n(s)]\; \mathrm{d}\hat{\mu} \; = \\ \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\hat{t}(s)] \quad + \quad [1-\hat{f}(s)]\, V[\hat{t}(s)]\; \mathrm{d}\hat{\mu} \quad = \quad \hat{V}^{**} \tag{A.8}$$

Hence, in this case there is nothing further to prove. So, it suffices to show that $\{\hat{t}_n\}_{n=1}^{\infty}$ contains a subsequence that converges in the sense above.

By way of contradiction, suppose that no subsequence of $\{\hat{t}_n\}_{n=1}^{\infty}$ converges for a set of $s \in \hat{\mathcal{S}}$ of $\hat{\mu}$-measure 1. This implies that $\{\hat{t}_n\}_{n=1}^{\infty}$ does not converge in measure. Hence, we can conclude that there exists an $\epsilon > 0$ such that for every positive integer M there are $n,\ m > M$ such that

$$\hat{\mu}\{s \in \hat{\mathcal{S}} \text{ such that } |\hat{t}_n(s) - \hat{t}_m(s)| > \epsilon\} > \epsilon \tag{A.9}$$

Now let and n and m as in (A.9) be given. Using our assumptions on V (concavity and Inada) it is now clear that if, for some $\lambda \in (0,1)$, we let $\tilde{t}_{n,m}(s) = \lambda\hat{t}_m(s) + (1-\lambda)\hat{t}_n(s)$ for every $s \in \hat{\mathcal{S}}$, (A.9) implies that there exists $\delta > 0$ such that either

$$\int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\tilde{t}_{n,m}(s)] \quad + \quad [1-\hat{f}(s)]\, V[\tilde{t}_{n,m}(s)]\; \mathrm{d}\hat{\mu} \quad \geq \\ \delta \; + \; \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\hat{t}_m(s)] \quad + \quad [1-\hat{f}(s)]\, V[\hat{t}_m(s)]\; \mathrm{d}\hat{\mu} \tag{A.10}$$

or

$$\int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\tilde{t}_{n,m}(s)] \;\; + \;\; [1-\hat{f}(s)]\, V[\tilde{t}_{n,m}(s)]\; \mathrm{d}\hat{\mu} \;\; \geq$$
$$\delta \; + \; \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\hat{t}_n(s)] \;\; + \;\; [1-\hat{f}(s)]\, V[\hat{t}_n(s)]\; \mathrm{d}\hat{\mu} \tag{A.11}$$

or both must hold. But since n and m can be chosen arbitrarily large, (A.10) and (A.11) together with the fact that $\delta > 0$ clearly contradict (A.7). ■

Lemma A.5: *Any solution to the auxiliary problem (17) satisfies the first order conditions specified in Proposition 6 up to a set of states of $\hat{\mu}$-measure zero, and the reservation expected utility constraint with equality.*

Hence the solution to the auxiliary problem is in fact unique, up to a set of states of $\hat{\mu}$-measure zero.

Proof: From Lemma A.4, we know that a solution to the auxiliary problem exists. The assertion that the constraint in problem (17) must be satisfied with equality follows trivially from the state-by-state monotonicity of the maximand and of the left-hand side of the constraint.

We now proceed by contradiction. Suppose now that there exists a set of states of positive $\hat{\mu}$-measure over which (19) is violated in some solution $\hat{t}^{**}$ to problem (17). It follows from the concavity of V, and from the fact that $\hat{t}^{**}$ must satisfy the constraint in (17) with equality, that we can find *two* sets of states A' and A'' in $\hat{\mathcal{A}}$ with $\hat{\mu}(A') > 0$ and $\hat{\mu}(A'') > 0$ which are such that, whenever $s' \in A'$ and $s'' \in A''$ we have that

$$\frac{\hat{f}(s')\, V'[1+\hat{t}^{**}(s')] \;\; + \;\; [1-\hat{f}(s')]\, V'[\hat{t}^{**}(s')]}{\hat{f}(s')\, V'[-\hat{t}^{**}(s')] \;\; + \;\; [1-\hat{f}(s')]\, V'[1-\hat{t}^{**}(s')]} >$$
$$\frac{\hat{f}(s'')\, V'[1+\hat{t}^{**}(s'')] \;\; + \;\; [1-\hat{f}(s'')]\, V'[\hat{t}^{**}(s'')]}{\hat{f}(s'')\, V'[-\hat{t}^{**}(s'')] \;\; + \;\; [1-\hat{f}(s'')]\, V'[1-\hat{t}^{**}(s'')]} \tag{A.12}$$

Next, define a new solution candidate as $\tilde{t}^{**}$ as follows. For every $s \in \hat{\mathcal{S}} \cap \overline{A}' \cap \overline{A}''$ let $\tilde{t}^{**}(s) = \hat{t}^{**}(s)$. For every $s \in A'$ set $\tilde{t}^{**}(s) = \hat{t}^{**}(s) + \epsilon$, and for every $s \in A''$ set $\tilde{t}^{**}(s) = \hat{t}^{**}(s) - \xi$, where $\epsilon > 0$ is an arbitrarily small number and ξ is chosen so that

$$\int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[-\tilde{t}^{**}(s)] \; + \; [1-\hat{f}(s)]\, V[1-\tilde{t}^{**}(s)]\; \mathrm{d}\hat{\mu} \; =$$
$$\int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[-\hat{t}^{**}(s)] \;\; + \;\; [1-\hat{f}(s)]\, V[1-\hat{t}^{**}(s)]\; \mathrm{d}\hat{\mu} \tag{A.13}$$

Using (A.12), the fact that ϵ and ξ are chosen so that (A.13) holds, and the concavity of V, completely standard arguments can now be used to show that

$$\int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\tilde{t}^{**}(s)] \;+\; [1-\hat{f}(s)]\, V[\tilde{t}^{**}(s)]\, \mathrm{d}\hat{\mu} \;> \\ \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\hat{t}^{**}(s)] \;+\; [1-\hat{f}(s)]\, V[\hat{t}^{**}(s)]\, \mathrm{d}\hat{\mu} \tag{A.14}$$

Since (A.14) together with (A.13) clearly contradicts the assumption that $\hat{t}^{**}$ solves the auxiliary problem (17) the proof is now complete. ■

Lemma A.6: *Recall that V^{**} denotes the supremum of expected utility that agent 1 can achieve with any finite contract, while $\hat{V}^{**}$ denotes the value of the maximand of the auxiliary problem (17) under the (unique up to equivalence) optimal contract $\hat{t}^{**}$. Then $V^{**} = \hat{V}^{**}$.*

Proof: To prove the claim, note that if $V^{**} > \hat{V}^{**}$, then we can find a finite contract $t : \mathcal{S} \to \mathbb{R}$ such that its translation to the state space $\hat{\mathcal{S}}$ improves on $\hat{V}^{**}$, immediately contradicting the definition of $\hat{V}^{**}$. Hence $V^{**} \leq \hat{V}^{**}$

Suppose now that $V^{**} < \hat{V}^{**}$. Since the solution to the auxiliary problem $\hat{t}^{**}$ is measurable, we can find a finite contract $\hat{t} : \hat{\mathcal{S}} \to \mathbb{R}$ that approximates $\hat{t}^{**}$ in the sense that

$$V^{**} \;<\; \int_{\hat{\mathcal{S}}} \hat{f}(s)\, V[1+\hat{t}(s)] \;+\; [1-\hat{f}(s)]\, V[\hat{t}(s)]\, \mathrm{d}\hat{\mu} \;<\; \hat{V}^{**} \tag{A.15}$$

However, (A.15) immediately implies that the translation to the state space $\mathcal{S}$ of the contract $\hat{t} : \hat{\mathcal{S}} \to \mathbb{R}$ yields an expected utility to agent 1 that exceeds V^{**}. Since this contradicts directly the definition of V^{**}, the proof is now complete. ■

Proof of Proposition 5: A proof of 1 is not required since the claim is a direct consequence of Lemmas A.4 and A.5.

To prove the sufficiency claim in 2, let $\hat{t}^{**} : \hat{\mathcal{S}} \to \mathbb{R}$ be a finite contract that solves the auxiliary problem. Clearly, its translation $t^{**} : \mathcal{S} \to \mathbb{R}$ to the state space $\mathcal{S}$ is feasible in the original problem, and yields an expected utility to agent 1 equal to $\hat{V}^{**}$. But, by Lemma A.6 we know that $\hat{V}^{**} = V^{**}$, and hence the claim is proved.

To prove the necessity claim in 2 assume that there is no finite contract that solves the auxiliary problem, and that finite contract $t^{**} : \mathcal{S} \to \mathbb{R}$ is a solution to the original problem. Clearly, the translation of $t^{**} : \mathcal{S} \to \mathbb{R}$ to the state space $\hat{\mathcal{S}}$ — denoted $\hat{t}^{**} : \hat{\mathcal{S}} \to \mathbb{R}$ — is a finite contract that is feasible in the auxiliary problem. Moreover, the value of the maximand of the auxiliary problem under $\hat{t}^{**} : \hat{\mathcal{S}} \to \mathbb{R}$ is V^{**}. Hence, by Lemma A.6, the finite contract $\hat{t}^{**} : \hat{\mathcal{S}} \to \mathbb{R}$ solves the auxiliary problem, establishing a contradiction.

To prove 3, it is sufficient to repeat the argument used in the proof of the sufficiency claim in 2 above.

To prove 4, we proceed by contradiction. Suppose then that for some optimizing sequence $\{t_n^{**}\}_{n=1}^{\infty}$, some A in $\mathcal{A}$ and corresponding $\hat{A} \in \hat{\mathcal{A}}$ with $\mu(A) > 0$, equation (18) is violated. Since the sequence on the left-hand side of (18) is clearly bounded, passing to subsequences if necessary, we can assume without loss of generality that the limit on the left-hand side of (18) is well defined and (by the contradiction hypothesis) it is not equal to the right-hand side.

Notice that $\{t_n^{**}\}_{n=1}^{\infty}$ is an optimizing sequence of finite contracts with $t_n^{**} : \mathcal{S} \to \mathbb{R}$ for every n. Now consider a sequence of contracts $\{\hat{t}_n\}_{n=1}^{\infty}$ with each $\hat{t}_n : \hat{\mathcal{S}} \to \mathbb{R}$ being the translation of t_n^{**} to the state space $\hat{\mathcal{S}}$. Using Lemma A.6, we know that equation (A.7) holds for the sequence $\{\hat{t}_n\}_{n=1}^{\infty}$. Hence, exactly as in the proof of Lemma A.4, we know that there exists a subsequence $\{\hat{t}_{n_m}\}_{m=1}^{\infty}$ that converges pointwise on a set of states $s \in \hat{\mathcal{S}}$ of $\hat{\mu}$-measure 1. Let $\tilde{t} : \hat{\mathcal{S}} \to \mathbb{R}$ be the limit of this subsequence. Clearly, $\tilde{t}$ is measurable.

Next, observe that since $\{t_n^{**}\}_{n=1}^{\infty}$ is an optimizing sequence, using Lemma A.6, and the way that $\tilde{t}$ has been constructed, we know that it must be a solution to the auxiliary problem.

Now recall that by our contradiction hypothesis there exists an A in $\mathcal{A}$ and corresponding $\hat{A} \in \hat{\mathcal{A}}$ with $\mu(A) > 0$, such that the limit on left-hand side of (18) is well defined and is different from the right-hand side. Because of the way we have constructed $\tilde{t}$, it is also clear that

$$\lim_{n\to\infty} E[U_1(t_n^{**})|A] \quad = \quad \frac{1}{\hat{\mu}(\hat{A})} \int_{\hat{A}} \hat{f}(s)\, V[1 + \tilde{t}(s)] \quad + \quad [1 - \hat{f}(s)]\, V[\tilde{t}(s)]\, \mathrm{d}\hat{\mu} \tag{A.16}$$

Therefore, our contradiction hypothesis leads us to conclude that

$$\begin{aligned} \int_{\hat{A}} \hat{f}(s)\, V[1 + \tilde{t}(s)] \quad + \quad [1 - \hat{f}(s)]\, V[\tilde{t}(s)]\, \mathrm{d}\hat{\mu} \quad \neq & \\ \int_{\hat{A}} \hat{f}(s)\, V[1 + \hat{t}^{**}(s)] \quad + \quad [1 - \hat{f}(s)]\, V[\hat{t}^{**}(s)]\, \mathrm{d}\hat{\mu} & \end{aligned} \tag{A.17}$$

Hence $\tilde{t}$ and $\hat{t}^{**}$ differ on a set of positive $\hat{\mu}$-measure. However, since both $\tilde{t}$ and $\hat{t}^{**}$ are solutions to the auxiliary problem this directly contradicts Lemma A.5. Hence the proof is now complete. ■

Proof of Proposition 6: The claim is proved in Lemma A.5. ■

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