The Butterfly Effect: Understanding the Unpredictable

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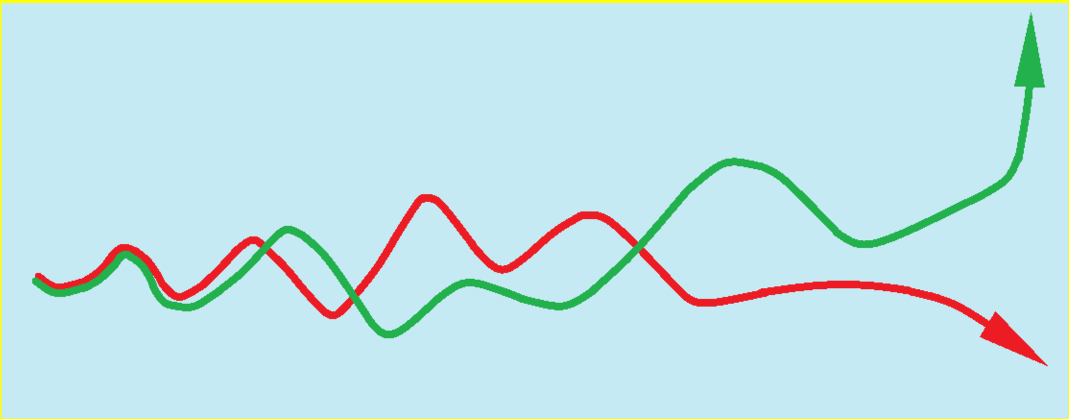
The Norah Boyce Lecture
Tuesday 10 May, St Edmund's College, Cambridge

Lorenz, 1963

"Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

Signature of chaos:
Small changes lead to bigger changes

So what is Chaos?



- An unexpected ‘random‘ response of a precise deterministic system
- With extreme sensitivity to how it is started … THE BUTTERFLY EFFECT
- Mathematically unsolvable, and computer simulations prone to large errors
- Long term prediction is impossible, but there is order within chaos
- Challenge: how to cope and maybe use it

Double Pendulum: a taste of chaos

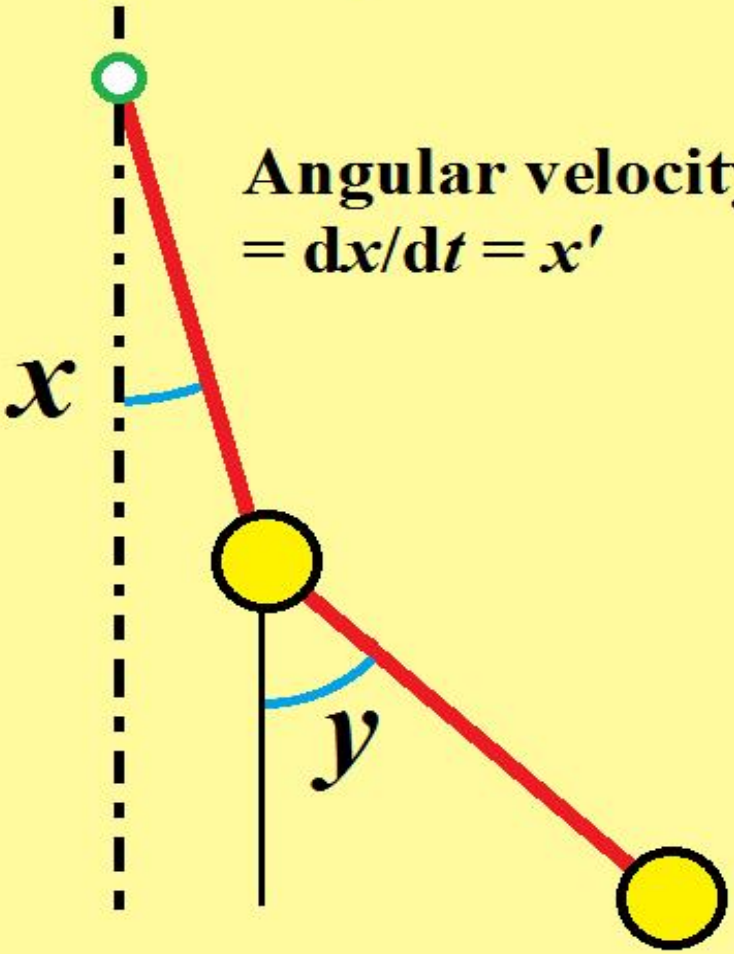
The double pendulum gives us a first glimpse of chaos.

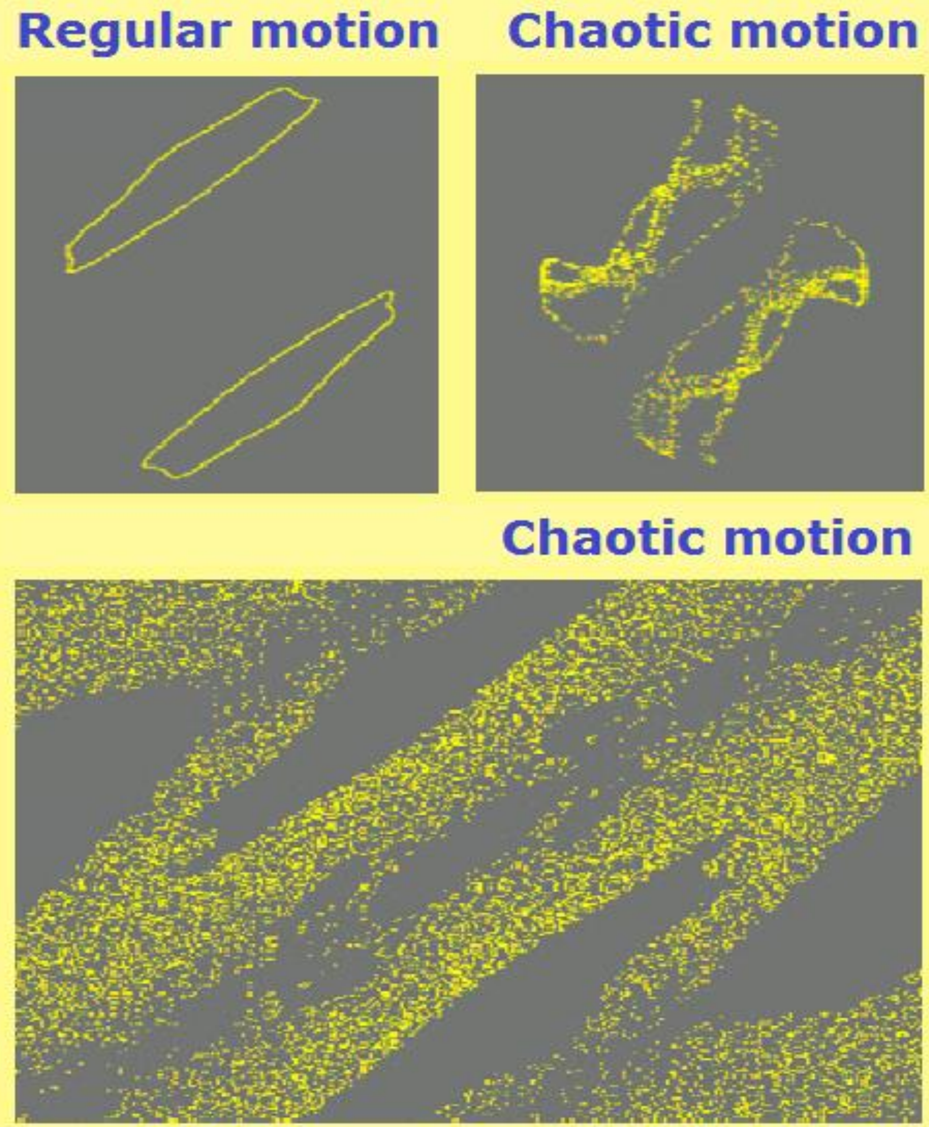
A real pendulum will have friction, air resistance, etc, which will dissipate energy: it will eventually stop.

But often it is useful to imagine a non-dissipative pendulum which oscillates for ever.

Undamped Double Pendulum in a Poincaré Section

Response depends on the start





Poincaré dots show *x* and *y* when $dx/dt = 0$

Outline of Talk

Impact of Newton

A little about phase space

Pioneers: Poincaré, Lorenz

Spaceflight for free

Introduction to fractals

Populations, Logistic map

Four concluding examples

Newton

1643-1727

Principia

1686

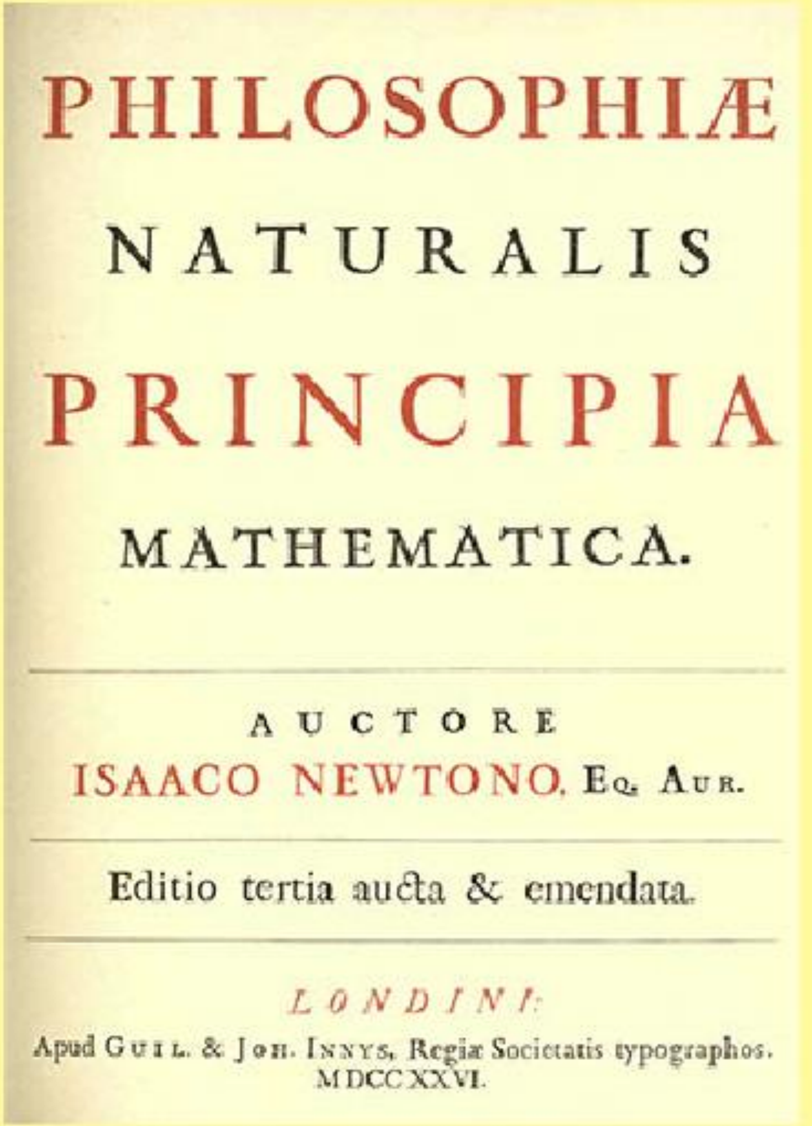
Newton's Laws of Motion revolutionised science.

Giving a dynamical system described by the calculus.

Still the ideal way to model a system evolving in time.

Starting conditions for the equations define a unique future.

But CHAOS has been found WITHIN this system.



PHILOSOPHIÆ

NATURALIS

PRINCIPIA

MATHEMATICA.

AUCTORE

ISAACO NEWTONO, Eq. Aur.

Editio tertia aucta & emendata.

LONDINI:

Apud Guil. & Joh. Innys, Regiæ Societatis typographos.

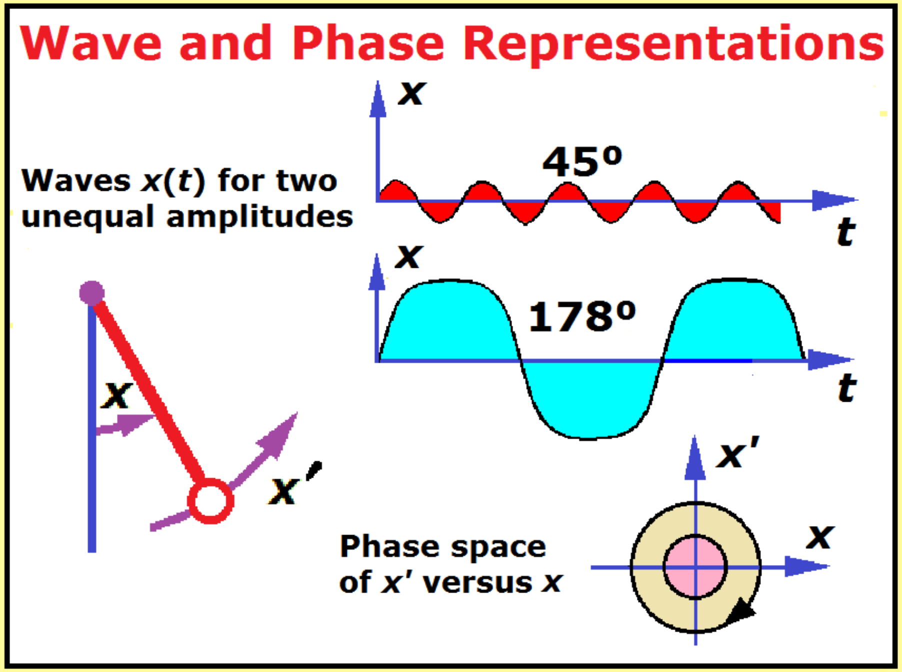
MDCCXXVI.

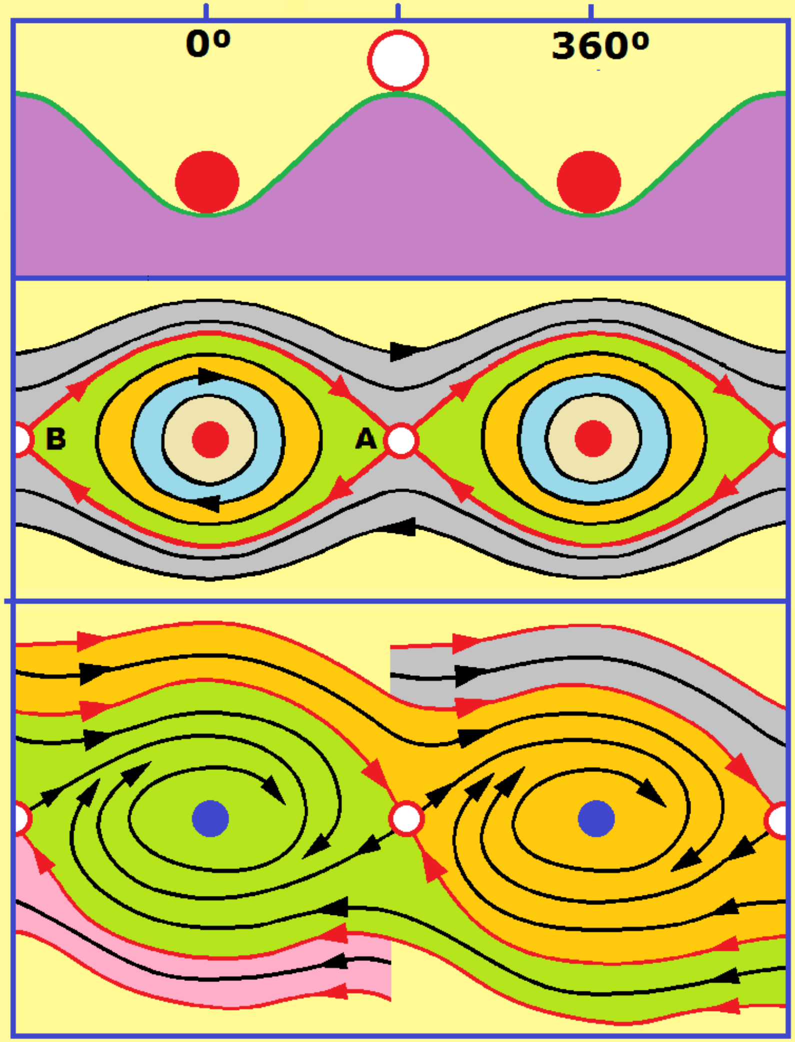
A quick look at Newtonian Dynamics

The Importance of Phase Space

The space of the starting conditions (x, x') full of trajectories, a unique path at every point

A pendulum is equivalent to a ball rolling on a wavy surface.

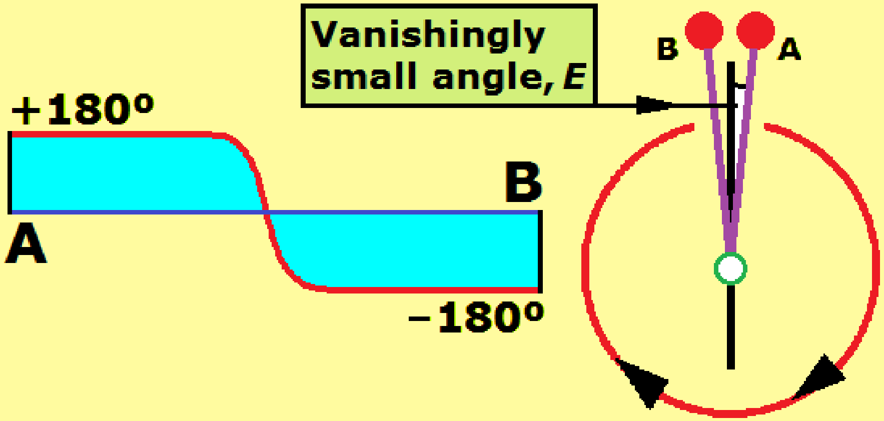


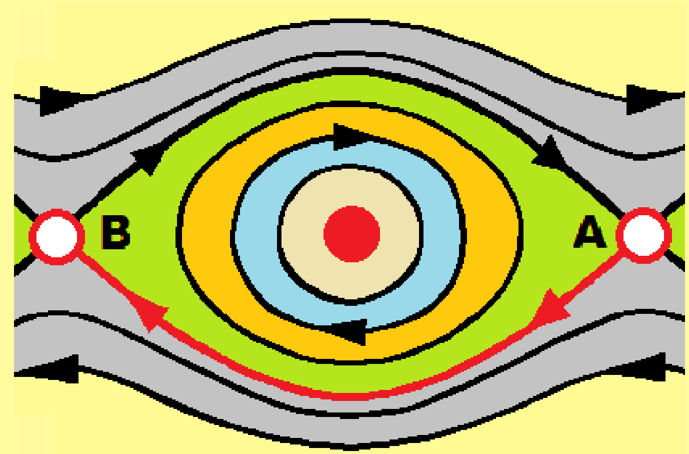


With Damping: Energy Dissipated

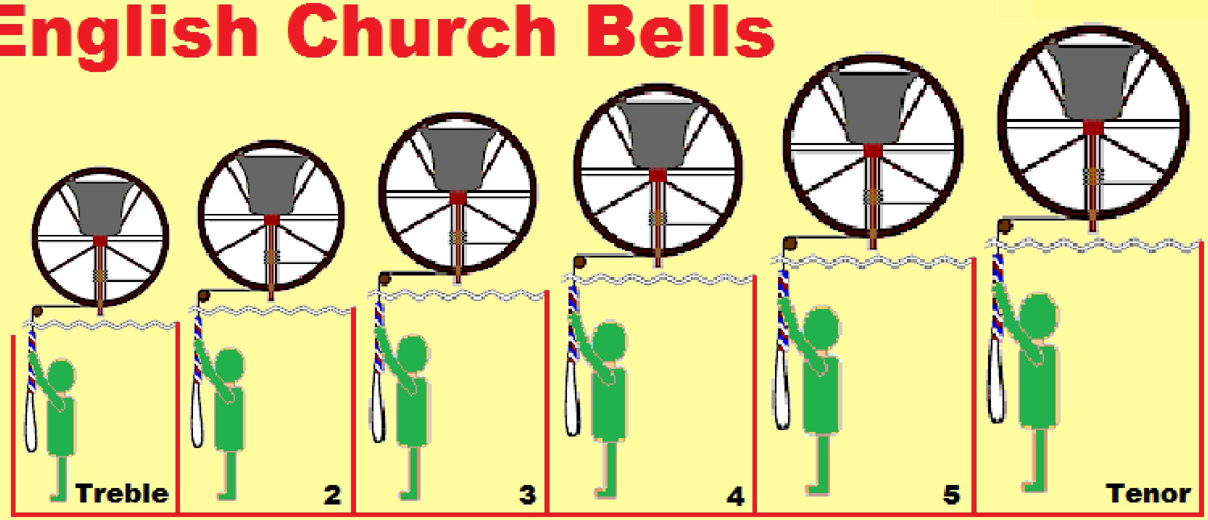
- **Stable equilibrium states are point attractors, sitting in coloured basins. Note the basin boundaries in red.**

Saddle connections play key roles in Generating Chaos

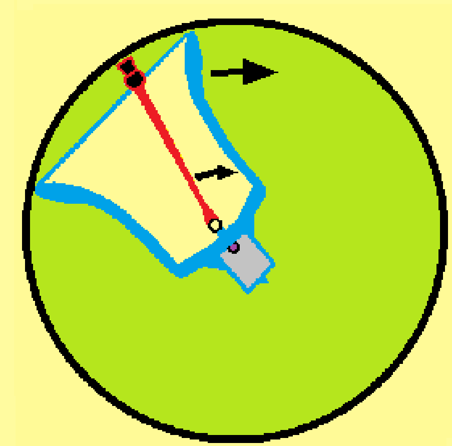




English Church Bells



A ringer rests the bell close to its inverted state. Then with little effort, but considerable skill, it can be tugged into its homoclinic trajectory. It swings through ~ 360° and back to 'rest'.



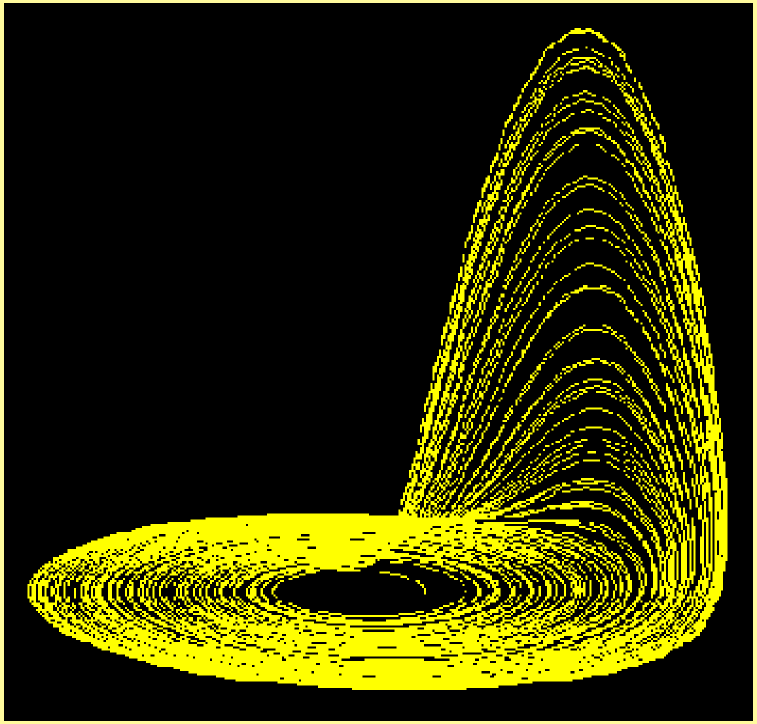
The clapper, rotating faster, is about to strike and ring the bell.

Phase Spaces with Higher Dimensions

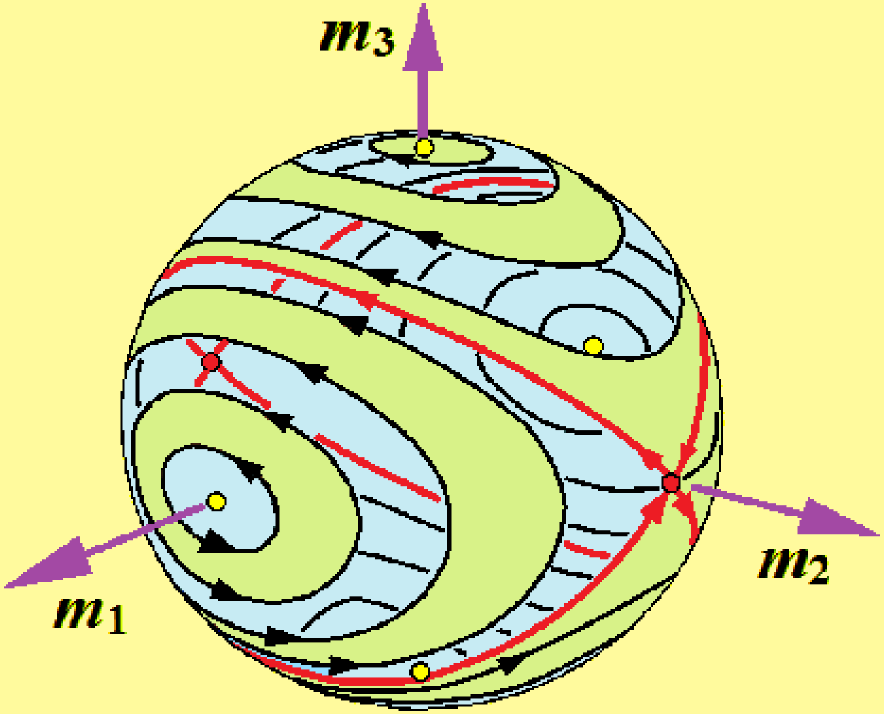
Chaos only exists in 3D or higher.
The double pendulum has 4D.

In 3D Rossler shows chaos created by repeated folding and mixing.

This creates an infinite number of thin layers: a fractal structure.



Rossler's Band (1976)



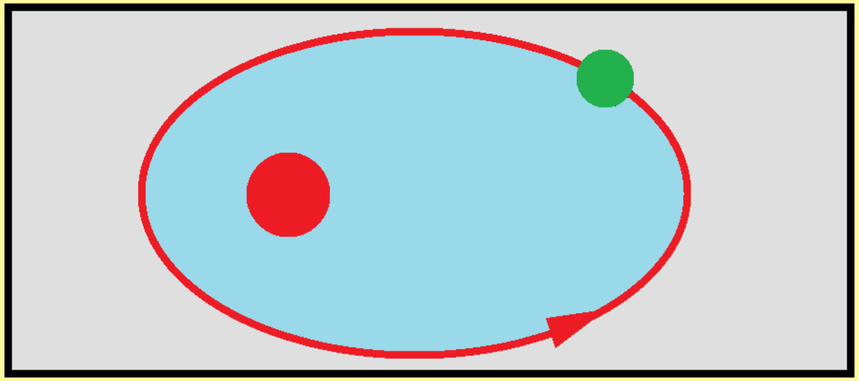
A Spinning Satellite

This has a 2D phase space, but it lies on the surface of a sphere (as on the left).

From Newton to Chaos

Newton's Laws allow precise solutions to two problems:

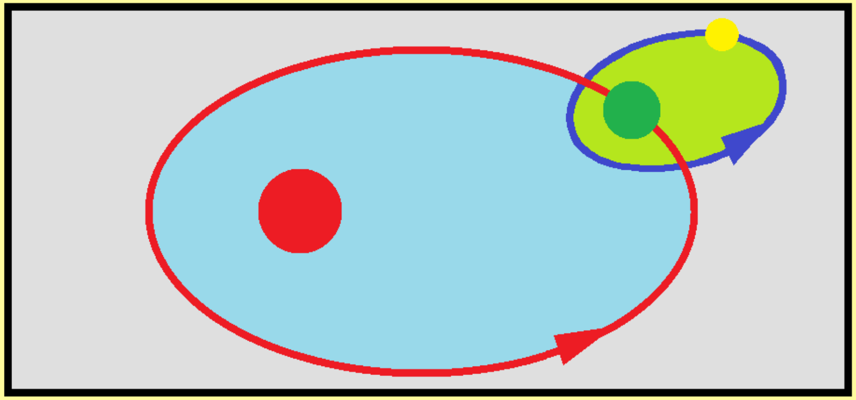
Simple pendulum
Sun-Earth (2 body) system



Newton solved 2-body case

One small step in complexity gives 'un-solvable' chaos for:

Double pendulum
Sun-Earth-Moon system



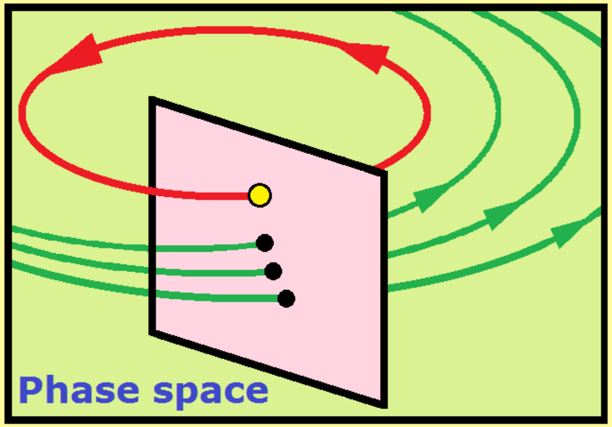
Henri Poincaré: 3-body case

BIRTH OF CHAOS

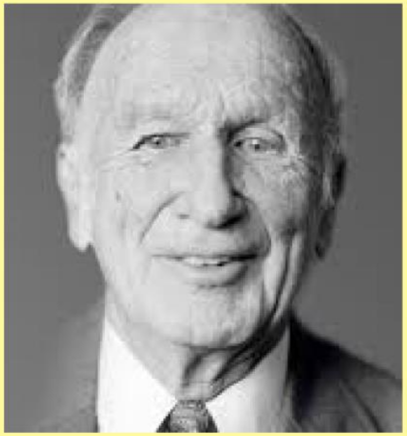
In 1887 the King of Sweden offered a prize for an answer to the issue 'is the solar system stable?'

Poincaré won for his 3-body studies, using his 'section' to prove that 'tangles' (our chaos) must occur.

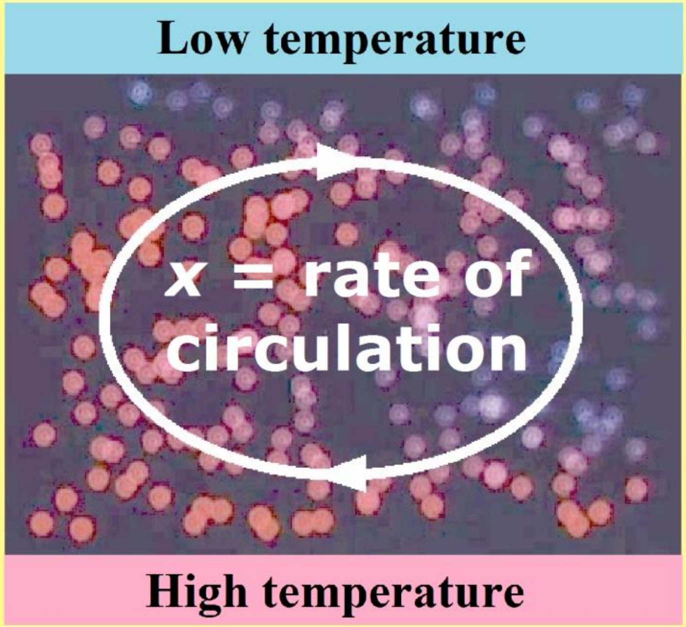
"I shall not even try to draw it ..."



Lorenz: Atmospheric convection in a box



(1917-2008)



Finds the BUTTERFLY EFFECT

In 1963 Ed Lorenz was trying to improve weather forecasting.

Using a recently available **computer** he discovered the first **chaotic attractor.**

He used three phase-variables (x, y, z) and a controlled thermal gradient, R.

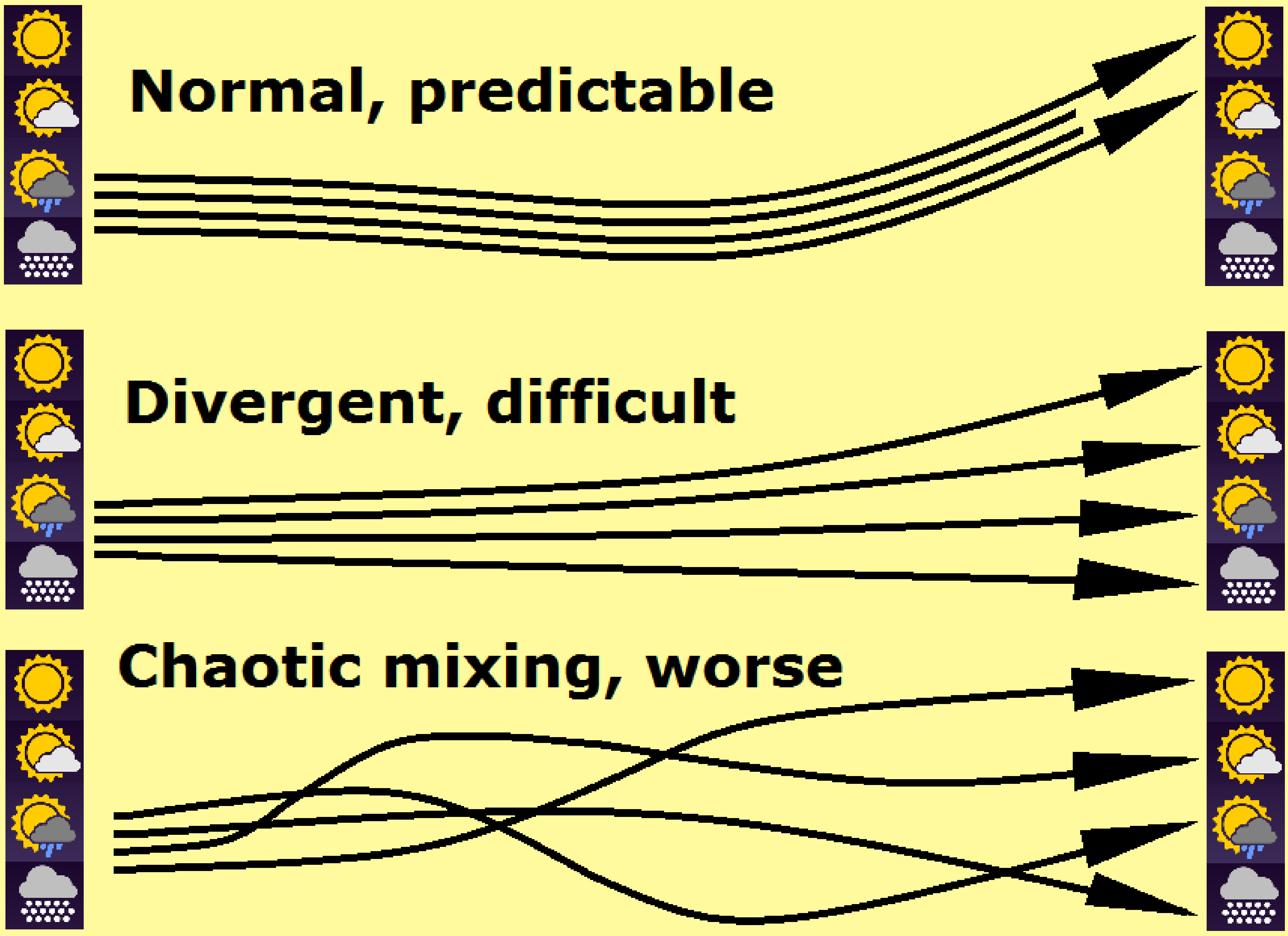
At fixed R, convection is represented by trajectories in the 3D phase space.

Convection starts at $R=1$.

At $R=28$, Lorenz showed that **all starts** settle onto a strange, chaotic attractor.

Divergence and mixing within the attractor make prediction impossible.

Headache for the Weather Man



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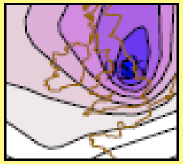
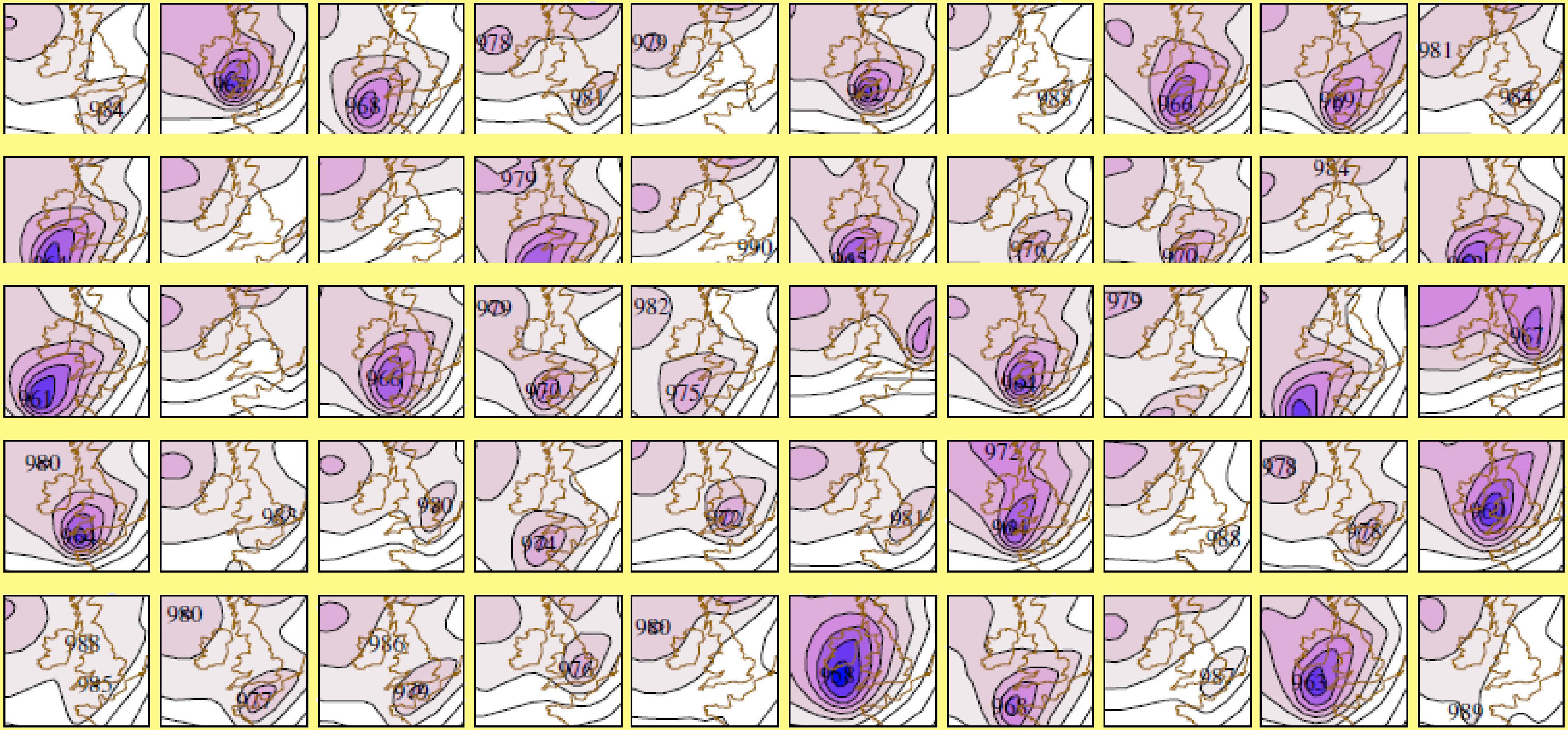
Chaos Theory now at the Heart of Weather Forecasting

BEFORE



FORECAST

Since Michael Fish (15-16 Oct, 1987)
Met Office uses probability forecasting



An ensemple of 50 forecasts with randomly perturbed starting winds, temperatures, pressures, etc.
Many develop deep depressions after 66 hours.

NASA's zero fuel highways: natural chaotic trajectories

Interplanetary Transport Network (ITN) IEEE 2002

Created and accessed by halo orbits around 3-body Lagrange points

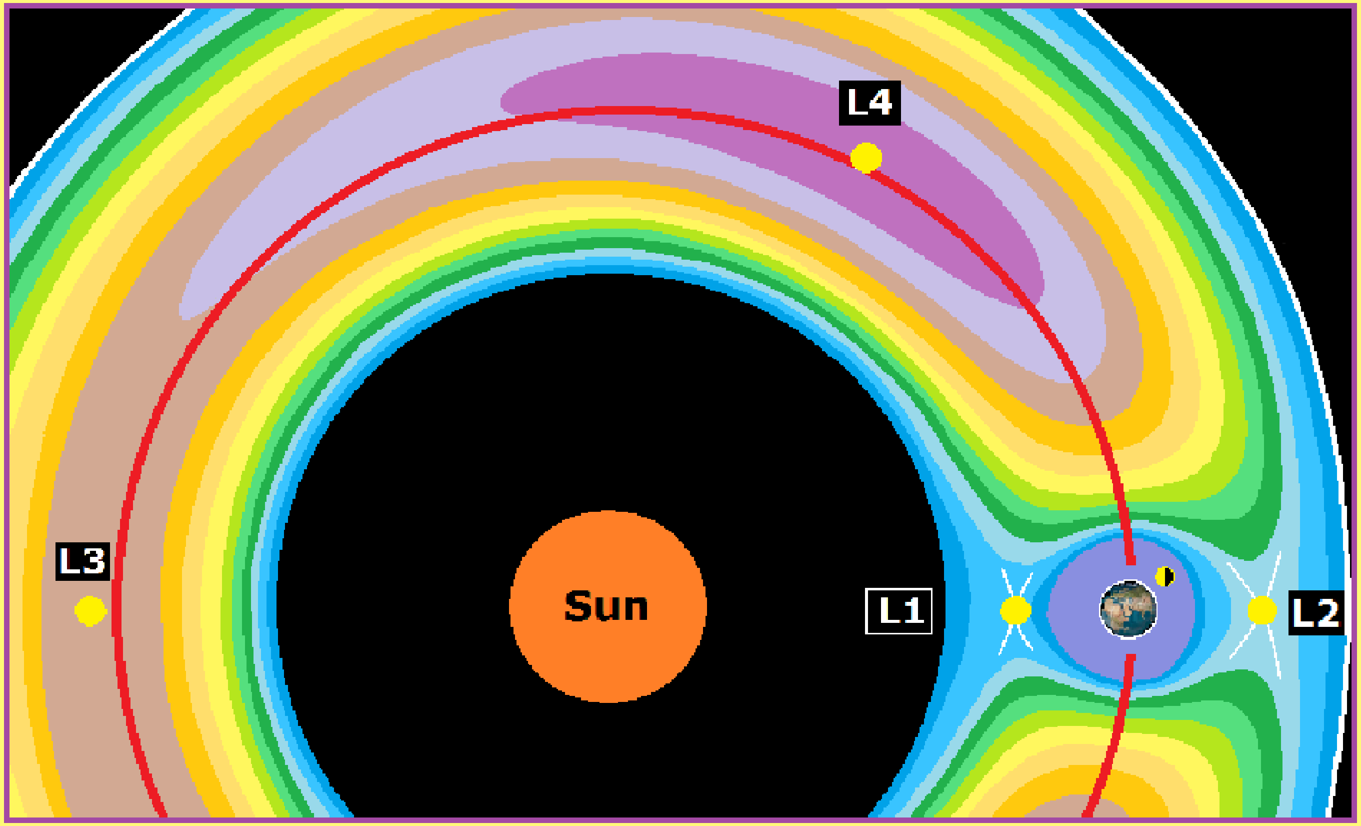
(A) Parking

Sun-Earth-Craft
in rotating frame

PE contours
(gravitational
+ centrifugal)

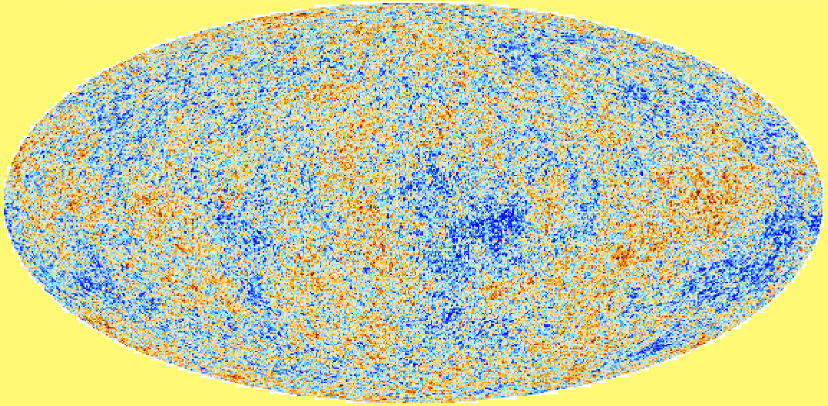
L1-3 unstable
saddles

L4-5 hill-tops,
stabilized
by coriolis



**L1 hosts 2 sun observatories,
Soho (1995) and Ace (1997).**

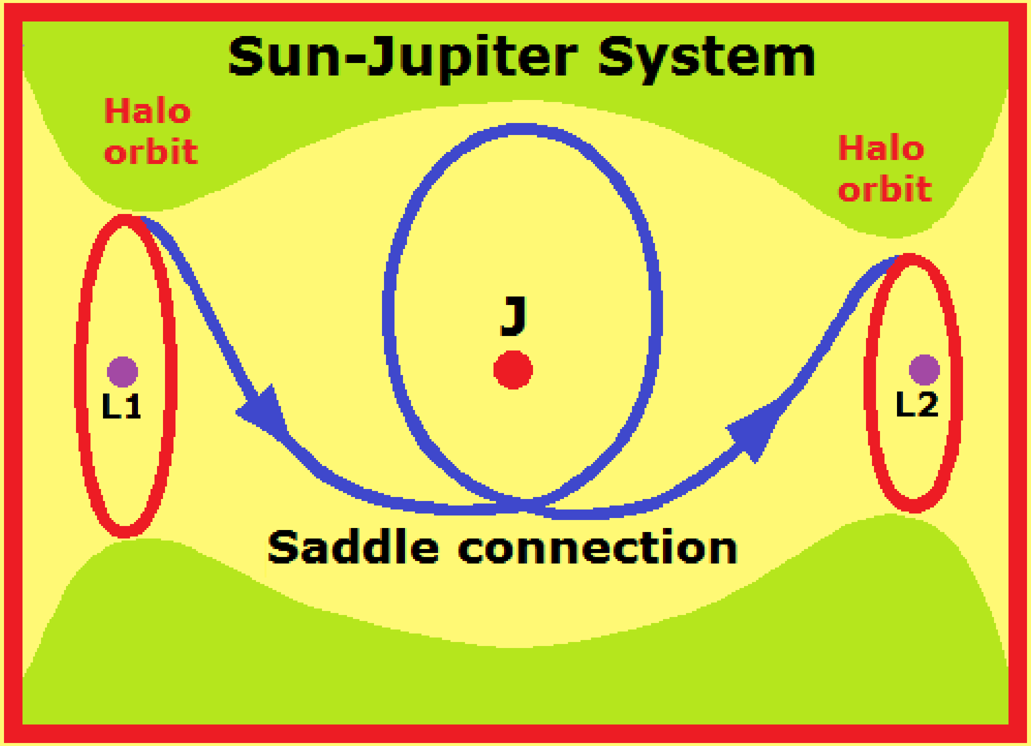
**L2 hosts Planck (2009, ESA):
Cosmic Microwave Background.**



(B) Chaotic trajectories of the ITN

These use saddle-connections between Lagrange halo orbits.

Being chaotic, they are easily controlled to different destinations.

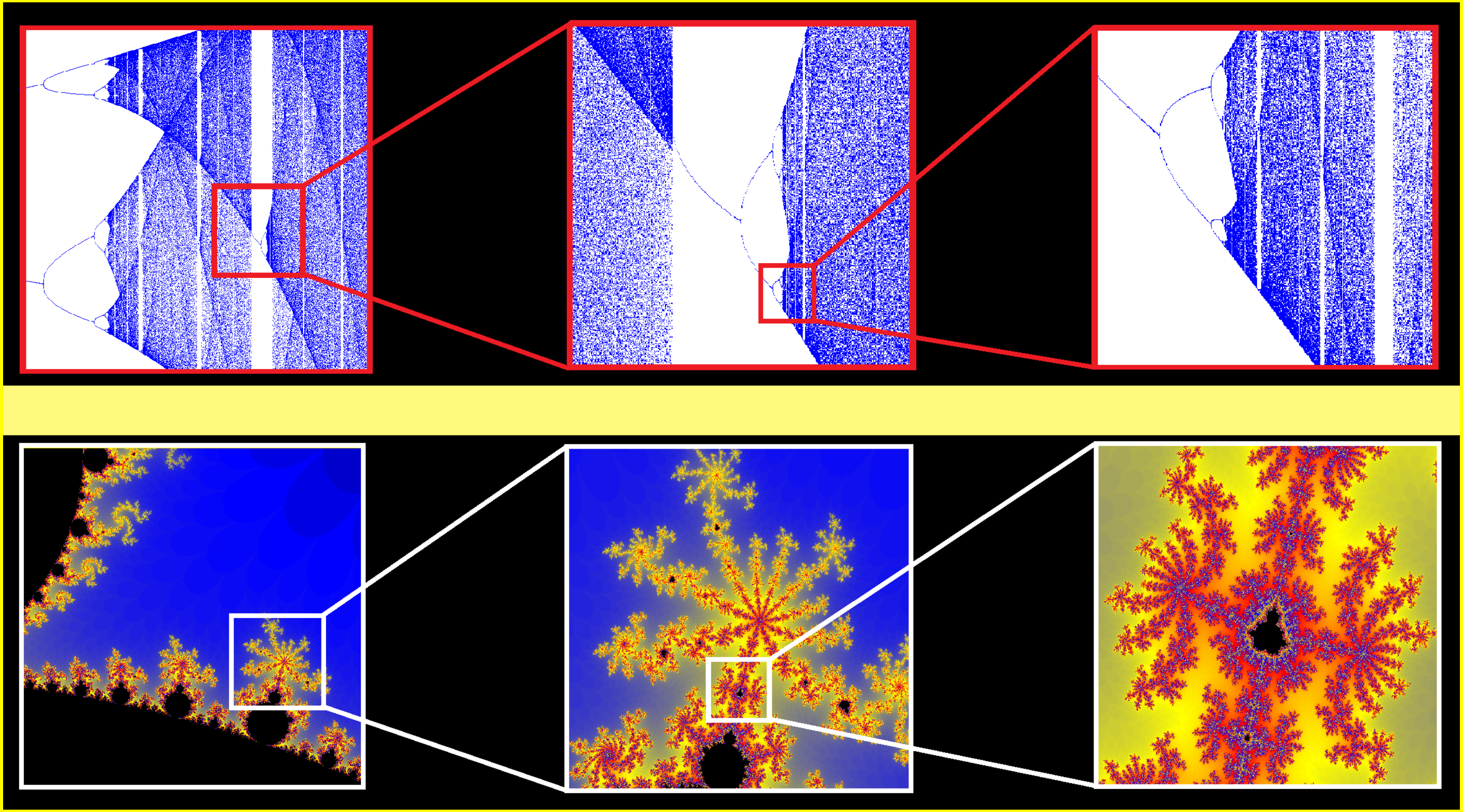


ITN Missions

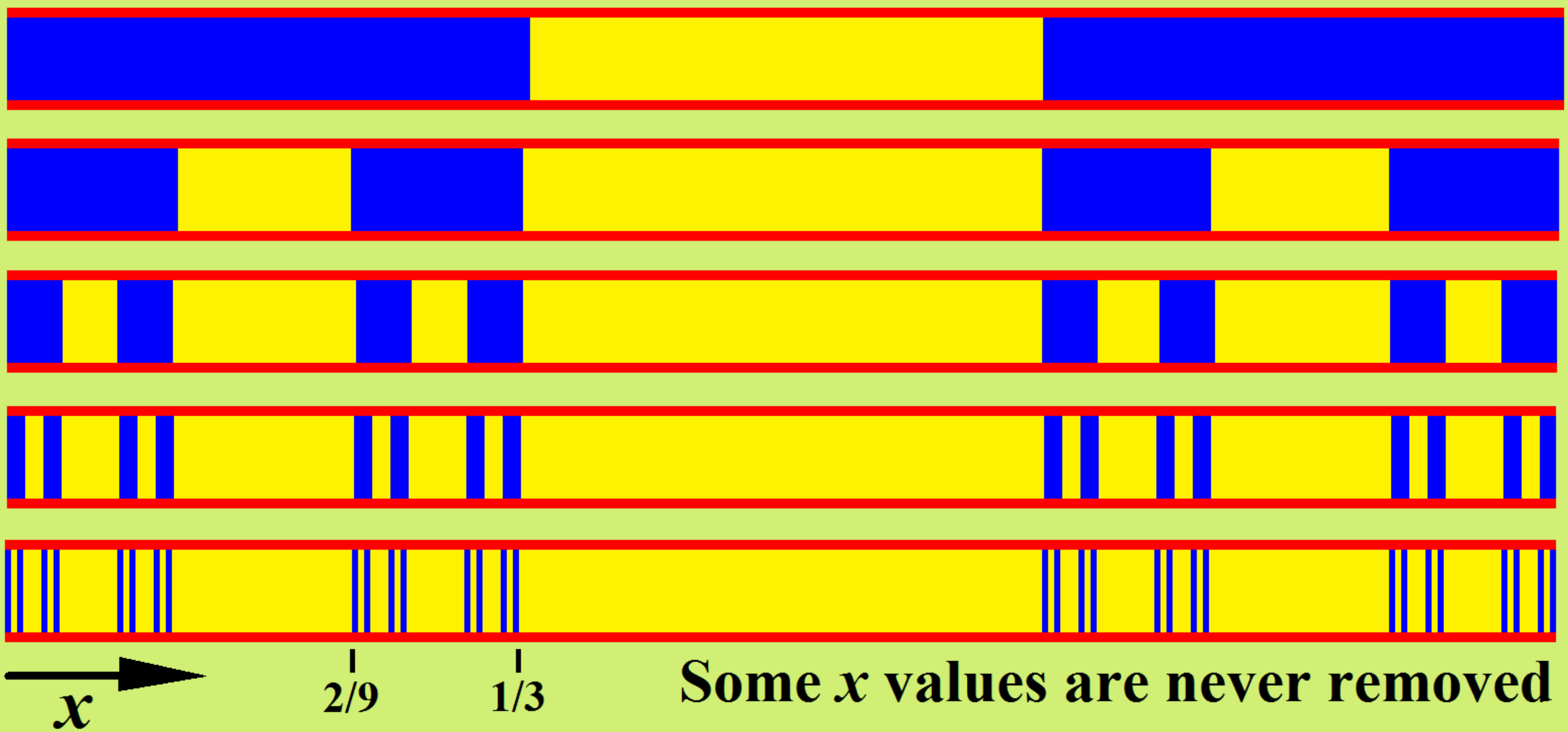
1990 Japan Hiten (moon)
2001 NASA Genesis (solar)
2003 Europe Smart 1 (moon)
2010 China Changé 2 (moon)

What is a Fractal?

A pattern that can be magnified for ever, revealing more and more detail



The Cantor Set (a simple example of a fractal)



Draw line 0 to 1, delete the (open) middle-third.

Do the same for each remaining segment.

Repeat this for ever, leaving the 'Cantor Set'.

Removed = $1/3 + 2(1/3)^2 + 4(1/3)^3 + \ldots = 1$.

All length is gone, but an infinity of points remain.

The Fractal Mandelbrot Set

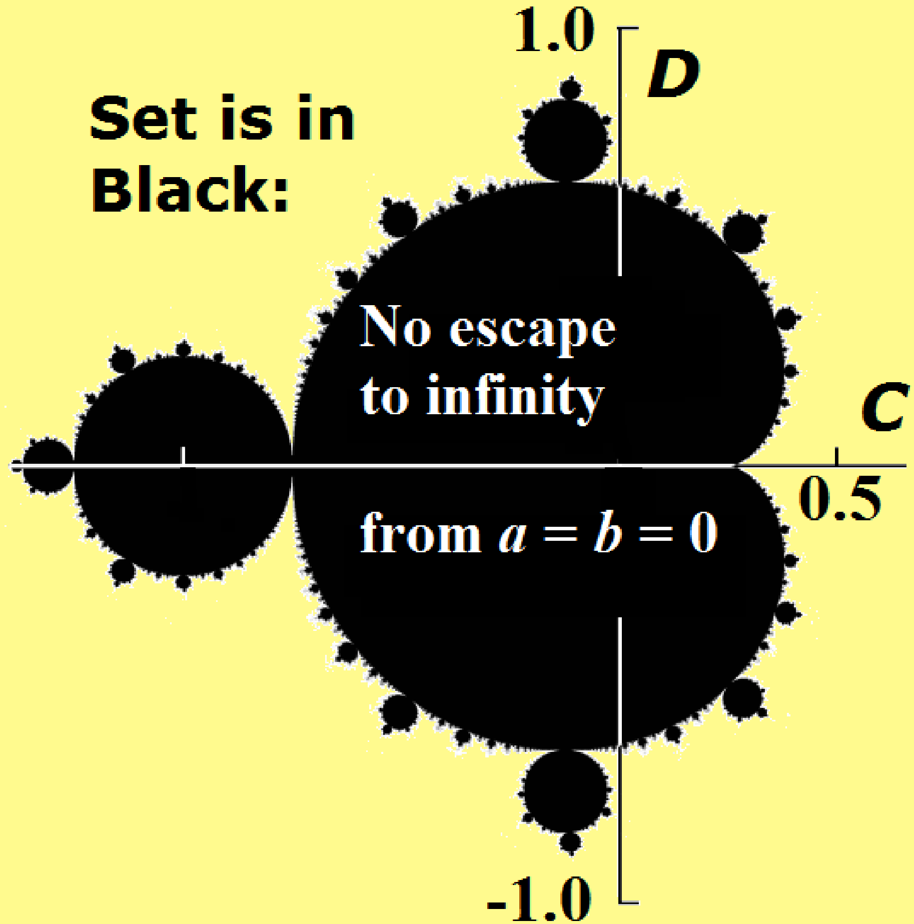
New a = Old $(a^2 - b^2 + C)$

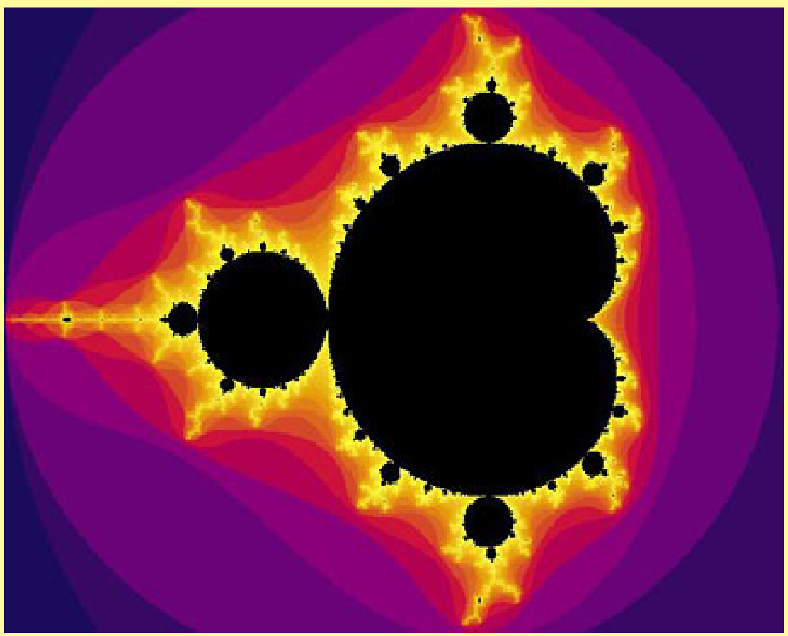
New b = Old $(2ab + D)$

C and D are held fixed during each calculation.

The set is determined by 'runs' from $a = 0$, $b = 0$

Set is in Black:

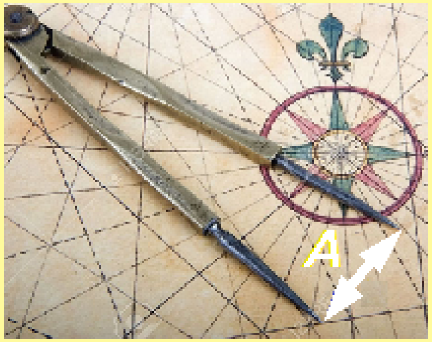




Colours define rate of divergence

Image of the set can be magnified indefinitely

Fractal Coastline of Britain

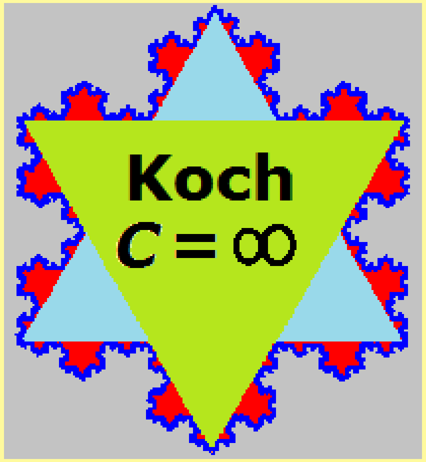


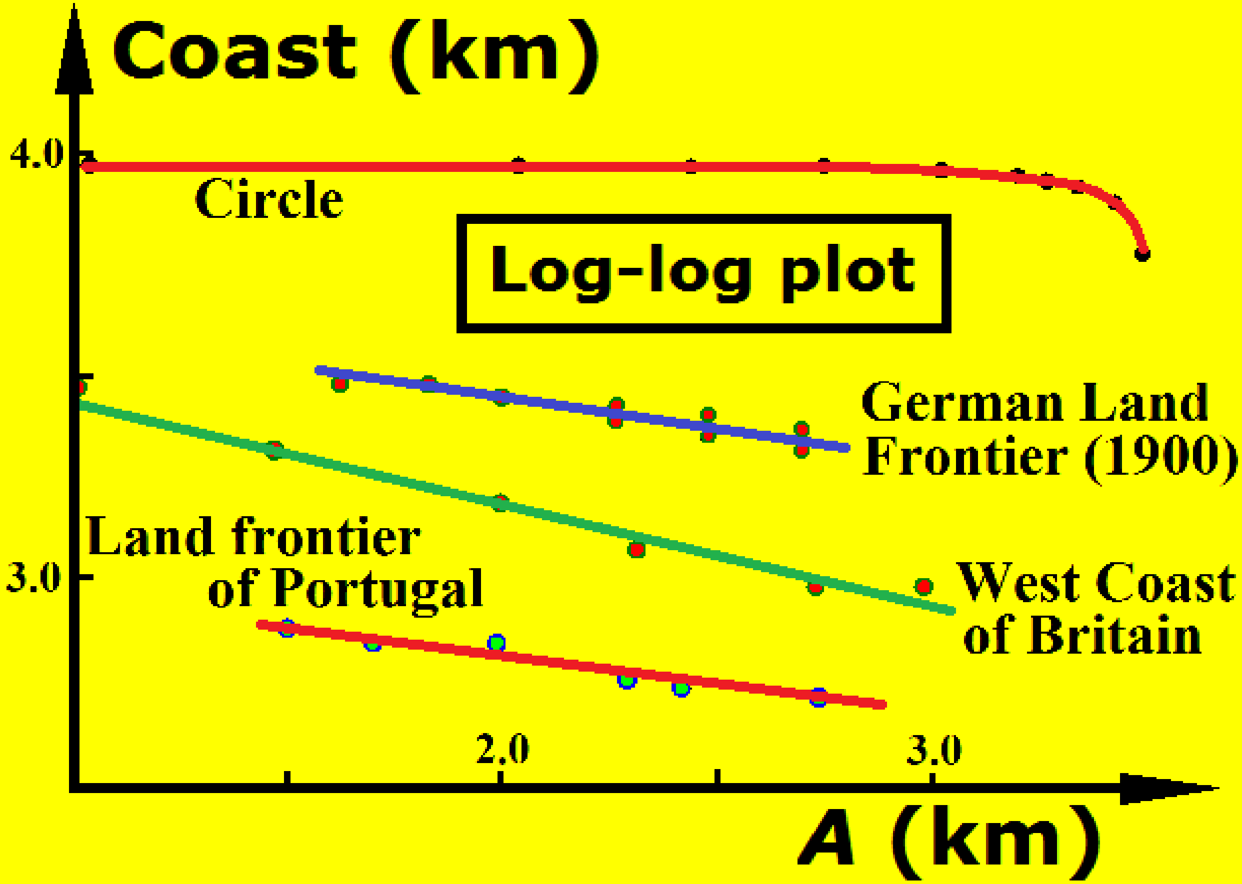
The more detailed a map, the greater is the length estimate

On the coast, a string will wind round stones and molecules

Coast is like a fractal. Its length tends to infinity as *A* goes to zero.

This is not the case for a circle.





Population Growth

Suppose the number of mayflies, *x*, increases by ratio *a* each year.
We have the 'discrete-time map':

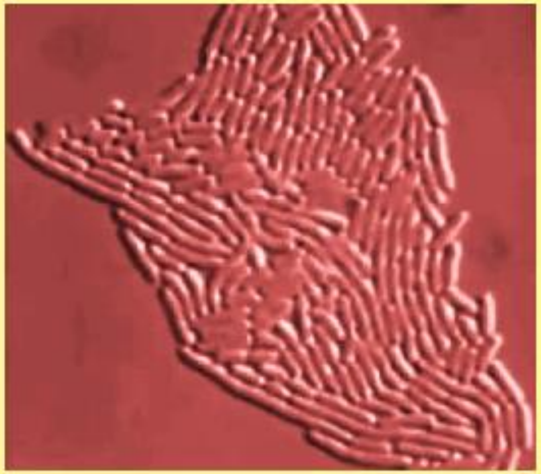
$$\text{New } x = a\,x$$

If a=2 and x=12 (million, say) yearly numbers explode:
12 ... 24 ... 48 ... 96 ... We have exponential increase!

Applied to humans, this alarmed Thomas Malthus.
His *Principle of Population* (1798) influenced Darwin and his thoughts on natural selection.

Taking $a=0.5$ (less than 1) we have:
12 ... 6 ... 3 ... decaying to zero.

We need a better model, including (say) competition and a limited food supply.



Explosive bacterial growth in a dish

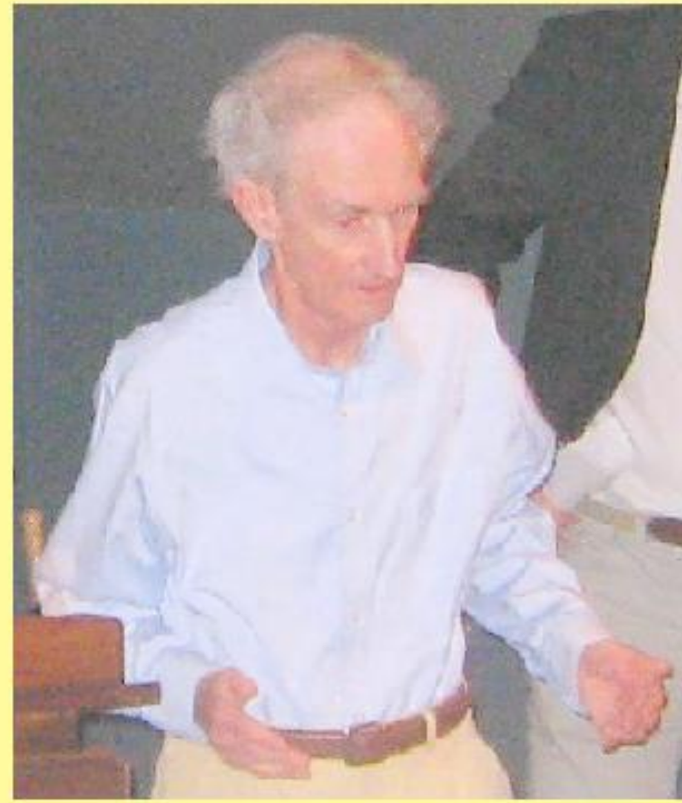
m36 BacGrow

Chaos in Logistic Map

New $x = a\,x\,(1 - x)$

x is fraction of the maximum population

$$x_{n+1} = r x_n (1 - x_n)$$

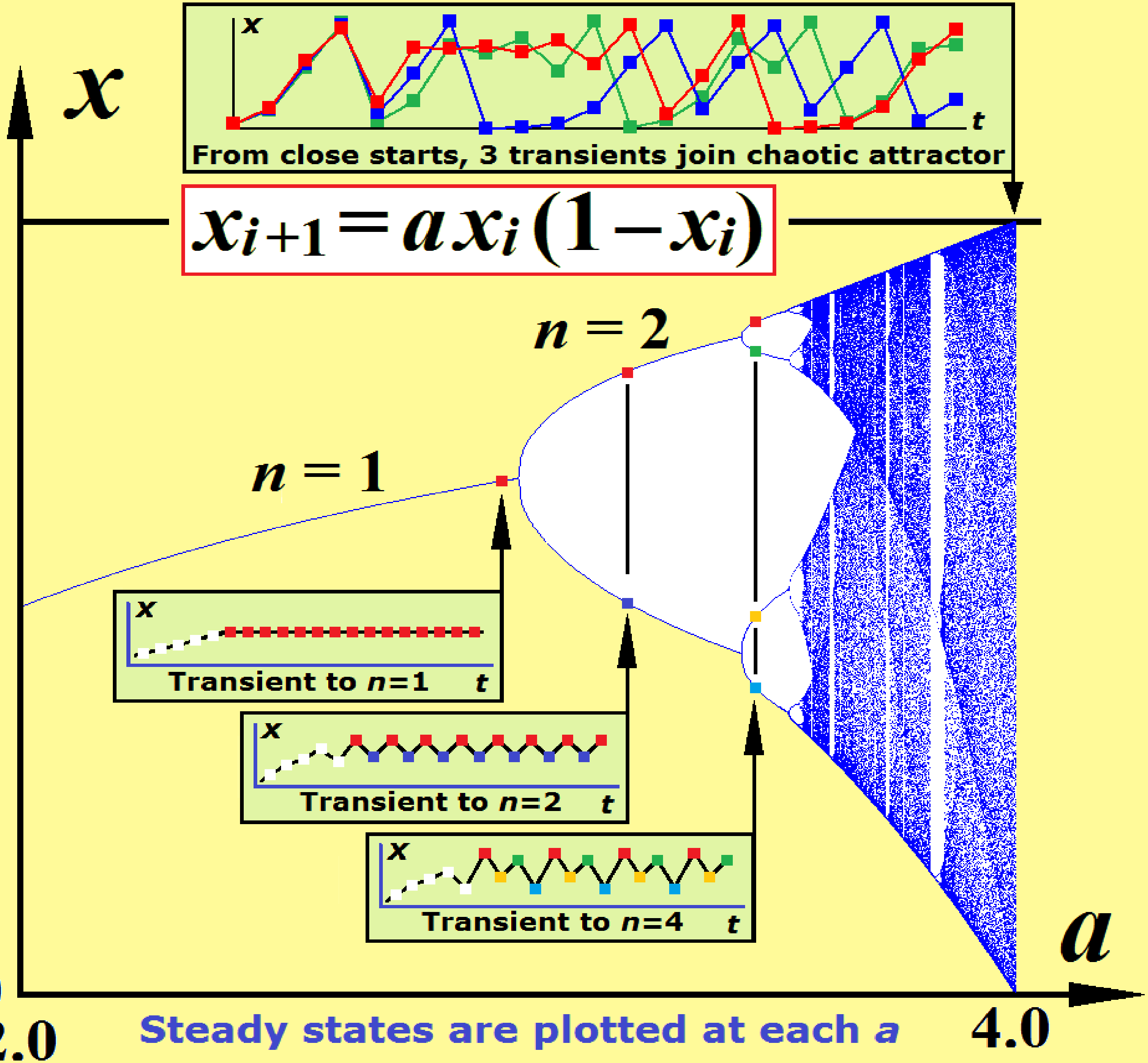


An improved model of population growth. (1-x) admits the constraint of limited food.

One-time President of the Royal Society, Lord Robert May (*Nature*, 1976) showed this gives sensitivity to initial conditions.

THE BUTTERFLY EFFECT AGAIN !!!

Cascade to chaos in the logistic map



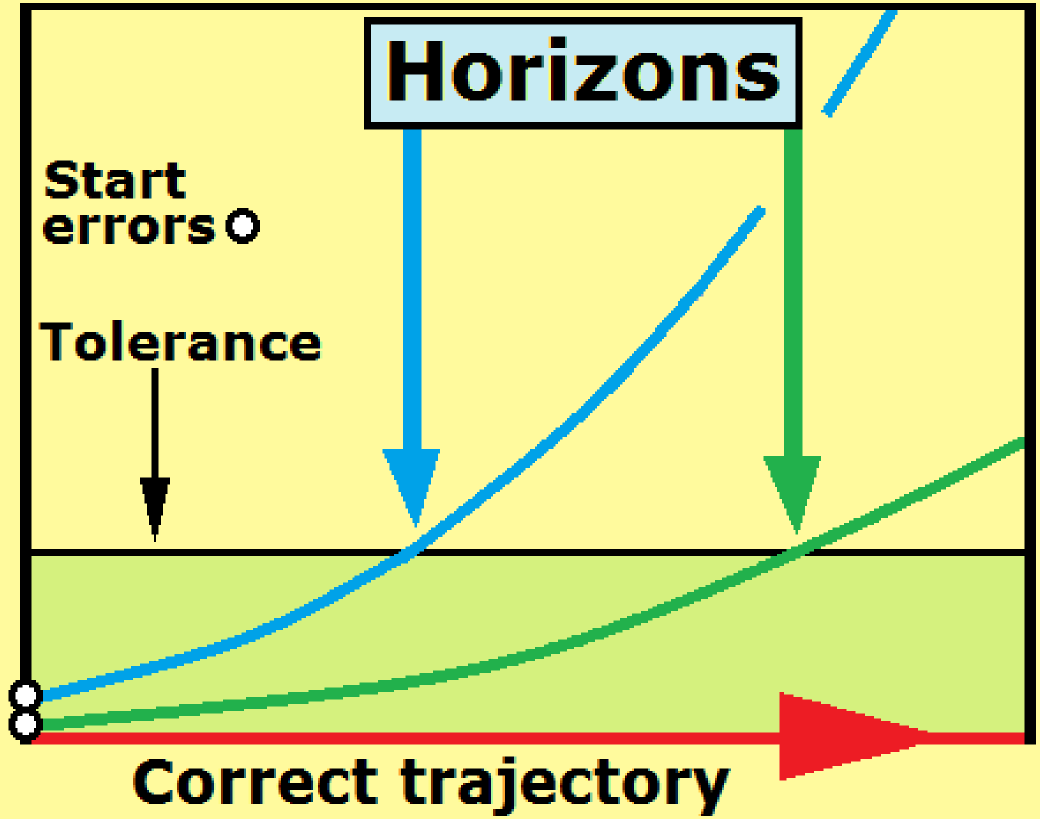
Universal features were discovered by Feigenbaum

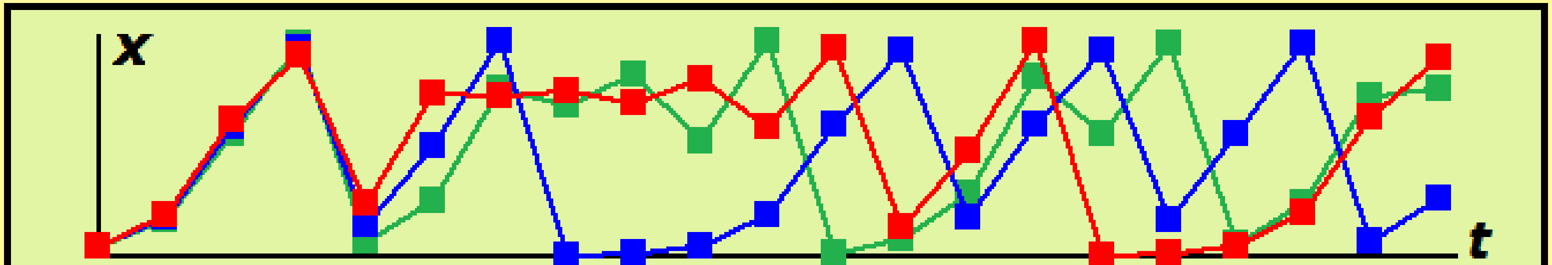
Predictability Horizons with Exponential Growth

We consider the multiplier 4 of the logistic map chaos.

Then, to double the horizon, we need to quarter the error.

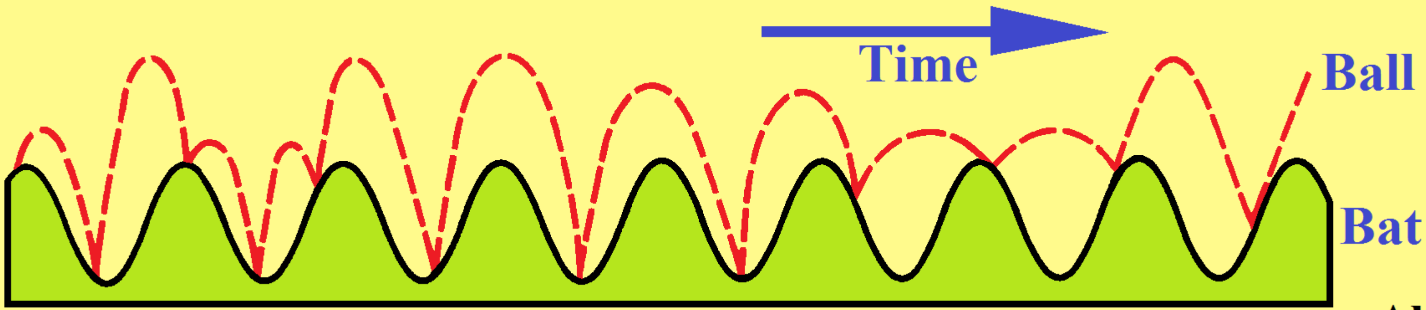
$$x_{i+1} = 4\,x_i\,(1 - x_i)$$

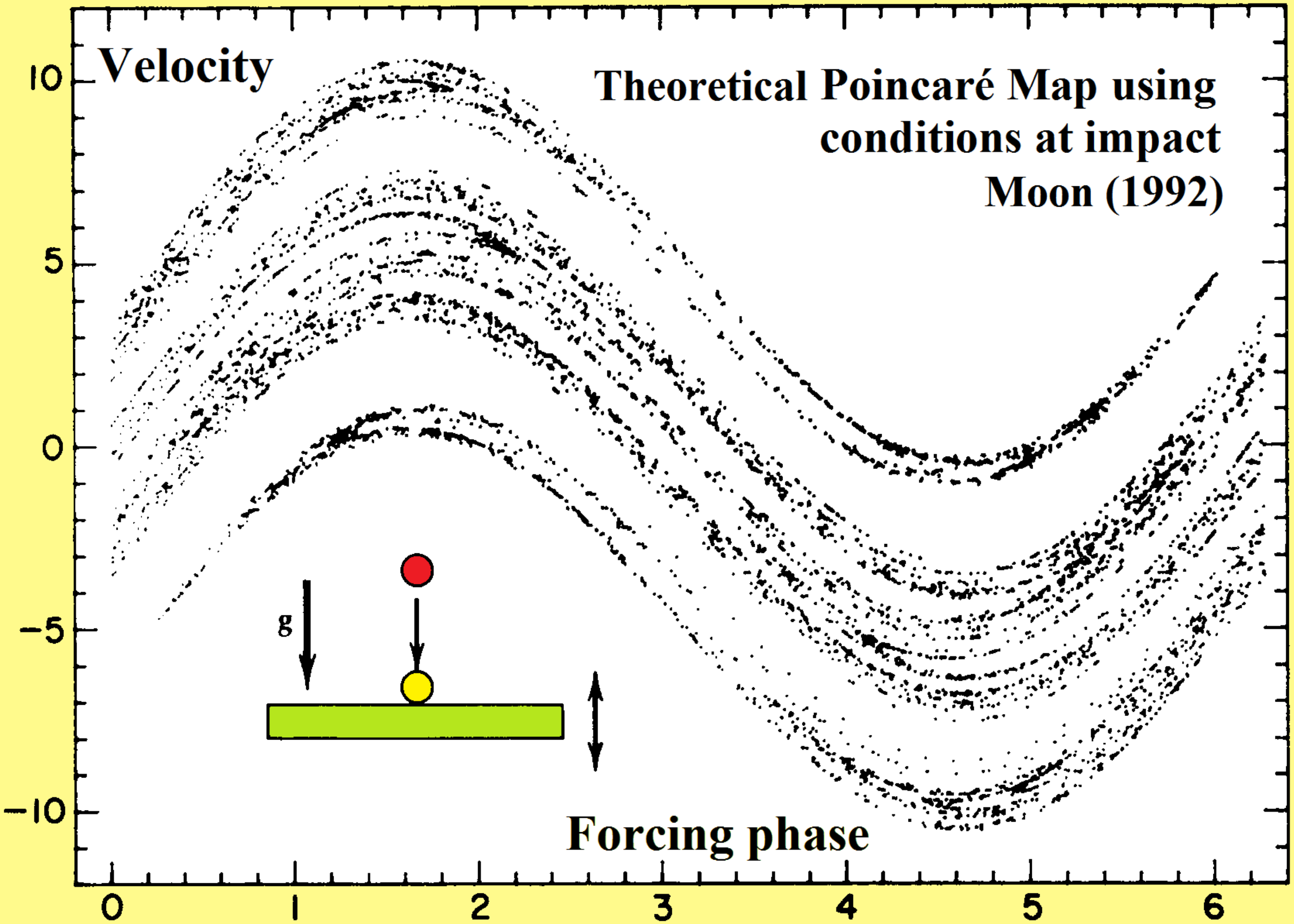




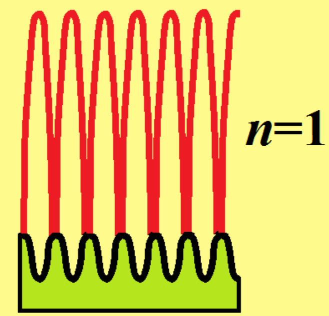
From close starts, 3 transients join chaotic attractor

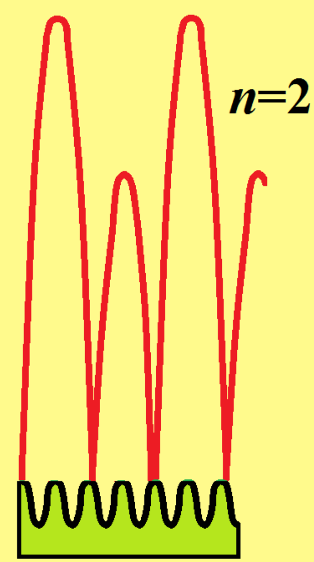
Chaotic bouncing on a periodically vibrating bat





Alternative steady states ...

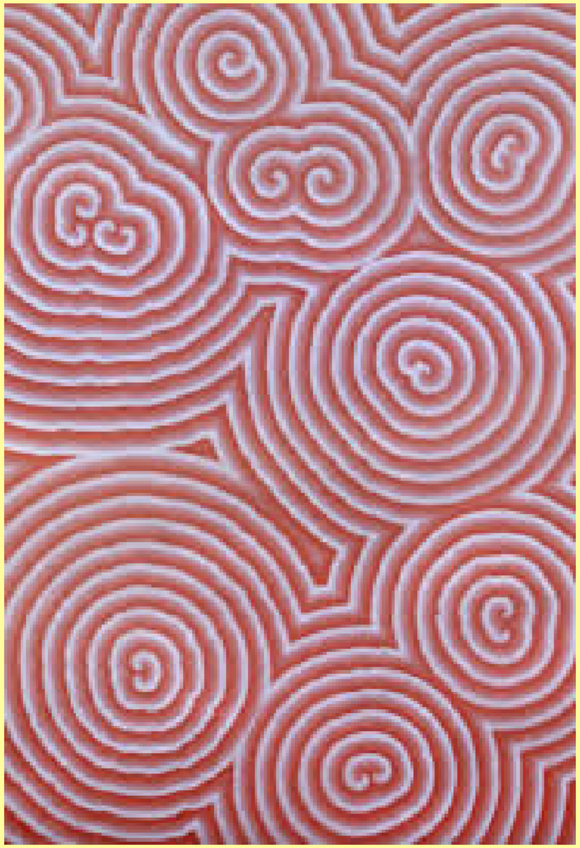


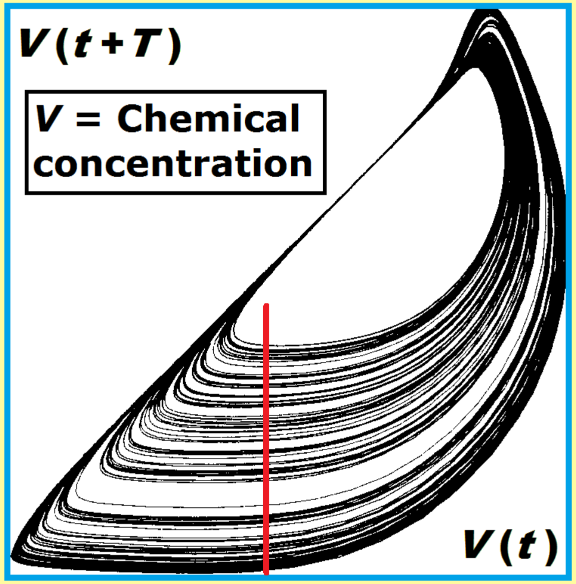


clck

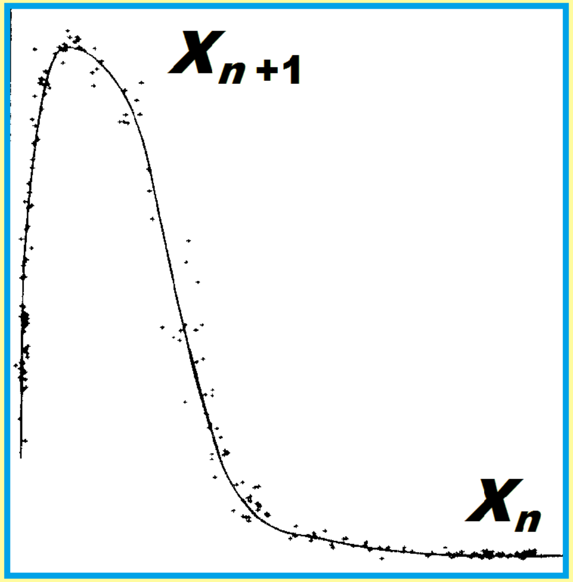
Chaos in oscillating chemical reaction (Belousov-Zhabotinsky)

Spontaneous pattern formation as conceived by Alan Turing



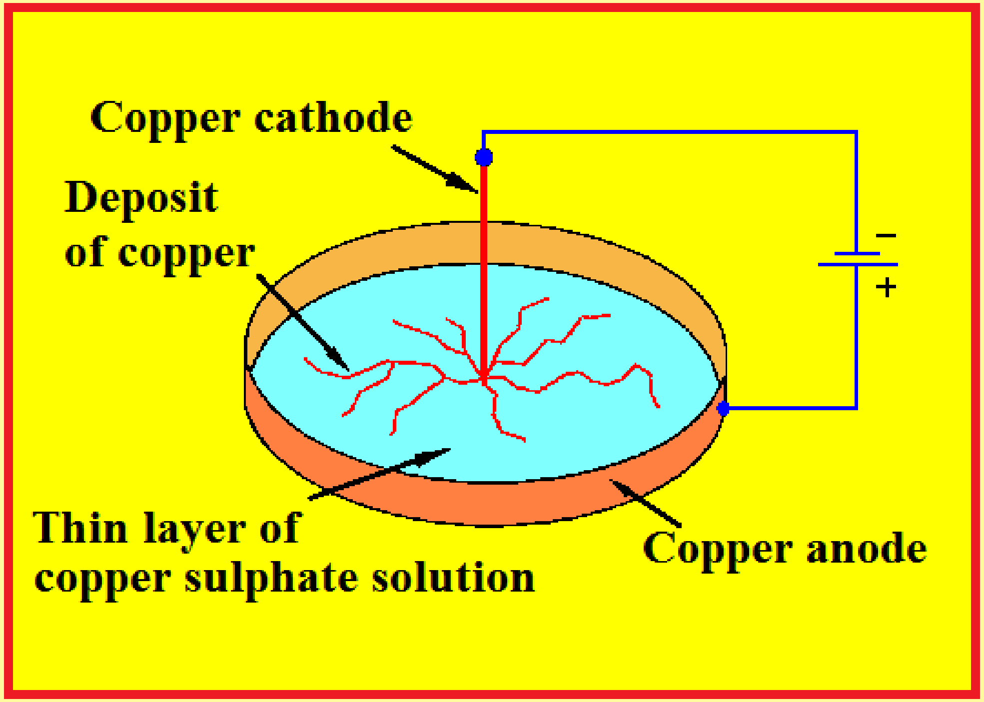


Chaos in a 2D projection of a 3D phase space using $V(t)$, $V(t+T)$, $V(t+2T)$

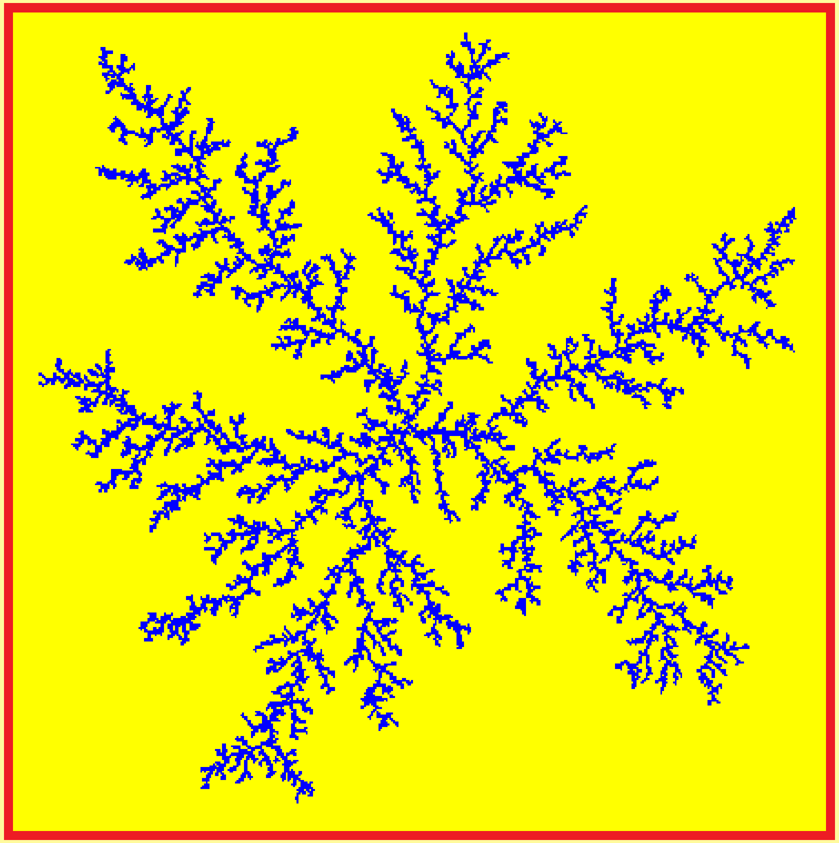


A 1D map from crossings of the red line in the left-hand figure

Fractal growth of a copper deposit during electrolysis



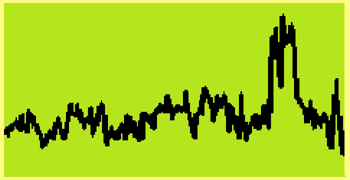
The experimental set-up in the laboratory



A theoretical prediction using 'diffusion-limited aggregation' modelling

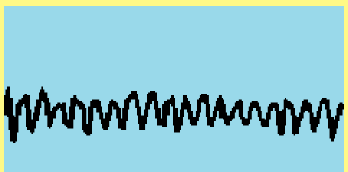
Chaotic Heart-Beat is Best

Plots of heart-beat rate over a 15 min. interval of three real hospital patients



Healthy
Chaotic wave

Unhealthy
Nearly constant



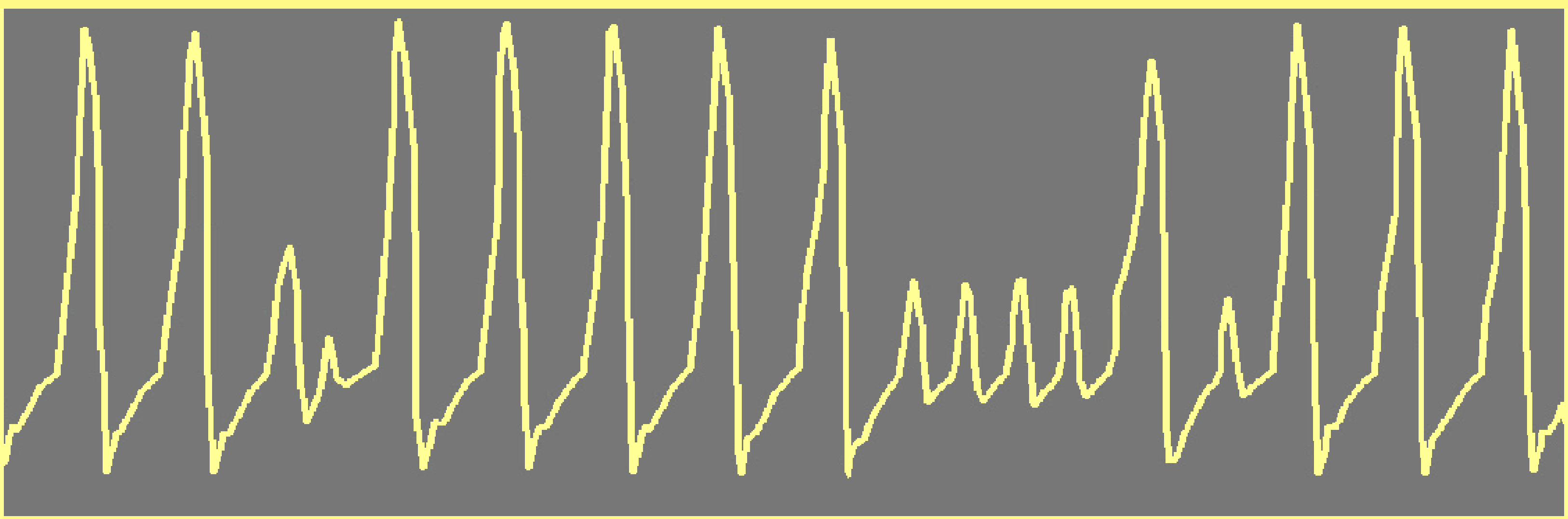
Unhealthy
Nearly periodic

Chaos is Good for You

It is said that we work best in a chaotic state

So if your brain is in chaos, the lecture was a success!

Chaotic firing of a periodically excited brain neuron



CONCLUDING REMARKS

- Chaos is a 'random' output of a deterministic system
- With extreme sensitivity to initial conditions
- Technically unsolvable, and numerically tricky
- No long-term prediction, but 'order within chaos'

WHY 300 YEARS FROM NEWTON TO CHAOS?

(1) There were no computers or video displays

(2) Researchers were looking for order

(3) Random results were thought wrong: into the waste bin

FURTHER INFORMATION

Movies & Simulation apps (and many more) are on U3AC Website: under Tutors, Handouts, SCE 25

SCE 25 'Chaos & Fractals' will be given again next year

My homepage: http://www.ucl.ac.uk/~ucess21/