Mathematics for Economists, Third Edition
Malcolm Pemberton and Nicholas Rau

ANSWERS TO EXERCISES

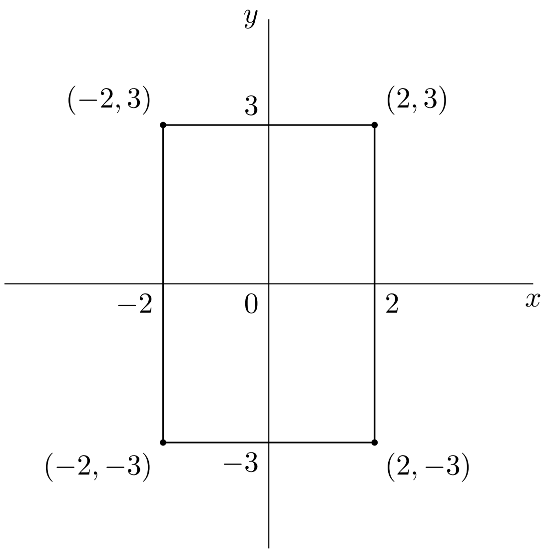
Revised August 23, 2015

If you find any errors in these answers,
please notify the authors at n.rau@ucl.ac.uk

1 LINEAR EQUATIONS

1.1 Straight line graphs

1.1.1



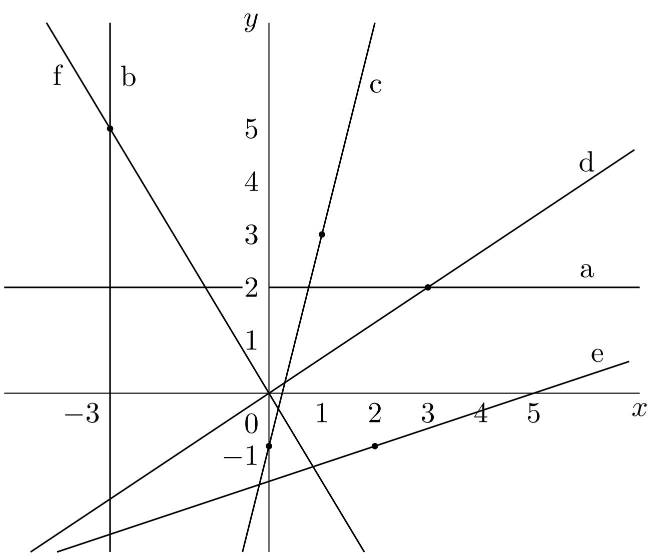
1.1.2 (a) passes through $(-1,0)$ and $(0,1)$, (b) through $(3,0)$ and $(0,-3)$, (c) through $(-8,0)$ and $(0,8)$. All lines have slope 1.

$y = x + 6$.

1.1.3 (a) passes through $(-\frac{3}{2},0)$ and $(0,3)$, (b) through $(3,0)$ and $(0,3)$, (c) through $(\frac{3}{8},0)$ and $(0,3)$. All lines have intercept 3.

$y = -3x + 3$.

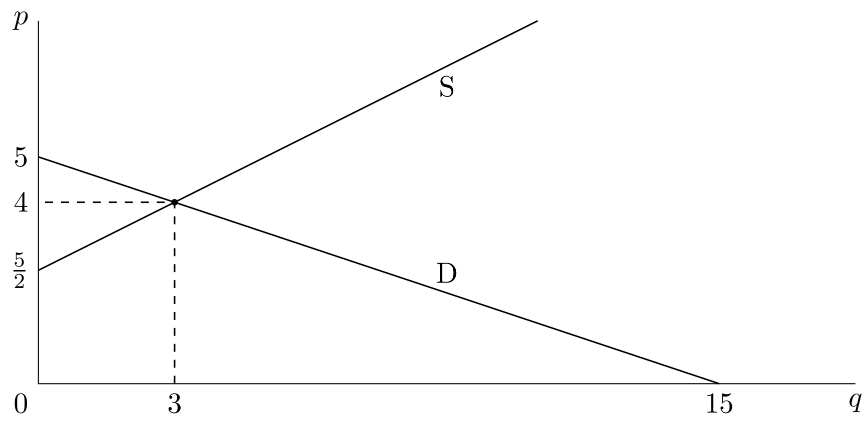
1.1.4



1.1.5 (a) $y = -4x - 1$, (b) $y = 7x - 5$, (c) $y = -4x$, (d) $x = 7$.

1.2 An economic application: supply and demand

1.2.1

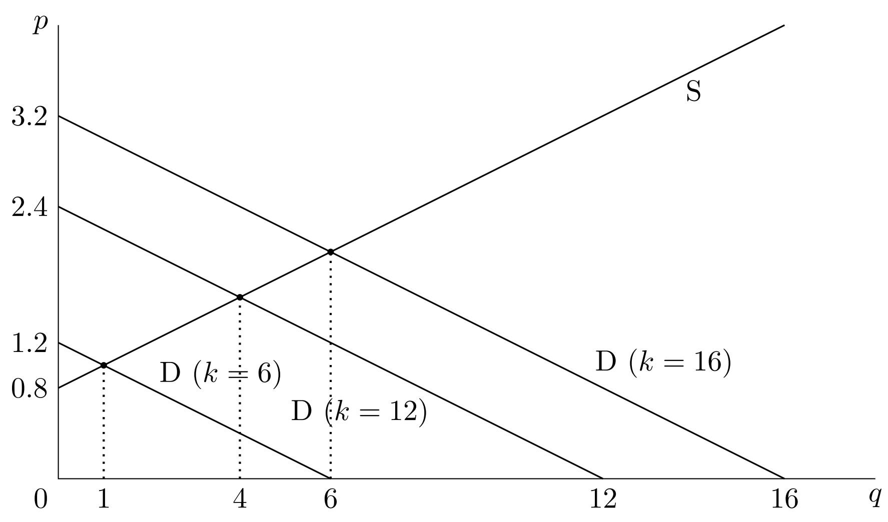


Equilibrium at $p = 4$, $q = 3$.

1.2.2 In equilibrium with positive price and quantity,

$$p = \frac{4+k}{10}, \quad q = \frac{k}{2} - 2.$$

If $k = 2$ none of the good is supplied or demanded; price must be less than 0.4 for any positive quantity to be demanded, and such a price is too low to elicit any supply.



1.3 Simultaneous equations

1.3.1 $x = 5$, $y = -1$.

1.3.2 $x = \frac{1}{4}(1 + 10s)$, $y = s$, where s is any number.

1.3.3 $x = \frac{1}{5}$, $y = -\frac{2}{5}$, $z = \frac{4}{5}$.

1.3.4 $x = -1$, $y = 0$, $z = 1$.

1.4 Input-output analysis

1.4.1 55 of X, 80 of Y, 80 of Z.

1.4.2 50 of X, 75 of Y.

2 LINEAR INEQUALITIES

2.1 Inequalities

2.1.1 (a) $x > -\frac{1}{2}$, (b) $x \geq \frac{8}{3}$, (c) $x \geq -12$, (d) $x > -\frac{2}{5}$.

2.1.2 The required region is on the same side of $x + 2y = 3$ as the origin and on the opposite side of $2x - 3y = 13$ to the origin.

2.2 Economic applications

2.2.1 Denoting by x_1 and x_2 the amounts consumed of fish and chips respectively, the budget set consists of the points satisfying $2x_1 + 3x_2 \leq 10$ and $x_1 \geq 0$, $x_2 \geq 0$.

If the prices are reversed, the budget set consists of the points satisfying $3x_1 + 2x_2 \leq 10$ and $x_1 \geq 0$, $x_2 \geq 0$.

2.2.2 The budget set consists of the points satisfying $x_1 + 32x_2 \leq 18$ and $x_1 \geq 0$, $x_2 \geq 0$.

In (a), (b) and (c) the budget set is identical to the original one.

In general, for income $18a$ and prices a and $3a$, where a is any positive constant, the budget set is identical to the original one.

2.2.3 Denoting by x and y the amounts produced per day of products X and Y respectively, the feasible set consists of the points satisfying $16x + 8y \leq 240$, $10x + 20y \leq 300$ and $x \geq 0$, $y \geq 0$.

2.2.4 The feasible set consists of the points satisfying the constraints in 2.2.3 together with $2x + 3y \leq 48$. In (a) the additional constraint does not restrict the original feasible set any further. In (b) the feasible set reduces to that defined by the carbon emissions constraint and $x \geq 0$, $y \geq 0$.

2.3 Linear programming

2.3.1 10 of X, 10 of Y; 15 of X, 0 of Y.

2.3.2 10.5 of X, 9 of Y; 12 of X, 0 of Y.

3 SETS AND FUNCTIONS

3.1 Sets and numbers

3.1.1 (a) $A \subset B$, (b) $B \subset A$, (c) $A \subset B$, (d) neither, (e) $A \subset B$, (f) neither.

3.1.2 (a) $x^2 - 12x + 36$, (b) $4x^2 - 9y^2$, (c) $12a^2 + 6ab$, (d) $x^2 + 2x - 3$.

3.1.3 (a) $(x - 9)^2$, (b) $(4a - 5b)(4a + 5b)$, (c) $5x(x - 3y)$, (d) $(x - 5)(x + 2)$.

3.1.4 (a) $(x+6)^2-33$, (b) $4(x-\frac{3}{2})^2$ [or $(2x-3)^2$], (c) $-(x-4)^2+9$.

3.1.5 (a) 18, $\mathbb{R}^2$; (b) 2, $\mathbb{R}^{18}$.

3.2 Functions

3.2.1 -3, 0, 0, $2a^2+5a-3$, $2b^2-5b-3$, $2(a-b)^2+5(a-b)-3$ [or $2a^2-4ab+2b^2+5a-5b-3$].

3.2.2 All are V-shaped with the corner at the origin. The graph of $y=|2x|$ rises most steeply, then $y=|x|$ and $y=|\frac{1}{2}x|$ is the least steep.

3.2.3 All are U-shaped with the bottom of the U at the origin. The graph of $y=2x^2$ rises most steeply, then $y=x^2$ and $y=\frac{1}{2}x^2$ is the least steep.

3.2.4 5, 5, $x^2+y^2=25$.

3.2.5 Original function is $f(x_1,x_2,x_3)=4x_1+2x_2+x_3$ and the new function is $F(x_1,x_2,x_3)=3x_1+3x_2+2x_3$.

(a) $f(g_1,g_2,g_3)$, i.e. $4g_1+2g_2+g_3$.
(b) $F(g_1,g_2,g_3)$, i.e. $3g_1+3g_2+2g_3$.
(c) $F(h_1,h_2,h_3)$, i.e. $3h_1+3h_2+2h_3$.
(d) $f(h_1,h_2,h_3)$, i.e. $4h_1+2h_2+h_3$.

3.3 Mappings

3.3.1 The image of (x,y) under h is its reflexion in the x–axis.

Points of the form $(x,0)$ i.e. the x–axis.

3.3.2 $(-y,-x)$, (y,x).

4 QUADRATICS, INDICES AND LOGARITHMS

4.1 Quadratic functions and equations

4.1.1 See next page.

4.1.2 (a) 2, 4; (b) $\frac{1}{4}(5\pm\sqrt{17})$; (c) 2, $-\frac{2}{5}$.

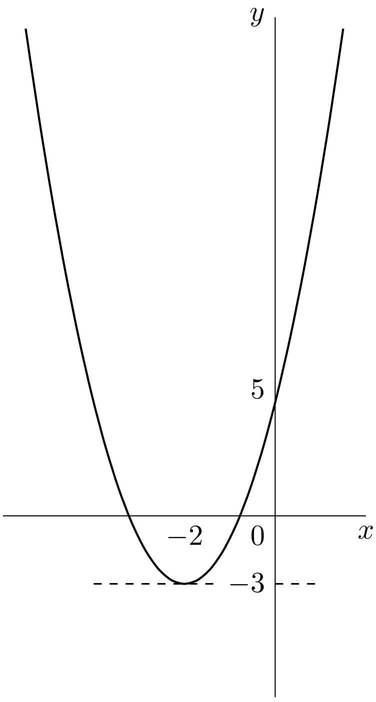
4.1.3 (a) $y=x^2-4$: U-shaped with vertex at $(0,-4)$. $(2,0)$ and $(-2,0)$. $|x|>2$.
(b) $y=x^2-8x+16$: U-shaped with vertex at $(4,0)$. $(4,0)$. $x\neq 4$.
(c) $y=x^2+2x+4$: U-shaped with vertex at $(-1,3)$. Does not meet x–axis. Every real number.

4.1.4 (a) $6x^2-7x-5=0$. $\frac{5}{3}$, $-\frac{1}{2}$.
(b) $5x^2-13x+8=0$. 8/5, 1.
(c) Same as (b).
(d) $x^2-3=0$. $\pm\sqrt{3}$.

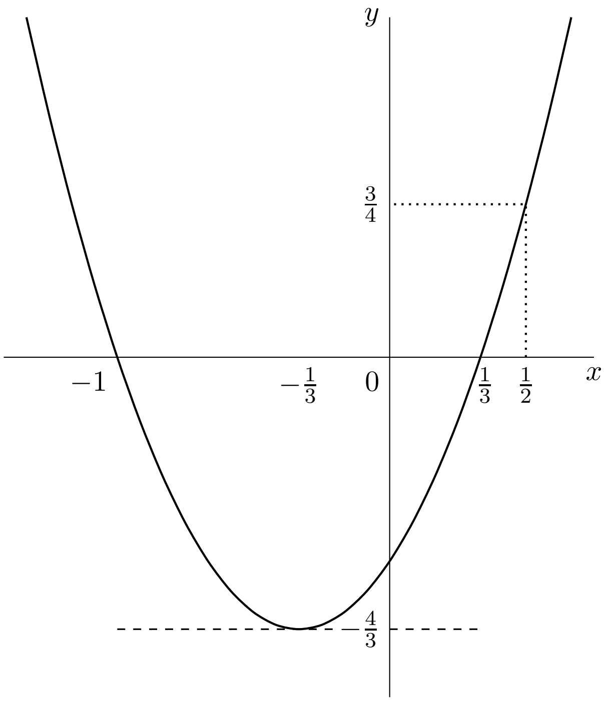
4.1.5 Let $d=\sqrt{b^2-4ac}$. Then

$$p+q=\frac{-b+d-b-d}{2a}=\frac{-2b}{2a}=-\frac{b}{a},\quad pq=\frac{(-b+d)(-b-d)}{4a^2}=\frac{b^2-d^2}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

4.1.1 (a)



(a)



(b)

4.2 Maximising and minimising quadratic functions

4.2.1 (a) U-shaped with vertex at $(0.5, -4)$. -4.

(b) U-shaped with vertex at $(3, -8)$. -8.

(c) U-shaped with vertex at $(-0.5, -2)$. -2.

-4, -8, -1.

4.2.2 $\dfrac{9}{4}$, $\dfrac{57}{8}$.

4.2.3 $\dfrac{9-t}{4}$, $\dfrac{9t-t^2}{4}$, $\dfrac{9}{2}$.

4.2.4 100 cm^2.

4.3 Indices

4.3.1 (a) 3.728×10^2, (b) 3.728×10^{-3}, (c) 3.728×10^0.

4.3.2 (a) $(x+y)^3 = (x+y)(x+y)^2$. Now

$$x(x+y)^2 = x(x^2+2xy+y^2) = x^3+2x^2y+xy^2,$$

$$y(x+y)^2 = (x^2+2xy+y^2)y = x^2y+2xy^2+y^3.$$

Hence by addition,

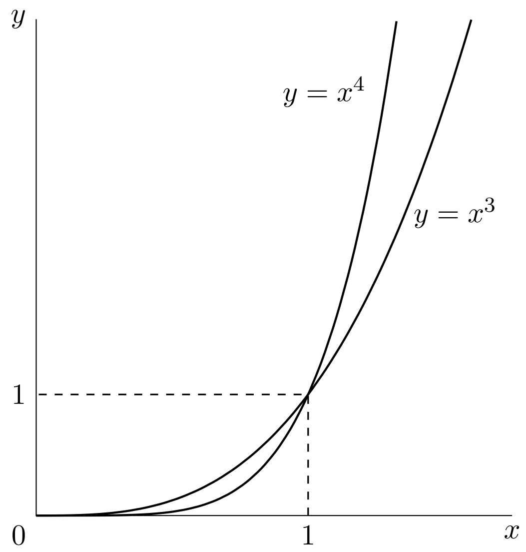
$$(x+y)^3 = x^3+(2+1)x^2y+(1+2)xy^2+y^3.$$

(b) $x^3 - y^3 = x(x^2 - y^2) + xy^2 - y^3 = x(x+y)(x-y) + (x-y)y^2 = (x-y)(x^2+xy+y^2)$.

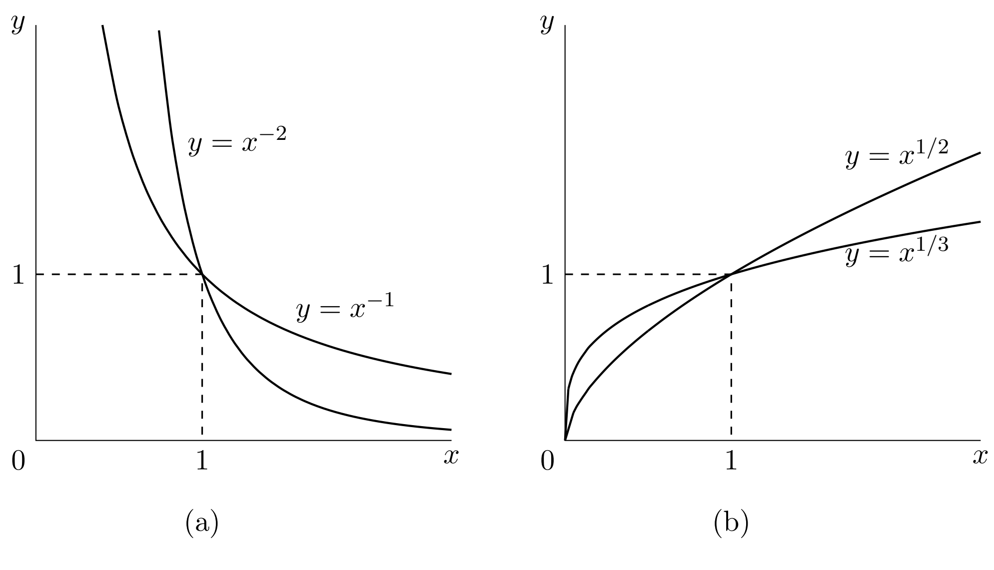
4.3.3 (a) $\dfrac{\text{LHS}}{\text{RHS}} = -\dfrac{xy(x^{-1} - y^{-1})}{x - y} = -\dfrac{y - x}{x - y} = 1$.

(b) $\dfrac{\text{LHS}}{\text{RHS}} = -\dfrac{x^2y^2(x^{-2} - y^{-2})}{(x - y)(x + y)} = -\dfrac{y^2 - x^2}{x^2 - y^2} = 1$.

4.3.4

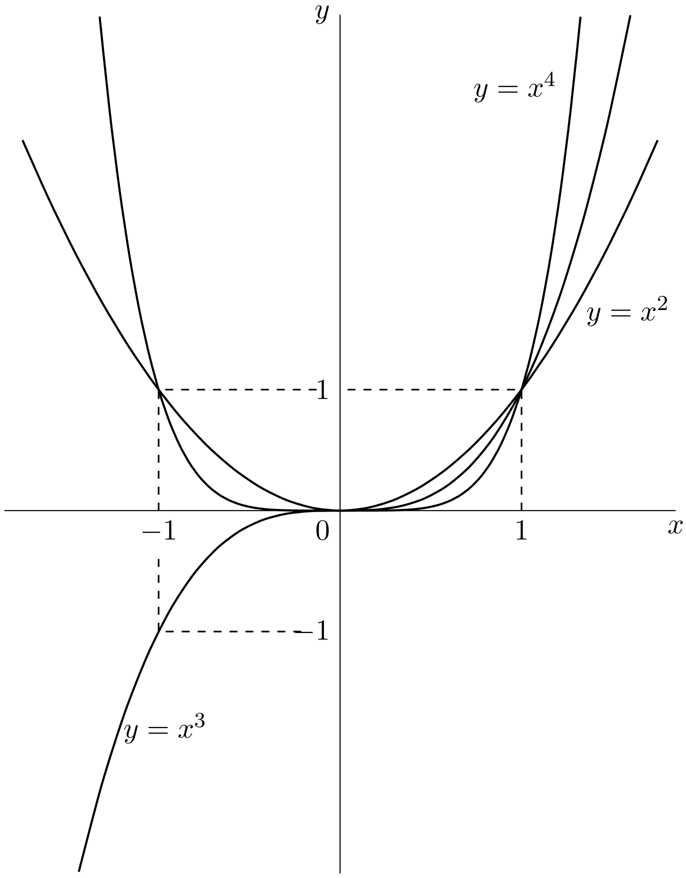


4.3.5

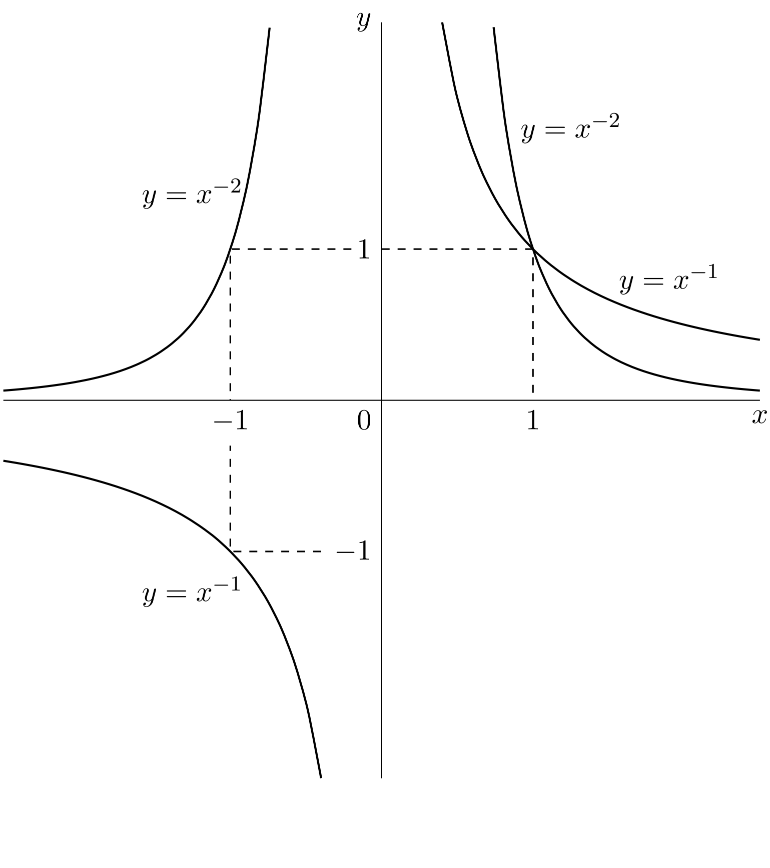


4.3.6 Since $-x = (-1)\times x$,

$$(-x)^3 = (-1)^3 \times x^3 = (-1)\times x^3 = -x^3, \quad (-x)^4 = (-1)^4 \times x^4 = 1\times x^4 = x^4.$$



4.3.7



4.3.8 (a) x^{10}, (b) $x^{5/3}$, (c) y^8/x, (d) $\dfrac{5x^2}{16y^2}$.

4.3.9 $x = 2^{-1/3}z^{4/3}$, $y = 2^{2/3}z^{4/3}$.

4.4 Logarithms

4.4.1 (a) 3, (b) -3, (c) 2, (d) $\frac{2}{3}$, (e) $\frac{8}{3}$.

4.4.2 $\log_a x \times \log_x a = \log_a a$ by the change-of-base formula **L5**, and $\log_a a = 1$ because $a^1 = a$.

4.4.3 $\log Y = \log 2 + \frac{1}{2}\log K + \frac{1}{3}\log L + \frac{1}{6}\log R$.

5 SEQUENCES AND SERIES

5.1 Sequences

5.1.1 (a) 7, 10, 13, arithmetic progression;

(b) -1, -7, -13, arithmetic progression;

(c) 4, 16, 64, geometric progression;

(d) -10, 20, -40, geometric progression;

(e) 3, 18, 81, neither.

5.1.2 (a) 2, 7, 12, nth term $5n-3$; (b) 4, 12, 36, nth term $4\times 3^{n-1}$.

5.1.3 (a) No limit ($u_n \to \infty$), (b) no limit ($u_n \to -\infty$), (c) 0, (d) 0.

5.2 Series

5.2.1 5050.

5.2.2 (a) 87, $\frac{1}{2}n(7n-13)$; (b) -87, $\frac{1}{2}n(13-7n)$; (c) $\frac{1}{2}(7^6-1)$, $\frac{1}{2}(7^n-1)$;

(d) $-\dfrac{21}{8}\times\dfrac{7^6-1}{7^6}$, $-\dfrac{21}{8}\times\dfrac{7^n-(-1)^n}{7^n}$.

5.2.3 (a) No; (b) no; (c) no; (d) yes, $-21/8$.

5.3 Geometric progressions in economics

5.3.1 (a) 196 (Usurian dollars), (b) 214.36, (c) 140, (d) 100.

5.3.2 0.072.

5.3.3 (a) £563.71, (b) 7.

5.3.4 £357.71.

5.3.5 (a) £839.20, (b) £805.23.

6 INTRODUCTION TO DIFFERENTIATION

6.1 The derivative

6.1.1

$$\frac{f(x+h)-f(x)}{h} = \frac{x^2-(x+h)^2}{2h} = -x-\tfrac{1}{2}h,$$

which is close to $-x$ if $|h|$ is small. Hence $f'(x) = -x$.

(a) -4, (b) 5.

$y = 7 - 2x$.

6.1.2 Using the result of Exercise 4.3.2(a),

$$\frac{f(x+h)-f(x)}{h} = \frac{3x^2h+3xh^2+h^3}{h} = 3x^2 + h(3x+h),$$

which is close to $3x^2$ if $|h|$ is small. Hence $f'(x) = 3x^2$.

Alternatively, one can use the result of Exercise 4.3.2(b):

$$f(x+h)-f(x) = h\left((x+h)^2 + x(x+h) + x^2\right).$$

Thus $(f(x+h)-f(x))/h$ is the sum of the three terms $(x+h)^2$, $x(x+h)$ and x^2, each of which approaches x^2 as $h \to 0$. Hence $f'(x) = 3x^2$.

Tangents: $y = a^2(3x-2a)$, $y = a^2(3x+2a)$. Two parallel lines.

6.1.3 Using the result of Exercise 4.3.3(b),

$$\frac{f(x+h)-f(x)}{h} = -\frac{2x+h}{(x+h)^2x^2}.$$

In the fraction on the right-hand side, the numerator is close to $2x$ and the denominator is close to x^4 if $|h|$ is small. Hence $f'(x) = -2x^{-3}$.

$-\dfrac{1}{4}$, $-\dfrac{2}{125}$.

6.2 Linear approximations and differentiability

6.2.1 (a) -0.02, (b) 0.04.

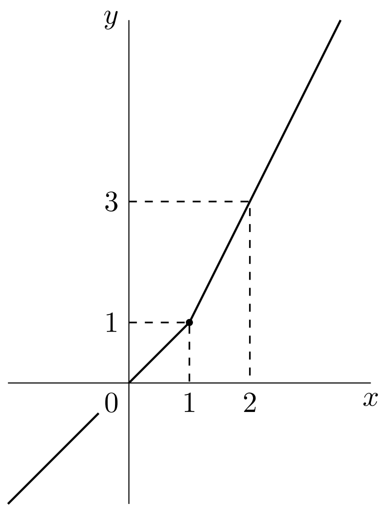
6.2.2 (a) 0.0125 (exact value 0.012985 to six decimal places), (b) -0.0016 (-0.001553).

6.2.3 Let $f(x) = x^5 + 3x - 12$. Then $f(1) = -8 < 0$ and $f(2) = 26 > 0$; the result now follows from the intermediate value theorem.

(a) Yes: since $243 > 240$, $f(1.5)$ is slightly greater than 0.

(b) No. Since x^5 and $3x-12$ increase when x increases, the curve $y = f(x)$ is upward-sloping. Hence the equation $f(x) = 0$ has at most one solution. Since we already know that the equation has a solution which is slightly less than 1.5, it has no solution which is greater than or equal to 1.5.

6.2.4



The graph can be drawn without lifting the pencil from the paper, but cannot be approximated by a straight linethrough the point $(1, 1)$.

6.3 Two useful rules

6.3.1 (a) $6x$, (b) $7x^6 - 4x^{-5}$, (c) $21x^2 - 4x + 5$, (d) $8x^3 + 7x^{-2}$, (e) $4x^5 - \frac{1}{2}x^{-4}$, (f) $-2.8x^{-5} - 9.3x^2$, (g) $3ax^2$, (h) $8ax + 6bx^{-3}$.

6.3.2 (a) $16x^3 - 6x^2$, (b) $3x^2 + 6x + 2$, (c) $\dfrac{3}{2} - \dfrac{1}{2x^2}$, (d) $2 + x^{-2}$, (e) $2x + 2a^2x^{-3}$, (f) $-\dfrac{2}{bx^3} + \dfrac{3a}{bx^4}$.

6.3.3 $f'(x) = 5x^4 + 3 \geq 3 > 0$ for all x. This confirms that the curve $y = f(x)$ is upward-sloping, which was obvious anyway.

6.4 Derivatives in economics

6.4.1 $5x - \frac{1}{2}x^2$, $5 - x$.

6.4.2 $2x + 3$.

6.4.3 2, $\frac{3}{2}$, $\frac{4}{3}$.

6.4.4 2, 2, 2. The elasticity is constant.

6.4.5 $0.2 + 0.1Y$.

7 METHODS OF DIFFERENTIATION

7.1 The product and quotient rules

7.1.1 (a) $(4x^3 - 6x)(5x + 1) + 5(x^4 - 3x^2)$,

(b) $(18x^2 + 1)(x^6 - 3x^4 - 2) + (6x^5 - 12x^3)(6x^3 + x)$,

(c) $mx^{m-1}(5x^2 + 2x^{-n}) + (10x - 2nx^{-n-1})(x^m + 8)$,

(d) $(16x^3 + 4x)(x^{n+1} + 5x^n) + ([n + 1]x^n + 5nx^{n-1})(4x^4 + 2x^2 - 1)$.

7.1.2 (a) $\dfrac{4}{(1-2x)^2}$, (b) $\dfrac{2x - 6x^2 - 2x^4}{(2x^3+1)^2}$,

(c) $-\dfrac{2ax + bx^2 + 8ax^3 + 6bx^4}{0.3(x^2 + 2x^4)^2}$, (d) $\dfrac{3b - 2ax - 3x^2}{(x^2+b)^2}$.

7.1.3 $-5(1 + 4t)^{-2}$.

7.2 The composite function rule

7.2.1 $(x^4 - 2)^3 + 1$, $(x^3 + 1)^4 - 2$; $12x^3(x^4 - 2)^2$, $12x^2(x^3 + 1)^3$.

7.2.2 (a) $30(3x - 7)^9$, (b) $15x^2(x^3 + 1)^4$,

(c) $\dfrac{2}{(4x+9)^{1/2}}$, (d) $\dfrac{4x^5}{(x^6-1)^{1/3}}$, (e) $\dfrac{3(x^{1/4}+5)^5}{2x^{3/4}}$,

(f) $\dfrac{4x^3 - 6x + 5}{4(x^4 - 3x^2 + 5x + 1)^{3/4}}$, (g) $-\dfrac{14x}{(x^2-1)^8}$, (h) $-\dfrac{20}{\sqrt{x}\,(\sqrt{x}+2)^6}$.

7.2.3 (a) $2x(x^3 + 1)^5 + 15x^2(x^3 + 1)^4(x^2 - 1)$.

(b) $(30x^4 - \frac{44}{3}x^{13/3} - \frac{2}{3}x^{-2/3})(x^5 - 2)^{-4}$.

7.2.4 $-\frac{1}{4}$.

7.2.5 $\frac{6}{5}(4 + 3t)^{-3/5}$.

7.3 Monotonic functions

7.3.1 (a) monotonic (↑); (b) monotonic (↑); (c) neither; (d) monotonic (↓); (e) neither; (f) monotonic (↓); (g) weakly monotonic (↑); (h) monotonic (↑).
[↑ means increasing, ↓ decreasing.]

7.3.2 (a) none, (b) $a \geq 20^{1/4}$, (c) $a \geq 40^{1/3}$.

7.3.3

$$\text{RHS} = \frac{1 - x - (1 + x)}{(1 + x)(1 - x)} = -\frac{2x}{1 - x^2} = \text{LHS}.$$

As x increases, $1 + x$ rises and $1 - x$ falls; hence $(1 + x)^{-1}$ falls and $(1 - x)^{-1}$ rises, so the difference between them falls.

7.4 Inverse functions

7.4.1 (a) $\frac{1}{3}(x+1)^{-1}$, (b) $(45x^4+3x^2+4)^{-1}$, (d) $-\frac{1}{6}x^{-5}$, (f) $-\frac{1}{2}(x-1)^2$;
(h) 1 if $x<0$, $\frac{1}{3}$ if $x>0$, not differentiable at $x=0$.

7.4.2 Demand function: $x=\sqrt{3-p}$ $(0\le p\le 3)$.

Using equation (7.5) in the text, the elasticity is

$$\frac{3-x^2}{x\times(-2x)}=-\frac{p}{2(3-p)}.$$

7.4.3 $p=\dfrac{10}{(x+1)^{1/3}},\ \dfrac{10(2x+3)}{3(x+1)^{4/3}}.$

7.4.4 Letting $y=f(x)$, the inverse function is

$$x=\begin{cases}\left(1-\sqrt{1+y^2}\right)y^{-1}, & \text{if } y\neq 0;\\ 0, & \text{if } y=0.\end{cases}$$

The negative square root of $1+y^2$ is chosen, so as to ensure that x and y have opposite signs.

$$\frac{dx}{dy}=-\frac{1}{\sqrt{1+y^2}}-\frac{1-\sqrt{1+y^2}}{y^2}=-\frac{\left(x^2-1\right)^2}{2\left(x^2+1\right)},$$

where the first expression (call it a) is found by direct calculation and the second (b) by the inverse function rule. To verify that $a=b$, note that

$$-a=\frac{1}{1-xy}+\frac{x}{y}=\frac{x^2-1}{x^2-1-2x^2}+\frac{x^2-1}{2}=\frac{x^2-1}{2}\left(1-\frac{2}{x^2+1}\right)=\frac{\left(x^2-1\right)^2}{2(x^2+1)}=-b.$$

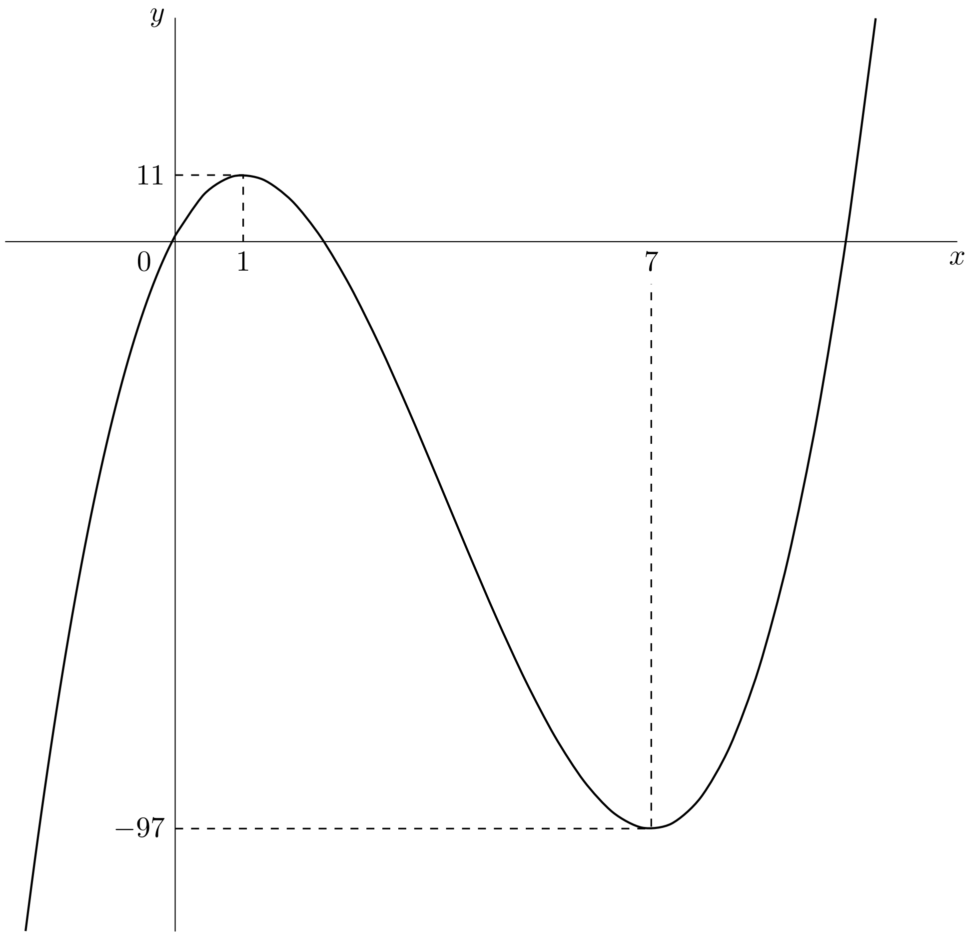
8 MAXIMA AND MINIMA

8.1 Critical points

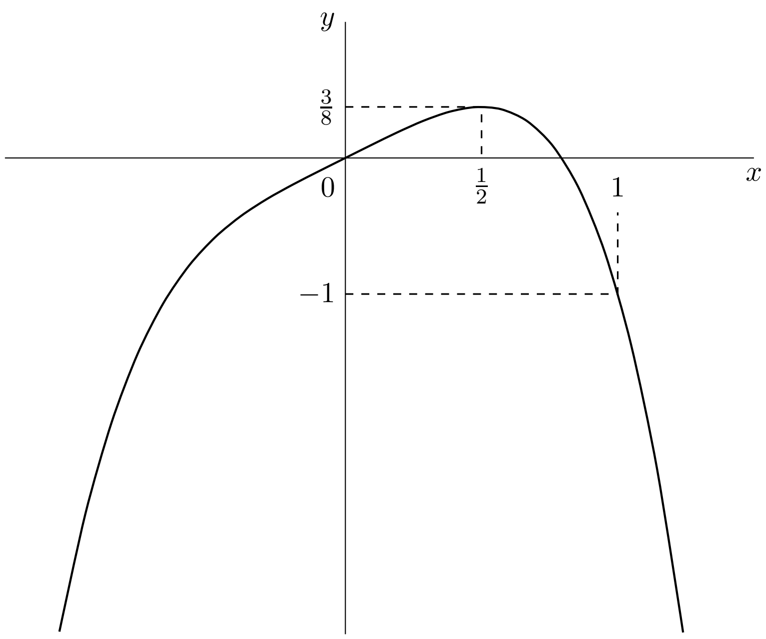
8.1.1 If $y=x^2$ then $dy/dx=2x$. Hence $dy/dx=0$ if $x=0$, $dy/dx<0$ if $x=0-$, $dy/dx>0$ if $x=0+$. Hence the graph has a minimum point at the origin; the same is true for $y=x^n$, where n is any even positive integer.

If $y=x^3$ then $dy/dx=3x^2$. Hence $dy/dx=0$ if $x=0$, $dy/dx>0$ if $x=0-$, $dy/dx>0$ if $x=0+$. Hence the graph has a critical point of inflexion at the origin; the same is true for $y=x^n$, where n is any odd integer greater than 1.

8.1.2 $(1, 11)$ is a maximum point, $(7, -97)$ is a minimum point.

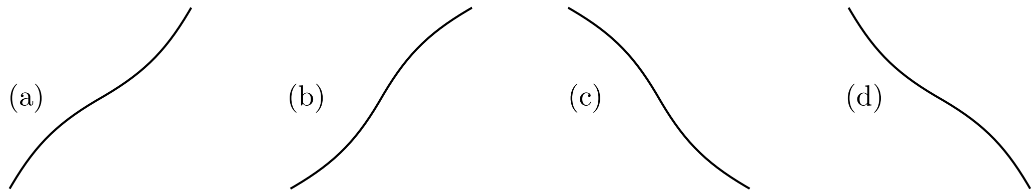


8.1.3 $3/8$ is a maximum value.



8.2 The second derivative

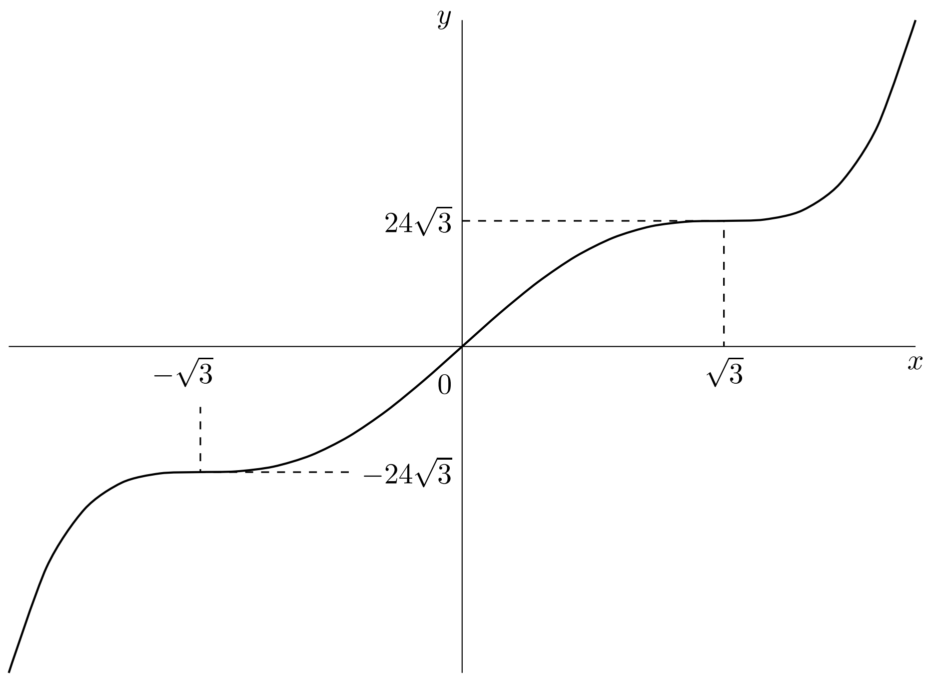
8.2.1



8.2.2 $(4, -43)$ is a point of inflexion.

8.2.3 No points of inflexion.

8.2.3 Critical points of inflexion at $(\sqrt{3}, 24\sqrt{3})$ and $(-\sqrt{3}, -24\sqrt{3})$. Non-critical point of inflexion at $(0, 0)$.



Graph of inverse function is the same but with axes reversed.

8.3 Optimisation

8.3.1 $(1, 11)$ is a local maximum; $(7, -97)$ is a local minimum. There are no global maxima and no global minima.

When $x \geq 0$ is imposed, $(1, 11)$ is a local maximum, while $(0, 1)$ and $(7, -97)$ are local minima. There are no global maxima but $(7, -97)$ is now the global minimum.

8.3.2 (a) $(1, -11)$ is a local minimum; $(7, 97)$ is a local maximum. There are no global maxima and no global minima.

(b) $(1, -11)$ is a local minimum; $(0, -1)$ and $(7, 97)$ are local maxima. There are no global minima; $(7, 97)$ is the global maximum.

(c) $(1, 11^5)$ is a local maximum; $(7, -97^5)$ is a local minimum; there are no global maxima and no global minima. When $x \geq 0$ is imposed, $(1, 11^5)$ is a local maximum, while $(0, 1)$ and $(7, -97^5)$ are local minima. There are no global maxima; $(0, -97^5)$ is the global minimum.

8.3.3 (a) Marginal cost is $x^2 - 12x + 160$. By completing the square, $\mathrm{MC} = (x-6)^2 + 124 \geq 124 > 0$.

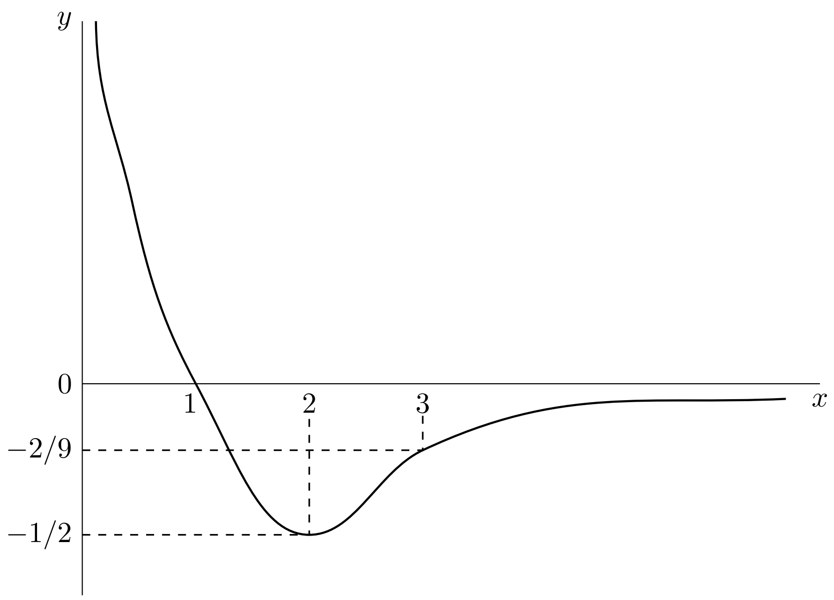
(b) 8.

(24,24)

8.3.4 $(a, f(a))$ is the global maximum.

8.3.5 $\dfrac{p}{\mathrm{MC}} = \dfrac{\eta}{1+\eta}$ (or $\dfrac{\varepsilon}{\varepsilon - 1}$, where $\varepsilon = |\eta|$).

8.3.6 $y = (1-x)x^{-2}$ and is therefore the product of something which is close to 1 if x is small and something which is very large if x is small. The only critical point is $(2, -1/2)$; $dy/dx < 0$ if $x < 2$, $dy/dx > 0$ if $x > 2$. The (non-critical) point of inflexion is at $(3, -2/9)$. The asymptotes are the axes.



8.4 Convexity and concavity

8.4.1 (a) and (d) are convex; (e) and (h) are concave.

(b) is neither convex nor concave because it has a critical point of inflexion at $(0, 1)$; this is easily shown using the method of Section 8.1.

(c) is neither convex nor concave because the second derivative has the same sign as x.

(f) is neither convex nor concave because it has a local minimum at the origin and local maxima where $x = \pm\frac{1}{2}$.

(g) is neither convex nor concave because it has a local minimum at the origin and local maxima where $x = \pm\sqrt{2.5}$.

8.4.2 $(2, 11)$.

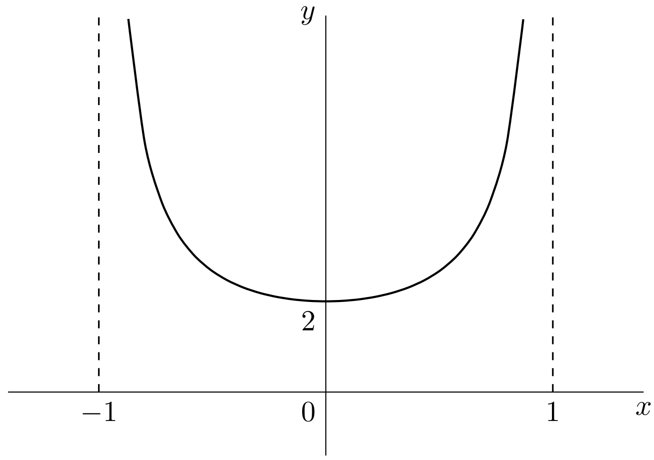
8.4.3 $(0,3)$, $(2,11)$.

(a) Local minimum point is $(1,8)$, global maximum is $(2,11)$.

(b) No local minimum, global maximum is $(2,11)$.

(c) No local minimum, global maximum is $(3,0)$.

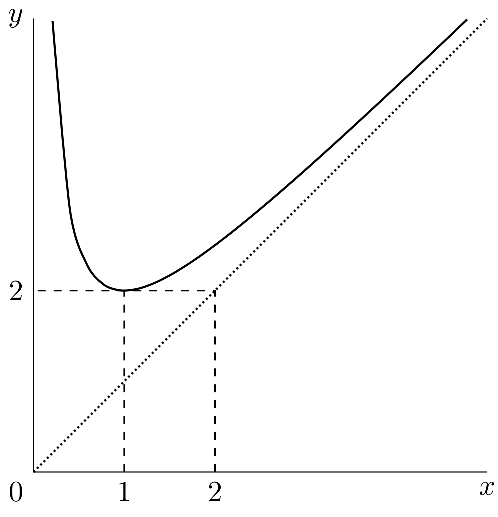
8.4.4 Assume $-1 < x < 1$. It is easiest to begin by noticing that $y = (1+x)^{-1} + (1-x)^{-1}$. Thus $d^2y/dx^2 = 2(1+x)^{-3} + 2(1-x)^{-3} > 0$, so the function is convex. The global minimum is at $(0,2)$.



8.4.5 Profit Π is $50 - 2x - \dfrac{50}{1+x}$. $\dfrac{d^2\Pi}{dx^2} = -100(1+x)^{-3} < 0$, so Π is concave in x.
At optimum, $x = 4$ and $p = 10$.

8.4.6 Assume $x > 0$ throughout. $d^2y/dx^2 = 2x^{-3} > 0$, so the function is convex. Global minimum at $(1,2)$. Asymptotes are y–axis and $y = x$.

$x \geq \frac{1}{2}$: global minimum at $(1,2)$, local maximum at $(\frac{1}{2}, \frac{5}{2})$.
$x \geq 2$: global minimum at $(2, \frac{5}{2})$, no local maximum.
$0 < x \leq 2$: global minimum at $(1,2)$, local maximum at $(2, \frac{5}{2})$.



9 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

9.1 The exponential function

9.1.2 (a) £740.12, (b) £745.42, (c) £745.91. The more frequent the compounding, the greater is the value.

With the shorter time period, we have (a) £540.80, (b) £552.49, (c) £552.58.

9.1.3 (a) £285.19, (b) £282.16, (c) £281.88. The more frequent the discounting, the smaller is the value.

9.1.4 (a) $2e^{2x} - 12e^{-4x}$, (b) $(2x+1)e^{2x}$, (c) $\dfrac{1+(1-x)e^x}{1+e^x)^2}$, (d) $12e^{3x}(e^{3x}-1)^3$.

9.2 Natural logarithms

9.2.1 $\ln(1+s)$.

9.2.2 (a) $\dfrac{1}{x}$, (b) $\dfrac{4x^3}{x^4+1}$, (c) $x+\ln x$, (d) $x^x(1+\ln x)$, (e) $\dfrac{e^x}{e^x+1}$, (f) $\exp(e^x+2)$, (g) $\dfrac{2(1-x^4)}{x(1+x^4)}$.

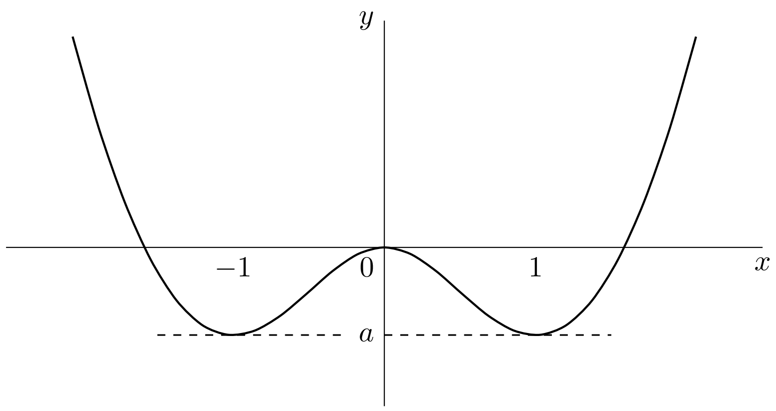
9.2.3 (a) Let $y = c^x$, so that $\ln y = x \ln c$. Differentiating, and using the composite function rule on the left-hand side,

$$\frac{1}{y}\frac{dy}{dx} = \ln c, \quad \text{so} \quad \frac{dy}{dx} = y \ln c = c^x \ln c.$$

(b) Let $y = \exp(-\frac{1}{2}x^2)$, so that $\ln y = -\frac{1}{2}x^2$. Differentiating,

$$\frac{1}{y}\frac{dy}{dx} = -x, \quad \text{so} \quad \frac{dy}{dx} = -xy = -x\exp(-\tfrac{1}{2}x^2).$$

9.2.4 Critical points are $(0,0)$, $(1,a)$ and $(-1,a)$, where $a = 1 - 2\ln 2 = -0.3863$ to four decimal places. The origin is a local maximum, the other two points are local minima.



9.2.5 $dy/dx = (1-x)e^{-x}$, which always has the same sign as $1-x$. Therefore, the only critical point is $(1, 1/e)$ and this is the global maximum.

$d^2y/dx^2 = (x-2)e^{-x}$, which always has the same sign as $x-2$. So there is one point of inflexion at $(2, 2/e^2)$, and the function is (a) convex for all $x > 2$, (b) concave for all $x < 2$.

9.2.6 $(Aap^a + Bbp^b)\big/(Ap^a + Bp^b)$, which $\to b$ when $p \to 0$ and $\to a$ when $p \to \infty$.

9.3 Time in economics

9.3.1 (a) $(b+2ct)/(a+bt+ct^2)$, (b) $(b+c+2ct)/(a+bt+ct^2)$.

9.3.2 $\dfrac{b}{a+bt}$, m, $\dfrac{b}{a+bt}-m$.

9.3.3 (a) Apply 'the economist's favourite approximation' where x is the rate of growth in discrete time, as usually defined.

(b) Since $\ln(y/z) = \ln y - \ln z$,

$$\ln(C_{t+1}/L_{t+1}) - \ln(C_t/L_t) = (\ln C_{t+1} - \ln C_t) - (\ln L_{t+1} - \ln L_t).$$

(c) In the notation of the text,

$$\frac{C_{t+1}}{L_{t+1}} = \frac{1+g_t}{1+h_t} \times \frac{C_t}{L_t}.$$

If g_t and h_t are small then $(1+g_t)/(1+h_t) \approx 1$, so C/L also grows slowly. But then

$$\begin{aligned} p_t &\approx \ln(C_{t+1}/L_{t+1}) - \ln(C_t/L_t) \quad \text{by (a)} \\ &= (\ln C_{t+1} - \ln C_t) - (\ln L_{t+1} - \ln L_t) \quad \text{by (b)} \\ &\approx g_t - h_t \quad \text{by (a) again.} \end{aligned}$$

9.3.4 (a) r, (b) $e^r - 1$, (c) r.

9.3.5 (a) $\ln c$, (b) $c-1$, (c) $\ln c$.

9.3.6 (a) yes, (b) no.

10 APPROXIMATIONS

10.1 Linear approximations and Newton's method

10.1.1 (a) $y = 12x - 16$, (b) $y = 27x - 54$.

x	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
$12x-16$	9.2	10.4	11.6	12.8	14.0	15.2	16.4	17.6	18.8
$27x-54$	2.7	5.4	8.1	10.8	13.5	16.2	18.9	21.6	24.3
x^3	9.3	10.7	12.2	13.8	15.6	17.6	19.7	22.0	24.4

As x increases from 2 to 3, the tangent at $x=2$ becomes a worse approximation to the true function, while the tangent at $x=3$ becomes a better one.

10.1.2 1.414. With the other starting point, the method leads to -1.414, an approximation to $-\sqrt{2}$.

10.1.3 $\frac{2}{3}$, 0.678.

10.2 The mean value theorem

10.2.1 2.

10.2.2 $\pm 1/\sqrt{3}$.

10.2.3 (a) 0.64, (b) $\dfrac{\ln(1+r)-\ln(1+s)}{r-s}$, (c) 0.5.

10.3 Quadratic approximations and Taylor's theorem

10.3.1 (a) $L(x) = 4\ln 2 + (4\ln 2 + 2)(x-2)$,
$Q(x) = 4\ln 2 + (4\ln 2 + 2)(x-2) + (\ln 2 + \frac{3}{2})(x-2)^2$.

x	1.80	1.95	2.02	2.10	2.25
$L(x)$	1.8181	2.5340	2.8680	3.2498	3.9657
$Q(x)$	1.9058	2.5394	2.8689	3.2718	4.1028
$f(x)$	1.9044	2.5394	2.8689	3.2719	4.1053

(b) $C(x) = \frac{1}{2}(x-1)(3x-1) + \frac{1}{3}(x-1)^3$.

x	0.80	0.95	1.02	1.10	1.25
$C(x)$	-0.1427	-0.0463	0.0206	0.1153	0.3490

10.3.2 $4x^3 - 9x^2 + 16x + 5$, $12x^2 - 18x + 16$, $24x - 18$, 24; all derivatives higher than the fourth are zero.

5, 16, -18, 24; the coefficients of x, x^2, x^3, x^4 are these values divided by 1!, 2!, 3!, 4! respectively. The constant term is $f(0) =$.

For a polynomial of degree n, the nth order Taylor 'approximation' is the exact function.

10.4 Taylor series

10.4.1 (a) 2.718 and 0.368, taking 7 terms in each case; (b) 0.1, 0.095, 0.0953.

True value is 0.0953 to 4 decimal places. The accuracy of the approximation is particularly good because the terms of the expansion alternate in sign.

10.4.2 $\dfrac{e^x - 1}{x} = 1 + x\left[\dfrac{1}{2!} + \dfrac{x}{3!} + \dfrac{x^2}{4!} + \dots\right]$.
The expression in square brackets approaches $\frac{1}{2}$ as $x \to 0$.

10.4.3 (a) From the series for e^x ,

$$\frac{e^x}{x} = \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots$$

As $x \to \infty$, $x^{-1} + 1 \to 1$ and the terms $x/2!$, $x^2/3!$, $x^3/4!$, … all $\to \infty$; hence $e^x/x \to \infty$, so $xe^{-x} \to 0$.

(b) A similar argument to (a) shows that as $x \to \infty$, $e^x/x^2 \to \infty$, so $x^2e^{-x} \to 0$. Again by a similar argument, $x^n e^{-x} \to 0$ as $x \to \infty$ for any positive integer n; even more generally, $x^a e^{-x} \to 0$ as $x \to \infty$ for any positive real number a.

(c) Let $y = -\ln x$. Then $\ln x = -y$ and $x = e^{-y}$. Further, $y \to \infty$ as $x \downarrow 0$. Thus

$$\lim_{x\downarrow 0} x \ln x = -\lim_{y\to\infty} y e^{-y} = 0,$$

by (a). Since

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

for all $x > 0$, $x^x \to e^0 = 1$ as $x \downarrow 0$.

10.4.4 $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots$, valid for $|x| < 1$. $\ln 3 = 1.099$, by taking 5 terms.

10.4.5 (a) $1 + 2x + 2x^2 + \ldots + \dfrac{2^n}{n!}x^n + \ldots$, valid for all x.

(b) $3x - \dfrac{9}{2}x^2 + 9x^3 - \ldots - \dfrac{(-3)^n}{n}x^n + \ldots$, valid for $-\frac{1}{3} < x \leq \frac{1}{3}$.

(c) $1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \ldots + \dfrac{\frac{1}{2}(\frac{1}{2}-1)\ldots(\frac{1}{2}-[n-1])x^n}{n!} + \ldots$, valid for $|x| < 1$.

(d) $1 + 5x + 25x^2 + \ldots + 5^n x^n + \ldots$, valid for $|x| < \frac{1}{5}$.

10.4.6 (a) $1 + 3x + 3x^2 + x^3$, (b) $1 + 4x + 6x^2 + 4x^3 + x^4$, (c) $1 - 6x + 12x^2 - 8x^3$, (d) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

11 MATRIX ALGEBRA

11.1 Vectors

11.1.1 Components of $\mathbf{a} + \mathbf{b}$ are the sums of Anne's and Bill's weekly expenditures on food, clothing and housing; components of $52\mathbf{a}$ are Anne's annual expenditures on food, clothing and housing.

11.1.2 The vectors are

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

$\mathbf{a} + \mathbf{b}$ is at the fourth vertex of the parallelogram of which the lines from the origin to $\mathbf{a}$ and to $\mathbf{b}$ form two sides. $3\mathbf{a}$ is at the end of the line obtained by stretching the line from the origin to $\mathbf{a}$ by a factor of 3. $-4\mathbf{b}$ is the reflexion of the end of the line obtained by stretching the line from the origin to $\mathbf{b}$ by a factor of 4. $3\mathbf{a} - 4\mathbf{b}$ is at the fourth vertex of the parallelogram of which the lines from the origin to $3\mathbf{a}$ and $-4\mathbf{b}$ form two sides.

11.1.3 $p = -\frac{1}{2}$, $q = -5$, $r = 1$.

11.1.4 In each part, denote the vectors by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(a) Yes: $\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$. (b) Linearly independent.

(c) Yes: $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \mathbf{0}$. (d) Yes: $0\mathbf{a} + 1\mathbf{b} + 0\mathbf{c} = \mathbf{0}$.

(e) Linearly independent. (f) Yes: $2\mathbf{a} + \mathbf{b} - 2\mathbf{c} = \mathbf{0}$.

11.2 Matrices

11.2.1

$$\begin{bmatrix} 6 & 0 & 7 \\ -1 & 7 & 4 \end{bmatrix}, \quad \begin{bmatrix} 10 & -5 & 0 \\ 5 & 15 & 25 \end{bmatrix}, \quad \begin{bmatrix} -8 & -2 & -14 \\ 4 & -8 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -7 & -14 \\ 9 & 7 & 27 \end{bmatrix}.$$

11.2.2 $\mathbf{wa}$ is Anne's total expenditure; $\mathbf{w}(\mathbf{a}-\mathbf{b})$ is the difference between Anne's total expenditure and Bill's.

11.2.3 Answers to (a), (b), (c) are respectively

$$\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}, \quad \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}, \quad \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}.$$

In (a), $\mathbf{A}$ maps $\mathbf{x}$ into its reflexion in the origin. In (b), $\mathbf{A}$ maps $\mathbf{x}$ into the end of the line obtained by stretching the line from the origin to $\mathbf{x}$ by a factor of 3. In (c), $\mathbf{A}$ maps $\mathbf{x}$ into the reflexion in the origin of the end of the line obtained by stretching the line from the origin to $\mathbf{x}$ by a factor of 2.

11.3 Matrix multiplication

11.3.1

$$\begin{bmatrix} 5 & 2 & 7 \\ 2 & 0 & -2 \\ -3 & -2 & -9 \\ 1 & -2 & -13 \end{bmatrix}, \quad \begin{bmatrix} 6 & -4 & 1 & 1 \\ 8 & -8 & 6 & 2 \\ 2 & -4 & 5 & 1 \\ 18 & -20 & 17 & 5 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}.$$

11.3.3 $\begin{bmatrix} \mathbf{A}_1\mathbf{B}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_2\mathbf{B}_2 \end{bmatrix}$

11.4 Square matrices

11.4.1 Any square matrices of the same order which satisfy $\mathbf{AB} \neq \mathbf{BA}$ will do.

11.4.2 Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can ensure that the off-diagonal entries of $\mathbf{A}^2$ are zero by letting $d = -a$, in which case both diagonal entries are equal to $a^2 + bc$. Thus an answer to (a) is given by setting $a = b = 1$, $c = d = -1$, and an answer to (b) is given by setting $a = b = 1$, $c = -2$, $d = -1$.

11.4.3 Any matrix of the form

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad \begin{bmatrix} a & 0 \\ a+d & d \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & a+d \\ 0 & d \end{bmatrix}.$$

11.4.4

$$\begin{bmatrix} 3a & -a+2b & 6a+b-5c \\ 0 & 2d & d-5e \\ 0 & 0 & -5f \end{bmatrix}$$

The product of two upper [lower] triangular matrices of the same order is upper [lower] triangular.

12 SYSTEMS OF LINEAR EQUATIONS

12.1 Echelon matrices

12.1.1 Examples of matrices (a) and (b) are respectively

$$\begin{bmatrix} \star & \cdot & 0 & \cdot \\ 0 & \star & 0 & \cdot \\ 0 & 0 & 0 & \cdot \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & \star & \cdot \\ 0 & 0 & \star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\star$ denotes a non-zero number and $\cdot$ denotes a number which may be either zero or non-zero.

In (a), x_3 does not occur in the system of equations. In (b), x_1 does not occur in the system of equations.

12.1.2 (a) $x_1 = -\frac{1}{2} + \frac{5}{6}\lambda - \frac{1}{3}\mu$, $x_2 = \frac{1}{2}(1+\lambda)$, $x_3 = \lambda$, $x_4 = \mu$.

(b) $x_1 = -2$, $x_2 = 2$, $x_3 = -5$, $x_4 = -2$.

(c) No solution.

(d) No solution.

(e) $x_1 = \frac{7}{3} - \frac{1}{2}\lambda - \frac{3}{2}\mu$, $x_2 = \lambda$, $x_3 = \mu$, $x_4 = -4$.

12.2 More on Gaussian elimination

12.2.1 (a) $x_1 = -\frac{3}{2}\lambda - \mu - \frac{1}{10}$, $x_2 = \lambda - \mu - \frac{1}{5}$, $x_3 = \lambda$, $x_4 = \mu$.

(b) No solution.

(c) $x_1 = \lambda$, $x_2 = \frac{1}{3} - 2\lambda$, $x_3 = \lambda$.

12.2.2 $x_1 = -3\lambda$, $x_2 = \lambda$, $x_3 = 0$.

After each elimination step, the right-hand sides stay at 0. Hence they can be omitted.

12.2.3 The left-hand sides of the systems are the same.
(a) $x_1 = 2$, $x_2 = -1$, $x_3 = 5$. (b) $x_1 = 20$, $x_2 = -5$, $x_3 = 17$.

12.3 Inverting a matrix

12.3.1 If $\mathbf{A}$ has a row of zeros, reduction to echelon form will lead to a Type 4 matrix. If $\mathbf{A}$ has a column of zeros, there is a vector $\mathbf{x}$ with one component equal to 1 and all others equal to zero which satisfies the equation $\mathbf{Ax} = \mathbf{0}$; hence $\mathbf{A}$ is singular.

12.3.2 Nonsingular, singular, singular, nonsingular.

12.3.3

$$\begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix}, \quad \begin{bmatrix} -11/8 & -1/8 & 1/2 \\ -1/4 & 1/4 & 0 \\ 5/8 & -1/8 & -1/10 \end{bmatrix}, \quad \begin{bmatrix} 3 & -4 & -9 \\ 3 & -4 & -8 \\ -2 & 3 & 6 \end{bmatrix}.$$

(a) $x_1 = 4$, $x_2 = 7$. (b) $x_1 = 1$, $x_2 = 3$, $x_3 = -1$.

12.3.4 $\mathbf{A}^{-1} - \mathbf{A}$, $3\mathbf{B} + 4\mathbf{A}$, $\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1}$.

12.3.5 (a) If $\mathbf{A}$ were invertible we could pre-multiply $\mathbf{AB} = \mathbf{O}$ by $\mathbf{A}^{-1}$; this gives $\mathbf{B} = \mathbf{O}$, contrary to hypothesis. If $\mathbf{B}$ were invertible we could post-multiply $\mathbf{AB} = \mathbf{O}$ by $\mathbf{B}^{-1}$; this gives $\mathbf{A} = \mathbf{O}$, contrary to hypothesis.

(b) $\mathbf{I} + \mathbf{A}$ is invertible with inverse $\mathbf{I} - \mathbf{A}$ and vice versa.

12.3.6 $\dfrac{1}{t^2-1}\begin{bmatrix} t & -1 \\ -1 & t \end{bmatrix}$.

If $t = \pm 1$, $\mathbf{A}$ is singular. To see why, let $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If $t = 1$, $\mathbf{Ay} = \mathbf{0}$; if $t = -1$, $\mathbf{Az} = \mathbf{0}$.

12.4 Linear dependence and rank

12.4.1 (a) $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. The columns of $\mathbf{A}$ are linearly independent.

(b) $x_1 = 4\lambda$, $x_2 = -4\lambda$, $x_3 = \lambda$ for any λ. The columns of $\mathbf{A}$ are linearly dependent. For example, $\alpha_1 = 4$, $\alpha_2 = -4$, $\alpha_3 = 1$.

12.4.2 (a) 2, (b) 1, (c) 2.

12.4.3 (a) 2, (b) 3, (c) 2, (d) 2, (e) 3, (f) 2.

12.4.4 (a) If $\mathbf{Bx} = \mathbf{0}$ for some non-zero vector $\mathbf{x}$, then $\mathbf{ABx} = \mathbf{0}$.

(b) Yes: apply (a), with $\mathbf{B}$ replaced by the relevant submatrix.

(c) Choose the corresponding columns. For example, if columns 2, 4, 5 and 9 of $\mathbf{AB}$ are linearly independent, so are columns 2, 4, 5 and 9 of $\mathbf{B}$.

(d) rank of $\mathbf{B} \geq$ rank of $\mathbf{AB}$.

13 DETERMINANTS AND QUADRATIC FORMS

13.1 Determinants

13.1.1 -40, $1 + abc$, -16.

13.1.2 All values except 0 and -3.

13.1.3 **D5** Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be 3×3 matrices. Let the second row of $\mathbf{A}$ be $[u\ v\ w]$, the second row of $\mathbf{B}$ $[u'\ v'\ w']$ and the second row of $\mathbf{C}$ $[u+u'\ \ v+v'\ \ w+w']$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be otherwise identical; then they all have the same cofactors for the second row, say $\tilde{u}$, $\tilde{v}$, $\tilde{w}$. Expanding determinants by their second row,

$$\begin{aligned} \det \mathbf{C} &= (u+u')\tilde{u} + (v+u')\tilde{v} + (w+w')\tilde{w} \\ &= (u\tilde{u} + v\tilde{v} + w\tilde{w}) + (u'\tilde{u} + v'\tilde{v} + w'\tilde{w}) \\ &= \det \mathbf{A} + \det \mathbf{B}. \end{aligned}$$

The same argument applies when 'second' is replaced by 'first' or 'third'.

D6 Let $\mathbf{A}$ and $\mathbf{B}$ be 3×3 matrices. Let the second row of $\mathbf{A}$ be $[u\ \ v\ \ w]$, and let the second row of $\mathbf{B}$ be $[\lambda u\ \ \lambda v\ \ \lambda w]$. Let $\mathbf{A}$ and $\mathbf{B}$ be otherwise identical; then they both have the same cofactors for the second row, say $\tilde{u}$, $\tilde{v}$, $\tilde{w}$. Expanding determinants by their second row,

$$\det \mathbf{B} = \lambda u\tilde{u} + \lambda v\tilde{v} + \lambda w\tilde{w} = \lambda(u\tilde{u} + v\tilde{v} + w\tilde{w}) = \lambda \det \mathbf{A}.$$

The same argument applies when 'second' is replaced by 'first' or 'third'.

13.2 Transposition

13.2.1

$$\begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & -1 \\ 1 & -4 & 1 \end{bmatrix}$$

13.2.2 (a) Direct calculation.

(b) Taking determinants in the equation $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$, using the facts that $\det \mathbf{A}^{\mathrm{T}} = \det \mathbf{A}$ and $\det \mathbf{I} = 1$, gives $(\det \mathbf{A})^2 = 1$. The first matrix in (a) has determinant 1 and the second has determinant -1.

13.2.3 The determinant is -1 and the adjoint is

$$\begin{bmatrix} -3 & 4 & 9 \\ -3 & 4 & 8 \\ 2 & -3 & -6 \end{bmatrix}.$$

13.2.4 Letting $p = (1 + abc)^{-1}$ we have $x = (1 - b + ab)p$, $y = (1 - c + bc)p$, $z = (1 - a + ca)p$.

13.2.5 The three-equation system may be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 0 \\ -c_1 & 1 & c_1 \\ -t_1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I + G \\ c_0 \\ t_0 \end{bmatrix}.$$

Solution by Cramer's rule gives the same answers as for Problem 2.1: see "Solutions to Problems".

13.3 Inner products

13.3.1 For example

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

13.3.2 Apply **L3**.

13.3.3

$$\|\lambda \mathbf{a}\|^2 = (\lambda a_1)^2 + \ldots + (\lambda a_n)^2 = \lambda^2(a_1^2 + \ldots + a_n^2) = \lambda^2 \|\mathbf{a}\|^2.$$

Now take square roots.

13.3.2 Apply **L3**. Direct calculation, using the fact that $\mathbf{b}^{\mathrm{T}}\mathbf{a} = \mathbf{a}^{\mathrm{T}}\mathbf{b}$.

13.3.5 $|\mathbf{x}^{\mathrm{T}}\mathbf{y}| \leq 1$ by **L4**, so $-1 \leq \mathbf{x}^{\mathrm{T}}\mathbf{y} \leq 1$. Examples:

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; (b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$; (c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; (d) $\begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$.

13.4 Quadratic forms and symmetric matrices

13.4.1 If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ then $\mathbf{aa}^{\mathrm{T}} = \begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}$.
If $\mathbf{a}$ is an n–vector, $\mathbf{aa}^{\mathrm{T}}$ is a symmetric $n \times n$ matrix.

13.4.2

$$\begin{bmatrix} \sum_{i=1}^{n} x_{1i}^2 & \sum_{i=1}^{n} x_{1i}x_{2i} \\ \sum_{i=1}^{n} x_{1i}x_{2i} & \sum_{i=1}^{n} x_{2i}^2 \end{bmatrix}$$

13.4.3 Let $\mathbf{C} = \mathbf{B}^{\mathrm{T}}\mathbf{AB}$. Then $\mathbf{C}^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}})^{\mathrm{T}}$. $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ by assumption and $(\mathbf{B}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{B}$ always, so $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$ as required.

13.4.4 $q(x_1, x_2, x_3) = x_1^2 + (x_2 - \frac{1}{2}x_3)^2 + \frac{3}{4}x_3^2 \geq 0$. If $q(x_1, x_2, x_3) = 0$ then x_1, $x_2 - \frac{1}{2}x_3$ and x_3 are all 0, so x_2 is also 0. Hence q is positive definite.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

13.4.5 The matrix $\mathbf{A}$ is $\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$, which has positive diagonal entries and determinant 2.

13.4.6 (a) $t > \sqrt{2}$, (b) $t = \sqrt{2}$, (c) $t < -\sqrt{2}$, (d) $t = -\sqrt{2}$, (e) $-\sqrt{2} < t < \sqrt{2}$.

13.4.7 Positive definite, indefinite, negative semidefinite.

14 FUNCTIONS OF SEVERAL VARIABLES

14.1 Partial derivatives

14.1.1 (a) $\begin{bmatrix} 3 \\ 12y^2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 24y \end{bmatrix}$.

(b) $\begin{bmatrix} 3x^2 \ln y + 12xy^3 + 2e^{2x}y \\ x^3/y + 18x^2y^2 + e^{2x} \end{bmatrix}$,

$$\begin{bmatrix} 6x \ln y + 12y^3 + 4e^{2x}y & 3x^2/y + 36xy^2 + 2e^{2x} \\ 3x^2/y + 36xy^2 + 2e^{2x} & -x^3/y^2 + 36x^2y \end{bmatrix}.$$

(c) $-(x^2 + 4y^2)^{-3/2} \begin{bmatrix} x \\ 4y \end{bmatrix}$, $\quad 2(x^2 + 4y^2)^{-5/2} \begin{bmatrix} x^2 - 2y^2 & 6xy \\ 6xy & -2x^2 + 16y^2 \end{bmatrix}$.

(d) $\begin{bmatrix} (1 - 2x - 8y)e^{-2x} + e^{3y} \\ 4e^{-2x} + (4 - 3x - 12y)e^{-3y} \end{bmatrix}$,

$$\begin{bmatrix} 4(-1 + x + 4y)e^{-2x} & -8e^{-2x} - 3e^{-3y} \\ -8e^{-2x} - 3e^{-3y} & 3(-8 + 3x + 12y)e^{-3y} \end{bmatrix}.$$

14.1.2 (a) $\begin{bmatrix} 6x \\ 10y^4 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 \\ 0 & 40y^3 \end{bmatrix}$; $\begin{bmatrix} 6 \\ 160 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 \\ 0 & -320 \end{bmatrix}$.

(b) $\begin{bmatrix} 6xy^3 + 6x^2y^2 \\ 9x^2y^2 + 4x^3y \end{bmatrix}$,

$$\begin{bmatrix} 6y^3 + 12xy^2 & 18xy^2 + 12x^2y \\ 18xy^2 + 12x^2y & 18x^2y + 4x^3 \end{bmatrix}; \quad \begin{bmatrix} -24 \\ 28 \end{bmatrix}, \quad \begin{bmatrix} 0 & 48 \\ 48 & -32 \end{bmatrix}.$$

(c) $(x^2+y^2)^{-2} \begin{bmatrix} -3x^2 + 3y^2 + 4xy \\ -2x^2 + 2y^2 - 6xy \end{bmatrix}$,

$$(x^2+y^2)^{-2} \begin{bmatrix} 6x^3 - 12x^2y - 18xy^2 + 4y^3 & 4x^3 + 18x^2y - 12xy^2 - 6y^3 \\ 4x^3 + 18x^2y - 12xy^2 - 6y^3 & -6x^3 + 12x^2y + 18xy^2 - 4y^3 \end{bmatrix};$$

$$\begin{bmatrix} 1/25 \\ 18/25 \end{bmatrix}, \quad \begin{bmatrix} -74/125 & -32/125 \\ -32/125 & 74/125 \end{bmatrix}.$$

(d) $\begin{bmatrix} \ln(1+y^2) \\ 2xy/(1+y^2) \end{bmatrix}$, $2(1+y^2)^{-2} \begin{bmatrix} 0 & y + y^3 \\ y + y^3 & x(1-y^2) \end{bmatrix}$;

$$\begin{bmatrix} \ln 5 \\ -0.8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0.8 \\ 0.8 & -0.24 \end{bmatrix}.$$

14.1.3 $-2x^2 - y^2 + 2xy + 25x + 20y$, $\;-4x + 2y + 25$, $\;-2y + 2x + 20$.

You would have needed first to find p_{X} and p_{Y} in terms of x and y.

14.1.4 -2, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$.

14.2 Approximations and the chain rule

14.2.1 (a) 0.17, (b) 0.36, (c) 0.53, (d) 0.33.

14.2.2 $6(3x^2y^4 + e^y) - 3(4x^3y^3 + xe^y)$.

14.2.3

$$\begin{bmatrix} y^2 \\ 2xy + z^2 \\ 2yz \end{bmatrix}, \quad 2\begin{bmatrix} 0 & y & 0 \\ y & x & z \\ 0 & z & y \end{bmatrix}.$$

$$\begin{bmatrix} x_2^2 \\ 2x_1x_2 + x_3^2 \\ 2x_2x_3 + x_4^2 \\ 2x_3x_4 + x_5^2 \\ 2x_4x_5 \end{bmatrix}, \quad 2\begin{bmatrix} 0 & x_2 & 0 & 0 & 0 \\ x_2 & x_1 & x_3 & 0 & 0 \\ 0 & x_3 & x_2 & x_4 & 0 \\ 0 & 0 & x_4 & x_3 & x_5 \\ 0 & 0 & 0 & x_5 & x_4 \end{bmatrix}.$$

14.2.4 (a) $-16p_1^{-3}p_2^{1/2}t^{-1/2} + 18p_1^{-2}p_2^{-1/2}t^{1/2}$, $\;2p_1^{-1/2}p_2^{-1/2}t^{-1/2} - 9p_1^{1/2}p_2^{-1/2}t^{1/2}$.

(b) $-16p_1^{-3}p_2^{1/2}ut^{-1/2} + 18p_1^{-2}p_2^{-1/2}t^{1/2}$, $\;2p_1^{-1/2}p_2^{-1/2}ut^{-1/2} - 9p_1^{1/2}p_2^{-1/2}t^{1/2}$, $\;-32p_1^{-3}p_2^{1/2}t^{1/2}$, $\;4p_1^{-1/2}p_2^{-1/2}t^{1/2}$.

What the rate of change of x_1 would be if the exchange rate were constant.

14.2.5 $\dfrac{\partial f}{\partial Y} + \dfrac{\partial f}{\partial T}\, g'(Y)$.

14.3 Production functions

14.3.1 $K/(K+L)$, $K^2/(K+L)^2$.

14.3.2 (a) Letting $Z = \delta K^\gamma + (1-\delta)L^\gamma$, we have $Q^\gamma = A^\gamma Z$, so by the composite function rule

$$\gamma Q^{\gamma-1}\frac{\partial Q}{\partial K} = A^\gamma \frac{\partial Z}{\partial K} = A^\gamma \delta\gamma K^{\gamma-1}.$$

Simplifying, $\partial Q/\partial K = \delta A^\gamma (Q/K)^{1-\gamma}$. Similarly, the marginal product of labour is $(1-\delta)A^\gamma(Q/L)^{1-\gamma}$.

(b) Let Z be as in the answer to (a) and let $W = ZK^{-\gamma} = \delta + (1-\delta)(L/K)^\gamma$. Then $Q/K = AZ^{1/\gamma}K^{-1} = AW^{1/\gamma}$. If $0 < \gamma < 1$, Q/K is an increasing function of W and W is an increasing function of L/K; if $\gamma < 0$, Q/K is a decreasing function of W and W is a decreasing function of L/K; so in both cases, Q/K is an increasing function of L/K. Using the answer to (a) and the fact that $\gamma < 1$, we see that $\partial Q/\partial K$ is also an increasing function of L/K. In particular, $\partial Q/\partial K$ is a decreasing function of K for given L, so we have diminishing returns to capital. Diminishing returns to labour is proved similarly.

14.3.3 $\alpha < 1$, $\beta < 1$.

14.3.4 $\alpha m + \beta n$.

14.4 Homogeneous functions

14.4.1 Decreasing if $\alpha + \beta < 1$, constant if $\alpha + \beta = 1$, increasing if $\alpha + \beta > 1$.

$Q = AK^\alpha L^{1-\alpha}$ $(A > 0,\ 0 < \alpha < 1)$.

14.4.2 Constant.

Decreasing if $\nu < 1$, constant if $\nu = 1$, increasing if $\nu > 1$.

14.4.3 Let $x_1 = f_1(p_1, p_2, m)$ and denote the own-price elasticity, the cross-price elasticity and the income elasticity by a, b, c respectively. Then

$$a = \frac{p_1}{x_1}\frac{\partial f_1}{\partial p_1}, \quad \text{so } p_1\frac{\partial f_1}{\partial p_1} = ax_1.$$

Similarly, $p_2\dfrac{\partial f_1}{\partial p_2} = bx_1$ and $m\dfrac{\partial f_1}{\partial m} = cx_1$. Applying Euler's theorem with $r = 0$ we see that $ax_1 + bx_1 + cx_1 = 0$, whence $a + b + c = 0$.

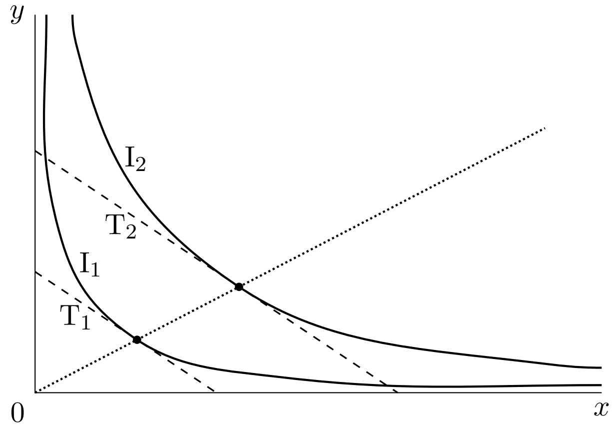
15 IMPLICIT RELATIONS

15.1 Implicit differentiation

15.1.1 (a) $-x(4-x^2)^{-1/2}$, (b) $-x/y$.

15.1.2 Similar to Example 1.

15.1.3



The tangents T_1 and T_2 are parallel.

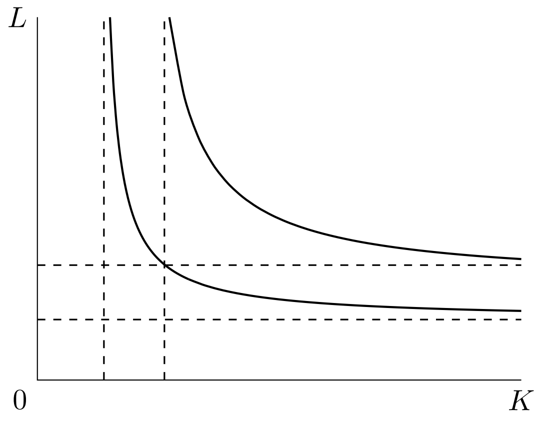
Answer to last part is yes. If the utility function is homogeneous of degree r, its partial derivatives are homogeneous of degree $r-1$, so their ratio is homogeneous of degree 0. Hence the slope of the indifference curve at any point on the line $y = cx$ is the same as the slope of the indifference curve at the point $(1, c)$.

15.1.4 $Q^{-2} = aK^{-2} + bL^{-2}$, so $-2Q^{-3}(\partial Q/\partial K) = -2aK^{-3}$, whence $\partial Q/\partial K = a(Q/K)^3$. Similarly $\partial Q/\partial L = b(Q/L)^3$. The slope of an isoquant is therefore given by

$$\frac{dL}{dK} = -\frac{a}{b}\left(\frac{L}{K}\right)^3,$$

which is obviously negative. Notice also that $|dL/dK|$ is an increasing function of L/K: so as we move rightward along an isoquant, increasing K and decreasing L, $|dL/dK|$ falls. Therefore, isoquants are convex.

Asymptotes are $K = \overline{Q}\sqrt{a},\ L = \overline{Q}\sqrt{b}$.



15.2 Comparative statics

15.2.1 $Y = m(a + I)$, $C = m(a + bI)$, $\Delta Y = m(I_1 - I_0)$ where $m = 1/(1-b)$. Assuming $I_1 > I_0$, ΔY is positive and in fact greater than $I_1 - I_0$.

15.2.2 $1/(1-f'(Y))$, which is greater than one.

15.2.3 (a) Letting g be the inverse function of f, we may write the equation $x/s = f(p)$ as $p = g(x/s)$. Hence revenue px is equal to $xg(x/s)$. Since f is a decreasing function, so is g. Profit-maximising output x is given by $g(x/s) + (x/s)g'(x/s) = c_1$. This determines x/s, given c_1; so when s increases, x increases by the same proportion and p does not change.

(b) It is easiest to work with the variable $z = x/s$. We know from the answer to (a) that MR is $g(z)+zg'(z)$, which we denote by $h(z)$. Hence the first-order condition for a maximum, MR = MC, may be written $h(z) = c_1 + 2c_2sz$.

Suppose s increases. Since $c_2 > 0$, the profit-maximising z decreases, as may be seen from the second-order condition and/or a diagram. Hence p increases. Under the usual assumption that h is a decreasing function, sz increases, so x increases (but by a smaller proportion than s).

15.3 Generalising to higher dimensions

15.3.1 (a) $\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$, (b) $\begin{bmatrix} 2x & -2y \\ 2yz & 2xz \end{bmatrix}$.

15.3.2 (a) $p = y^{(3+4\alpha)/\gamma}$, $P = y^{(12+\beta)/\gamma}$, where $\gamma = 18 - 2\alpha\beta$.

(b) Sufficient, but not necessary, conditions for dp/dY and dP/dY to be positive are $\alpha > 0$, $\beta > 0$ and $\alpha\beta < 9$.

(c) $\alpha > 0$ corresponds to $\partial f/\partial P > 0$; $\beta > 0$ corresponds to $\partial F/\partial p > 0$; $\alpha\beta < 9$ corresponds to cross-price effects being small.

16 OPTIMISATION WITH SEVERAL VARIABLES

16.1 Critical points and their classification

16.1.1 Writing the function as $-(x-2y)(2x-y)$, we see that the contour corresponding to $k = 0$ is the pair of straight lines $y = \frac{1}{2}x$ and $y = 2x$. To draw the full contour map, consider first the positive quadrant. If $0 < \frac{1}{2}x < y < 2x$, then $f(x, y) > 0$ and $f(x, y)$ increases from 0 to ∞ as we move out along a ray from the origin. If $0 \leq y < \frac{1}{2}x$, or $0 \leq 2x < y$, then $f(x, y) < 0$ and $f(x, y)$ decreases from 0 to $-\infty$ as we move out along a ray from the origin. The behaviour in the other three quadrants is symmetrical.

16.1.2 (a) Local minimum, (b) local maximum, (c) saddle point, (d) saddle point.

By considering small movements away from $x = 0$, $y = 0$ and using the fact that, for instance, $x^2 > 0$ for $x \neq 0$.

16.1.3 In each case, the Hessian is the zero matrix at $x = 0$, $y = 0$, so the test in terms of the Hessian fails. The alternative method of Exercise 16.1.2 gives the following results: (a) local minimum, (b) local maximum, (c) saddle point, (d) saddle point.

16.1.4 The gradient is obviously zero. In the direction of the y–axis, the function has a point of inflexion at the origin. Thus the origin is not a maximum or a minimum point. Nor is it a saddle point, because the function does not have a local maximum at the origin in any direction. To prove this, it suffices to consider directions other than that of the y–axis. Let a and b be constants such that $a \neq 0$. If $(x, y) = (\lambda a, \lambda b)$, then $z = \lambda^2(a^2 + \lambda b^3)$, which is positive if λ is sufficiently close, but not equal, to zero.

16.1.5 (a) Saddle point at $(4, -2, 32)$, local minimum point at $(12, -6, 0)$.

(b) Saddle point at $(0, 0, 0)$, local minimum points at $(-1, -1, -1)$ and $(1, 1, -1)$.

(c) Local minimum points at $(-1, -1, -2)$ and $(1, 1, -2)$, saddle point at $(0, 0, 0)$.

16.2 Global optima, concavity and convexity

16.2.1 In Exercises 16.1.2a and 16.1.3a, $(0, 0, 0)$ is the global minimum. In Exercises 16.1.2b and 16.1.3b, $(0, 0, 0)$ is the global maximum.

16.2.2 $\begin{bmatrix} -4x + 4y + 10 \\ 4x - 6y - 14 \end{bmatrix}$, $\begin{bmatrix} -4 & 4 \\ 4 & -6 \end{bmatrix}$, 27/2.

16.2.3 9 of X, 6 of Y.

16.2.4 0, 1.

(a) $D^2 f(x, y)$ is a diagonal matrix with negative diagonal entries.

(b) $H(u) = \exp u$ and exp is a strictly increasing function.

[For any constant k, the surface $z = g(x, y)$ intersects the plane $y = kx$ in a bell-shaped curve; hence g is not a concave function.]

16.2.5 (a) Let $0 \leq \alpha \leq 1$. Also let

$$\begin{aligned} A &= f\big(\alpha x_1 + (1-\alpha)x_2,\ \alpha y_1 + (1-\alpha)y_2,\ \alpha z_1 + (1-\alpha)z_2\big) \\ &\qquad\qquad\qquad - \alpha f(x_1, y_1, z_1) - (1-\alpha) f(x_2, y_2, z_2), \\ B &= g\big(\alpha x_1 + (1-\alpha)x_2,\ \alpha y_1 + (1-\alpha)y_2\big) - \alpha g(x_1, y_1) - (1-\alpha) g(x_2, y_2). \end{aligned}$$

f is concave if and only if $A \geq 0$ for all values of the arguments, while g is concave if and only if $B \geq 0$ for all values of the arguments. But $A = B$, by assumption.

(b) Let $0 \leq \alpha \leq 1$. Then

$$\begin{aligned} &u\big(\alpha x_1 + (1-\alpha)x_2,\ \alpha y_1 + (1-\alpha)y_2\big) - \alpha u(x_1, y_1) - (1-\alpha) u(x_2, y_2) \\ &\quad = 3\left[f\big(\alpha x_1 + (1-\alpha)x_2,\ \alpha y_1 + (1-\alpha)y_2\big) - \alpha f(x_1, y_1) - (1-\alpha) f(x_2, y_2)\right] \\ &\qquad + 4\left[g\big(\alpha x_1 + (1-\alpha)x_2,\ \alpha y_1 + (1-\alpha)y_2\big) - \alpha g(x_1, y_1) - (1-\alpha) g(x_2, y_2)\right] \end{aligned}$$

If f and g are concave, then the RHS of this equation is non-negative for all values of the arguments, so u is concave. If g is also linear then $-g$ is also concave; in this case concavity of v may be proved in the same way as concavity of u, with g replaced by $-g$.

Answer to last part is no: if g is concave and nonlinear and $f = 2g$, then v and f are the same concave function.

16.2.6 The sum of three convex functions is convex.

16.2.7 (a) Denote the Hessian by $\mathbf{H}$. Then one diagonal entry of $\mathbf{H}$ has the same sign as $\alpha(\alpha-1)$, the other has the same sign as $\beta(\beta-1)$, and $\det \mathbf{H}$ has the sign of

$$\alpha\beta(\alpha-1)(\beta-1)-\alpha^2\beta^2=\alpha\beta(1-\alpha-\beta).$$

Since α and β are positive, $\det \mathbf{H}$ has the same sign as $1-\alpha-\beta$. Thus if $\alpha+\beta>1$ the function is not concave; if $\alpha+\beta\le 1$ the function will be concave provided the diagonal entries of $\mathbf{H}$ are non-positive. But the three inequalities $\alpha>0$, $\beta>0$ and $\alpha+\beta\le 1$ imply that $0<\alpha<1$ and $0<\beta<1$, and hence that the diagonal entries of $\mathbf{H}$ are negative; therefore $\alpha+\beta\le 1$ is sufficient as well as necessary for concavity.

(b) U is concave because its Hessian is a diagonal matrix with negative diagonal entries, V is concave if and only if $\alpha+\beta\le 1$.

16.3 Non-negativity constraints

16.3.1 19/2.

16.3.2 25/3 of X, 0 of Y.

16.3.3 $x_1=0$, $x_2=9$, profit is 28.5. For any given $x\ge 0$, revenue is independent of how x is split between x_1 and x_2, cost is lowest when $x_1=0$, $x_2=x$.

17 PRINCIPLES OF CONSTRAINED OPTIMISATION

17.1 Lagrange multipliers

17.1.1 3. The optimum is where the line $3x+4y=12$ is tangent to the highest attainable member of the family of curves $xy=k$.

17.1.2 $4\sqrt{6}$. The optimum is where the curve $xy=2$ is tangential to the lowest attainable member of the family of straight lines $3x+4y=k$.

With the alternative constraint, the optimum is attained at the same values of x and y as in Exercise 17.1.1.

17.1.3 375/7.

17.1.4 $2\sqrt{5}$, $-2\sqrt{5}$. The maximum is at the point of tangency of the circle $x^2+y^2=4$ and the highest attainable member of the family of straight lines $2x+y=k$. The minimum is at the point of tangency of the same circle and the lowest attainable member of the same family of straight lines.

17.1.5 (a) The circle $x^2+y^2=k$ meets the straight line $2x+y=a$ for arbitrarily large k. Hence there is no solution.

(b) The solution is where the straight line $2x+y=a$ is tangential to the smallest attainable member of the family of circles $x^2+y^2=k$.

17.2 Extensions and warnings

17.2.1 $3\sqrt{14}$, $-3\sqrt{14}$.

17.2.2 32/3.

17.2.3 4/3.

17.3 Economic applications

17.3.1 (a) Maximise $x_1^\alpha x_2^\beta$ subject to $p_1x_1 + p_2x_2 = m$.

(b) $\alpha x_1^{\alpha-1}x_2^\beta = \lambda p_1$, $\beta x_1^\alpha x_2^{\beta-1} = \lambda p_2$. They are sufficient because the indifference curves are negatively sloped and convex.

$$x_1 = \frac{\alpha m}{(\alpha+\beta)p_1}, \quad x_2 = \frac{\beta m}{(\alpha+\beta)p_2}.$$

(c) $\alpha/(\alpha+\beta)$, $\beta/(\alpha+\beta)$.

17.3.2 (a) Minimise $rK + wL$ subject to $AK^\alpha L^\beta = q$.

(b) $r = \mu\alpha AK^{\alpha-1}L^\beta$, $w = \mu\beta AK^\alpha L^{\beta-1}$. They are sufficient because the isoquants are negatively sloped and convex.

$$K = \left(\left[\frac{\alpha w}{\beta r}\right]^\beta \frac{q}{A}\right)^{\frac{1}{\alpha+\beta}}, \quad L = \left(\left[\frac{\beta r}{\alpha w}\right]^\alpha \frac{q}{A}\right)^{\frac{1}{\alpha+\beta}}.$$

$C = \left[\gamma r^\alpha w^\beta (q/A)\right]^{1/(\alpha+\beta)}$, where γ is a constant depending on α and β.

17.4 Quasi-concave functions

17.4.1 The isoquants are negatively sloped and convex. The function is concave for $\nu \le 1$.

17.4.2 U is concave, $\tilde{U}$ is quasi-concave.

18 FURTHER TOPICS IN CONSTRAINED OPTIMISATION

18.1 The meaning of the multipliers

18.1.1 $k^2/48$, $k/24$.

(a) 49/48, (b) 4/3, (c) 27/25, (d) 7/24.

The increase in the maximum value when k increases from 7 to 7.2 is approximately 0.2 times the value of the Lagrange multiplier when $k = 7$.

18.1.2 $b_1 \ln(\beta m'/p_1) + b_2 \ln((1-\beta)m'/p_2)$, where $\beta = b_1/(b_1+b_2)$ and $m' = m - p_1c_1 - p_2c_2$.

18.1.3 Let a constrained maximum be attained at (x^*, y^*, z^*, w^*). Then the function

$$f(x,y,z,w) - v\big(g(x,y,z,w),\ h(x,y,z,w)\big)$$

attains its unconstrained maximum at (x^*, y^*, z^*, w^*). The first-order conditions for this unconstrained maximum give the Lagrange multiplier rule.

18.2 Envelope theorems

18.2.1 (a) Upward-sloping convex curves in the non-negative quadrant, hitting the vertical axis at the points $(0, b/4)$, $(0, b)$ and $(0, 4b)$ respectively. For any given $Q > 0$ the slope of the curve is less, the greater is K. Crossing points: $C(\frac{1}{2}, Q) = C(1, Q)$ at $Q = 1/\sqrt{2}$, $C(1, Q) = C(2, Q)$ at $Q = \sqrt{2}$ and $C(\frac{1}{2}, Q) = C(2, Q)$ at $Q = 1$,

(b) Minimising $C(K, Q)$ with respect to K we have $K = Q$, in which case $C = 2bQ^2$.

(c) The slope of the short-run cost curve is $4bK^{-2}Q^3$. At the point where the curve meets the long-run cost curve, $Q = K$. At that point the slope of the short-run cost curve is equal to $4bQ$, which is the slope of the long-run cost curve.

18.2.2 (a) $A = 3(\alpha\beta^2 Q/4)^{1/3}$. The curve lies in the non-negative quadrant. It is positively sloped, concave and passes through the origin, where its slope is infinite.

(b) $A = 2(\alpha\beta^2 Q)^{1/3}$. The curve is similar to that in (a) but is above it except at the origin where the two curves meet. $[3 \times 4^{-1/3} \approx 1.89 < 2.]$

18.2.3 Denote the Lagrangian by $L(x_1, \dots, x_n, \lambda, p_1, \dots, p_n, m)$. By the envelope theorem,

$$\partial V/\partial p_i = \partial L/\partial p_i = -\lambda x_i, \quad \partial V/\partial m = \partial L/\partial m = \lambda;$$

Roy's identity follows by division.

18.2.4 Let $\beta = b_1/(b_1 + b_2)$ and $m' = m - p_1 c_1 - p_2 c_2$ as in the earlier exercise. Then

$$\frac{\partial V}{\partial p_1} = -\frac{b_1}{p_1} + (b_1 + b_2)\frac{\partial}{\partial p_1}(\ln m') = -(b_1 + b_2)\left(\frac{\beta}{p_1} + \frac{c_1}{m'}\right),$$

$$\frac{\partial V}{\partial p_2} = -(b_1 + b_2)\left(\frac{1 - \beta}{p_2} + \frac{c_2}{m'}\right), \qquad \frac{\partial V}{\partial m} = (b_1 + b_2)\frac{\partial}{\partial m}(\ln m') = \frac{b_1 + b_2}{m'}.$$

Hence by Roy's identity,

$$x_1 = \frac{\beta m'}{p_1} + c_1, \quad x_2 = \frac{(1 - \beta)m'}{p_2} + c_2.$$

18.3 Non-negativity constraints again

18.3.1 (a) 40 at $(5, 1)$, (b) 5 at $(1, 0)$.

18.3.2 Let the utility function be U and let $\beta = b_1/(b_1 + b_2)$. If $m > 3p_1 + \dfrac{5\beta}{1 - \beta}p_2$, quantities demanded are

$$x_1 = 3 + \frac{\beta}{p_1}(m - 3p_1 + 5p_2), \quad x_2 = -5 + \frac{1 - \beta}{p_2}(m - 3p_1 + 5p_2).$$

If $3p_1 < m \leq 3p_1 + \dfrac{5\beta}{1 - \beta}p_2$, quantities demanded are $x_1 = m/p_1$, $x_2 = 0$.

Now let the utility function be $\tilde{U}$. Again let $\beta = b_1/(b_1 + b_2)$; also let

$$a_1 = \frac{3(1 - \beta)}{\beta}p_1 - 5p_2, \quad a_2 = \frac{5\beta}{1 - \beta}p_2 - 3p_1.$$

Then a_1 and a_2 are of opposite signs. If $m > a_1 \geq 0 \geq a_2$ or $m > a_2 > 0 > a_1$, quantities demanded are

$$x_1 = -3 + \frac{\beta}{p_1}(m + 3p_1 + 5p_2), \quad x_2 = -5 + \frac{1-\beta}{p_2}(m + 3p_1 + 5p_2).$$

If $a_1 < 0 < m \leq a_2$, quantities demanded are $x_1 = m/p_1$, $x_2 = 0$. If $a_2 < 0 < m \leq a_1$, quantities demanded are $x_1 = 0$, $x_2 = m/p_2$.

18.3.3 (a) 25, 200; (b) 12.5, 0.

18.4 Inequality constraints

18.4.1 (a) 0 at $(1, 1)$, (b) 4/25 at (19/25, 17/25).

18.4.2 (a) Lagrangian is $f(x, y) + \lambda x + \mu y$, where the multipliers λ and μ are required to be non-negative.

(b) Lagrangian is $f(x, y) - \lambda g(x, y) + \mu x + \nu y$, where the multipliers μ and ν are required to be non-negative.

(c) $\partial L/\partial x \leq 0$ with equality if $x > 0$, $\partial L/\partial y \leq 0$ with equality if $y > 0$, $\lambda \geq 0$, $g(x, y) \leq 0$ and $\lambda g(x, y) = 0$.

18.4.3 (a) Yes.

(b) No: $z = xy$ is not a concave function. Conditions are satisfied by $x^* = y^* = \lambda^* = 0$, which is obviously not optimal.

(c) $x = y = -1 - \sqrt{2.5}$, solution value 6.66 to 2 decimal places.

(d) $x = y = \sqrt{2.5} - 1$, solution value 0.34 to 2 decimal places.

19 INTEGRATION

19.1 Areas and integrals

19.1.1 $\frac{1}{2}\left(3^2 - (-2)^2\right) = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$.

This is the difference between the areas of two right-angled triangles, one with base = height = 3, the other with base = height = 2.

19.1.2 $\frac{1}{2}c(b^2 - a^2)$.

19.1.3 17, the sum of the areas of a 5×3 rectangle and a right-angled triangle with base = height = 2.

19.1.4 $2x^6$, 1330.

19.2 Rules of integration

19.2.1 (a) $\dfrac{x^8}{8} + C$, (b) $2\sqrt{x} + C$, (c) $C - \dfrac{1}{4e^{4t}}$.

19.2.2 (a) 31/5, (b) $-5/2$, (c) $4(e^{3/4} - 1)$.

19.2.3 (a) $\frac{1}{2}x^4 + \frac{3}{2}x^2 - x + C$, (b) $2x^{3/2} - 4\ln|x| - x + C$, (c) $\frac{2}{5}e^{5t} - e^{-5t} - \frac{5}{2}t^2 - x + C$.

19.2.4 (a) 11, (b) $4(\sqrt{2}-\ln 2)-3$, (c) $\frac{2}{5}(e^{10}-e^5)-e^{-10}+e^{-5}-\frac{15}{2}$.

19.2.5 (a) $\frac{1}{3}x^3-x^2-3x+C$, (b) $\frac{4}{3}x^{3/4}-6\ln|x|+C$, (c) $\frac{1}{5}(e^{5x}-e^{-5x})+e^x-e^{-x}+C$.

19.2.6 $2\ln(1+3)-2\ln(0+3)=2\ln\frac{4}{3}$.

Range of integration $[-5,-4]$: the limits are on the same side of -3 and the integral is $-2\ln 2$.

Range of integration $[-4,-2]$: the limits are on opposite sides of -3, so the integral is not defined.

19.2.7 $\displaystyle\int\left(4+\frac{13}{x-3}\right)dx = 4x+13\ln|x-3|+C$.

19.3 Integration in economics

19.3.1 $6x-x^2$, $p=6-x$.

19.3.2 8, $8(3^{3/2}-1)$, $8\times 3^{3/2}$, $8t^{3/2}+25$.

19.3.3 (a) $(Y/r)(e^{rT}-1)$, (b) $(Y/r)(1-e^{-rT})$.

19.4 Numerical integration

19.4.1 11.5.

19.4.2 0.835.

19.4.3 $\ln 2$.

(a) 0.708, (b) 0.694. The true value is 0.693 to 3 decimal places, so approximation (b) is much more accurate than (a).

19.4.4 $1/(n+1)$, $\left(1+2^{2-n}\right)/6$.

4.12%, 12.5%.

19.4.5 3666.67, 0.27.

19.4.6 $(400-2q_1-q_2-2q_3-q_4-2q_5)/450$, 8/9. The approximation is more accurate than the one given in the text if the true value of the Gini coefficient is close to 1.

20 ASPECTS OF INTEGRAL CALCULUS

20.1 Methods of integration

20.1.1 (a) $\frac{2}{15}(3x-2)(x+1)^{3/2}+C$, (b) $\frac{18}{125}$.

20.1.2 (a) $(x+a)(\ln(x+a)-1)+C$, (b) $\frac{1}{4}(x+a)^2(2\ln(x+a)-1)+(b-a)(x+a)(\ln(x+a)-1)+C$.

20.1.3 (a) $\displaystyle\int(t^2-1)t(2t)dt=\int(2t^4-2t^2)dt=\frac{2}{15}(3t^2-5)t^3+C$. Substituting $t=(x+1)^{1/2}$ into this expression, we obtain the same answer as before.

(b) $\displaystyle\int_0^1(u^2+1)u(2u\,du)=\int_0^1(2u^4+2u^2)du=\frac{2}{5}+\frac{2}{3}=\frac{16}{15}$.

20.1.4 (a) $(x^2+1)^{11}/22$, (b) $\frac{1}{4}\exp(x^4+1)+C$, (c) $3\ln(x^2+1)+C$.

20.1.5 $\frac{3}{2}\left(\exp\left(-a^{2/3}\right)-\exp\left(-b^{2/3}\right)\right)$.

20.1.6 $\ln(1+e^x)+C$, $2+\ln 2-\ln(1+e^2)$, $\ln(1+e^2)-\ln 2$.

20.2 Infinite integrals

20.2.1 (a) $2(1-X^{-1/2})$, 2.

(b) $\ln X \to \infty$ as $X \to \infty$.

(c) Integral exists for $\alpha > 1$.

20.2.2 (a) $\frac{3}{2}(1-\delta^{2/3})$, $\frac{3}{2}$.

(b) $\ln\delta \to -\infty$ as $\delta \downarrow 0$.

(c) Integral exists for $\alpha < 1$.

20.2.3 c.

20.2.4 $\frac{3}{2}(e^{-\gamma}-e^{-1})$ where $\gamma=\delta^{2/3}$, $\frac{3}{2}(e^{-1}-e^{-Y})$ where $Y=X^{2/3}$, $3/2$.

20.3 Differentiation under the integral sign

20.3.1 LHS $=\dfrac{d}{dr}\left(\dfrac{Y}{r}\left[1-e^{-rT}\right]\right)=-\dfrac{Y}{r^2}\left(1-e^{-rT}\right)+\dfrac{YT}{r}e^{-rT}$. By integration by parts,

$$\text{RHS}=\frac{YT}{r}e^{-rT}-\int_0^T\frac{Y}{r}e^{-rt}dt=\frac{YT}{r}e^{-rT}-\frac{Y}{r^2}\left(1-e^{-rT}\right)=\text{LHS}.$$

A similar but simpler argument applies when T is replaced by ∞.

20.3.2 (a) $-\int_1^5(x+y)^{-2}f(x)\,dx$,

(b) $f(y\exp y)\exp y+\int_1^{\exp y}xf'(xy)\,dx$,

(c) $-f(0)-\int_y^1 f'(x-y)\,dx$, (d) $f(1-y)$.

By making the substitution $x=y+u$, we may evaluate the answer to (c) as $-f(1-y)$, so the answers to (c) and (d) sum to zero. The reason for this is as follows. Substituting $x=y+u$ in the integral of (c), and $x=u$ in the integral of (d), we may write the sum of these integrals as $\int_0^1 f(u)\,du$, which is independent of y.

20.4 Double integrals

20.4.1 $\dfrac{9}{5}$.

20.4.2 1.

20.4.3 $-\dfrac{1568}{15}$.

20.4.4 $\dfrac{1}{18}$.

21 INTRODUCTION TO DYNAMICS

21.1 Differential equations

21.1.1 $y = \frac{1}{4}t^4 + C$. The solution curves are U-shaped with vertex at $(0, C)$.

(a) $y = \frac{1}{4}t^4 + 4$, (b) $y = \frac{1}{4}t^4 - 64$.

21.1.2 (a) $y = \frac{1}{6}t^6 + C$, (b) $y^4 = (A - 4t)^{-1}$.

21.1.3 $y = 2\exp(\frac{3}{2}x^2)$.

21.1.4 (a) $p = 3/(A - t^3)$, $p = 3/(1 - t^3)$; (b) $p = A\exp(\frac{1}{3}t^3)$, $p = 3\exp(\frac{1}{3}t^3)$.

21.1.5 We separate the variables and integrate, using the 'generalised Rule 2' of Section 19.2:

$$at = \int \frac{1}{y}\,dy + \int \frac{b}{a - by}\,dy = \ln|y| - \ln|a - by| + \text{constant}.$$

Taking exponentials,

$$e^{at} = C\left|\frac{y}{a - by}\right|, \qquad (*)$$

where C is a constant. Since the left-hand side of $(*)$ is finite and positive for all t, the expression inside the | signs is nonzero for all t; hence, by continuity, $y/(a - by)$ never changes sign. By our assumptions on y_0, this sign is positive, so $0 < y < a/b$ for all t. But then the | signs in $(*)$ may be suppressed. Solving $(*)$ for y then gives

$$y = a\big/(b + Ce^{-at}),$$

while setting $t = 0$ in $(*)$ gives $C = (a - by_0)/y_0$. Since $a > 0$, $y \to a/b$ as $t \to \infty$.

21.2 Linear equations with constant coefficients

21.2.1 (a) $y = 2 + Ae^{-7t}$; $y \to 2$ as $t \to \infty$.

(b) $y = -2 + Ae^{7t}$. When $A > 0$, $y \to \infty$ as $t \to \infty$; when $A < 0$, $y \to -\infty$ as $t \to \infty$, when $A = 0$, $y = -2$ for all t.

21.2.2 $y = 3 + Ae^{-4t}$.

(a) $y = 3 - e^{-4t}$: y increases as t increases. The graph meets the vertical axis at $y = 2$; as $t \to \infty$, $y \to 3$.

(b) $y = 3$: the graph is a horizontal line.

(c) $y = 3 + e^{-4t}$: y decreases as t increases. The graph meets the vertical axis at $y = 4$; as $t \to \infty$, $y \to 3$.

21.2.3 $y = 14 + Ae^{-t/7}$, $y = 14 - 9e^{-t/7}$.

21.2.4 (a) $y = 3e^{-t} + Ae^{-2t}$, (b) $y = (3t + A)e^{-2t}$.

21.2.5 $y = \frac{1}{9}(10e^{3t} - 12t - 1)$.

21.2.6 $\dfrac{dp}{dt} + 3p = 2$, $p = \frac{2}{3} + Ae^{-3t}$, $p \to \frac{2}{3}$ as $t \to \infty$.

21.3 Harder first-order equations

21.3.1 $y = 3e^{-t} + Ae^{-2t}$.

21.3.2 (a) $y = 2t^2 + At^{1/2}$, (b) $y = \frac{6}{5}t^2 + At^{-1/2}$.

When t is small and positive, (a) $|y|$ is small and has the same sign as A, (b) $|y|$ is large and has the same sign as A.

21.3.3 $y = \left[\dfrac{t}{4} + \dfrac{1}{32} + Ae^{8t}\right]^{-1/2}$.

21.3.4 $\dfrac{dy}{dt} - ay = -by^2$; hence $\dfrac{dx}{dt} + ax = +b$, where $x = y^{-1}$. The general solution is $y^{-1} = (b/a) + ke^{-at}$, where k is a constant. Letting $C = ka$ and rearranging, we obtain the same solution as in Exercise 21.1.5.

21.4 Difference equations

21.4.1 Putting $\Delta y_t = 0$ gives the constant particular solution $Y_t = b/a$. In the text the equation is written in the form $y_{t+1} + (a-1)y_t = b$; the constant particular solution is obtained by setting $y_{t+1} = y_t = Y$ and solving for Y. Putting $\Delta y_t = 0$ is equivalent to this but more directly analogous to finding the constant particular solution of a first order differential equation by setting $dy/dt = 0$.

21.4.2 $y_t = 3 + A(-3)^t$, $y_t = 3 - (-3)^t$.

21.4.3 (a) Not equivalent, $y_t = 3 + A(-1/3)^t$.

(b) Equivalent, $y_t = 3 + A(-3)^t$.

21.4.4 $u_n = 2 + A(-2/3)^n$, $u_n = 2\,[1 + (-2/3)^n]$.

21.4.5 (a) $y_t = 2 + A(-5/3)^t$, $y_t = 2 - 2(-5/3)^t$.

(b) $y_t = 2 + A(-3/5)^t$, $y_t = 2 - 2(-3/5)^t$.

21.4.6 (a) $y_t = A \times 2^t - 5t - 6$, (b) $y_t = A \times 2^t + 3^t$, (c) $y_t = (A + \frac{1}{2}t)2^t$.

22 THE CIRCULAR FUNCTIONS

22.1 Cycles, circles and trigonometry

22.1.1 (a) 0.175, (b) 1.484, (c) 0.332.

22.1.2 (a) 68.75°, (b) 48.70°, (c) 19.10°.

22.1.3 (a) 2, (b) 2.

(c) If the straight line $y = ax + b$ makes an angle θ with the x–axis, then $\tan\theta = a$.

22.1.4 $3/\sqrt{10}$, 3.

22.2 Extending the definitions

22.2.1 Sines: $1/\sqrt{2}$, $-1/2$, $-\sqrt{3}/2$, $1/2$, $-\sqrt{3}/2$. Cosines: $-1/\sqrt{2}$, $-\sqrt{3}/2$, $1/2$, $\sqrt{3}/2$, $1/2$. Tangents: -1, $1/\sqrt{3}$, $-\sqrt{3}$, $1/\sqrt{3}$, $-\sqrt{3}$.

22.2.2 (a) The graphs are similar to those of $\sin x$, $\cos x$ and $\tan x$ but with periods π, π and $\pi/2$ respectively.

(b) Again similar to those of $\sin x$, $\cos x$ and $\tan x$ but with periods $2\pi/3$, $2\pi/3$ and $\pi/3$ respectively.

(c) As (b), with 3 replaced by n.

(d) As (b), with 3 replaced by a. The difference between this case and the others is that, if a is not a natural number, then the original periods 2π, 2π and π no longer contain a whole number of the new periods.

22.2.3 Since $\tan\alpha < 1$ and $\sin\beta < 1/\sqrt{2}$, each of α and β is less than $\pi/4$, so $\alpha+\beta < \pi/2$.

$$\cos(\alpha+\beta) = \frac{3}{\sqrt{10}}\times\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}}\times\frac{1}{\sqrt{5}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}},$$

so $\alpha+\beta = \pi/4$.

22.2.4 For the first part, use the addition formulae with $\beta = \alpha$. For the second part, note that

$$\sin 3\alpha = \sin(2\alpha+\alpha) = \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha.$$

Now use the first part and the fact that $\cos^2\alpha = 1-\sin^2\alpha$:

$$\begin{aligned}\sin 3\alpha &= 2\sin\alpha\cos^2\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha\\ &= 2\sin\alpha\,(1-\sin^2\alpha) + \sin\alpha\,(1-2\sin^2\alpha)\\ &= 3\sin\alpha - 4\sin^3\alpha.\end{aligned}$$

22.2.5 $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

22.3 Calculus with circular functions

22.3.1 (a) $a\cos ax$, (b) $-a\sin ax$, (c) $a/\cos^2 ax$, (d) $5\sin^4 x\cos x$, (e) $5x^4\cos(x^5)$, (f) $\sin x + x\cos x$, (g) $5x^4\tan 2x + 2x^5/\cos^2 2x$, (h) $-(x\sin x+\cos x)/x^2$.

22.3.2 $dy/dx = Am\cos mx - Bm\sin mx$, whence

$$d^2y/dx^2 = -Am^2\sin mx - Bm^2\cos mx = -m^2y.$$

22.3.3 (a) $\dfrac{\sin^7 x}{7} + A$, (b) $\dfrac{\pi+4}{4\sqrt{2}} - 1$.

22.3.4 0.841, 0.540.

22.3.5 (a) $\pi/3$, (b) $2\pi/3$, (c) $\pi/4$, (d) $-\pi/6$.

22.3.6 $(1-x^2)^{-1/2}$, $3/(1+9x^2)$.

22.3.7 $\dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta} \times \dfrac{1}{\cos\theta}$. As $\theta \to 0$, both $\dfrac{\sin\theta}{\theta}$ and $\cos\theta$ approach 1, so $\dfrac{\tan\theta}{\theta} \to 1$.

Let $\theta = \arctan x$. As $x \to 0$, $\theta \to 0$, so

$$\lim_{x\to 0} \frac{\arctan x}{x} = \lim_{\theta\to 0} \frac{\theta}{\tan\theta} = 1.$$

22.4 Polar coordinates

22.4.1 (a) $(2,\ \pi/3)$, (b) $(\sqrt{8},\ 3\pi/4)$, (c) $(1,\ -2\pi/3)$, (d) $(\sqrt{2},\ -\pi/4)$.

22.4.2 (a) $(\frac{1}{2},\ \frac{1}{2}\sqrt{3})$, (b) $(-\sqrt{2},\ \sqrt{2})$, (c) $(\frac{1}{4}\sqrt{3},\ -\frac{1}{4})$, (d) $(-0.42,\ 0.91)$.

22.4.3 (a) Circle of radius 2 and centre (0,0).

(b) Straight line parallel to y–axis, 4 units to the right of it.

(c) Straight line parallel to x–axis, 3 units above it.

23 COMPLEX NUMBERS

23.1 The complex number system

23.1.1 $1,\ -32i,\ -1,\ i,\ -i$.

23.1.2 $1+5i,\ -17i,\ 8+25i$.

23.1.3 (a) $-2 \pm 3i$, (b) $\frac{1}{2}(5 \pm i\sqrt{11})$.

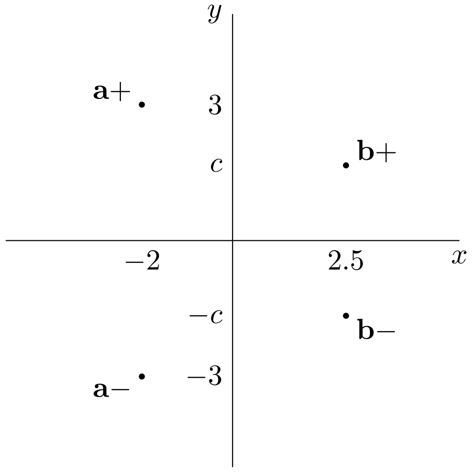
23.1.4 $\dfrac{u}{v} = -\dfrac{20-17i}{13},\ \dfrac{v}{u} = -\dfrac{20+17i}{53}$.

23.1.5 Let $v = w/z$. Then $vz = w$, so $|v||z| = |w|$, whence $|v| = |w|/|z|$.

23.1.6 $z = \frac{1}{2}(u - iv)$, $w = \frac{1}{2}(u + iv)$. If u and v are real, $w = \bar{z}$.

23.2 The trigonometric form

23.2.1 $c = \frac{1}{2}\sqrt{11} = 1.66$ to two decimal places.



23.2.2 $(-y, x)$, $(x, -y)$.

23.2.3 (a) $1 + i\sqrt{3}$, 2, $\frac{\pi}{3}$, $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.
(b) $-2 + 2i$, $2\sqrt{2}$, $\frac{3\pi}{4}$, $2\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$.
(c) $-\frac{1}{2}(1 + i\sqrt{3})$, 1, $-\frac{2\pi}{3}$, $\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3})$.
(d) $1 - i$, $\sqrt{2}$, $-\frac{\pi}{4}$, $\sqrt{2}\left[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right]$.

23.2.4 $2(\cos 0 + i\sin 0)$, $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, $\cos\pi + i\sin\pi$,
$\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$, $\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$, $\sqrt{2}\left[\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})\right]$.

23.2.5 -2^{20}.

23.3 Complex exponentials and polynomials

23.3.1 $\sqrt{2}e^{\pi i/4}$, $\sqrt{2}e^{3\pi i/4}$, $2e^{\pi i/3}$, $e^{-2\pi i/3}$.

23.3.2 (a) $\frac{1}{2}(1 + i\sqrt{3})$, (b) $-\sqrt{2}(1 + i)$, (c) $\sqrt{3}$, (d) $i\sqrt{3}$.

23.3.3 $(1 + 2i)z^2 + (3 - i)z - 4 - 3i$.

23.3.4 1, $\frac{1}{2}(-1 \pm i\sqrt{3})$; -2, $1 \pm i\sqrt{3}$; $-i$, $\frac{1}{2}(i \pm \sqrt{3})$.

23.3.5 $3i \pm (1 - i)\sqrt{2}$.

23.3.6 By direct calculation, using the fact that $\dfrac{d^2}{dx^2}e^{imx} = (im)^2e^{imx} = -m^2e^{imx}$, and similarly when m is replaced by $-m$. Alternatively, set $A = P + Q$, $B = i(P - Q)$ and proceed as in Exercise 22.3.2.

24 FURTHER DYNAMICS

24.1 Second-order differential equations

24.1.1

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\left[\frac{dy}{dt} - py\right]e^{-pt}\right) = \left(\frac{d^2y}{dt^2} - 2p\frac{dy}{dt} + p^2y\right)e^{-pt}.$$

Multiplying through by e^{pt} and recalling that $2p = -b$ and $p^2 = c$ by definition of p, we get the desired result.

The differential equation reduces to $d^2u/dt^2 = 0$, This implies that du/dt is a constant, say B. Integrating again, we have the general solution $u = A + Bt$, where A and B are arbitrary constants. (24.7) now follows from the fact that $y = ue^{pt}$.

24.1.2 (a) $y = Ae^{3t} + Be^{-2t}$, (b) $y = Ae^{3t} + Be^{-2t} - \frac{1}{2}$, (c) $y = Ce^{-2t}\cos(t + \alpha)$,
(d) $y = Ce^{-2t}\cos(t + \alpha) + 2$, (e) $y = (A + Bt)e^{-5t}$, (f) $y = (A + Bt)e^{-5t} + \frac{2}{5}$.

24.1.3 (a) $y = Ae^{t/3} + Be^{-t} - 6$, (b) $y = (At + B)e^{-3t} - \dfrac{3t - 5}{27}$.

24.1.4 From (24.5), $y = e^{gt}[Ae^{iht} + Be^{-iht}]$. The expression in square brackets may be written $A\cos ht + iA\sin ht + B\cos ht - iB\sin ht$. Hence (24.6′) holds with $A' = A + B$ and $B' = i(A - B)$. If $A = a + ib$ and $B = a - ib$, where a and b are real numbers, then A' is the real number $2a$ and B' is the real number $-2b$.

24.1.5 (a) $y = 3\cos 2t + 4\sin 2t + 5$, (b) $y = -\frac{1}{2}e^t - e^{2t} + \frac{3}{2}e^{3t}$.

24.2 Qualitative behaviour

24.2.1 (a) **UN**, (b) **UN**, (c) **SO**, (d) **SO**, (e) **SN**, (f) **SN**.

24.2.2 Let $v = u/c$, $p = |c|^{1/2}$. If $c < 0$, the general solution is $y = Ae^{pt} + Be^{-pt} + v$, where A and B are constants; if $A \neq 0$ then, as $t \to \infty$, $y \to \pm\infty$ depending on the sign of A. If $c > 0$, the general solution is $y = C\cos(pt + \alpha) + v$, where C and α are constants; if $C \neq 0$ then y displays regular oscillations around v.

24.2.3 (a) $\theta < 2\sqrt{\alpha/\beta}$, (b) $\sigma < 2\sqrt{\beta/\alpha}$.

24.3 Second-order difference equations

24.3.1 (a) $y_t = \dfrac{1}{\sqrt{5}}\left(\left[\dfrac{1+\sqrt{5}}{2}\right]^t - \left[\dfrac{1-\sqrt{5}}{2}\right]^t\right)$.

(b) $y_t = \frac{2}{3}((-4)^t + 2^{1+t})$, (c) $y_t = \frac{1}{3}(5\times 2^t - 2^{1-t})$.

24.3.2 (a) $y_t = 2^t A + (-3)^t B - \frac{3}{2}$. **UN**: y_t alternates eventually and $|y_t| \to \infty$ as $t \to \infty$.

(b) $y_t = 2^{t/2} C\cos(\frac{3}{4}\pi t + \alpha) + \frac{6}{5}$. **UO**: y_t oscillates explosively about $\frac{6}{5}$.

(c) $y_t = 4^t(A + Bt) + \frac{2}{3}$. **UN**: eventual monotonic behaviour, $|y_t| \to \infty$ as $t \to \infty$.

24.3.3 $y_t = C\cos(\frac{2}{3}\pi t + \alpha) + \frac{4}{3}t - 1$.

24.3.4 (a) $p_{t+1} + 2p_{t-1} = 2$, $p_t = 2^{t/2} C\cos(\frac{1}{2}\pi t + \alpha) + \frac{2}{3}$, $|p_t| \to \infty$ as $t \to \infty$.

(b) It is the same except that the price variable in q_t^S is now p_{t-1}. In both models, the equilibrium value of p is $\frac{2}{3}$, and in both models $|p_t| \to \infty$ as $t \to \infty$. In the model of Exercise 21.4.6, p_t eventually alternates whereas here p_t oscillates.

24.3.5 $Y_t - 2Y_{t-1} + \frac{4}{3}Y_{t-2} = 20, 60$.

$Y_t = C\left(\dfrac{4}{3}\right)^{t/2}\cos\left(\dfrac{\pi t}{6} + \alpha\right) + 60$, explosive oscillations..

25 EIGENVALUES AND EIGENVECTORS

25.1 Diagonalisable matrices

25.1.1 -2, 7.

The eigenvectors corresponding to -2 are non-zero multiples of $[1 \;\; -1]^{\mathrm{T}}$. The eigenvectors corresponding to 7 are non-zero multiples of $[4 \;\; 5]^{\mathrm{T}}$.

25.1.2 The eigenvalues of $\theta\mathbf{A}$ are θ times the eigenvalues of $\mathbf{A}$. For $\theta \neq 0$, the eigenvectors are the same as those of $\mathbf{A}$. For $\theta = 0$, any non-zero vector is an eigenvector.

The eigenvalues of $\mathbf{A} + \theta\mathbf{I}$ are θ plus the eigenvalues of $\mathbf{A}$. The eigenvectors are the same as those of $\mathbf{A}$.

25.1.3 Possibilities are $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

$$\mathbf{A}^k = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^{k-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

25.2 The characteristic polynomial

25.2.1 The eigenvalues are the diagonal entries.

25.2.2 The sum of all entries in the ith row of $\mathbf{P}$ is the ith diagonal entry of $\mathbf{BC}$; hence $\operatorname{tr}\mathbf{BC}$ is the sum of all entries of $\mathbf{P}$. The sum of all entries in the jth column of $\mathbf{P}$ is the jth diagonal entry of $\mathbf{CB}$; hence $\operatorname{tr}\mathbf{CB}$ is also the sum of all entries of $\mathbf{P}$.

25.2.3 (a) Possibilities are $\mathbf{D} = \begin{bmatrix} 1+i\sqrt{3} & 0 \\ 0 & 1-i\sqrt{3} \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 1 & 1 \\ -i\sqrt{3} & i\sqrt{3} \end{bmatrix}$.

$$\mathbf{A}^k = \frac{2^k}{\sqrt{3}} \begin{bmatrix} \sqrt{3}\cos(k\pi/3) & -\sin(k\pi/3) \\ 3\sin(k\pi/3) & \sqrt{3}\cos(k\pi/3) \end{bmatrix}.$$

(b) Possibilities are $\mathbf{D} = \begin{bmatrix} 2+2i & 0 \\ 0 & 2-2i \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 1 & 1 \\ -1-2i & -1+2i \end{bmatrix}$.

$$\mathbf{A}^k = 2^{(3k/2)-1} \begin{bmatrix} 2\cos(k\pi/4) - \sin(k\pi/4) & -\sin(k\pi/4) \\ 5\sin(k\pi/4) & 2\cos(k\pi/4) + \sin(k\pi/4) \end{bmatrix}.$$

25.2.4 (a) $\alpha\mathbf{I}$.

(b) The result of (a) implies that the only 2×2 d–matrix with eigenvalues $0, 0$ is $\mathbf{O}$. The given matrix has eigenvalues $0, 0$ but is not $\mathbf{O}$; therefore it is not a d–matrix.

25.2.5 All eigenvectors are multiples of $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

25.2.6 If the eigenvalues are p, p then $a = -p^2$, $b = 2p$ and eigenvectors are multiples of $\begin{bmatrix} 1 \\ p \end{bmatrix}$.

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$. If $\mathbf{A}$ is diagonalisable it has two linearly independent eigenvectors and hence, by first part, two distinct eigenvalues. The converse is true for every matrix. The matrix $\begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix}$ has the same characteristic polynomial as $\mathbf{A}$; if its eigenvalues are p, p then all eigenvectors are multiples of $\begin{bmatrix} p \\ 1 \end{bmatrix}$. Second part follows as before.

25.3 Eigenvalues of symmetric matrices

25.3.1 Possibilities are $\mathbf{D} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$, $\mathbf{S} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

25.3.2 2×2 matrices with real entries whose off-diagonal entries are both non-negative or both non-positive.

25.3.3 They have at least one positive and one negative eigenvalue.

25.3.4 $\mathbf{A}^{1/2} = \mathbf{S}\mathbf{D}^{1/2}\mathbf{S}^{\mathrm{T}}$.

$\mathbf{B}$ and $\mathbf{B}^2$ are positive definite symmetric matrices, and $\mathbf{B}^2 = \mathbf{A}^{-1}$. This last fact, together with the definition of $\mathbf{B}$, makes it reasonable to refer to $\mathbf{B}$ as $\mathbf{A}^{-\frac{1}{2}}$.

26 DYNAMIC SYSTEMS

26.1 Systems of difference equations

26.1.1 (a) $\mathbf{y}(t) = (\tfrac{1}{2})^t c_1 \begin{bmatrix} 5 \\ -1 \end{bmatrix} + (-\tfrac{1}{4})^t c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$.

(b) $\mathbf{x}(t) = \begin{bmatrix} 18 \\ -2 \end{bmatrix} + (\tfrac{1}{2})^t c_1 \begin{bmatrix} 5 \\ -1 \end{bmatrix} + (-\tfrac{1}{4})^t c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$\mathbf{x}(t) \to \begin{bmatrix} 18 \\ -2 \end{bmatrix}$ as $t \to \infty$.

26.1.2 (a) $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (\tfrac{1}{2})^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\mathbf{x}(t) \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as $t \to \infty$.

(b) $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (\tfrac{1}{2})^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} - (-2)^t \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

As $t \to \infty$, the components of $\mathbf{x}(t)$ display explosive alternations.

26.1.3 (a)

$$\mathbf{y}(t) = \begin{bmatrix} y_{t+2} \\ y_{t+1} \\ y_t \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -f & -g & -h \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b)

$$\mathbf{x}(t) = \begin{bmatrix} x_{t+3} \\ x_{t+2} \\ x_{t+1} \\ x_t \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -b_1 & -b_2 & -b_3 & -b_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_5 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

26.2 Systems of differential equations

26.2.1 General solution is $\mathbf{y}(t) = c_1 e^{2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The boundary condition implies that $c_1 = 1,\ c_2 = 3$.

26.2.2 General solution is

$$\mathbf{y}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

The boundary condition implies that $c_1 = c_2 = 1$.

26.3 Qualitative behaviour

26.3.1 (a) $(-1, 1)$, saddle point. (b) $(2.6, -1.4)$, centre.
(c) $(4, -1)$, spiral sink. (d) $(0, 0)$, source.

26.3.2 In Exercise 26.2.1, $(0, 0)$ is a source. In Exercise 26.2.2, $(-3, -1)$ is a saddle point: the stable branch is the straight line through $(-3, -1)$ of slope $\frac{1}{6}$.

26.3.3 (a) The eigenvalues are $1 + 5i$ and $1 - 5i$.

(b) $a = p + q,\ b = i(p - q)$.

(c) From the first differential equation of the system, $y = [\dot{x} - x]/5$. But by differentiating the solution for x given in (b) using the product rule,

$$\dot{x} = x + e^t \frac{d}{dt}(a \cos 5t + b \sin 5t).$$

Hence

$$y = \frac{e^t}{5} \frac{d}{dt}(a \cos 5t + b \sin 5t) = (-a \sin 5t + b \cos 5t)e^t.$$

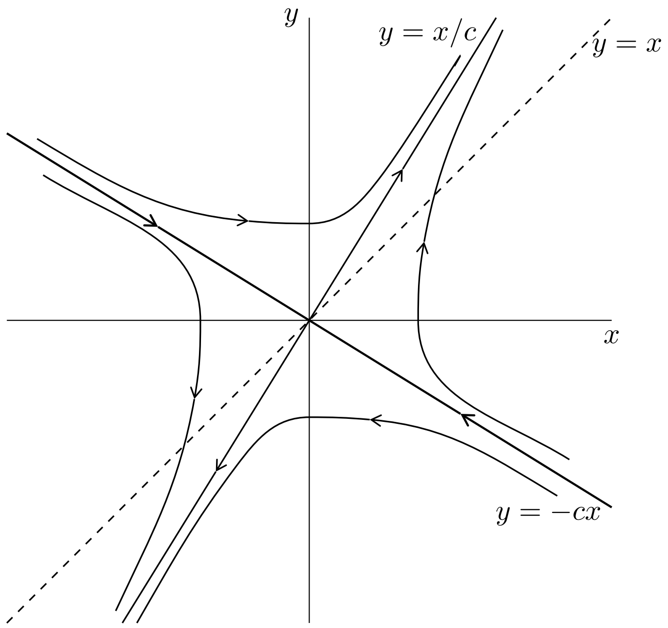
The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = ae^t \begin{bmatrix} \cos 5t \\ -\sin 5t \end{bmatrix} + be^t \begin{bmatrix} \sin 5t \\ \cos 5t \end{bmatrix}.$$

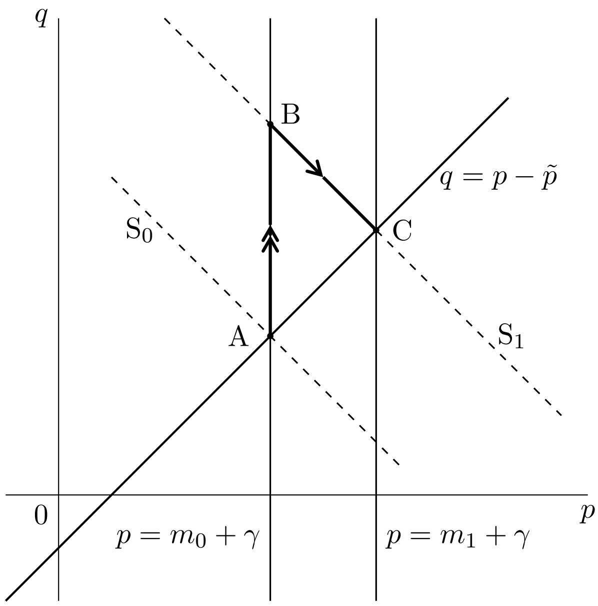
26.3.4 It is obtained by replacing x by $-x$ in the system depicted in Figure 26.5. Therefore the phase portrait is obtained by reflecting that of Figure 26.5 in the y–axis. Hence the origin is a spiral sink approached via clockwise spirals.

$\dot{x} = x - 5y,\ \dot{y} = 5x + y$.

26.3.5 The equation of the stable branch is $y = -cx$, where $c = \frac{1}{2}(\sqrt{5}-1)$. The phase diagram is very similar to Figure 26.9 in the text. The phase portrait is as follows:



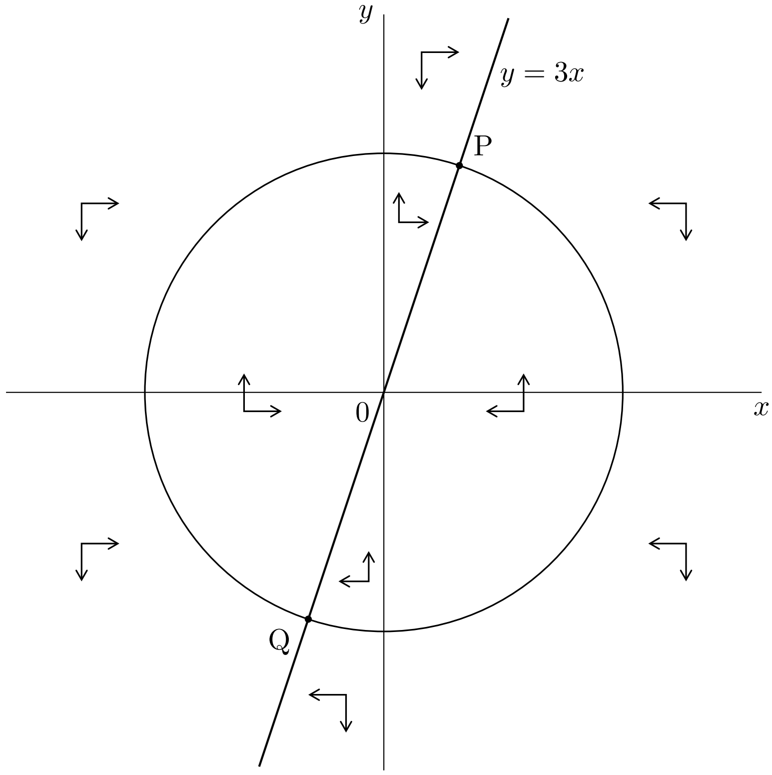
26.3.6 The fixed points for the old and the new systems of differential equations lie on the line $q = p - \tilde{p}$, with the new fixed point (C in diagram on next page) to the right of the old one (A). S_0 is the old stable branch; the new stable branch S_1 is a downward-sloping line through the new fixed point. The economy's reaction to the increase in m from m_0 to m_1 is an immediate move from A to the point B on S_1 with the same p–coordinate. Therefore the value to which q tends as $t \to \infty$ (i.e. the vertical coordinate of C) is higher than the value of q before time 0, but the value of q immediately after time 0 is higher still.



26.4 Nonlinear systems

26.4.1 The fixed points are $(0,0)$ and $(0,1/a)$. At $(0,0)$ the product of the eigenvalues of the Jacobian is -1, so we have a local saddle point. At $(0,1/a)$ the eigenvalues of the Jacobian are $\frac{1}{2}(-1 \pm i\sqrt{3}$, so we have a locally stable focus.

26.4.2 The fixed point in the positive quadrant (P in the diagram) has coordinates $(1,3)$. The other fixed point Q is the point $(-1,-3)$. At P, the eigenvalues of the Jacobian are -4 and -5: P is a locally stable node. At Q,the product of the eigenvalues of the Jacobian is -20, so we have a local saddle point.



27 DYNAMIC OPTIMISATION IN DISCRETE TIME

27.1 The basic problem

27.1.1 The control conditions are

$$\frac{\partial H_t}{\partial w_t} = 0, \quad \frac{\partial H_t}{\partial x_t} = 0 \qquad (t = 0, 1, \ldots, T)$$

The costate equations are

$$\frac{\partial H_t}{\partial y_t} = \lambda_{t-1} - \lambda_t, \quad \frac{\partial H_t}{\partial z_t} = \mu_{t-1} - \mu_t \qquad (t = 1, \ldots, T)$$

27.1.2 Equation (27.4) of the text, together with the fact that $u'(c) > 0$ for all c, implies that $u'(c_t) \gtreqless u'(c_{t-1})$ according as $\rho \gtreqless r_t$. Now replace t by $t+1$: if $r_{t+1} > \rho$ then $u'(c_{t+1}) < u'(c_t)$; since u' is a decreasing function, it follows that $c_{t+1} > c_t$. Similarly, $c_{t+1} < c_t$ if $r_{t+1} < \rho$.

27.1.3 Equation (27.5) is replaced by $c_t = \nu^t c_0$ $(t = 1, \dots, T)$ where

$$\nu = \left[\frac{1+r}{1+\rho}\right]^{1/\gamma}.$$

Hence the term $(1+\rho)^{-t}$ on the right-hand side of (27.6) is replaced by $\nu^t(1+r)^{-t}$. It follows that

$$c_0 = \tilde{\rho} \sum_{t=0}^{T} (1+r)^{-t} w_t,$$

where

$$\tilde{\rho} = \frac{1-(1+r)^{-1}\nu}{1-(1+r)^{-T-1}\nu^{T+1}} = \frac{1-(1+\rho)^{-1/\gamma}(1+r)^{(1-\gamma)/\gamma}}{1-(1+\rho)^{-(1+T)/\gamma}(1+r)^{(1+T)(1-\gamma)/\gamma}}.$$

27.1.4 (a) We proceed as in the text until the equation before (27.6), whose RHS is now B rather than 0. Hence $(1+r)^{-T}B$ must be subtracted from the LHS of (27.6). The optimal path of consumption is given by (27.5) and

$$c_0 = \rho^* \sum_{t=0}^{T} (1+r)^{-t} w_t - \rho^*(1+r)^{-T} B,$$

where θ and ρ^* are as in the text.

(b) Consider first the general problem of this section, with the difference that $\Phi(y_{T+1})$ is added to the maximand and y_{T+1} is now chosen by the agent. Then $\Phi(y_{T+1})$ is added to the Lagrangian at the top of page 601, and we have the additional first-order condition $\Phi'(y_{T+1}) = \lambda_T$. In the case at hand, this first-order condition becomes $\beta/B = \lambda_T$. Now $1/\lambda_T = (1+\rho)^T c_T$ by the control condition. Hence $B = \beta(1+\rho)^T c_T$. Using (27.5) and the definition of θ, we see that $B = \beta(1+r)^T c_0$. The solution is therefore as in (a), but with $(1+r)^{-T}B = \beta c_0$. The optimal path of consumption is given by (27.5) and

$$c_0 = \frac{\rho^*}{1+\beta\rho^*} \sum_{t=0}^{T} (1+r)^{-t} w_t,$$

where θ and ρ^* are as in the text.

27.2 Variants of the basic problem

27.2.1 Since $\pi_t = \pi_0$ for all t,

$$\mu_t = \frac{\pi_0}{1+r} \frac{1-\theta^{T-t}}{1-\theta} = \frac{\pi_0(1-\theta^{T-t})}{\delta + r}.$$

In particular, $\mu_T = 0$, $\mu_{T-1} = \dfrac{\pi_0}{1+r}$ and $\mu_t \approx \dfrac{\pi_0}{\delta+r}$ if $T-t$ is large. Also,

$$I_t = \max\left(\frac{\mu_t - a}{2b}, 0\right)$$

for all t. Thus if $a < \dfrac{\pi_0}{\delta+r}$ and T is sufficiently large,

$$I_0 > I_1 > \dots > I_{\tau-1} > 0 = I_\tau = \dots = I_T$$

for some $\tau \le T$. If also $a > \dfrac{\pi_0}{1+r}$, then $\tau < T$.

27.2.2 Let $\beta = \dfrac{(1+g)(1-\delta)}{1+r}$. Then $I_t = \max\left[\dfrac{\mu_t - a}{2b}, 0\right]$ for all t, where

$$\frac{\mu_t}{(1+g)^t} = \pi_0 \frac{1+g}{1+r}\left(1 + \beta + \ldots + \beta^{T-t-1}\right).$$

Hence by the geometric series formula,

$$\mu_t = \frac{(1+g)^{t+1}(1-\beta^{T-t})}{r - g + (1+g)\delta}.$$

Here there is no reason for μ (and hence I) to be falling monotonically over time, though investment will eventually be zero. Indeed if $\beta < 1$, which will be true if but not only if $g < r$, μ_t will be growing at a rate close to g when $T - t$ is large.

27.3 Dynamic programming

27.3.1 For $1 \leq t \leq T-1$,

$$\begin{aligned} v_t(K) &= q_t K + w_t \\ &= \max_{I \geq 0}\left\{\pi_t K - C(I) + (1+r)^{-1} q_{t+1}((1-\delta)K + I) + (1+r)^{-1} q_{t+1}\right\}, \end{aligned}$$

where $q_T = \pi_T$ and $w_T = 0$. The first-order condition for maximisation is

$$C'(I_t) = (1+r)^{-1} q_{t+1} \text{ if } q_{t+1} > (1+r)C'(0),\ \ I_t = 0 \text{ otherwise.}$$

Thus I_t is given by (27.7) in the text, except that μ_t is now equal to $(1+r)^{-1}q_{t+1}$. It remains to show that this μ_t is the same as the one in Section 27.2.

By direct substitution,

$$q_t = \pi_t + \frac{1-\delta}{1+r} q_{t+1}, \quad w_t = \frac{q_{t+1} I_t + w_{t+1}}{1+r} - C(I_t).$$

Letting $\theta = (1-\delta)/(1+r)$ as in Section 27.2,

$$q_T = \pi_T, \quad q_{T-1} = \pi_{T-1} + \theta \pi_T, \quad q_{T-2} = \pi_{T-2} + \theta \pi_{T-1} + \theta^2 \pi_T$$

and in general

$$q_t = \pi_t + \theta \pi_{t+1} + \ldots + \theta^{T-t} \pi_T \qquad (t = 0, 1, \ldots, T).$$

Recalling that $\mu_t = (1+r)^{-1} q_{t+1}$, we see that

$$\mu_t = (1+r)^{-1}\left[\pi_{t+1} + \theta \pi_{t+2} + \ldots + \theta^{T-t-1} \pi_T\right] \qquad (t = 0, 1, \ldots, T-1).$$

This is the same solution for μ_t as in Section 27.2: see the equation at the bottom of page 607 of the text.

27.3.2 $x_4 = x_7 = 0$, $x_i = 1$ otherwise (or $x_6 = x_7 = 0$, $x_i = 1$ otherwise), solution value 16.

27.3.3 From the first-order condition for maximisation,

$$I = \max\left[\frac{(1+r)^{-1}q - a}{2b},\ 0\right].$$

Equating coefficients, $q = \pi_0 + \dfrac{1-\delta}{1+r}q$. Hence $q = \dfrac{1+r}{r+\delta}\pi_0$ and

$$I = \max\left[\frac{(r+\delta)^{-1}\pi_0 - a}{2b},\ 0\right].$$

Thus the result of Exercise 27.2.1 implies that if $T-t$ is large then $I_t \approx I^*$, where I^* is optimal investment for the corresponding infinite-horizon problem.

28 DYNAMIC OPTIMISATION IN CONTINUOUS TIME

28.1 The basic problem and its variants

28.1.1 The Euler equation can be written as $\dfrac{d^2y}{dt^2} - y = -3e^{2t}$; this has general solution

$$y = Ae^t + Be^{-t} - e^{2t}.$$

(a) Boundary conditions are $A + B = 1$, $Ae + Be^{-1} = 2 + e^2$. Solution is

$$y = Ae^t + (1-A)e^{-t} - e^{2t},$$

where $A = (e^3 + 2e - 1)/(e^2 - 1)$.

(b) Transversality condition is $\dot{y}(1) = 0$, so boundary conditions are $A + B = 1$, $Ae - Be^{-1} = 2e^2$. Solution is as in (a), except that now $A = (2e^3 + 1)/(e^2 + 1)$.

28.1.2 (a) The problem is equivalent to maximising $-\displaystyle\int_1^2 \dot{y}^2\,dt$ subject to the same endpoint conditions. The Euler equation is $d^2y/dt^2 = 0$, with general solution $y = At + B$. From the endpoint conditions, $A + B = 1$ and $2A + B = 5$; hence $A = 4$, $B = -3$ and the solution is $y = 4t - 3$.

(b) $y = 1$ makes the integral 0 and satisfies the left-endpoint condition; hence it is the solution. Recall from (a) that the Euler equation is $d^2y/dt^2 = 0$ for all t; the transversality condition says that $dy/dt = 0$ if $t = 2$. If $y = 1$ for all t then $dy/dt = 0$ for all t, so the Euler equation and the transversality condition are both satisfied.

28.1.3 (a) The Hamiltonian is

$$H(c, a, \lambda, t) = e^{-\rho t}u(c) + \lambda(ra + w - c).$$

The control condition is $e^{-\rho t}u'(c) = \lambda$ and the costate equation is $\lambda r = -\dot{\lambda}$.

(b) From the control condition, $\ln u'(c) = \rho t + \ln\lambda$. Differentiating both sides with respect to t and using the costate condition gives the required result.

(c) In this case $\ln u'(c) = -\ln c$; so by the result of (b),

$$\frac{d}{dt}\ln c = r - \rho.$$

Integrating, $c = Ae^{(r-\rho)t}$ where A is a constant. To find A, we multiply the state equation by the integrating factor e^{-rt}:

$$\frac{d}{dt}(e^{-rt}a) = e^{-rt}(w - c) = e^{-rt}w - Ae^{-\rho t}.$$

Integrating from $t = 0$ to $t = T$ and using the endpoint conditions,

$$0 = \int_0^T e^{-rt}w(t)\,dt - A\int_0^T e^{-\rho t}dt.$$

Therefore

$$A = \frac{\rho}{1 - e^{-\rho T}}\int_0^T e^{-rt}w(t)\,dt.$$

(d) In this case, the result of (b) becomes

$$\frac{d}{dt}\ln c = \frac{r - \rho}{\gamma}.$$

Integrating, $c = Be^{(r-\rho)t/\gamma}$ where B is a constant. To find B, we again multiply the state equation by the integrating factor e^{-rt}: in this case

$$\frac{d}{dt}(e^{-rt}a) = e^{-rt}w - Be^{-\nu t},$$

where $\nu = \gamma^{-1}\rho + (1 - \gamma^{-1})r$. The constant B is determined by proceeding as in (c) with ρ replaced by ν.

28.2 The maximum principle

28.2.1 $H(w, x, y, z, \lambda, \mu, t) = f(w, x, y, z, t) + \lambda g(w, x, y, z, t) + \mu h(w, x, y, z, t)$,

$$\mathcal{H}(y, z, \lambda, \mu, t) = \max_{w,x} H(w, x, y, z, \lambda, \mu, t).$$

Along the optimal path,

$$H(w(t), x(t), y(t), z(t), \lambda(t), \mu(t), t) = \mathcal{H}(y(t), z(t), \lambda(t), \mu(t), t) \quad \text{for all } t$$

and

$$\dot{y} = \frac{\partial\mathcal{H}}{\partial\lambda}, \quad \dot{\lambda} = -\frac{\partial\mathcal{H}}{\partial y}, \quad \dot{z} = \frac{\partial\mathcal{H}}{\partial\mu}, \quad \dot{\mu} = -\frac{\partial\mathcal{H}}{\partial z}.$$

28.2.2 (a) $\widetilde{H}(x, y, \mu, t) = 4y - 10y^2 - x^2 + \mu x$. The maximised current-value Hamiltonian is

$$\widetilde{\mathcal{H}}(x, y, \mu, t) = 4y - 10y^2 + \tfrac{1}{4}\mu^2.$$

(b) In this case, the system (28.5) is $\dot{y} = \frac{1}{2}\mu$, $\dot{\mu} = 3\mu - (4 - 20y)$.

(c) Eliminating μ between the two equations in (b) gives

$$2\frac{d^2y}{dt^2} = 6\frac{dy}{dt} - 4 + 20y,$$

which is the required differential equation.

28.2.3 The problem is equivalent to maximising $\int_0^T f(x, y, t)\, dt$ subject to the state equation $\dot{y} = x$, and fixed endpoints. The Hamiltonian is

$$H(x, y, \lambda, t) = f(x, y, t) + \lambda x.$$

Since f is a concave function of the two variables x, y for any given t, it follows that H is concave in x, y for any given λ, t. Hence, by the sufficiency condition stated in the text, the Euler equation is sufficient for a maximum.

28.2.4 (a) $\widetilde{H}(I, K, \mu, t) = \pi(t)K - C(I) + \mu(I - \delta K)$. The control condition is

$$C'(I) \geq \mu \text{ with equality if } I > 0.$$

The costate equation is $\dot{\mu} = (r + \delta)\mu - \pi$.

(b) Identical to Figure 27.1.

(c) Multiplying the costate equation by the integrating factor $e^{-(r+\delta)t}$ and rearranging,

$$\frac{d}{dt}\left(e^{-(r+\delta)t}\mu(t)\right) = -e^{-(r+\delta)t}\pi(t).$$

This, together with the transversality condition $\mu(T) = 0$, gives

$$\mu(t) = \int_t^T e^{(r+\delta)(t-s)}\pi(s)\, ds.$$

(d) $\mu(t) = \dfrac{\bar{\pi}}{r+\delta}\left[1 - e^{-(r+\delta)(T-t)}\right], \quad I(t) = \max\left[\dfrac{\mu - a}{2b}, 0\right].$

28.2.5 $\widetilde{H}(I, K, \mu, t) = \bar{\pi}K - aI + \mu(I - \delta K)$. The costate equation and the solution for $\mu(t)$ $(0 \leq t \leq T)$ are as in part (d) of Exercise 28.2.4. By the maximum principle,

$$I(t) = \begin{cases} \theta a^{-1}\bar{\pi}K(t) & \text{if } \mu(t) > a, \\ 0 & \text{if } \mu(t) < a. \end{cases}$$

Assuming that $\bar{\pi} > (\delta + r)a$, there is exactly one time τ $(0 < \tau < T)$ such that $\mu(\tau) = a$. Notice that we do not need to worry about how $I(t)$ is determined when $\mu(t) = a$, since this happens only instantaneously at $t = \tau$. Thus for $0 \leq t < \tau$, $K(t)$ grows at rate $\theta a^{-1}\bar{\pi} - \delta$ and $I(t) = \theta a^{-1}\bar{\pi}K(t)$; for $\tau \leq t \leq T$, $K(t)$ grows at rate $-\delta$ and $I(t) = 0$.

28.3 Two problems in resource economics

28.3.1 (a) $q^* = \sqrt{a/b}$, $\gamma = \sqrt{ab}$. The result for $\bar{\lambda}$ follows from the result

$$p(T) - \gamma = \bar{\lambda}e^{rT}$$

of the text. Since

$$p - \bar{\lambda}e^{rt} > p - \bar{\lambda}e^{rT} = \sqrt{ab} > 0$$

for $0 \leq t < T$, $C'(\phi(t, \bar{\lambda})) = p - \bar{\lambda}e^{rt}$. The expression for $\phi(t, \bar{\lambda})$ follows from the fact that $C'(q) = bq$.

(b) Use $R(0) = \int_0^T \phi(t, \bar{\lambda})\, dt$. Setting

$$F(t,p) = \frac{pt - bR(0)}{p - \sqrt{ab}}, \quad G(t,r) = \frac{1 - e^{-rt}}{r},$$

we may draw the graphs of $u = F(t,p)$ and $u = G(t,r)$ for $t \geq 0$ and given p, r. The graph of F is a straight line with slope > 1 and negative intercept. The graph of G is strictly concave, increasing, contains the origin and tends to $1/r$ as $t \to \infty$. There is therefore exactly one point of intersection, in the positive quadrant; at that point, $t = T(p,r)$.

(c) Using the notation of (b), $\partial G/\partial r < 0$; it is clear from the diagram that $\partial T/\partial r < 0$. By the quotient rule, $\partial F/\partial p$ has the same sign as $t - F(t,p)$. But $G(t,r) < t$ for all $t > 0$. Hence $\partial F/\partial p > 0$ when $t = T(p,r)$. It can be seen from the diagram that $\partial T/\partial p < 0$.

28.3.2 If the condition holds, then

$$p(t) - \bar{\lambda}e^{rt} > p(T) - \gamma + C'(0) - \bar{\lambda}e^{rt} = \bar{\lambda}(e^{rT} - e^{rt}) + C'(0) > +C'(0) \quad (0 \leq t < T).$$

It follows from condition (ii) of the text that $q(t) > 0$.

28.3.3 If $q(t) > 0$ then $n(t) = \bar{\lambda}e^{rt}$ and the result follows.

28.3.4 (a) $\dot{P}(t) = g'(\tilde{s}(t))$, $\dot{M}(t) = g'(s^*(t))$. In Figure 28.2, $\tilde{s}(t)) < s^*(t)$ when $0 \leq t < t_1$. Since g is strictly concave, it follows that $\dot{M}(t) < \dot{P}(t)$ for such t. Thus $P(t) - M(t)$ is strictly increasing in t for $0 \leq t < t_1$; but $P(t_1) = M(t_1)$; hence $P(t) - M(t) < 0$ if $0 \leq t < t_1$, and the result follows.

(b) In Figure 28.2, $\tilde{s}(t)) > s^*(t)$ when $t_2 < t \leq T$. Using the strict concavity of g as in (a), we infer that $M(t) - P(t)$ is strictly increasing in t for $t_2 < t \leq T$; but $M(t_2) = P(t_2)$; hence $M(t) - P(t) > 0$ if $t_2 < t \leq T$, and the result follows.

28.3.5 Putting $\dot{p} = 0$ in (28.8) leads to $\tilde{s} = \dfrac{k(r - \rho)}{2r}$.

28.3.6 (a) The resource manager's problem is to

$$\text{maximise} \int_0^T e^{-\rho t}[p(t)h(t) + \tau s(t)]\, dt$$

$$\text{subject to } \dot{s}(t) = g(s(t)) - h(t) \quad (0 \leq h(t) \leq \bar{h},\ 0 < t < T)$$

and the endpoint conditions $s(0) = s_0$, $s(T) = s_1$. Here $\bar{h}$ is the maximal feasible harvest rate.

(b) $\widetilde{H}(h, s, \mu, t) = ph + \tau s + \mu[g(s) - h]$.

(c) The costate equation is $\dot{\mu} = \rho\mu - \tau - \mu g'(s)$. By the maximum principle $h(t)$ is chosen to

$$\text{maximise } [p(t) - \mu(t)]h(t) \text{ subject to } 0 \leq h(t) \leq \bar{h}.$$

The solution may be split into time-intervals of the same type as in the example in the text. Inside an interval of type (iii), $\mu(t) = p(t)$; therefore, by the costate equation,

$$\dot{p} = \rho p - \tau - pg'(s).$$

This is the equation for the singular solution.

(d) If p is constant, the equation for the singular solution reduces to $g'(s) = -\theta$, where

$$\theta = (\tau/p) - \rho.$$

In the logistic case, $g'(s) = r(1 - 2k^{-1}s)$ and the singular solution is

$$\tilde{s} = \frac{(r+\theta)k}{2r}.$$

(e) (i) If $\theta \geq r$, i.e. $\tau \geq (\rho + r)p$, then $\tilde{s} \geq k$. In this case, the singular solution corresponds to not harvesting at all.

(ii) If $\theta \leq -r$, i.e. $\tau \leq (\rho - r)p$, then $\tilde{s} \leq 0$. In this case, the singular solution corresponds to extinction.

28.4 Problems with an infinite horizon

28.4.1 For parts (a)–(c), answers are as in Exercise 28.2.4 with T replaced by ∞, provided the relevant integral converges (if it doesn't, the problem has no solution). For part (d), $\mu(t) = \bar{\mu}$ for all t, where $\bar{\mu} = \bar{\pi}/(r + \delta)$. Assuming that $\bar{\mu} > a$, $I(t) = \dfrac{\bar{\mu} - a}{2b}$ for all t. The transversality condition for this problem is similar to (28.11), with ρ replaced by r. The condition is satisfied because $r > 0$, μ is constant and $K(t) \to \dfrac{\bar{\mu} - a}{2b\delta}$ as $t \to \infty$.

28.4.2 The maximum principle gives the same expression for $I(t)$, given $K(t)$ and $\mu(t)$, as in Exercise 28.2.5. As in Exercise 28.4.1, $\mu(t) = \bar{\pi}/(r + \delta)$ for all t. Hence $I(t)/K(t) = J$ for all t, where the constant J is given by

$$J = \begin{cases} \theta\bar{\pi}/a & \text{if } \bar{\pi} > (r+\delta)a, \\ 0 & \text{otherwise.} \end{cases}$$

28.4.3 Let K^* be as in the model of the text, i.e.

$$K^* = \left[\frac{\alpha A}{\rho + \delta}\right]^{1/(1-\alpha)}.$$

If $K = K^*$, $C = C^*$, where C^* is as in the text; then K remains at K^*. This 'singular solution', in the sense of Section 28.3, corresponds to μ taking the value 1. If $K < K^*$, $\mu > 1$ and $C = 0$; K then rises, attaining the value K^* in finite time. If $K > K^*$, $\mu < 1$ and $C = AK^\alpha$; K then falls, attaining the value K^* in finite time.

28.4.4 (a) Recall from the text that

$$\psi'(K) = -\frac{U'(C)}{U''(C)} \frac{\alpha A K^{\alpha-1} - (\rho+\delta)}{AK^\alpha - \delta K - C}.$$

Hence $\psi'(K^*) = \ell C^*/\gamma$, where

$$\begin{aligned} \ell &= \lim_{K \to K^*} \frac{\alpha A K^{\alpha-1} - (\rho+\delta)}{AK^\alpha - \delta K - C} \\ &= \lim_{K \to K^*} \frac{\alpha(\alpha-1)AK^{\alpha-2}}{\alpha A K^{\alpha-1} - \delta - (dC/dK)} \quad \text{by l'Hôpital's rule} \\ &= \frac{\alpha - 1}{K^*} \cdot \frac{\alpha A (K^*)^{\alpha-1}}{\alpha A (K^*)^{\alpha-1} - \delta - \psi'(K^*)}. \end{aligned}$$

Setting $z = \psi'(K^*)$ and recalling that

$$A(K^*)^{\alpha-1} = \frac{\rho+\delta}{\alpha} = \frac{C^*}{K^*} + \delta,$$

we see that

$$z = \frac{\ell C^*}{\gamma} = \frac{\alpha-1}{\gamma} \cdot \frac{C^*}{K^*} \cdot \frac{\rho+\delta}{\rho-z} = \frac{1-\alpha}{\gamma} \cdot \frac{\rho+(1-\alpha)\delta}{\alpha} \cdot \frac{\rho+\delta}{z-\rho}.$$

Hence z satisfies the quadratic equation

$$z^2 - \rho z - \phi = 0,$$

where

$$\phi = \frac{1-\alpha}{\alpha\gamma}(\rho+\delta)(\rho+[1-\alpha]\delta).$$

Since $0 < \alpha < 1$, $\phi > 0$, so the quadratic equation has two real roots of opposite sign. We know from Figure 28.3 that the stable branch is upward-sloping in the vicinity of the point (K^*, C^*), so the positive root should be taken.

(b) $\psi(K) \approx C^* + z(K - K^*)$, where $z = \psi'(K^*)$ as in (a).

(c) Let $\zeta = \alpha C^*/K^* = \rho + (1-\alpha)\delta$. Then

$$s(K^*) = 1 - \frac{\zeta}{\alpha A(K^*)^{\alpha-1}} = 1 - \frac{\rho+(1-\alpha)\delta}{\rho+\delta} = \frac{\alpha\delta}{\rho+\delta}.$$

For any K, $s'(K) = \dfrac{1}{AK^\alpha}\left[\dfrac{\alpha C}{K} - \dfrac{dC}{dK}\right]$. Thus $s'(K^*)$ has the same sign as $\zeta - z$, where ζ is as above and z is as in (a) and (b). Now $\zeta - z$ has the same sign as $\zeta^2 - \rho\zeta - \phi$ (to see this, sketch the graph of the function $y = x^2 - \rho x - \phi$), and

$$\zeta^2 - \rho\zeta - \phi = (1-\alpha)\zeta\left\{\delta - \frac{\rho+\delta}{\alpha\gamma}\right\}.$$

Therefore $s'(K^*)$ has the same sign as $\gamma - \gamma_0$, where $\gamma_0 = \dfrac{\rho+\delta}{\alpha\delta} = \dfrac{1}{s(K^*)}$. Note that $\gamma_0 > 1$; thus in the case where $U(C) = \ln C$, i.e. $\gamma = 1$, $s(K)$ is a decreasing function of K for values of K close to K^*.

29 INTRODUCTION TO ANALYSIS

29.1 Rigour

29.1.1 $Q \Rightarrow P$, $P \Leftrightarrow R$ and $R \Rightarrow S$; hence $Q \Rightarrow R$ and $P \Rightarrow S$.

29.1.2 Let P_n be the propostion to be proved. P_1 is obvious, so it remains to prove that $P_n \Rightarrow P_{n+1}$. Let $a > 0$, $n \in \mathbb{N}$ and suppose P_n holds. Then

$$\begin{aligned}(1+a)^{n+1} &= (1+a)(1+a)^n \\ &\geq (1+a)(1+na+\tfrac{1}{2}n(n-1)a^2) \quad \text{by } P_n \\ &\geq (1+a)(1+na) + \tfrac{1}{2}n(n-1)a^2 \quad \text{since } a^3 > 0 \\ &= 1+(n+1)a + \left[1+\tfrac{1}{2}(n-1)\right]na^2 \\ &= 1+(n+1)a+\tfrac{1}{2}(n+1)na^2,\end{aligned}$$

so P_{n+1} holds as required.

29.1.3 Let P_n be the proposition to be proved. P_1 is obvious, so it remains to prove that $P_n \Rightarrow P_{n+1}$. Suppose P_n holds for some $n \geq 1$. Let $x_0, x_1, \ldots, x_n$ be members of I and let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be $n+1$ positive numbers that sum to 1. Let $\lambda = 1 - \alpha_0$; then $0 < \lambda < 1$. Let $\beta_i = \alpha_i/\lambda$ for $i = 1, \ldots, n$; then $\beta_1, \ldots, \beta_n$ are n positive numbers that sum to 1. Let $u = \beta_1 x_1 + \ldots + \beta_n x_n$; then by P_n,

$$f(u) \geq \beta_1 f(x_1) + \ldots + \beta_n f(x_n). \tag{$*$}$$

But then

$$\begin{aligned} f(\alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n) &= f((1-\lambda)x_0 + \lambda u) \\ &\geq (1-\lambda)f(x_0) + \lambda f(u) \quad \text{since } f \text{ is concave} \\ &\geq (1-\lambda)f(x_0) + \lambda\beta_1 f(x_1) + \ldots + \lambda\beta_n f(x_n) \quad \text{by } (*) \\ &= \alpha_0 f(x_0) + \alpha_1 f(x_1) + \ldots + \alpha_n f(x_n), \end{aligned}$$

and P_{n+1} holds as required.

29.1.4 $I \times J$ is the square with corners $(0,1)$, $(0,3)$, $(2,3)$ and $(2,1)$. $J \times I$ is the square with corners $(1,0)$, $(3,0)$, $(3,2)$ and $(1,2)$.

29.2 More on the real number system

29.2.1 A is bounded above and has a greatest member: $\max A = \sup A = 2$. A is also bounded below and has a least member: $\min A = \inf A = -2$. Answers for B are as for A.

C is not bounded above and therefore has no greatest member. C is bounded below and has a least member: $\min C = \inf C = 1$. D is not bounded above and therefore has no greatest member. D is bounded below, with $\inf D = 0$, but has no least member.

29.2.2 Let $u = \sup S$, $x \in \mathbb{R}$; we must show that $x \geq u$ if and only if x is an upper bound for S. 'If' is true because u is the *least* upper bound for S. 'Only if' is true because u is *an* upper bound for S: $u \geq s \; \forall \, s \in S$. If $x \geq u$, then $x \geq s \; \forall \, s \in S$, so x is indeed an upper bound for S.

29.2.3 $a^2 < 2 < b^2$ and $a^2 - b^2 = (a-b)(a+b)$; hence $(a-b)(a+b) < 0$. But since a and b are positive, $a + b > 0$. Therefore $a - b < 0$.

29.2.4 (a) Apply (29.2) with y replaced by $-y$, recalling that $|-y| = |y|$.

(b) $|x| = |x - y + y| \leq |x - y| + |y|$ by (29.2).

(c) Let $z = |x| - |y|$. By (b), $z \leq |x - y|$. Interchanging x and y, $-z \leq |y - x|$. But $|y - x| = |x - y|$. Therefore $\max(z, -z) \leq |x - y|$, as required.

(d) Suppose $y \neq 0$ and let $z = x/y$. Then $|x| = |yz| = |y|\,|z|$ by (29.1); now divide through by $|y|$.

29.3 Sequences of real numbers

29.3.1 Yes and no respectively. To prove the latter, let $u = 0$, $x_n = -n^{-1} \; \forall \, n \in \mathbb{N}$. To prove the former, let ε be any positive real number; then $u < x_n < x + \varepsilon$ for all sufficiently large n, so $u - x < \varepsilon$. Since this is true for any positive ε, however small, $u - x \leq 0$.

29.3.2 Let $\varepsilon > 0$. Since $a_n \to x$, we may choose a positive integer N_1 such that $x - \varepsilon < a_n < x + \varepsilon \; \forall n > N_1$. Since $b_n \to x$, we may choose a positive integer N_2 such that $x - \varepsilon < b_n < x + \varepsilon \; \forall n > N_2$. Let $N = \max(N_1, N_2)$. Then, for all $n > N$,

$$x - \varepsilon < a_n \leq x_n \leq b_n < x + \varepsilon.$$

Hence $|x_n - x| < \varepsilon \; \forall n > N$. Since this argument is valid for every positive ε, $x_n \to x$.

29.3.3 Suppose $0 < b < 1$; then $b^{-1} > 1$. Applying the given inequality with $a = b^{-1} - 1$, $b^{-n} > n(b^{-1} - 1)$. Hence

$$0 < b^n < \left(\frac{b}{1-b}\right)\frac{1}{n} \quad \text{for all } n,$$

so $b^n \to 0$ as required.

29.3.4 (a), (b) and (c) are true. (d) is false: if x_n is 0 for all even n and 1 for all odd n, then $\lim_{n\to\infty} x_{2n} = 1$ but the sequence $\{x_n\}$ does not converge.

29.3.5 Let $x_n \to x$. Let ε be any positive real number; then we may choose $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon/2 \;\; \forall n > N$. If $m > N$ and $n > N$, then

$$|x_m - x_n| \leq |x_m - x| + |x - x_n| < \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence.

29.4 Continuity

29.4.1 Let $y_n = |x_n| - |x|$. From Exercise 29.2.4(c), $|y_n|$ is squeezed between 0 and $|x_n - x|$ for all n and therefore converges to 0, so $|x_n| \to |x|$.

Let f be continuous and let $x_0 \in I$. By **SQ8**, $f(x_n) \to f(x_0)$ for every sequence $\{x_n\}$ of members of I that converges to x_0. Hence, by first part, $|f(x_n)| \to |f(x_0)|$ for every such sequence. Hence, by **SQ8**, $|f|$ is continuous.

29.4.2 Let $\{x_n\}$ be a sequence in $[a, b]$ that converges to a member x of $[a, b]$; by **SQ8**, it suffices to show that $F(x_n) \to f(x)$. Let $y_n = f(x_n)$ for $n = 1, 2, \ldots$, and let $y = f(x)$. Since f is continuous, it follows from **SQ8** that $\{y_n\}$ is a sequence in $[c, d]$ that converges to the member y of $[c, d]$; and since g is continuous, it follows from **SQ8** that $g(y_n) \to g(y)$. But then $g(f(x_n)) \to g(f(x))$, which is the required result.

29.4.3 (a) Since $\{x_n\}$ is a Cauchy sequence it converges, say to x_0. Since $1 \geq x_n \geq 0 \; \forall n \in \mathbb{N}$, $1 \geq x_0 \geq 0$; thus $x_0 \in I$. Since f is continuous, $f(x_n) \to f(x_0)$ by **SQ8**. Hence $\{f(x_n)\}$ is a Cauchy sequence.

(b) No. Let $x_n = n^{-1}$. Then the sequence $\{x_n\}$ is convergent and therefore Cauchy, but since the limit is not in I the argument of (a) no longer applies. Indeed, if we let $f(x) = 1/x$ then $|f(x_m) - f(x_n)| \geq 1$ whenever $m \neq n$, so $\{f(x_n)\}$ is not a Cauchy sequence.

30 METRIC SPACES AND EXISTENCE THEOREMS

30.1 Metric spaces

30.1.1 Putting $y = x$ in **M0** and applying **M2**, we see that $2d(x, z) \geq 0$ for all x, z in X; this implies **M1**. Putting $z = x$ in **M0** and applying **M2**, we see that $d(x, y) \leq d(y, x)$. Since this is so for all x, y in X it remains true if x and y are interchanged, so **M3** holds as required.

30.1.2 (a) The easiest method is to use the result of Exercise 30.1.1. **M2** clearly holds, so it remains to prove **M0**. Let x, y, z be points of X and let

$$p = d(x, y), \quad q = d(x, z) + d(y, z).$$

We wish to show that $p \leq q$. This is obvious if $p = 0$. If $p = 1$ then $x \neq y$; but then at least one of x and y is distinct from z, so $q \geq 1$.

(b) We proceed as in (a), with the same notation. If $p \leq 1$, the argument of (a) applies. If $p = 2$ then x, y, ω are all different and z is x, or y, or ω or none of them; the corresponding values of q are respectively 2, 2, 2 and 4.

30.1.3 Suppose $x_n \to x$ and $x_n \to y$; we want to show that $x = y$. Let $\varepsilon > 0$, and let M, N be integers such that

$$d(x_n, x) < \varepsilon \; \forall \, n > M, \quad d(x_n, y) < \varepsilon \; \forall \, n > N.$$

Let $n > \max(M, N)$: by **M3** and **M4**, $d(x, y) \leq d(x_n, x) + d(x_n, y) < 2\varepsilon$. Since $d(x, y) < 2\varepsilon$ for all positive ε, $d(x, y) \leq 0$. But then $x = y$ by **M1** and **M2**.

30.1.4 Each element of the sequence is a point on the circle of radius 1 whose centre is at the origin. The sequence is therefore bounded: by the Bolzano–Weierstrass theorem, it has a convergent subsequence. Since each member of the sequence is obtained from the previous one by a rotation through 1 radian, the distance between the nth and $n+1$th members of the sequence is the same positive number for all n. Hence the sequence is not a Cauchy sequence, and is therefore not convergent.

30.1.5 (a) We want to prove that

$$-d(y, z) \leq d(x, y) - d(x, z) \leq d(y, z). \tag{$*$}$$

The left-hand inequality in $(*)$ may be written $d(x, z) \leq d(x, y) + d(y, z)$; this is just **M4**, with y and z interchanged. The right-hand inequality in $(*)$ may be written $d(x, y) \leq d(x, z) + d(y, z)$; this follows immediately from **M3** and **M4**.

(b) By **M3**,

$$\begin{aligned} |d(x, y) - d(z, w)| &= |d(y, x) - d(y, z) + d(z, y) - d(z, w)| \\ &\leq |d(y, x) - d(y, z)| + |d(z, y) - d(z, w)|, \end{aligned}$$

and the result now follows from (a).

(c) From (b), $|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y)$. The result now follows from **SQ1** and **SQ2** in the preceding chapter.

30.1.6 $f(0) = 1$, $f(x) = 0$ if $0 < x \leq 1$. Yes. No.

30.1.7 Let $\varepsilon > 0$. Since convergence is uniform, we may choose an integer m such that $|f_m(x) - f(x)| < \varepsilon/3$ for all x in $[a, b]$. Since f_m is continuous at x_0 we may choose $\delta > 0$ with the property that $|f_m(x) - f_m(x_0)| < \varepsilon/3$ for all x such that $a \leq x \leq b$ and $|x - x_0| < \delta$. Then for all such x,

$$|f(x) - f(x_0)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since this argument is valid for every $\varepsilon > 0$, f is continuous at x_0.

If each f_n is a continuous function then the argument above is valid for every $x_0 \in [a, b]$, so f is also continuous.

30.1.8 If convergence were uniform then f would be continuous, and we know that it isn't.

30.2 Open, closed and compact sets

30.2.1 A set A in X that is *not* open must contain a point with a certain property (specifically, every open ball with that point as centre also contains a point of A^c). A set B in X that is *not* compact must contain a sequence of points with a certain property (specifically, no convergent subsequence). Since $\varnothing$ contains no points, it is both open and compact. Being compact, it is closed and bounded.

30.2.2 Denoting such a set by S, $\sup S$ is a boundary point. Since S is closed, $\sup S \in S$ and so S has a greatest member.

30.2.3 The complement of $(0, 1]$ in P is the set $Y = \{y \in P : y > 1\}$. If $y \in Y$ and $0 < \delta \leq y - 1$, then the open interval $(y - \delta, y + \delta)$ is contained in Y. Hence Y is open in P, so $(0, 1]$ is closed. Also $(0, 1]$ is contained in the open interval $(0, 2)$ and hence is bounded. The sequence $\{n^{-1}\}$ has no subsequence that converges to a point of $(0, 1]$, so $(0, 1]$ is not compact.

30.2.4 In each case, the boundary points are the points on the parabola $y = x^2$ with $-1 \leq x \leq 1$, and the points on the line $y = 1$ with $-1 \leq x \leq 1$.

(a) Closed since it contains all its boundary points.

(b) Open since it contains none of its boundary points.

(c) Neither since it contains some but not all its boundary points.

30.2.5 (a) Suppose that X is contained in the open ball in $\mathbb{R}^\ell$ with centre x_0 and radius r, and Y is contained in the open ball in $\mathbb{R}^m$ with centre y_0 and radius s. Then $X{\times}Y$ is contained in the open ball in $\mathbb{R}^{\ell+m}$ with centre (x_0, y_0) and radius $\sqrt{r^2 + s^2}$.

(b) Let X and Y be closed sets, and let $\{(x_n, y_n)\}$ be a sequence in $X{\times}Y$ converging to a point (x_0, y_0) of $\mathbb{R}^{\ell+m}$. To prove that $X{\times}Y$ is closed it suffices, by Proposition 2, to show that $(x_0, y_0) \in X{\times}Y$. Since each component sequence of $\{(x_n, y_n)\}$ converges to the corresponding component of (x_0, y_0), $x_n \to x_0$ and $y_n \to y_0$. But X and Y are closed sets. Therefore $x_0 \in X$ and $y_0 \in Y$, so $(x_0, y_0) \in X{\times}Y$ as required.

(c) By the Bolzano–Weierstrass theorem, a subset of $\mathbb{R}^k$ is compact if and only if it is closed and bounded. Since this is so for $k = \ell$, $k = m$ and $k = \ell + m$, (c) follows immediately from (a) and (b). [(c) can also be proved directly from the definition of compactness, using an argument similar to that used to obtain the Bolzano–Weierstrass theorem from **SQ6**.]

30.2.6 (a) The proof is by contraposition. Let x_1 be a point such that $d(x_1, x_0) \neq r$; we wish to show that x_1 is *not* a boundary point of the open ball B with centre x_0 and radius r. If $d(x_1, x_0) < r$ then $x_1 \in B$, and the result follows from the fact that B is an open set. It remains to consider the case where x_1 belongs to the set $\{ x \in X : d(x, x_0) > r \}$. Since this set is open, there exists an open ball C with centre x_1 such that $d(x, x_0) > r \; \forall \, x \in C$. Hence x_1 is not a boundary point of B.

(b) Let $S = \{ \mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| = 1 \}$. By (a), every boundary point of B is in S; we must prove the converse. Let $\mathbf{x} \in S$ and let D be an open ball with centre $\mathbf{x}$. Let the radius of D be ρ, and let α be a real number such that $0 < \alpha < \min(1, \rho)$. Then $(1 - \alpha)\mathbf{x}$ and $(1 + \alpha)\mathbf{x}$ are points in D, $(1 - \alpha)\mathbf{x} \in B$ and $(1 + \alpha)\mathbf{x} \in B^c$.

Generalisation: the boundary of the open ball $\{ \mathbf{x} \in \mathbb{R}^m : \|\mathbf{x} - \mathbf{x}^0\| < r \}$ is the set $\{ \mathbf{x} \in \mathbb{R}^m : \|\mathbf{x} - \mathbf{x}^0\| = r \}$.

(c) Let B be the open ball with centre x_0 and radius 1. As in any metric space, x_0 is not a boundary point of B. In this case, B has just one member, namely x_0. If $x_1 \neq x_0$, then the open ball with centre x_1 and radius 1 also has just one member, namely x_1, and therefore contains no member of B; thus x_1 is not a boundary point of B. It follows that B has no boundary points; by contrast, $\{ x \in X : d(x, x_0) = 1 \}$ is the non-empty set of all points in X other than x_0.

30.2.7 Immediate from Proposition 2.

30.2.8 Suppose A is closed in X. Then every Cauchy sequence in A converges to a point in X (by completeness of X) which is also in A (by Proposition 2). Hence the metric space A is complete. Conversely, suppose that A is not closed. Then by Proposition 2 we may choose a sequence of points in A which converges to a point of X that is not in A. We then have a Cauchy sequence in A that does not converge to a point of A, so the metric space A is not complete.

30.3 Continuous mappings

30.3.1 (a) Let $x_0 \in X$ and let $C \subset X'$ be an open ball with centre $f(x_0)$. Let the radius of C be r, and let $B \subset X$ be the open ball with centre x_0 and radius r/β. Then for any $x \in B$,

$$d'(f(x), f(x_0)) \leq \beta \, d(x, x_0) < r,$$

so $f(x) \in C$.

(b) $|x^2 - y^2| = |x + y||x - y|$. Since $|x + y|$ can be as large as we like, f does not have the property mentioned in (a), but f is continuous.

(c) If $X = [0, 1]$, $|x^2 - y^2| \leq 2|x - y|$; f now does have the property mentioned in (a), and is therefore continuous.

30.3.2 (a) Let $x \in X$ and suppose $x_n \to x$. Define a second sequence $\{w_n\}$ by $w_n = x_0 \; \forall \, n \in \mathbb{N}$. Then, by Exercise 30.1.5 part (c), $d(w_n, x_n) \to d(x_0, x)$, so $f(x_n) \to f(x)$. It follows that f is continuous at x and hence on all of X.

(b) Let f be as in (a); then f is continuous. Since K is compact we may apply Weierstrass's theorem: there exists $y_0 \in K$ such that $f(y_0) \leq f(y) \; \forall \, y \in K$. The result follows.

(c) Choose any $\mathbf{x}_1 \in A$ and let $r = \|\mathbf{x}_1 - \mathbf{x}_0\|$, $K = \{\, \mathbf{x} \in A : \|\mathbf{x} - \mathbf{x}_0\| \le r \,\}$. If we can show that K is compact, then the result follows from (b). Let $\{\mathbf{y}_n\}$ be a sequence of points in K that converges to a point $\mathbf{y}$ in $\mathbb{R}^m$. Then $\mathbf{y} \in A$ since A is closed, and $\|\mathbf{y} - \mathbf{x}_0\| \le r$ by (a), so $\mathbf{y} \in K$. Hence K is closed. Since K is obviously bounded, K is compact by the Bolzano–Weierstrass theorem.

30.3.3 For this exercise, it is helpful to use the notation for half-open intervals introduced in Exercise 30.2.3. In general, a half-open interval is a set of the form

$$[a, b) = \{\, x \in \mathbb{R} : a \le x < b \,\} \quad \text{or} \quad (a, b] = \{\, x \in \mathbb{R} : a < x \le b \,\},$$

where a, b are real numbers such that $a < b$. Such a set is neither open nor closed in the metric space $\mathbb{R}$, since it contains just one of its two boundary points.

(a) f is continuous and I is open. Sketching the graph of f using the methods of Chapter 8, one can see that $\{\, f(x) : x \in I \,\}$ is the half-open interval $[0, 4)$, which is not an open set in $\mathbb{R}$.

(b) f is continuous and Z is a closed subset of X. $\{\, f(x) : x \in Z \,\}$ is the half-open interval $(0, 1]$, which is not a closed subset of Y.

Thus generalisations replacing ‘compact’ by ‘open’ or ‘closed’ are not true.

30.4 Fixed point theorems

30.4.1 (a) $|F(x) - F(y)| = \frac{1}{2}|x - y|$; therefore F is a contraction mapping.

(b) $\frac{1}{2}x = x$ only for $x = 0$, and 0 is not a member of X.

(c) The metric space X is not complete.

30.4.2 Let $\mathbf{a}$, $\mathbf{x}$ and $\mathbf{y}$ be m–vectors, and suppose $0 \le \alpha \le 1$. Then

$$\begin{aligned} \|\alpha\mathbf{x} + (1-\alpha)\mathbf{y} - \mathbf{a}\| &= \|\alpha(\mathbf{x} - \mathbf{a}) + (1-\alpha)(\mathbf{y} - \mathbf{a})\| \\ &\le \|\alpha(\mathbf{x} - \mathbf{a})\| + \|(1-\alpha)(\mathbf{y} - \mathbf{a})\| \quad \text{by (30.1)} \\ &= \alpha\|\mathbf{x} - \mathbf{a}\| + (1-\alpha)\|\mathbf{y} - \mathbf{a}\|. \end{aligned}$$

If $\mathbf{x}$ and $\mathbf{y}$ belong to the closed ball B with centre $\mathbf{a}$ and radius r, then $\|\mathbf{x} - \mathbf{a}\| = r - \beta$ and $\|\mathbf{y} - \mathbf{a}\| = r - \gamma$ for some non-negative numbers β and γ. Therefore

$$\|\alpha\mathbf{x} + (1-\alpha)\mathbf{y} - \mathbf{a}\| \le \alpha(r - \beta) + (1-\alpha)(r - \gamma) = r - [\alpha\beta + (1-\alpha)\gamma] \le r.$$

This shows that B is a convex set. The convexity of the open ball with centre $\mathbf{a}$ and radius r is proved similarly; in this case β and γ are positive, so $\alpha\beta + (1-\alpha)\gamma > 0$.

30.4.3 The rotation of a circle about its centre by, say, $\pi/3$ radians is a continuous mapping that has no fixed point.

30.4.4 X is nonempty because $\mathbf{0} \in X$, closed because it contains all its boundary points and bounded because it is contained in any open ball with centre $\mathbf{0}$ and radius greater than 1. Since X is closed and bounded, X is compact.

Suppose $\mathbf{x} \in X$, $\mathbf{y} \in X$ and $0 \le \alpha \le 1$. Let $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$. Clearly each component of $\mathbf{z}$ is non-negative, and

$$z_1 + \ldots + z_m = \alpha(x_1 + \ldots + x_m) + (1-\alpha)(y_1 + \ldots + y_m) = 1.$$

Thus X is convex. A similar argument shows that Y is a nonempty, compact, convex set in $\mathbb{R}^n$. The functions u and v are quadratic forms and therefore continuous. For each $\mathbf{y} \in Y$, the function $u(\cdot, \mathbf{y})\colon X \to \mathbb{R}$ is linear and therefore quasi-concave. Similarly, for each $\mathbf{x} \in X$, the function $v(\mathbf{x}, \cdot)\colon Y \to \mathbb{R}$ is quasi-concave. Thus the conditions of the theorem are all met and the result follows.

30.4.5 If $(\bar{x}, \bar{y})$ is a Nash equilibrium, then

$$|\bar{x} - y| \geq |\bar{x} - \bar{y}| \geq |x - \bar{y}|$$

for all $x, y \in [0, 1]$. But if the left-hand inequality is true for all $y \in [0, 1]$, then $\bar{x} = \bar{y}$, in which case the right-hand inequality is false whenever $x \neq \bar{y}$. Therefore, no Nash equilibrium exists.

$u(x, y)$ is not a quasi-concave function of x for given y.