A Resurrection of the Condorcet Jury Theorem

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September 25, 2007

Abstract

This paper analyzes the optimal size of a deliberating committee where, (i) there is no conflict of interest among individuals, and (ii) information acquisition is costly. The committee members simultaneously decide whether or not to acquire information, and then, they make the ex-post efficient decision. The optimal committee size, k^*, is shown to be bounded. The main result of this paper is that any arbitrarily large committee aggregates the decentralized information more efficiently than the committee of size k^*-2. This result implies that oversized committees generate only *small* inefficiencies.

1 Introduction

The classical Condorcet Jury Theorem (CJT) states that large committees can aggregate decentralized information more efficiently than small ones. Its origin can be traced to the dawn of the French Revolution when Marie Jean Antoine Nicolas Caritat le Marquis de Condorcet [1785, translation 1994] investigated decision-making processes in societies.[1] Recently, a growing literature on committee design pointed out that if the information acquisition is costly, the CJT fails to hold. The reason is that if the size of a committee is large, a committee member realizes that the probability that she can influence the final decision is too small compared to the cost of information acquisition. As a result, she might prefer to save this cost and free-ride on the information of others. Therefore, largeer committees might generate lower social welfare than smaller ones. These results suggest that in the presence of costly information acquisition, optimally choosing the size of a committee is an important and delicate issue. On the one hand, we also identify a welfare loss associated to oversized committees. On the other hand, we show that this loss is surprisingly small in certain environments. Therefore, the careful design of a committee might not be as important

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[1]Summaries of the history of the CJT can be found in, for example, Grofman and Owen [1986], Miller [1986], and Gerling, Grüner, Kiel, and Schulte [2003].

an issue as it was originally thought to be, as long as the committee size is large enough. In fact, if either the information structure is ambiguous, or the committee has to make decisions in various informational environments, it might be optimal to design the committee to be as large as possible.

The reason committee design receives a considerable attention by economists is that in many situations, groups rather than individuals make decisions. Information about the desirability of the possible decisions is often decentralized: individual group members must separately acquire costly information about the alternatives. A classical example is a trial jury where a jury has to decide whether a defendant is guilty or innocent. Each juror may individually obtain some information about the defendant at some effort cost (paying attention to the trial, investigating evidences, etc.). Another example is a firm facing the decision whether or not to implement a project. Each member of the executive committee can collect information about the profitability of the project (by spending time and exerting effort). Yet another example is the hiring decisions of academic departments. Each member of the recruiting committee must review the applications individually before making a collective decision. What these examples have in common is the fact that information acquisition is costly and often unobservable.

The specific setup analyzed in this paper is as follows. A group of individuals has to make a binary decision. There is no conflict of interest among the group members, but they have imperfect information about which decision is the best. First, k individuals are asked to serve in a committee. Then, the committee members simultaneously decide whether or not to invest in an informative signal. Finally, the committee makes the optimal decision given the acquired information. We do not explicitly model how the committee members communicate and aggregate information. Instead, we simply assume that they end up making the ex-post efficient decision.[2] The only strategic choice an individual must make in our model is the choice whether or not to acquire a signal upon being selected to serve in the committee.

The central question of our paper is the following: how does the committee size, k, affect social welfare? First, for each k, we fully characterize the set of equilibria (including asymmetric and mixed-strategy equilibria). We show that there exists a k^p ($\in \mathbb{N}$) such that whenever $k \leq k^p$, there is a unique equilibrium in which each committee member invests in information with probability one. In addition, the social welfare generated by these equilibria is an increasing function of k. If $k > k^p$, then there are multiple equilibria and many of them involve randomizations by the members. We also show that the social welfare generated by the *worst* equilibrium in the game, where the committee size is k, is decreasing in k if $k > k^p$. The optimal committee size, k^*, is defined such that (i) if the committee size is k^*, then there exists an equilibrium that maximizes social welfare, and (ii) in this equilibrium, each member invests in information with positive probability. We prove that the optimal committee size, k^*, is either k^p or $k^p + 1$. This implies that the CJT fails to

[2] Since there is no conflict of interest among the individuals, it is easy to design a mechanism which is incentive compatible and efficiently aggregates the signals. Alternatively, one can assume that the collected information is hard.

hold: large committees can generate smaller social welfare than smaller committees. Nevertheless, we show that if the size of the committee is larger than k^*, even the worst equilibrium generates higher social welfare than the unique equilibrium in the committee of size $k^* - 2$. That is, the welfare loss due to an oversized committee is *quite small*.

Literature Review

Although the Condorcet Jury Theorem provides important support for the basis of democracy, many of the premises of the theorem have been criticized. Perhaps most importantly, Condorcet assumes sincere voting. That is, each individual votes as if she were the only voter in the society. This means that an individual votes for the alternative that is best conditional on her signal. Austen-Smith and Banks [1996] showed that in general, sincere voting is not consistent with equilibrium behavior. This is because a rational individual votes not only conditional on her signal, but also on her being pivotal. Feddersen and Pesendorfer [1998] have shown that as the jury size increases, the probability of convicting an innocent can actually increase under the unanimity rule.

A variety of papers have shown, however, that even if the voters are strategic, the outcome of a voting converges to the efficient outcome as the number of voters goes to infinity in certain environments. Feddersen and Pesendorfer [1997] investigate a model in which preferences are heterogeneous and each voter has a private signal concerning which alternative is best. They construct an equilibrium for each population size, such that the equilibrium outcome converges to the *full information* outcome as the number of voters goes to infinity. The full information outcome is defined as the result of a voting game, where all information is public. Myerson [1998] has shown that asymptotic efficiency can be achieved even if there is population uncertainty; that is, a voter does not know how many other voters there are.

In contrast, the Condorcet Jury Theorem might fail to hold if the information acquisition is costly. Mukhopadhaya [2003] has considered a model, similar to ours, where voters have identical preferences but information acquisition is costly. He has shown by numerical examples that mixed-strategy equilibria in large committees may generate lower expected welfare than pure-strategy equilibria in small committees.[3]

The paper most related to ours in spirit is probably Martinelli [2006]. The author also introduced cost of information acquisition and in addition, he allows the precision of the signals to depend continuously on the amount of investment. Martinelli [2006] proves that if the cost and the marginal cost of the precision are zero at zero level of precision, then the decision is asymptotically efficient. More precisely, if the size of the committee converges to infinity, then there is a sequence of symmetric equilibria in which each member invests only a little, and the probability of a correct decision converges to one. Our paper emphasizes the fixed cost aspect of information acquisition and should be viewed as a complementary result to Martinelli [2006].[4]

[3] The results are quite different if the voting, rather than the information acquisition, is costly, see e.g. Borgers [2004].

[4] In his accompanying paper, Martinelli [2007] analyzes a model in which information has a fixed cost, voters

Numerous papers have analyzed the optimal decision rules in the presence of costly information. Persico [2004] discusses the relationship between the optimal decision rules and the accuracy of the signals. He shows that a voting rule that requires a strong consensus in order to upset the status quo is only optimal if the signals are sufficiently accurate. The intuition for the extreme case, where the decision rule is the unanimity rule, is the following: under the unanimity rule, the probability of being pivotal is small. However, this probability increases as the signals become more accurate. Therefore, in order to provide a voter with an incentive to invest in information, the signals must be sufficiently accurate. Li [2001], Gerardi and Yariv [2006], and Gershkov and Szentes [2004] have shown that the optimal voting mechanism sometimes involves ex-post inefficient decisions. That is, the optimal mechanism might specify inefficient decisions for certain signal profiles. As we mentioned earlier, we simply restrict attention to ex post efficient decision rules. We believe that this is the appropriate assumption in the context of a deliberating committee in which there is no conflict of interest among individuals.

Section 2 describes the model. The main theorems of the paper are stated and proved in Section 3. Section 4 concludes. Some of the proofs are relegated to the Appendix.

2 The Model

There is a population consisting of $N(>1)$ individuals. The state of the world, ω, can take one of two values: 1 and -1. Furthermore, $\Pr[\omega = 1] = \pi \in (0,1)$. The society must make a decision, d, which is either 1 or -1. There is no conflict of interest among individuals. Each individual has a benefit of $u(d,\omega)$ if decision d is made when the state of the world is ω. In particular,

$$u(d,\omega) = \begin{cases} 0 & \text{if } d = \omega, \\ -q & \text{if } d = -1 \text{ and } \omega = 1, \\ -(1-q) & \text{if } d = 1 \text{ and } \omega = -1, \end{cases}$$

where $q \in (0,1)$, indicates the severity of type-I error[5]. Each individual can purchase a signal at a cost $c\,(>0)$ at most once. Signals are iid across individuals conditional on the realization of the state of the world. The ex-post payoff of an individual who invests in information is $u - c$. Each individual maximizes her expected payoff.

There are two stages of the decision-making process. At Stage 1, $k\,(\leq N)$ members of the society are designated to serve in the committee at random. At Stage 2, the committee members decide simultaneously and independently whether or not to invest in information. Then, the efficient decision is made given the signals collected by the members.

are heterogeneous in their costs, and abstention is not allowed. On the one hand, the author shows that if the support of the cost distribution is not bounded away from zero, asymptotic efficiency can be achieved. On the other hand, if the cost is bounded away form zero and the number of voters is large, nobody acquires information in any equilibrium.

[5] In the jury context, where $\omega = 1$ corresponds to the innocence of the suspect, q indicates how severe error it is to covict an innocent.

We do not model explicitly how committee members deliberate at Stage 2. Since there is no conflict of interest among the members, it is easy to design a communication protocol that efficiently aggregates information. Alternatively, one can assume that the acquired information is *hard*. Hence, no communication is necessary for making the ex-post efficient decision. We focus merely on the committee members' incentives to acquire information.

Next, we turn our attention to the definition of social welfare. First, let μ denote the ex-post efficient decision rule. That is, μ is a mapping from sets of signals into possible decisions. If the signal profile is $(s_1, ..., s_n)$, then

$$\mu(s_1, ..., s_n) = 1 \Leftrightarrow \mathbb{E}_\omega [u(1, \omega) : s_1, ..., s_n] \geq \mathbb{E}_\omega [u(0, \omega) : s_1, ..., s_n].$$

The social welfare is measured as the expected sum of the payoffs of the individuals, that is,

$$N\mathbb{E}_{s_1,,,.s_n,\omega} [u(\mu(s_1, ..., s_n), \omega) - cn], \tag{1}$$

where the expectation also takes into account the possible randomization of the individuals. That is, n can be a random variable.

If the committee is large, then a member might prefer to save the cost of information acquisition and choose to rely on the opinions of others. On the other hand, if k is too small, there is too little information to aggregate, and thus the final decision is likely to be inefficient. The question is: What is the optimal k that maximizes ex-ante social welfare? To be more specific, the optimal size of the committee is k, if (i) the most efficient equilibrium, in the committee with k members, maximizes the social surplus among all equilibria in any committee, and (ii) each member acquires information with positive probability in this equilibrium.

Since the signals are iid conditional on the state of the world, the expected benefit of an individual from the ex post efficient decision is a function of the number of signals acquired. We define this function as follows:

$$\eta(k) = \mathbb{E}_{s_1,\cdots,s_k,\omega} [u(\mu(s_1, \cdots, s_k), \omega)].$$

We assume that the signals are informative about the state of the world, but only imperfectly. That is, as the number of signals goes to infinity, the probability of making the correct decision is strictly increasing and converges to one. Formally, the function η is strictly increasing and $\lim_{k\to\infty} \eta(k) = 0$. An individual's marginal benefit from collecting an additional signal, when k signals are already obtained, is

$$g(k) = \eta(k+1) - \eta(k).$$

Note that $\lim_{k\to\infty} g(k) = 0$. For our main theorem to hold, we need the following assumption.

Assumption 1 *The function g is log-convex.*

This assumption is equivalent to $g(k+1)/g(k)$ being increasing in $k\,(\in \mathbb{N})$. Whether or not this assumption is satisfied depends only on the primitives of the model, that is, on the distribution of the signals and on the parameters q and π. An immediate consequence of this assumption is the following.

Remark 1 *The function g is decreasing.*

Proof. Suppose by contradiction that there exists an integer, $n_0 \in \mathbb{N}$, such that, $g(n_0+1) > g(n_0)$. Since $g(k+1)/g(k)$ is increasing in k, it follows that $g(n+1) > g(n)$ whenever $n \geq n_0$. Hence, $g(n) > g(n_0) > 0$ whenever $n > n_0$. This implies that $\lim_{k\to\infty} g(k) \neq 0$, which is a contradiction. ■

Next, we explain that Assumption 1 essentially means that the marginal value of a signal decreases *rapidly*. Notice that the function g being decreasing means that the marginal social value of an additional signal is decreasing. We think that this assumption is satisfied in most economic and political applications. How much more does Assumption 1 require? Since g is decreasing and $\lim_{k\to\infty} g(k) = 0$, there always exists an increasing sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$, such that $g(n_k) - g(n_{k+1})$ is decreasing in k. Hence, it is still natural to restrict attention to information structures where the second difference in the social value of a signal, $g(k) - g(k+1)$, is decreasing. Recall that Assumption 1 is equivalent to $(g(k) - g(k+1))/g(k)$ being decreasing. That is, Assumption 1 requires that the second difference in the value of a signal does not only decrease, but decreases at an increasing rate.

In general, it is hard to check whether this assumption holds because it is often difficult (or impossible) to express $g(k)$ analytically. The next section provides examples where Assumption 1 is satisfied.

2.1 Examples for Assumption 1

First, suppose that the signals are normally distributed around the true state of the world. The log-convexity assumption is satisfied for the model where $\pi + q = 1$. That is, the society would be indifferent between the two possible decisions if information acquisition were impossible. The assumption is also satisfied even if $\pi + q \neq 1$ if the signals are sufficiently precise. Formally:

Proposition 1 *Suppose that $s_i \sim N(\omega, \sigma)$.*

(i) If $q + \pi = 1$ then Assumption 1 is satisfied.

(ii) For all q, π, there exists an $\varepsilon > 0$, such that Assumption 1 is satisfied if $\varepsilon > \sigma$.

Proof. See the Appendix. ■

In our next example the signal is trinary, that is, its possible values are $\{-1, 0, 1\}$. In addition,

$$\Pr(s_i = \omega|\omega) = pr, \ \ \Pr(s_i = 0|\omega) = 1 - r, \ \text{ and } \ \Pr(s_i = -\omega|\omega) = (1-p)\,r.$$

Notice that $r\,(\in (0,1))$ is the probability that the realization of the signal is informative, and p is the precision of the signal conditional on being informative.

Proposition 2 *Suppose that the signal is trinary. Then, there exists a* $\overline{p}(r) \in (0,1)$ *such that, if* $p > \overline{p}(r)$*, Assumption 1 is satisfied.*

Proof. See the Appendix. ■

Next, we provide an example where the logconvexity assumption is not satisfied. Suppose that the signal is binary, that is, $s_i \in \{-1, 1\}$ and

$$\Pr(s_i = \omega|\omega) = p, \ \ \Pr(s_i = -\omega|\omega) = 1 - p.$$

Proposition 3 *If the signal is binary then Assumption 1 is not satisfied.*

Proof. See the Appendix. ■

3 Results

This section is devoted to the proofs of our main theorems. To that end, we first characterize the set of equilibria for all $k\,(\in \mathbb{N})$. The next subsection shows that if k is small, the equilibrium is unique and each member incurs the cost of information (Proposition 4). Section 3.2 describes the set of mixed-strategy equilibria for k large enough (Propositions 5). Finally, Section 3.3 proves the main theorems (Theorems 1 and 2).

3.1 Pure-strategy equilibrium

Suppose that the size of the committee is k. If the first $k-1$ members acquire information, the expected *gain* from collecting information for the kth member is $g(k-1)$. She is willing to invest if this gain exceeds the cost of the signal, that is, if

$$c < g(k-1). \tag{2}$$

This inequality is the incentive compatibility constraint guaranteeing that a committee member is willing to invest in information if the size of the committee is k.[6]

Proposition 4 *Let k denote the size of the committee. There exists a $k^P \in \mathbb{N}$, such that, there exists a unique equilibrium in which each member invests in a signal with probability one if and only if $k \leq \min\{k^p, N\}$. Furthermore, the social welfare generated by these equilibria is monotonically increasing in k $\left(\leq \min\left\{k^P, N\right\}\right)$.*

Proof. Recall from Remark 1 that g is decreasing and $\lim_{k\to\infty} g(k) = 0$. Therefore, for any positive cost $c < g(0)$[7], there exists a unique $k^P \in \mathbb{N}$ such that

$$g\left(k^P\right) < c < g\left(k^P - 1\right). \tag{3}$$

[6] In what follows, we ignore the case where there exists a $k \in \mathbb{N}$ such that $c = g(k)$. This euqility does not hold generically, and would have no effect on our results.

[7] If $c > \eta(1) - \eta(0)$, then nobody has incentive to collect information, hence $k^P = 0$.

First, we show that if $k < k^P$ then there is a unique equilibrium in which each committee member invests in information. Suppose that in an equilibrium, the first $k-1$ members randomize according to the profile $(r_1, ..., r_{k-1})$, where $r_i \in [0,1]$ denotes the probability that the ith member invests. Let I denote the number of signals collected by the first $(k-1)$ members. Since the members randomize, I is a random variable. Notice that $I \leq k-1$, and

$$E_{r_1,\ldots,r_{k-1}}\left[g\left(I\right)\right] \geq g\left(k-1\right)$$

because g is decreasing. Also notice that from $k \leq k^p$ and (3), it follows that

$$g\left(k-1\right) > c.$$

Combining the previous two inequalities, we get

$$E_{r_1,\ldots,r_{k-1}}\left[g\left(I\right)\right] > c.$$

This inequality implies that no matter what the strategies of the first $(k-1)$ members are, the kth member strictly prefers to invest in information. From this observation, the existence and uniqueness of the pure-strategy equilibrium follow. It remains to show that if $k > k^P$, such a pure-strategy equilibrium does not exist. But if $k > k^p$, then $g\left(k\right) < c$. Therefore, the incentive compatibility constraint, (2), is violated and there is no equilibrium where each member incurs the cost of the signal.

Finally, we must show that the social welfare generated by these pure-strategy equilibria is increasing in $k\left(\leq \min\left\{k^p, N\right\}\right)$. Notice that since $N > 1$,

$$c < g\left(k-1\right) = \eta(k) - \eta(k-1) < N\left(\eta(k) - \eta(k-1)\right).$$

After adding $N\eta(k-1) - kc$, we get

$$N\eta(k-1) - c\left(k-1\right) < N\eta(k) - ck.$$

The left-hand side is the social welfare generated by the equilibrium in committee of size $k-1$, while the right-hand side is the social welfare induced by the committee of size k. ■

This is what Mukhopadhaya [2003] has shown in the case where the signal is binary. He has also shown by numerical examples that mixed-strategy equilibria in large committees can yield lower expected welfare than small committees. Our analysis goes further by analytically comparing the expected welfare of all mixed-strategy equilibria.

Figure 1 is the graph of $g\left(k\right)$ and c, where $s_i \sim N\left(\omega, 1\right)$, $\pi = .3$, $q = .7$, and $c = 10^{-4}$.

The amount of purchased information in equilibrium is inefficiently small. Although information is a public good, when a committee member decides whether or not to invest, she considers her private benefit rather than the society's benefit. Hence, the total number of signals acquired in an equilibrium is smaller than the socially optimal amount. This is why the social welfare is monotonically increasing in the committee size k, as long as $k \leq k^p$.

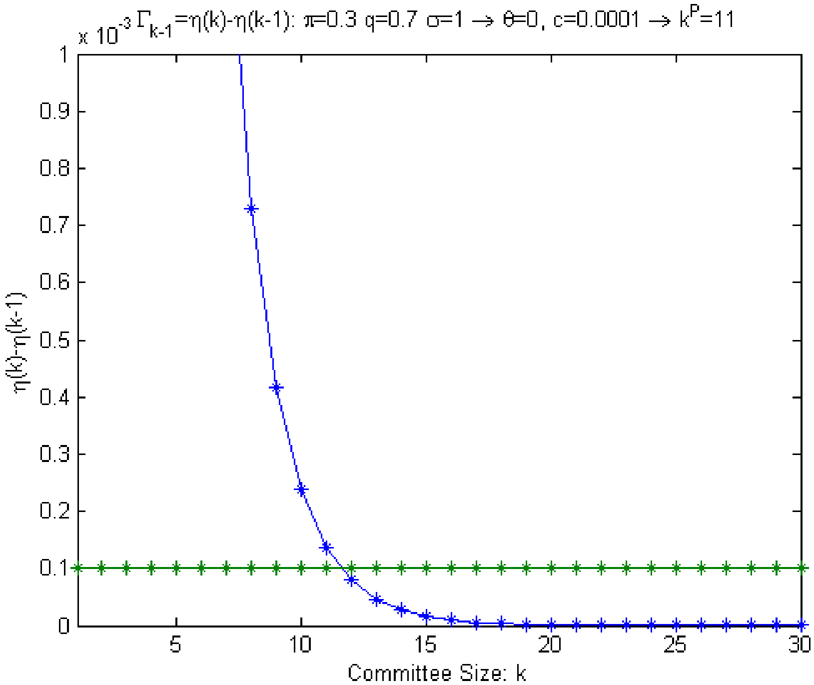

Figure 1: Expected gain $g(k)$ and the cost c

3.2 Mixed-strategy equilibrium

Suppose now that the size of the committee is larger than k^p. We consider strategy profiles in which the committee members can randomize when making a decision about incurring the cost of information acquisition. The following proposition characterizes the set of mixed-strategy equilibria (including asymmetric ones).

We show that each equilibrium is characterized by a pair of integers (a, b). In the committee, a members invest in a signal with probability one, and b members acquire information with positive but less than one probability. The rest of the members, a $k-(a+b)$ number of them, do not incur the cost. We call such an equilibrium a *type-(a, b) equilibrium.*

Proposition 5 *Let the committee size be $k\,(> k^p)$. Then, for all equilibria there is a pair (a, b), such that a members invests for sure, b members invests with probability $r \in (0,1)$, and $k-(a+b)$ members do not invest. In addition*

$$a \leq k^P \leq a+b \leq k, \tag{4}$$

where the first two inequalities are strict whenever $b > 0$.

Proof. First, we explain that if, in an equilibrium in which one member invests with probability $r_1 \in (0,1)$ and another invests with probability $r_2 \in (0,1)$, then $r_1 = r_2$. Since the marginal benefit from an additional signal is decreasing, our games exhibit strategic substitution. That is, the more information the others acquire, the less incentive a member has to invest. Hence, if $r_1 < r_2$ then the individual who invests with probability r_1 faces more information in expectation and has less incentive to invest than the individual who invests with probability r_2. On the other hand, since $r_1, r_2 \in (0,1)$ both individuals must be exactly indifferent between investing and not investing, a contradiction. Now, we formalize this argument. Let $r_i \in [0,1]$ $(i = 1, \cdots, k)$ be the probability that the ith member collects information in equilibrium. Suppose that r_1, $r_2 \in (0,1)$, and $r_1 > r_2$. Let I_{-1} and I_{-2} denote the number of signals collected by members $2, 3, ..., k$ and by members $1, 3, ..., k$, respectively. Notice that since $r_1 > r_2$ and g is decreasing,

$$E_{r_2, r_3, \ldots, r_k}\left(g\left(I_{-1}\right)\right) > E_{r_1, r_3, \ldots, r_k}\left(g\left(I_{-2}\right)\right). \tag{5}$$

On the other hand, a member who strictly randomizes must be indifferent between investing and not investing. Hence, for $j = 1, 2$

$$E_{r_j, r_3, \ldots, r_k}\left(g\left(I_{-j}\right)\right) = c. \tag{6}$$

This equality implies that (5) should hold with equality, which is a contradiction. Therefore, each equilibrium can be characterized by a pair (a, b) where a members collect information for sure, and b members randomize but collect information with the same probability.

It remains to show that there exists a type-(a, b) equilibrium if and only if (a, b) satisfies (4). First, notice that whenever $k > k^p$, in all pure-strategy equilibria k^p members invest with probability one, and the rest of the members never invests. In addition, the pair $(k^p, 0)$ satisfies (4). Therefore, we only have to show that there exists an equilibrium of type-(a, b) equilibrium where $b > 0$ if and only if (a, b) satisfies

$$a < k^P < a + b \leq k. \tag{7}$$

Suppose that in a committee, a members invest in information for sure and $b - 1$ invests with probability r. Let $G(r; a, b)$ denote the expected gain from acquiring information for the $(a + b)$th member. That is,

$$G(r; a, b) = \sum_{i=0}^{b-1} \binom{b-1}{i} r^i (1-r)^{b-1-i} g(a+i).$$

We claim that there exists a type-(a, b) equilibrium if and only if there exists an $r \in (0,1)$ such that $G(r; a, b) = c$. Suppose first that such an r exists. We first argue that there exists a type-(a, b) equilibrium in which b members invest with probability r. This means that the b members, those that are randomizing, are indifferent between investing and not investing. The a members, who invest for sure, strictly prefer to invest because the marginal gain from an additional signal exceeds $G(r; a, b)$. Similarly, those members who don't invest, a $k - (a + b)$ number of them, are strictly better off not investing because their marginal gains are strictly smaller than $G(r; a, b)$.

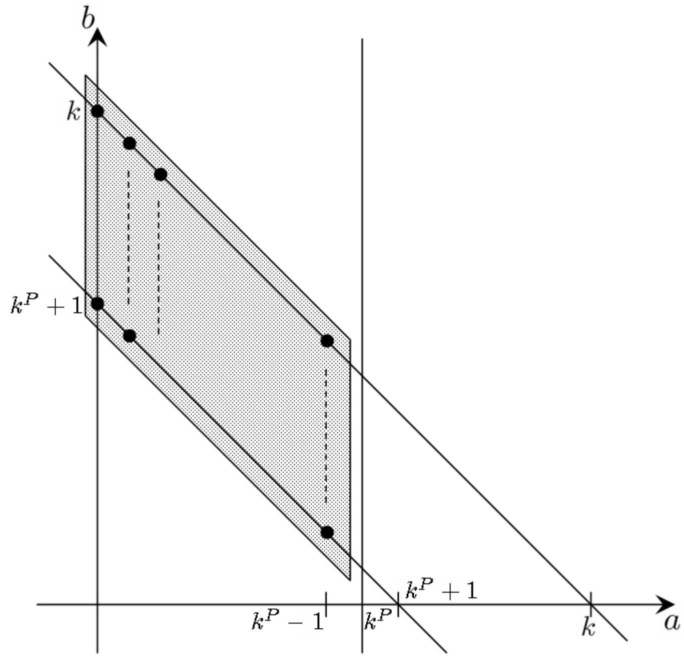

Figure 2: The set of mixed-strategy equilibria

Next, we argue that if $G(r; a, b) = c$ does not have a solution in $(0, 1)$ then there exists no type-(a, b) equilibrium. But this immediately follows from the observation that if b members are strictly randomizing, they must be indifferent between investing and not investing and hence $G(r; a, b) = c$. Therefore, it is sufficient to show that $G(r; a, b) = c$ has a solution in $(0, 1)$ if and only if (7) holds.

Notice that $G(r; a, b)$ is strictly decreasing in r because g is strictly decreasing. Also observe that $G(0; a, b) = g(a)$ and $G(1; a, b) = g(a + b - 1)$. By the Intermediate Value Theorem, $G(r; a, b) = c$ has a solution in $(0, 1)$ if and only if $G(1; a, b) < c < G(0; a, b)$, which is equivalent to

$$g(a + b - 1) < c < g(a). \tag{8}$$

Recall that k^P satisfies

$$g\left(k^P\right) < c < g\left(k^P - 1\right).$$

Since g is decreasing, (8) holds if and only if $a < k^P$ and $a + b > k^P$. That is, the two strict inequalities in (7) are satisfied. The last inequality in (7), must hold because $a + b$ cannot exceed the size of the committee, k. ■

Figure 2 graphically represents the set of pairs (a, b) which satisfy (4).

According to the previous proposition, there are several equilibria in which more than k^P members acquire information with positive probability. A natural question to ask is: can these mixed-strategy equilibria be compared from the point of view of social welfare? The next proposition partially answers this question. We show that if one fixes the number of members who acquire information for sure, then the larger the number of members who randomize is, the smaller the social welfare generated by the equilibrium is. This proposition plays an important role in determining the optimal size of the committee.

Proposition 6 *Suppose that $k \in \mathbb{N}$, such that there are both type-(a, b) and type-$(a, b+1)$ equilibria. Then, the type-(a, b) equilibrium generates strictly higher social welfare than the type-$(a, b+1)$ equilibrium.*

In order to prove this proposition we need the following results.

Lemma 1 *(i) $G(r; a, b) > G(r; a, b+1)$ for all $r \in [0, 1]$, and*

(ii) $r_{a,b} > r_{a,b+1}$, where $r_{a,b}$ and $r_{a,b+1}$ are the solutions for $G(r; a, b) = c$ and $G(r; a, b+1) = c$ in r, respectively.

Proof. See the Appendix. ■

Proof of Proposition 6. Suppose that a members collect information with probability one, and b members invest with probability r. Let $f(r; a, b)$ denote the benefit of an individual, that is,

$$f(r; a, b) = \sum_{i=0}^{b} \binom{b}{i} r^i (1-r)^{b-i} \eta(a+i).$$

Clearly

$$\frac{\partial f(r; a, b)}{\partial r} = \sum_{i=1}^{b} \binom{b}{i} i r^{i-1} (1-r)^{b-i} \eta(a+i) - \sum_{i=0}^{b-1} \binom{b}{i} r^i (b-i)(1-r)^{b-i-1} \eta(a+i).$$

Notice that $\binom{b}{i} i = b \binom{b-1}{i-1}$ and $\binom{b}{i}(b-i) = b\binom{b-1}{i}$. Therefore, the right-hand side of the previous equality can be rewritten as

$$\sum_{i=1}^{b} b \binom{b-1}{i-1} r^{i-1} (1-r)^{b-i} \eta(a+i) - \sum_{i=0}^{b-1} b \binom{b-1}{i} r^i (1-r)^{b-i-1} \eta(a+i).$$

After changing the notation in the first summation, this can be further rewritten:

$$\begin{aligned} & \sum_{i=0}^{b-1} b \binom{b-1}{i} r^i (1-r)^{b-i-1} \eta(a+i+1) - \sum_{i=0}^{b-1} b \binom{b-1}{i} r^i (1-r)^{b-i-1} \eta(a+i) \\ = \; & b \sum_{i=0}^{b-1} \binom{b-1}{i} r^i (1-r)^{b-i-1} \left[\eta(a+i+1) - \eta(a+i)\right]. \end{aligned}$$

This last expression is just $bG(r; a, b)$, and hence, we have

$$\frac{\partial f(r; a, b)}{\partial r} = bG(r; a, b).$$

Next, we show that

$$f\left(r_{a,b};a,b\right)-f\left(r_{a,b+1};a,b+1\right)>b\left(r_{a,b}-r_{a,b+1}\right)c. \tag{9}$$

Since $f(0;a,b)=f(0;a,b+1)=\eta(a)$

$$\begin{aligned}
f\left(r_{a,b};a,b\right)-f\left(r_{a,b+1};a,b+1\right) &= \left[f\left(r_{a,b};a,b\right)-f\left(0;a,b\right)\right]-\left[f\left(r_{a,b+1};a,b+1\right)-f\left(0;a,b+1\right)\right] \\
&= b\int_0^{r_{a,b}} G\left(r;a,b\right)dr - b\int_0^{r_{a,b+1}} G\left(r;a,b+1\right)dr.
\end{aligned}$$

By part (i) of Lemma 1, this last difference is larger than

$$b\int_0^{r_{a,b}} G\left(r;a,b\right)dr - b\int_0^{r_{a,b+1}} G\left(r;a,b\right)dr = b\int_{r_{a,b+1}}^{r_{a,b}} G\left(r;a,b\right)dr.$$

By part (ii) of Lemma, we know that $r_{a,b+1}<r_{a,b}$. In addition, since G is decreasing in r, this last expression is larger than

$$b\left(r_{a,b}-r_{a,b+1}\right)G\left(r_{a,b};a,b\right).$$

Recall that $r_{a,b}$ is defined such that $G\left(r_{a,b};a,b\right)=c$ and hence we can conclude (9).

Let $S(a,b)$ denote the social welfare in the type-(a,b) equilibrium, that is:

$$S(a,b)=Nf\left(r_{a,b};a,b\right)-c\left(a+br_{a,b}\right).$$

Then,

$$\begin{aligned}
S\left(a,b\right)-S\left(a,b+1\right) &= Nf\left(r_{a,b};a,b\right)-c\left(a+br_{a,b}\right)-\left[Nf\left(r_{a,b+1};a,b+1\right)-c\left(a+br_{a,b+1}\right)\right] \\
&> Nb\left(r_{a,b}-r_{a,b+1}\right)c-cb\left(r_{a,b}-r_{a,b+1}\right) \\
&= \left(N-1\right)cb\left(r_{a,b}-r_{a,b+1}\right)>0,
\end{aligned}$$

where the first inequality follows from (9), and the last one follows from part (ii) of Lemma 1. ■

3.3 The Proofs of the Theorems

First, we show that the optimal committee size is either k^P or k^P+1. Second, we prove that if $k>k^*$ then even the worst possible equilibrium yields higher social welfare than the unique equilibrium in the committee of size k^*-2.

Theorem 1 *The optimal committee size, k^*, is either k^P or k^P+1.*

We emphasize that for a certain set of parameter values, the optimal size is $k^*=k^p$, and for another set, $k^*=k^p+1$.

Proof. Suppose that k^* is the optimal size of the committee and the equilibrium that maximizes social welfare is of type-(a,b). By the definition of optimal size, $a+b=k^*$. If $b=0$, then all of the committee members invest in information in this equilibrium. From Proposition 4, $k^*\leq k^p$

follows. In addition, Proposition 4 also states that the social welfare is increasing in k as long as $k \leq k^p$. Therefore, $k^* = k^p$ follows. Suppose now that $b > 0$. If there exists an equilibrium of type-$(a, b-1)$, then, by Proposition 6, k^* is not the optimal committee size. Hence, if the size of the committee is k^*, there does not exist an equilibrium of type-$(a, b-1)$. By Proposition 5, this implies that the pair $(a, b-1)$ violates the inequality chain (4) with $k = k^*$. Since the first and last inequalities in (4) hold because there is a type-(a, b) equilibrium, it must be the case that the second inequality is violated. That is, $k^P \geq a + b - 1 = k^* - 1$. This implies that $k^* \leq k^p + 1$. Again, from Proposition 4, it follows that $k^* = k^p$ or $k^p + 1$. ■

Next, we turn our attention to the potential welfare loss due to oversized committees.

Theorem 2 *In any committee of size $k\,(> k^*)$, all equilibria induce higher social welfare than the unique equilibrium in the committee of size $k^* - 2$.*

The following lemma plays an important role in the proof. We point out that this is the only step of our proof that uses Assumption 1.

Lemma 2 *For all $k \geq 1$ and $i \in \mathbb{N}$,*

$$g\left(k-1\right)\left\{g\left(i\right)-g\left(k\right)\right\} \geq \left\{g\left(k\right)-g\left(k-1\right)\right\}\left\{\eta\left(i\right)-\eta\left(k\right)\right\}, \tag{10}$$

and it holds with equality if and only if $i = k$ or $k - 1$.

Proof of Theorem 2. Recall that $S\left(a,b\right)$ denotes the expected social welfare generated by an equilibrium of type-$(a.b)$. Using this notation, we have to prove that $S\left(k^*-2,0\right) < S\left(a,b\right)$. From Theorem 1, we know that $k^* = k^P$ or $k^P + 1$. By Proposition 4, $S\left(k^P-2,0\right) < S\left(k^P-1,0\right)$. Therefore, in order to establish $S\left(k^*-2,0\right) < S\left(a,b\right)$, it is enough to show that

$$S\left(k^P-1,0\right) < S\left(a,b\right), \tag{11}$$

for all pairs of (a, b) which satisfy (4).

Notice that if $a+i$ members invests in information, which happens with probability $\binom{b}{i} r_{a,b}^i (1-r_{a,b})^{b-i}$ in a type-(a, b) equilibrium, the social welfare is $N\eta(a + i) - c(a + i)$. Therefore,

$$\begin{aligned} S\left(a,b\right) &= \sum_{i=0}^{b}\binom{b}{i}{r_{a,b}}^{i}\left(1-r_{a,b}\right)^{b-i}\left[N\eta\left(a+i\right)-c\left(a+i\right)\right] \\ &= \left\{\sum_{i=0}^{b}\binom{b}{i}{r_{a,b}}^{i}\left(1-r_{a,b}\right)^{b-i}\left[N\eta\left(a+i\right)-ci\right]\right\}-ca \\ &= N\left\{\sum_{i=0}^{b}\binom{b}{i}{r_{a,b}}^{i}\left(1-r_{a,b}\right)^{b-i}\eta\left(a+i\right)\right\}-c\left(a+br_{a,b}\right). \end{aligned}$$

In the last equation, we used the identity $\sum_{i=0}^{b}\binom{b}{i}{r_{a,b}}^{i}\left(1-r_{a,b}\right)^{b-i}i = br_{a,b}$. Therefore, (11) can be rewrittes as

$$N\eta\left(k^P-1\right)-c\left(k^P-1\right) < N\left\{\sum_{i=0}^{b}\binom{b}{i}r_{a,b}^{i}\left(1-r_{a,b}\right)^{b-i}\eta\left(a+i\right)\right\}-c\left(a+br_{a,b}\right).$$

Since $a \leq k^P - 1$ by (4) and $b \leq N$, the right hand side of the previous inequality is larger than

$$N\left\{\sum_{i=0}^{b}\binom{b}{i}r_{a,b}^{i}\left(1-r_{a,b}\right)^{b-i}\eta\left(a+i\right)\right\}-c\left(k^{p}-1+Nr_{a,b}\right)$$

Hence it suffices to show that

$$N\eta\left(k^{P}-1\right)-c\left(k^{P}-1\right)<N\left\{\sum_{i=0}^{b}\binom{b}{i}r_{a,b}^{i}\left(1-r_{a,b}\right)^{b-i}\eta\left(a+i\right)\right\}-c\left(k^{p}-1+Nr_{a,b}\right).$$

After adding $c\left(k^{p}-1\right)$ to both sides and dividing through by N, we have

$$\eta(k^{P}-1)<\left\{\sum_{i=0}^{b}\binom{b}{i}r_{a,b}^{i}(1-r_{a,b})^{b-i}\eta(a+i)\right\}-cr_{a,b}. \tag{12}$$

The left-hand side is a payoff of an individual if $k^P - 1$ signals are acquired by others, while the right-hand side is the payoff of an individual who is randomizing in a type-(a, b) equilibrium with probability $r_{a,b}$. Since this individual is indifferent between randomizing and not collecting information, the right-hand side of (12) can be rewritten as

$$\sum_{i=0}^{b-1}\binom{b-1}{i}r_{a,b}^{i}(1-r_{a,b})^{b-1-i}\eta(a+i).$$

Hence (12) is equivalent to

$$\eta(k^{P}-1)<\sum_{i=0}^{b-1}\binom{b-1}{i}r_{a,b}^{i}(1-r_{a,b})^{b-1-i}\eta(a+i). \tag{13}$$

By Lemma 2

$$\begin{aligned} & g\left(k^{p}-1\right)\left\{\sum_{i=0}^{b-1}\binom{b-1}{i}r_{a,b}^{i}(1-r_{a,b})^{b-1-i}g(a+i)-g\left(k^{p}\right)\right\} \qquad (14)\\ > \quad & \left\{g\left(k^{p}\right)-g\left(k^{p}-1\right)\right\}\left\{\sum_{i=0}^{b-1}\binom{b-1}{i}r_{a,b}^{i}(1-r_{a,b})^{b-1-i}\eta(a+i)-\eta\left(k^{p}\right)\right\}. \end{aligned}$$

Notice that

$$\sum_{i=0}^{b-1}\binom{b-1}{i}r_{a,b}^{i}(1-r_{a,b})^{b-1-i}g(a+i)=c<g\left(k^{p}-1\right), \tag{15}$$

where the equality guarantees that a member who is randomizing is indifferent between investing and not investing, and the inequality holds by (3). Hence, from (14) and (15),

$$\begin{aligned} & g\left(k^{p}-1\right)\left\{g\left(k^{p}-1\right)-g\left(k^{p}\right)\right\}\\ > \quad & \left\{g\left(k^{p}\right)-g\left(k^{p}-1\right)\right\}\left\{\sum_{i=0}^{b-1}\binom{b-1}{i}r_{a,b}^{i}(1-r_{a,b})^{b-1-i}\eta(a+i)-\eta\left(k^{p}\right)\right\}. \end{aligned}$$

Since $g\left(k^P-1\right)-g\left(k^P\right)>0$, the previous inequality is equivalent to

$$g\left(k^p-1\right)>\eta\left(k^p\right)-\sum_{i=0}^{b-1}\binom{b-1}{i}r_{a,b}^i(1-r_{a,b})^{b-1-i}\eta(a+i).$$

Finally, since $\eta\left(k^p\right)-g\left(k^p-1\right)=\eta\left(k^p-1\right)$, this inequality is just (13). ■

Figure 3 is the graph of the expected social welfare, with parameters $N=100, \sigma=1, c=0.0001$.

4 Conclusion

In this paper, we have discussed the optimal committee size and the potential welfare losses associated with oversized committees. We have focused on environments in which there is no conflict of interest among individuals but information acquisition is costly. First, we have confirmed that the optimal committee size is bounded. In other words, the Condorcet Jury Theorem fails to hold, that is, larger committees might induce smaller social welfare. However, we have also showed that the welfare loss due to oversized committees is surprisingly small. In an arbitrarily large committee, even the worst equilibrium generates a higher welfare than an equilibrium in a committee in which there are two less members than in the optimal committee. Our results suggest that carefully designing committes might be not as important as it was thought to be.

5 Appendix

Lemma 3 *Suppose that $\eta\in C^1\left(\mathbb{R}_+\right)$ is absolutely continuous and $\eta'\left(k+1\right)/\eta'\left(k\right)$ is increasing for $k>\varepsilon$, where $\varepsilon\geq 0$. Let $g\left(k\right)=\eta\left(k+1\right)-\eta\left(k\right)$ for all $k\geq 0$. Then $g\left(k+1\right)/g\left(k\right)<g\left(k+2\right)/g\left(k+1\right)$ for $k\geq\varepsilon$.*

Proof. Fix a $k\left(\geq\varepsilon\right)$. For all $t\in\left(k,k+1\right)$,

$$\frac{\eta'\left(k+2\right)}{\eta'\left(k+1\right)}<\frac{\eta'\left(t+2\right)}{\eta'\left(t+1\right)}\Leftrightarrow\eta'\left(k+2\right)\eta'\left(t+1\right)<\eta'\left(k+1\right)\eta'\left(t+2\right).$$

Therefore,

$$\begin{aligned}
&\eta'\left(k+2\right)\int_k^{k+1}\eta'\left(t+1\right)dt<\eta'\left(k+1\right)\int_k^{k+1}\eta'\left(t+2\right)dt\\
\Leftrightarrow\quad&\eta'\left(k+2\right)\left[\eta\left(k+2\right)-\eta\left(k+1\right)\right]<\eta'\left(k+1\right)\left[\eta\left(k+3\right)-\eta\left(k+2\right)\right]\\
\Leftrightarrow\quad&\frac{\eta'\left(k+2\right)}{\eta'\left(k+1\right)}<\frac{\eta\left(k+3\right)-\eta\left(k+2\right)}{\eta\left(k+2\right)-\eta\left(k+1\right)}.
\end{aligned}$$

Similarly, for all $t\in\left(k,k+1\right)$,

$$\frac{\eta'\left(k+2\right)}{\eta'\left(k+1\right)}>\frac{\eta'\left(t+1\right)}{\eta'\left(t\right)}\Leftrightarrow\eta'\left(k+2\right)\eta'\left(t\right)>\eta'\left(k+1\right)\eta'\left(t+1\right).$$

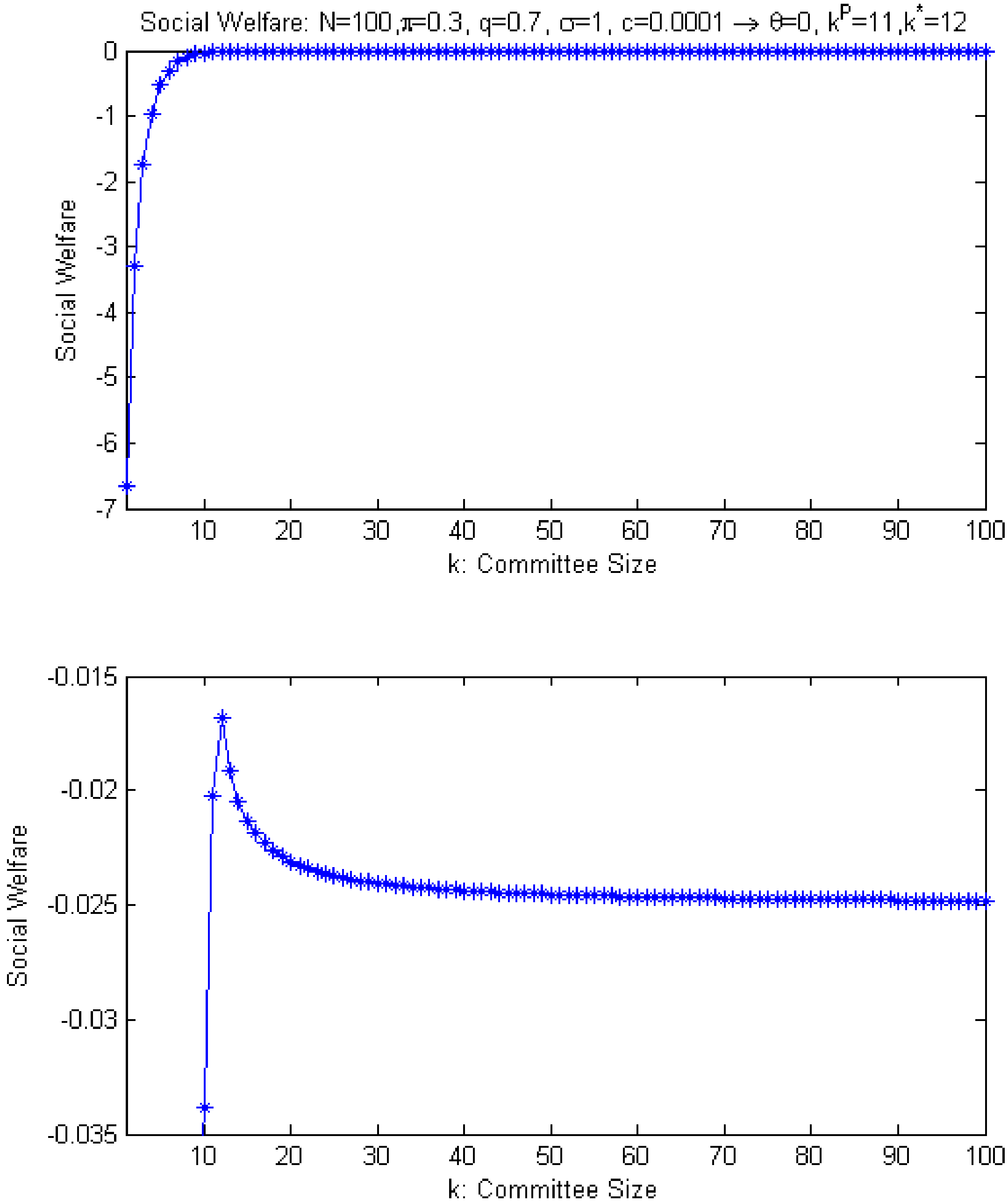

Figure 3: Social Welfare, as a function of the committee size k

Therefore,

$$\begin{aligned}
& \eta'(k+2)\int_k^{k+1}\eta'(t)\,dt > \eta'(k+1)\int_k^{k+1}\eta'(t+1)\,dt \\
\Leftrightarrow\quad & \eta'(k+2)\left[\eta(k+1)-\eta(k)\right] > \eta'(k+1)\left[\eta(k+2)-\eta(k+1)\right] \\
\Leftrightarrow\quad & \frac{\eta'(k+2)}{\eta'(k+1)} > \frac{\eta(k+2)-\eta(k+1)}{\eta(k+1)-\eta(k)}.
\end{aligned}$$

Hence we have

$$\frac{\eta(k+2)-\eta(k+1)}{\eta(k+1)-\eta(k)} < \frac{\eta'(k+2)}{\eta'(k+1)} < \frac{\eta(k+3)-\eta(k+2)}{\eta(k+2)-\eta(k+1)} \Rightarrow \frac{g(k+1)}{g(k)} < \frac{g(k+2)}{g(k+1)}$$

for any $k \geq \varepsilon$. ■

Proof of Proposition 1. **Part(i)** If $q+\pi=1$, then $q\pi=(1-q)(1-\pi)$. Hence,

$$-\frac{\eta(k)}{q\pi} = \Pr\left[\mu(s_1,\cdots,s_k)=-1|\omega=1\right]+\Pr\left[\mu(s_1,\cdots,s_k)=1|\omega=-1\right].$$

The ex post efficient decision rule is

$$\mu(s_1,\cdots,s_k)=\begin{cases} 1 & \text{if } s_1+\cdots+s_k \geq 0, \\ -1 & \text{if } s_1+\cdots+s_k < 0. \end{cases}$$

The sum of normally distributes signals is also normal; $\sum_{i=1}^k s_i \sim N\left(\omega k, \sigma\sqrt{k}\right)$. Hence,

$$\begin{aligned}
\Pr\left(s_1+\cdots+s_k|\,\omega=1\right) &= \frac{1}{\sigma}\phi\left(\frac{\left(\sum_{i=1}^k s_i\right)-k}{\sigma\sqrt{k}}\right), \\
\Pr\left(s_1+\cdots+s_k|\,\omega=-1\right) &= \frac{1}{\sigma}\phi\left(\frac{\left(\sum_{i=1}^k s_i\right)+k}{\sigma\sqrt{k}}\right)
\end{aligned}$$

where $\phi(x)=(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{x^2}{2}\right)$. Therefore,

$$-\frac{\eta(k)}{q\pi} = \Phi\left(\frac{-k}{\sigma\sqrt{k}}\right)+1-\Phi\left(\frac{k}{\sigma\sqrt{k}}\right) = 2\Phi\left(\frac{-\sqrt{k}}{\sigma}\right).$$

Hence,

$$\eta'(k)=\frac{q\pi}{\sigma}\frac{1}{\sqrt{k}}\phi\left(\frac{\sqrt{k}}{\sigma}\right) \text{ for } k>0.$$

Moreover,

$$\frac{\eta'(k+1)}{\eta'(k)}=\sqrt{\frac{k}{k+1}}\exp\left(-\frac{1}{2\sigma^2}\right),$$

which is increasing in $k\,(>0)$. By taking $\varepsilon=0$ in Lemma , $g(k+1)/g(k)$ is increasing in $k \in \mathbb{N}$.

Part(ii) The ex post efficient decision is again a cut-off rule:

$$\mu(s_1,\cdots,s_k)=\begin{cases} 1 & \text{if } s_1+\cdots+s_k \geq \theta, \\ -1 & \text{if } s_1+\cdots+s_k < \theta, \end{cases}$$

where $\theta = \left(\sigma^2/2\right)\log\left[(1-q)(1-\pi)/q\pi\right].$ Hence

$$\begin{aligned}\eta(k) &= -q\pi \Pr_{s_1,\cdots,s_k}\left[\mu(s_1,\cdots,s_k)=-1|\omega=1\right] \\ &\quad -(1-q)(1-\pi)\Pr_{s_1,\cdots,s_k}\left[\mu(s_1,\cdots,s_k)=1|\omega=-1\right] \\ &= -q\pi\Phi\left(\frac{\theta-k}{\sigma\sqrt{k}}\right)-(1-q)(1-\pi)\Phi\left(-\frac{\theta+k}{\sigma\sqrt{k}}\right).\end{aligned}$$

Next, we show that for any $\varepsilon > 0$, there exists $\delta > 0$, such that if $\sigma < \delta$, then $g(k+1)/g(k) < g(k+2)/g(k+1)$ for all $k > \varepsilon$. To this end, we first argue that for any $\varepsilon > 0$, there is a sufficiently small σ such that $\eta'(k+1)/\eta'(k)$ is increasing for all $k > \varepsilon$. Notice that

$$\begin{aligned}\frac{\eta'(k+1)}{\eta'(k)} &= \sqrt{\frac{k}{k+1}}\frac{\phi\left(\frac{\theta-(k+1)}{\sigma\sqrt{k+1}}\right)}{\phi\left(\frac{\theta-k}{\sigma\sqrt{k}}\right)} = \sqrt{\frac{k}{k+1}}\frac{\exp\left(-\frac{1}{2\sigma^2}\left(\frac{\theta^2}{k+1}-2\theta+k+1\right)\right)}{\exp\left(-\frac{1}{2\sigma^2}\left(\frac{\theta^2}{k}-2\theta+k\right)\right)} \\ &= \sqrt{\frac{k}{k+1}}\exp\left(\frac{1}{2\sigma^2}\left(\frac{\theta^2}{k(k+1)}-1\right)\right).\end{aligned}$$

Hence, $\eta'(k+1)/\eta'(k)$ is increasing if and only if

$$\varphi(x) \equiv \frac{x+1}{x}\exp\left(-\frac{\theta^2}{\sigma^2 x(x+1)}\right)$$

is decreasing in x. Also notice that

$$\varphi'(x) = \exp\left(-\frac{\theta^2}{\sigma^2 x(x+1)}\right)\left\{\frac{x+1}{x}\left(\frac{\theta^2}{\sigma^2}\right)\left(\frac{1}{x^2}-\frac{1}{(x+1)^2}\right)-\frac{1}{x^2}\right\}.$$

Therefore,

$$\begin{aligned}\varphi \text{ is decreasing} &\Leftrightarrow \frac{x+1}{x}\left(\frac{\theta^2}{\sigma^2}\right)\left(\frac{1}{x^2}-\frac{1}{(x+1)^2}\right)-\frac{1}{x^2} < 0 \\ &\Leftrightarrow \frac{\theta^2}{\sigma^2} < \frac{1}{\frac{x+1}{x}\left(\frac{1}{x^2}-\frac{1}{(x+1)^2}\right)x^2} = \frac{x(x+1)}{2x+1} \\ &\Leftrightarrow \frac{\sigma^2}{4}\left(\log\left(\frac{(1-q)(1-\pi)}{q\pi}\right)\right)^2 < \frac{x(x+1)}{2x+1}.\end{aligned}$$

The right-hand side is strictly increasing in $x\,(\geq 0)$ and $\lim_{x\to 0}\{x(x+1)/(2x+1)\} = 0$. Hence for sufficiently small σ, φ is decreasing. Therefore, $\eta'(k+1)/\eta'(k)$ is increasing for all $k > \varepsilon$. Then, by Lemma 3, $g(k+1)/g(k) < g(k+2)/g(k+1)$ for all $k \geq \varepsilon$. By taking $\varepsilon \in (0,1)$, we have shown that $g(k+1)/g(k) < g(k+2)/g(k+1)$ for all $k \geq 1$. What remains to be shown is $g(1)/g(0) < g(2)/g(1)$. From the argument above, $\eta'(2)/\eta'(1) < \eta'(t+1)/\eta'(t)$ for all $t \in (1,2)$ implies $\eta'(2)/\eta'(1) < g(2)/g(1)$. Hence it suffices to show that $g(1)/g(0) < \eta'(2)/\eta'(1)$ for sufficiently small σ. Define $L = \log\{(1-q)(1-\pi)/(q\pi)\}$. Then for $k > 0$,

$$\eta(k) = -q\pi\left\{\Phi\left(\frac{L\sigma}{2\sqrt{k}}-\frac{\sqrt{k}}{\sigma}\right)+e^L\Phi\left(-\frac{L\sigma}{2\sqrt{k}}-\frac{\sqrt{k}}{\sigma}\right)\right\}$$

and

$$\begin{aligned}\frac{\eta'(2)}{\eta'(1)} &= \frac{\frac{1}{\sqrt{2}}\phi\left(\frac{\theta-2}{\sigma\sqrt{2}}\right)}{\phi\left(\frac{\theta-1}{\sigma}\right)} = \frac{1}{\sqrt{2}}\exp\left(-\frac{1}{2}\left\{\left(\frac{\theta-2}{\sigma\sqrt{2}}\right)^2-\left(\frac{\theta-1}{\sigma}\right)^2\right\}\right) \\ &= \frac{1}{\sqrt{2}}\exp\left(-\frac{1}{2\sigma^2}\left(-\frac{\theta^2}{2}+1\right)\right) = \frac{1}{\sqrt{2}}\exp\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right).\end{aligned}$$

We show that

$$\lim_{\sigma\to 0}\frac{g(1)}{\eta'(2)/\eta'(1)} = 0.$$

Then

$$\lim_{\sigma\to 0}\frac{g(1)/g(0)}{\eta'(2)/\eta'(1)} = 0$$

because $\lim_{\sigma\to 0} g(0) = \min\{q\pi, (1-q)(1-\pi)\}$, which implies $g(1)/g(0) < \eta'(2)/\eta'(1)$ for sufficiently small σ. Notice that

$$\lim_{\sigma\to 0}\frac{g(1)}{\eta'(2)/\eta'(1)} = \sqrt{2}q\pi\lim_{\sigma\to 0}\left(\frac{\frac{\eta(1)}{-q\pi}-\frac{\eta(2)}{-q\pi}}{\exp\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right)}\right).$$

By l'Hôpital's Rule, for $k\in\{1,2\}$,

$$\begin{aligned}\lim_{\sigma\to 0}\left(\frac{\frac{\eta(k)}{-q\pi}}{\exp\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right)}\right) &= \lim_{\sigma\to 0}\left(\frac{\frac{\partial}{\partial\sigma}\left(\frac{\eta(k)}{-q\pi}\right)}{\frac{\partial}{\partial\sigma}\exp\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right)}\right) \\ &= \lim_{\sigma\to 0}\left(\frac{\frac{2\sqrt{k}}{\sigma^2}\phi\left(\frac{L\sigma}{2\sqrt{k}}-\frac{\sqrt{k}}{\sigma}\right)}{\left(\frac{L^2}{8}\sigma+\frac{1}{\sigma^3}\right)\exp\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right)}\right) \\ &= \lim_{\sigma\to 0}\left(\frac{2\sigma\sqrt{k}\phi\left(\frac{L\sigma}{2\sqrt{k}}-\frac{\sqrt{k}}{\sigma}\right)}{\left(\frac{L^2}{8}\sigma^4+1\right)\exp\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right)}\right)\end{aligned}$$

Now

$$\begin{aligned}\lim_{\sigma\to 0}\left(\frac{\phi\left(\frac{L\sigma}{2\sqrt{k}}-\frac{\sqrt{k}}{\sigma}\right)}{\exp\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right)}\right) &= \frac{1}{\sqrt{2\pi}}\lim_{\sigma\to 0}\exp\left(-\frac{1}{2}\left(\frac{L\sigma}{2\sqrt{k}}-\frac{\sqrt{k}}{\sigma}\right)^2-\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right)\right) \\ &= \frac{1}{\sqrt{2\pi}}\lim_{\sigma\to 0}\exp\left(-\left(\frac{1}{8k}+\frac{1}{16}\right)L^2\sigma^2+\frac{L}{2}+\frac{1-k}{2\sigma^2}\right) \\ &= \begin{cases} 0 & \text{if } k=2 \\ \frac{1}{\sqrt{2\pi}}e^{\frac{L}{2}} & \text{if } k=1\end{cases}\end{aligned}$$

In either case,

$$\lim_{\sigma\to 0}\left(\frac{\frac{\eta(k)}{-q\pi}}{\exp\left(\frac{L^2}{16}\sigma^2-\frac{1}{2\sigma^2}\right)}\right) = 0$$

Hence

$$\lim_{\sigma\to 0}\frac{g(1)}{\eta'(2)/\eta'(1)} = 0,$$

which completes the proof. ■

Proof of Proposition 2. First, we claim that the ex post efficient decision rule $\mu : \{-1,0,1\}^k \to \{-1,1\}$ is the following cut-off rule:

$$\mu(s_1,\cdots,s_k) = \begin{cases} 1 & \text{if } \sum_{i=1}^{k} s_i \geq \widehat{\theta}, \\ -1 & \text{if } \sum_{i=1}^{k} s_i < \widehat{\theta}. \end{cases} \tag{16}$$

where $\widehat{\theta} = \log\left((1-q)(1-\pi)/q\pi\right)/\log\left(p/(1-p)\right)$. Suppose that $(s_1,\cdots,s_k)$ is a permutation of

$$\left\{\underbrace{1,\cdots,1}_{a},\underbrace{0,\cdots,0}_{k-a-b},\underbrace{-1,\cdots,-1}_{b}\right\}.$$

Then $\mu(s_1,\cdots,s_k) = 1$ if

$$\begin{aligned} & \mathbb{E}_\omega\left[u(\omega,1)|s_1,\cdots,s_k\right] > \mathbb{E}_\omega\left[u(\omega,-1)|s_1,\cdots,s_k\right] \\ \Leftrightarrow\quad & -(1-q)(1-\pi)\frac{\Pr\left[s_1,\cdots,s_k|\omega=-1\right]}{\Pr\left[s_1,\cdots,s_k\right]} > -q\pi\frac{\Pr\left[s_1,\cdots,s_k|\omega=1\right]}{\Pr\left[s_1,\cdots,s_k\right]}, \end{aligned}$$

and

$$\begin{aligned} \Pr\left[s_1,\cdots,s_k|\omega=1\right] &= (pr)^a(1-r)^{k-a-b}(r(1-p))^b, \\ \Pr\left[s_1,\cdots,s_k|\omega=-1\right] &= (pr)^b(1-r)^{k-a-b}(r(1-p))^a. \end{aligned}$$

Hence $\mu(s_1,\cdots,s_k) = 1$ if

$$-(1-q)(1-\pi)p^b(1-p)^a > -q\pi p^a(1-p)^b \Leftrightarrow a-b > \frac{\log\left(\frac{(1-q)(1-\pi)}{q\pi}\right)}{\log\left(\frac{p}{1-p}\right)} = \widehat{\theta}.$$

By definition, $\sum_{i=1}^{k} s_i = a-b$, which gives the first line of (16). The opposite sign of inequality gives the second line of (16).

Recall that

$$\begin{aligned} \eta(k) = & -q\pi \Pr_{s_1,\cdots,s_k}\left[\mu(s_1,\cdots,s_k)=-1|\omega=1\right] \\ & -(1-q)(1-\pi)\Pr_{s_1,\cdots,s_k}\left[\mu(s_1,\cdots,s_k)=1|\omega=-1\right] \end{aligned}$$

Now, we consider the case where p converges to 1. Notice that $\left|\widehat{\theta}\right| < 1$ when p is close enough to one.

There are 3 distinct cases according to the parameter values of q and π.

(i) First, suppose that $q+\pi > 1$. Then $\log\left(\frac{(1-q)(1-\pi)}{q\pi}\right) < 0$, which gives $\widehat{\theta} < 0$. When p is close enough to one, $-1 < \widehat{\theta} < 0$. Then

$$\begin{aligned} \eta(k) = & -q\pi\left(\Pr\left[a-b=-1|\omega=1\right] + \Pr\left[a-b<-1|\omega=1\right]\right) \\ & -(1-q)(1-\pi)\left(\begin{array}{c} \Pr\left[a-b=0|\omega=-1\right] + \Pr\left[a-b=1|\omega=-1\right] \\ +\Pr\left[a-b>1|\omega=-1\right] \end{array}\right), \end{aligned}$$

and

$$\Pr\left[a-b=0|\omega=-1\right]=\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{k!}{j!\left(k-2j\right)!j!}\left(pr\right)^{j}\left(1-r\right)^{k-2j}\left(\left(1-p\right)r\right)^{j}.$$

Let $\varepsilon=1-p$. Then,

$$\Pr\left[a-b=0|\omega=-1\right]=\left(1-r\right)^{k}+k\left(k-1\right)r^{2}\left(1-r\right)^{k-2}\varepsilon+o\left(\varepsilon^{2}\right)$$

Similarly,

$$\begin{aligned}
\Pr\left[a-b=0|\omega=1\right] &= \left(1-r\right)^{k}+k\left(k-1\right)r^{2}\left(1-r\right)^{k-2}\varepsilon+o\left(\varepsilon^{2}\right),\\
\Pr\left[a-b=-1|\omega=1\right] &= kr\left(1-r\right)^{k-1}\varepsilon+o\left(\varepsilon^{2}\right),\\
\Pr\left[a-b=1|\omega=-1\right] &= kr\left(1-r\right)^{k-1}\varepsilon+o\left(\varepsilon^{2}\right),
\end{aligned}$$

and

$$\Pr\left[a-b<-1|\omega=1\right]=o\left(\varepsilon^{2}\right),\ \Pr\left[a-b>1|\omega=-1\right]=o\left(\varepsilon^{2}\right)$$

Therefore,

$$\begin{aligned}
\eta\left(k\right) &= -q\pi kr\left(1-r\right)^{k-1}\varepsilon-\left(1-q\right)\left(1-\pi\right)\left(\begin{array}{c}\left(1-r\right)^{k}+k\left(k-1\right)r^{2}\left(1-r\right)^{k-2}\varepsilon\\ +kr\left(1-r\right)^{k-1}\varepsilon\end{array}\right)+o\left(\varepsilon^{2}\right)\\
&= -\left(1-q\right)\left(1-\pi\right)\left(1-r\right)^{k}\\
&\quad -\left\{q\pi r\left(1-r\right)+\left(1-q\right)\left(1-\pi\right)\left[\left(k-1\right)r^{2}+r\left(1-r\right)\right]\right\}k\left(1-r\right)^{k-2}\varepsilon+o\left(\varepsilon^{2}\right)
\end{aligned}$$

(ii) Similarly, if $q+\pi<1$, then $\log\left(\frac{(1-q)(1-\pi)}{q\pi}\right)>0$, which gives $\widehat{\theta}>0$. When p is close enough to one, $0<\widehat{\theta}<1$. Then,

$$\begin{aligned}
\eta\left(k\right) &= -q\pi\left(1-r\right)^{k}\\
&\quad -\left\{\left(1-q\right)\left(1-\pi\right)r\left(1-r\right)+q\pi\left[\left(k-1\right)r^{2}+r\left(1-r\right)\right]\right\}k\left(1-r\right)^{k-2}\varepsilon+o\left(\varepsilon^{2}\right)
\end{aligned}$$

(iii) In the symmetric case, $q+\pi=1$, and $\widehat{\theta}=0$.

$$\begin{aligned}
\eta\left(k\right) &= -q\pi\left(\frac{1}{2}\Pr\left[a-b=0|\omega=1\right]+\Pr\left[a-b=-1|\omega=1\right]+\Pr\left[a-b<-1|\omega=1\right]\right)\\
&\quad -\left(1-q\right)\left(1-\pi\right)\left(\frac{1}{2}\Pr\left[a-b=0|\omega=-1\right]+\Pr\left[a-b=1|\omega=-1\right]+\Pr\left[a-b>1|\omega=-1\right]\right)
\end{aligned}$$

Since $q\pi=\left(1-p\right)\left(1-\pi\right)$,

$$\begin{aligned}
\eta\left(k\right) &= -q\pi\left(\Pr\left[a-b=0|\omega=1\right]+2\Pr\left[a-b=-1|\omega=1\right]+2\Pr\left[a-b<-1|\omega=1\right]\right)\\
&= -q\pi\left(\left(1-r\right)^{k}+k\left(k-1\right)r^{2}\left(1-r\right)^{k-2}\varepsilon+2kr\left(1-r\right)^{k-1}\varepsilon\right)+o\left(\varepsilon^{2}\right)
\end{aligned}$$

In the first 2 cases,

$$\begin{aligned}
& \eta\left(k+1\right)-\eta\left(k\right)\\
&= -q\pi r\left(1-r-rk\right)\left(1-r\right)^{k-1}\varepsilon\\
&\quad -\left(1-q\right)\left(1-\pi\right)\left(\left(1-r\right)^{k}+k\left(k-1\right)r^{2}\left(1-r\right)^{k-2}\varepsilon+kr\left(1-r\right)^{k-1}\varepsilon\right)+o\left(\varepsilon^{2}\right)\\
&= A\left(k\right)+B\left(k\right)\varepsilon+o\left(\varepsilon^{2}\right)
\end{aligned}$$

where

$$\begin{aligned} A(k) &= -(1-q)(1-\pi)(1-r)^k, \\ B(k) &= -q\pi r(1-r-rk)(1-r)^{k-1}-(1-q)(1-\pi)k(1-2r+kr)r(1-r)^{k-2} \end{aligned}$$

Then

$$\begin{aligned} \frac{g(k+1)}{g(k)} &= \frac{\eta(k+2)-\eta(k+1)}{\eta(k+1)-\eta(k)} = \frac{A(k+1)+B(k+1)\varepsilon+o\left(\varepsilon^2\right)}{A(k)+B(k)\varepsilon+o\left(\varepsilon^2\right)} \\ &= \frac{A(k+1)\left(1+\frac{B(k+1)}{A(k+1)}\varepsilon+o\left(\varepsilon^2\right)\right)}{A(k)\left(1+\frac{B(k)}{A(k)}\varepsilon+o\left(\varepsilon^2\right)\right)}. \end{aligned}$$

Since

$$\frac{1+\frac{B(k+1)}{A(k+1)}\varepsilon+o\left(\varepsilon^2\right)}{1+\frac{B(k)}{A(k)}\varepsilon+o\left(\varepsilon^2\right)} = 1+\left(\frac{B(k+1)}{A(k+1)}-\frac{B(k)}{A(k)}\right)\varepsilon+o\left(\varepsilon^2\right),$$

it follows that

$$\begin{aligned} \frac{g(k+1)}{g(k)} &= \frac{A(k+1)}{A(k)}\left(1+\left(\frac{B(k+1)}{A(k+1)}-\frac{B(k)}{A(k)}\right)\varepsilon\right)+o\left(\varepsilon^2\right) \\ &= (1-r)\left(1+\left(-\frac{q\pi r}{(1-q)(1-\pi)(1-r)}+\frac{r(1+2kr-r)}{(1-r)^2}\right)\varepsilon\right)+o\left(\varepsilon^2\right). \end{aligned}$$

Therefore, $g(k+1)/g(k)$ is increasing in k for sufficiently small ε, as long as r is strictly smaller than 1 (which is what we wanted to show).

In the symmetric case,

$$\eta(k+1)-\eta(k) = -q\pi\left(\begin{array}{c} -r(1-r)^k+k(2-r-rk)r^2(1-r)^{k-2}\varepsilon \\ +2kr(1-r)^{k-1}(1-r-rk)\varepsilon \end{array}\right)+o\left(\varepsilon^2\right).$$

Hence

$$\frac{g(k+1)}{g(k)} = (1-r)\left(1+\frac{2r(-1+r+rk)+2(1-r)(-1+2r+2rk)}{(1-r)^2}\varepsilon\right)+o\left(\varepsilon^2\right).$$

Therefore, $g(k+1)/g(k)$ is increasing in k for sufficiently small ε, as long as r is strictly smaller than 1. ■

Proof of Proposition 3. Recall that

$$\begin{aligned} \eta(k) &= \mathbb{E}_{\omega,s}[u(\omega,\mu(s))] \\ &= -q\pi\Pr_s[\mu(s)=-1|\omega=1]-(1-q)(1-\pi)\Pr_s[\mu(s)=1|\omega=-1] \end{aligned} \tag{17}$$

The ex post efficient rule is again a cutoff rule:

$$\begin{aligned} \mu(s)=1 &\Leftrightarrow \mathbb{E}_\omega[u(\omega,1)|s] \geq \mathbb{E}_\omega[u(\omega,-1)|s] \\ &\Leftrightarrow -(1-q)\Pr_\omega[\omega=-1|s] \geq -q\Pr_\omega[\omega=1|s] \\ &\Leftrightarrow -(1-q)(1-\pi)\Pr_s[s|\omega=-1] \geq -q\pi\Pr_s[s|\omega=1] \\ &\Leftrightarrow \frac{\Pr_s[s|\omega=1]}{\Pr_s[s|\omega=-1]} \geq \frac{(1-q)(1-\pi)}{q\pi} \end{aligned}$$

For the binary signal,

$$\frac{\Pr[s_i|\omega=1]}{\Pr[s_i|\omega=-1]}=\left(\frac{p}{1-p}\right)^{s_i}$$

for $s_i\in\{-1,1\}$. Hence

$$\mu(s)=1\Leftrightarrow\left(\frac{p}{1-p}\right)^{\sum s_i}\geq\frac{(1-q)(1-\pi)}{q\pi}\Leftrightarrow\sum s_i\geq\theta$$

where $\theta=\log((1-q)(1-\pi)/q\pi)\log(p/(1-p))$. By symmetry, we can assume $\theta\geq 0$ without loss of generality.

(i) Suppose $\theta>1$. Then for any $k<\theta$, no signal profile realization $(s_1,\cdots,s_k)$ can upset the status quo, thus $\eta(k)=-q\pi$. Then $g(k)$ is zero for $k<\theta-1$ and $g(k+1)/g(k)$ is not well-defined.

(ii) Suppose $0\leq\theta<1$. Then we claim that

$$\begin{aligned}\eta(2m) &= -\{q\pi+(1-q)(1-\pi)\}\left(\sum_{j=0}^{m}\binom{2m}{j}p^j(1-p)^{2m-j}\right) &(18)\\ &\quad+(1-q)(1-\pi)\binom{2m}{m}p^m(1-p)^m\\ \eta(2m+1) &= -\{q\pi+(1-q)(1-\pi)\}\left(\sum_{j=0}^{m}\binom{2m+1}{j}p^j(1-p)^{2m+1-j}\right) &(19)\end{aligned}$$

Suppose $k(=2m)$ is even. Then $\mu(s)=-1$ iff $\sum s_i\in\{-2m,-2m+2,\cdots,-2,0\}$. It happens when there are $j(\in\{0,\cdots,m\})$ signals of $s_i=1$ and the other signals are $s_i=-1$. Hence

$$\Pr_s[\mu(s)=-1|\omega=1]=\sum_{j=0}^{m}\binom{2m}{j}p^j(1-p)^{2m-j}.$$

Similarly, $\mu(s)=1$ iff $\sum s_i\in\{2,4,\cdots,2m\}$. It happens when there are $j(\in\{0,\cdots,m-1\})$ signals of $s_i=-1$ and the other signals are $s_i=1$. Hence,

$$\Pr_s[\mu(s)=1|\omega=-1]=\sum_{j=0}^{m-1}\binom{2m}{j}p^j(1-p)^{2m-j}.$$

Then, by (17), (18), follows.

Suppose $k(=2m+1)$ is odd. Then $\mu(s)=-1$ iff $\sum s_i\in\{-2m-1,-2m+1,\cdots,-3,-1\}$. It happens when there are $j(\in\{0,\cdots,m\})$ signals of $s_i=1$ and the other signals are $s_i=-1$. Hence,

$$\Pr_s[\mu(s)=-1|\omega=1]=\sum_{j=0}^{m}\binom{2m}{j}p^j(1-p)^{2m+1-j}.$$

Similarly, $\mu(s)=1$ iff $\sum s_i\in\{1,3,\cdots,2m+1\}$. It happens when there are $j(\in\{0,\cdots,m\})$ signals of $s_i=-1$ and the other signals are $s_i=1$. Hence

$$\Pr_s[\mu(s)=1|\omega=-1]=\sum_{j=0}^{m}\binom{2m}{j}p^j(1-p)^{2m+1-j}.$$

Then by (17), (19) follows.

Now, using (18) and (19),

$$\begin{aligned} g(2m) &= \{pq\pi - (1-p)(1-q)(1-\pi)\}\binom{2m}{m}p^m(1-p)^m, \\ g(2m+1) &= \{(1-q)(1-\pi) - q\pi\}\binom{2m+1}{m}p^m(1-p)^m. \end{aligned}$$

Recall that we assumed $0 \leq \theta < 1$, which is equivalent to $pq\pi - (1-p)(1-q)(1-\pi) > 0$ and $(1-q)(1-\pi) - q\pi \geq 0$. Now, if $\theta > 0$,

$$\frac{g(2m+2)}{g(2m+1)} = \frac{pq\pi - (1-p)(1-q)(1-\pi)}{(1-q)(1-\pi) - q\pi}2p(1-p)$$

is a constant, and Assumption 1 cannot be satisfied. If $\theta = 0$, $g(2m+2)/g(2m+1)$ is not well-defined. ■

Proof of Lemma 1. Part (i). Notice that

$$G(r;a,b) = \sum_{i=0}^{b-1}\binom{b-1}{i}r^i(1-r)^{b-i-1}g(a+i).$$

Since $r^i(1-r)^{b-i-1} = r^i(1-r)^{b-i} + r^{i+1}(1-r)^{b-i-1}$,

$$G(r;a,b) = \sum_{i=0}^{b-1}\binom{b-1}{i}\left[r^i(1-r)^{b-i} + r^{i+1}(1-r)^{b-i-1}\right]g(a+i).$$

Since g is decreasing

$$\begin{aligned} G(r;a,b) &> \sum_{i=0}^{b-1}\binom{b-1}{i}r^i(1-r)^{b-i}g(a+i) + \sum_{i=0}^{b-1}\binom{b-1}{i}r^{i+1}(1-r)^{b-i-1}g(a+i+1) \\ &= \sum_{i=0}^{b-1}\binom{b-1}{i}r^i(1-r)^{b-i}g(a+i) + \sum_{i=1}^{b}\binom{b-1}{i-1}r^i(1-r)^{b-i}g(a+i) \\ &= \sum_{i=0}^{b}\left[\binom{b-1}{i} + \binom{b-1}{i-1}\right]r^i(1-r)^{b-i}g(a+i), \end{aligned}$$

where the first equality holds because we have just redefined the notation in the second summation, and the second equality holds because, by convention, $\binom{n}{-1} = \binom{n}{n+1} = 0$ for all $n \in \mathbb{N}$. Finally, using $\binom{b-1}{i} + \binom{b-1}{i-1} = \binom{b}{i}$, we have

$$G(r;a,b) > \sum_{i=0}^{b}\binom{b}{i}r^i(1-r)^{b-i}g(a+i) = G(r;a,b+1).$$

Part (ii). By the definitions of $r_{a,b}$ and $r_{a,b+1}$, we have

$$c = G(r_{a,b};a,b) = G(r_{a,b+1};a,b+1),$$

and by part (i) of this lemma,

$$G\left(r_{a,b+1};a,b+1\right)<G\left(r_{a,b+1};a,b\right).$$

Therefore

$$G\left(r_{a,b};a,b\right)<G\left(r_{a,b+1};a,b\right)$$

Since $G\left(r;a,b\right)$ is strictly decreasing in r, $r_{a,b}>r_{a,b+1}$ follows. ■

Proof of Lemma 2. The statement of the lemma is obvious if $i\in\{k-1,k\}$. It remains to show that (10) hold with strict inequality whenever $i\notin\{k-1,k\}$. First, notice that for any positive sequence, $\{a_j\}_0^\infty$, if $a_{j+1}/a_j<a_{j+2}/a_{j+1}$ for all $j\in\mathbb{N}$, then

$$\frac{a_k}{a_{k-1}}>\frac{\sum_{j=i+1}^{k}a_j}{\sum_{j=i}^{k-1}a_j}\quad\text{for all } k\geq 1 \text{ and for all } i\in\{1,...,k-2\}.$$

Assumption 1 allows us to apply this result for the sequence $a_j=g\left(j\right)$, and hence, for all $k\geq 1$

$$\frac{g\left(k\right)}{g\left(k-1\right)}>\frac{\sum_{j=i+1}^{k}g\left(j\right)}{\sum_{j=i}^{k-1}g\left(j\right)}=\frac{\eta\left(k+1\right)-\eta\left(i+1\right)}{\eta\left(k\right)-\eta\left(i\right)}\quad\text{for all } i\in\{1,...,k-2\}.$$

Since $\eta\left(a\right)>\eta\left(b\right)$ if $a>b$, this implies that for all $i\in\{1,...,k-2\}$

$$g\left(k\right)\left(\eta\left(i\right)-\eta\left(k\right)\right)<g\left(k-1\right)\left(\eta\left(i+1\right)-\eta\left(k+1\right)\right).\tag{20}$$

Similarly, for a positive sequence $\{a_j\}_0^\infty$ if $a_{j+1}/a_j<a_{j+2}/a_{j+1}$ for all $j\in\mathbb{N}$, then

$$\frac{a_k}{a_{k-1}}<\frac{\sum_{j=k+1}^{i}a_j}{\sum_{j=k}^{i-1}a_j}\quad\text{for all } k\geq 1 \text{ and for all } i\geq k+1.$$

Again, by Assumption 1, we can apply this result to the sequence $a_j=g\left(j\right)$ and get

$$\frac{g\left(k\right)}{g\left(k-1\right)}<\frac{\sum_{j=k+1}^{i}g\left(j\right)}{\sum_{j=k}^{i-1}g\left(j\right)}=\frac{\eta\left(i+1\right)-\eta\left(k+1\right)}{\eta\left(i\right)-\eta\left(k\right)}\quad\text{for all } i>k.$$

Multiplying through by $g\left(k-1\right)\left(\eta\left(i\right)-\eta\left(k\right)\right)$, we get (20). That is, (20) holds whenever $i\notin\{k-1,k\}$. After substructing $g\left(k-1\right)\left(\eta\left(i\right)-\eta\left(k\right)\right)$ from both sides of (20) we get (10). ■

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