Pioneers of Chaos

- Newton's experiments: dissipation and attractors
- Building blocks of phase space
- Spinning and tumbling in space
- Henri Poincaré: the birth of chaos and homoclinic tangling
- Chaotic spinning of Hyperion
- Lorenz, convection and the Butterfly Effect
- Divergence, folding and mixing

Henri Poincaré 1854-1912

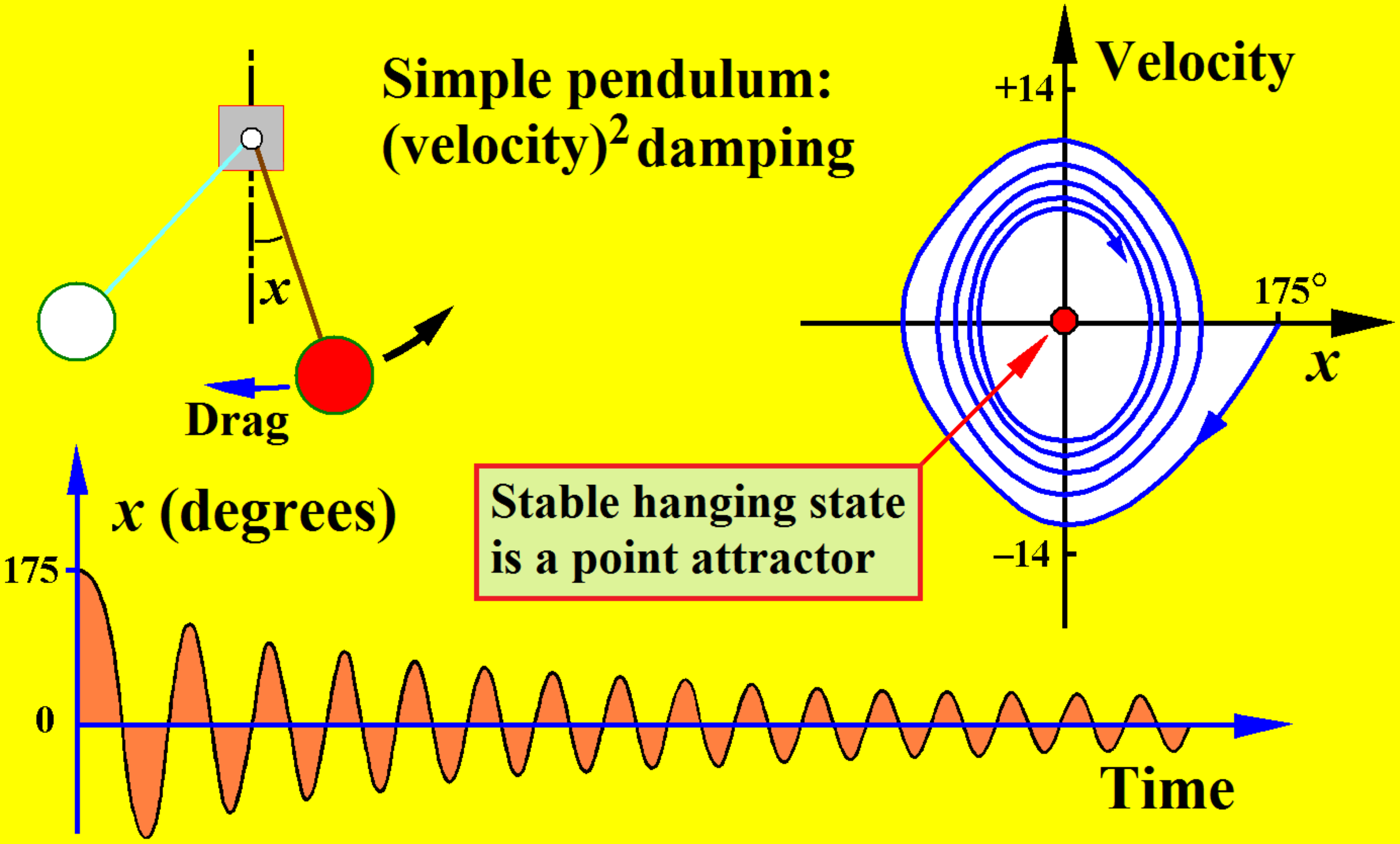
s01 ex-pend

Dissipation makes Attractors

- The pendulum that we have been discussing is not realistic!
- It oscillates for ever, with no decay of its swinging amplitude
- The mathematical model that we used was incomplete
- We ignored friction in the bearing and air-drag on the bob
- Both of these dissipate energy and slow the pendulum down
- Newton made experiments to estimate the drag force
- Closed phase-space orbits become spirals to a point attractor

Newton's Pendulum Experiments

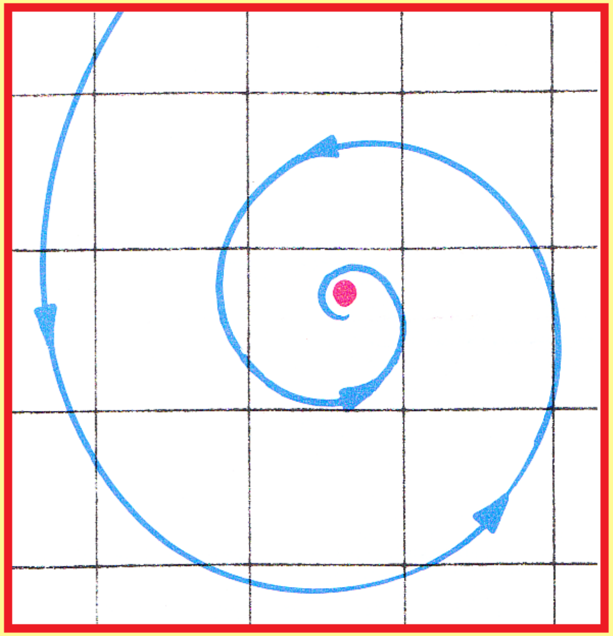
Newton made experiments with bobs in air, water and mercury.
He deduced, correctly, that drag is proportional to the fluid's density.
In rapid motion he found that drag varies as the square of the velocity.
This is roughly correct, but lacks the precision of his gravity theory.



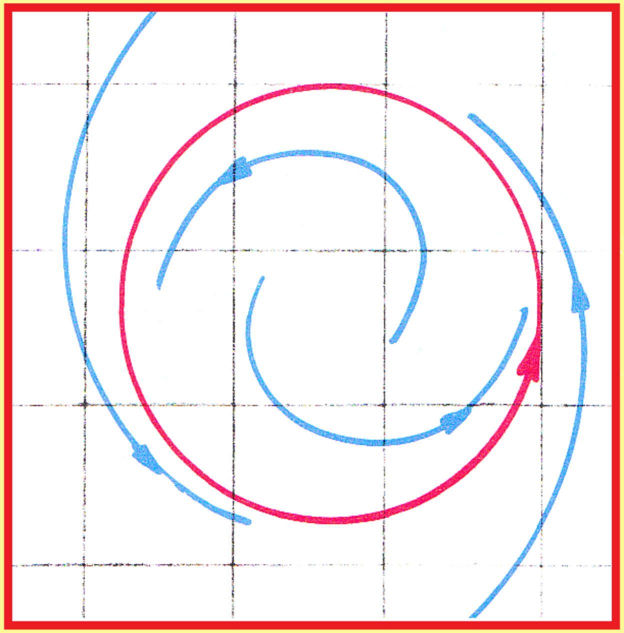
FEATURES OF A DISSIPATIVE PHASE SPACE

Building Blocks of a two-dimensional Phase Space

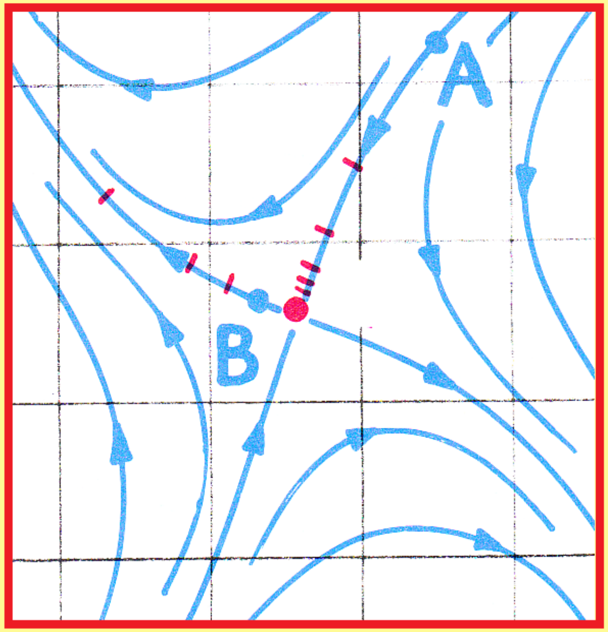
The only attractors in 2D are the fixed point and the periodic cycle. In higher D: quasi-periodic attractor, chaotic attractor.



Point attractor
(reverse arrows to get a point repellor)



Periodic attractor
(reverse arrows to get periodic repellor)



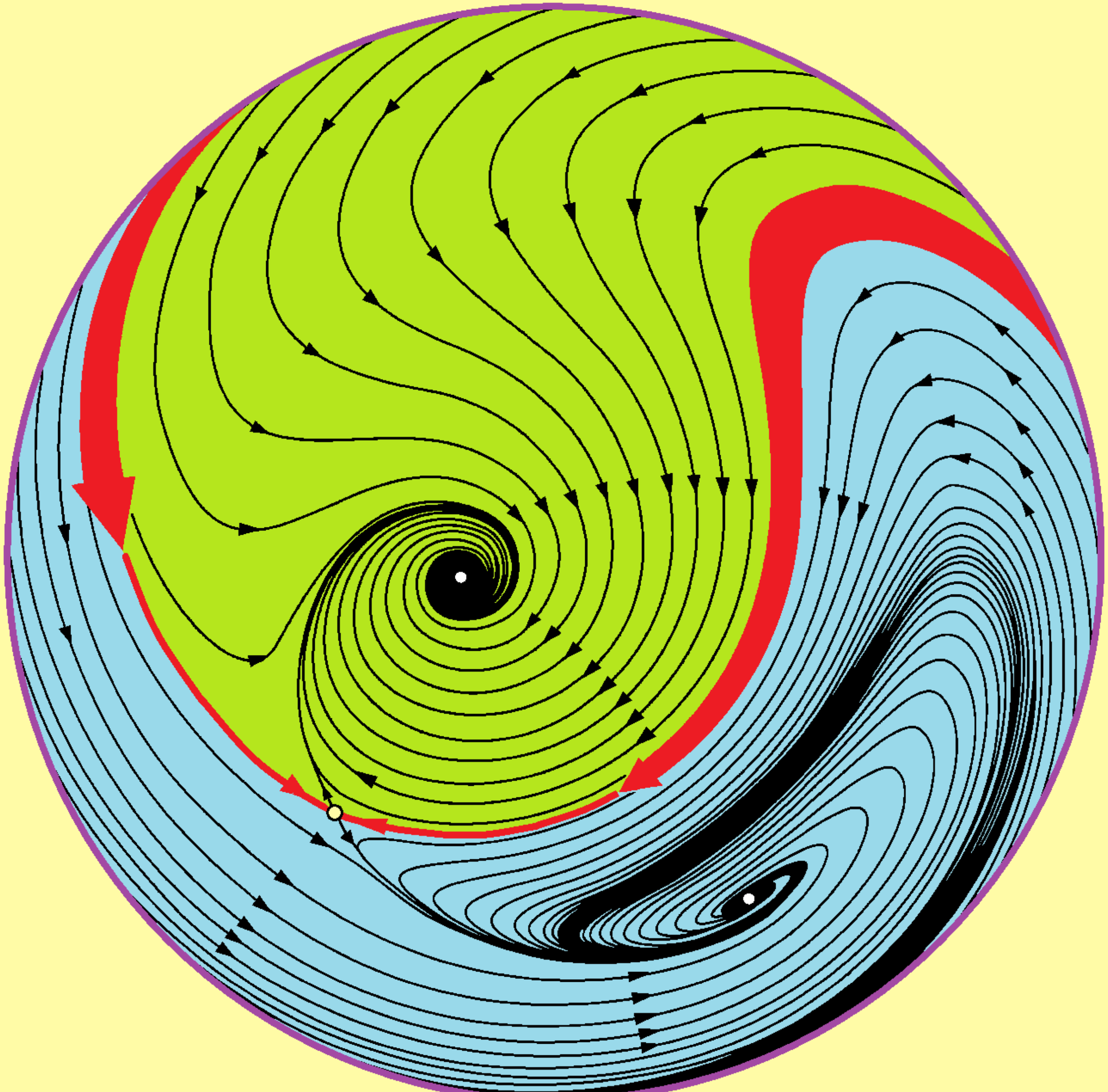
Saddle point
with inset (A) and outset (B)

Basins of Attraction

This phase space has 2 attractors

Each has its own basin of attraction

The boundary between these basins is the inset of an unstable saddle solution

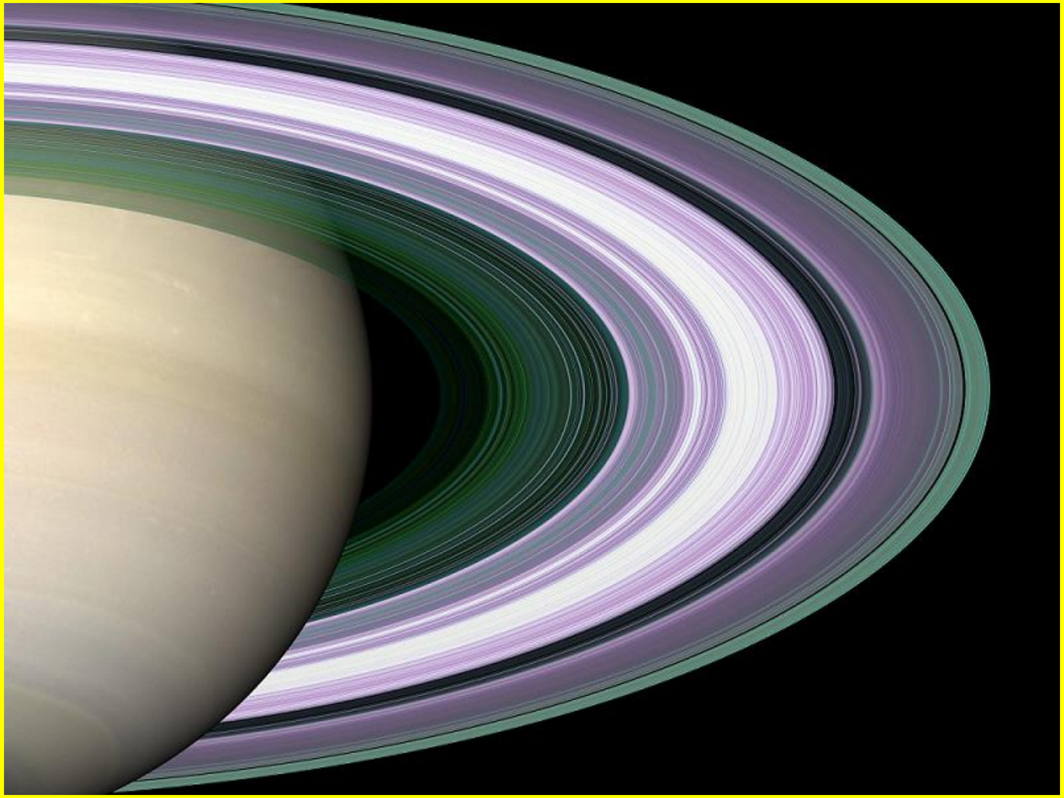


The basins of two spiral attractors are shown in green and blue.
The boundary between these basins is the inset of the saddle, show (thickened) in red.

Spinning and Tumbling in Space

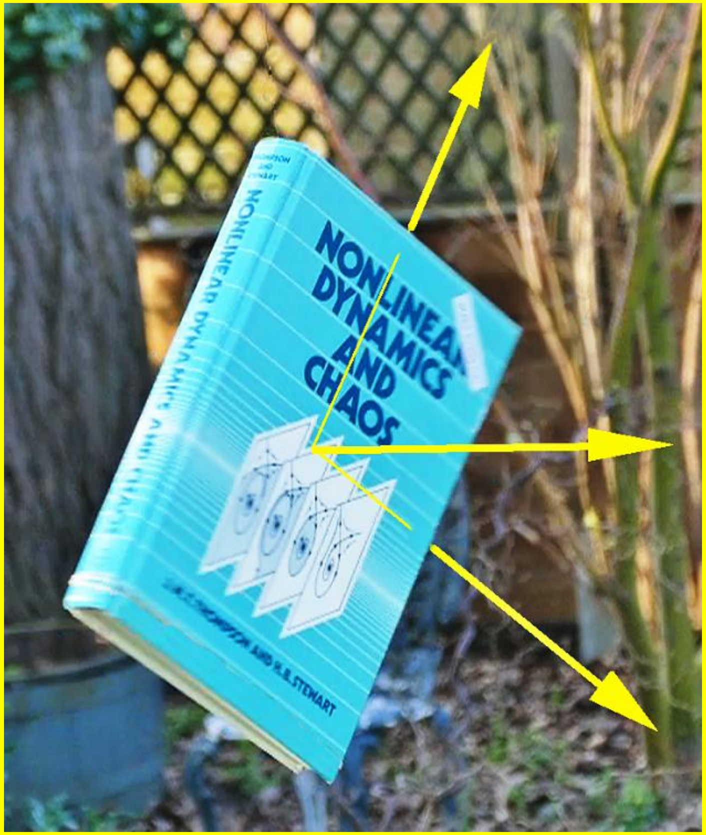
[m39 ed white]

Ed White, Gemini 4, 1965, NASA



We see later how Hyperion, a moon of Saturn tumbles chaotically due to its elliptical orbit

But first two movies



m31 BkHrt m32CatFeet

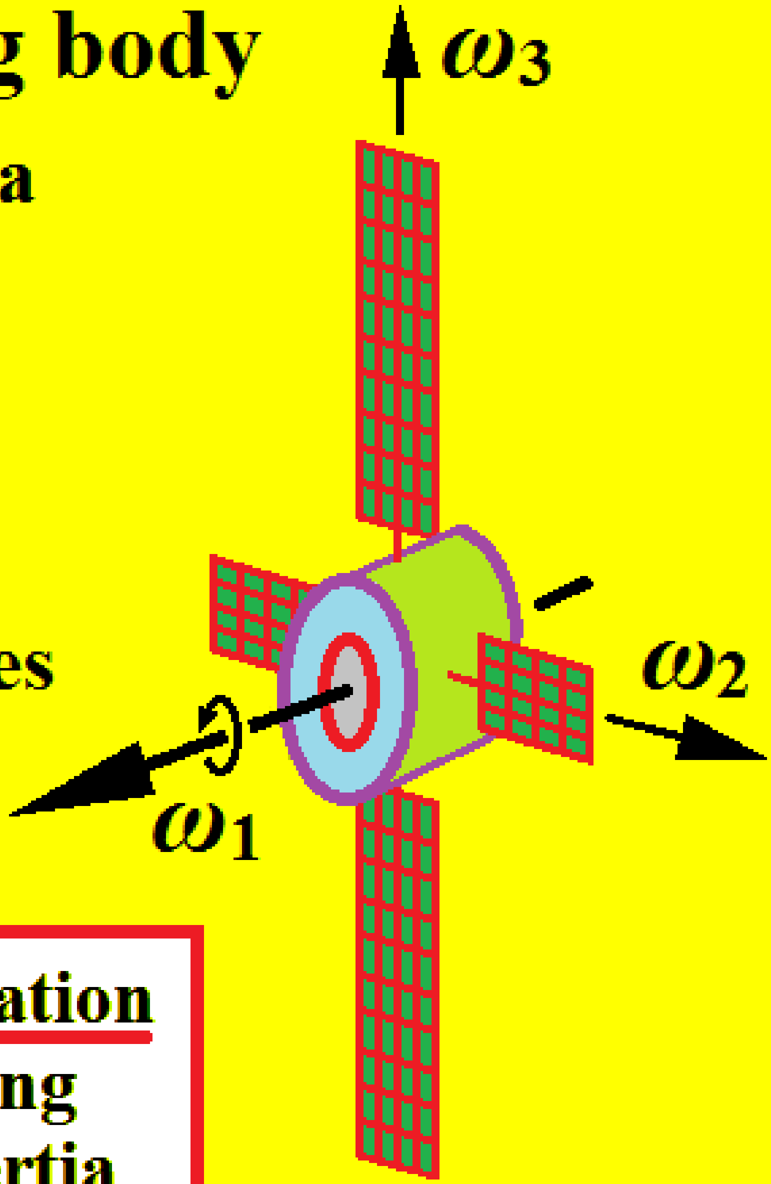
Mechanics of a spinning body

3 first-order Euler eqns give a 3D phase space.

Two conserved quantities:

Angular momentum (vector) gives 2D spherical surface.

Energy (if no dissipation) fixes trajectories on surface.



A satellite with <u>internal dissipation</u> is asymptotically stable spinning about axis (1) of maximum inertia

Phase portrait of spinning satellite (a) no dissipation

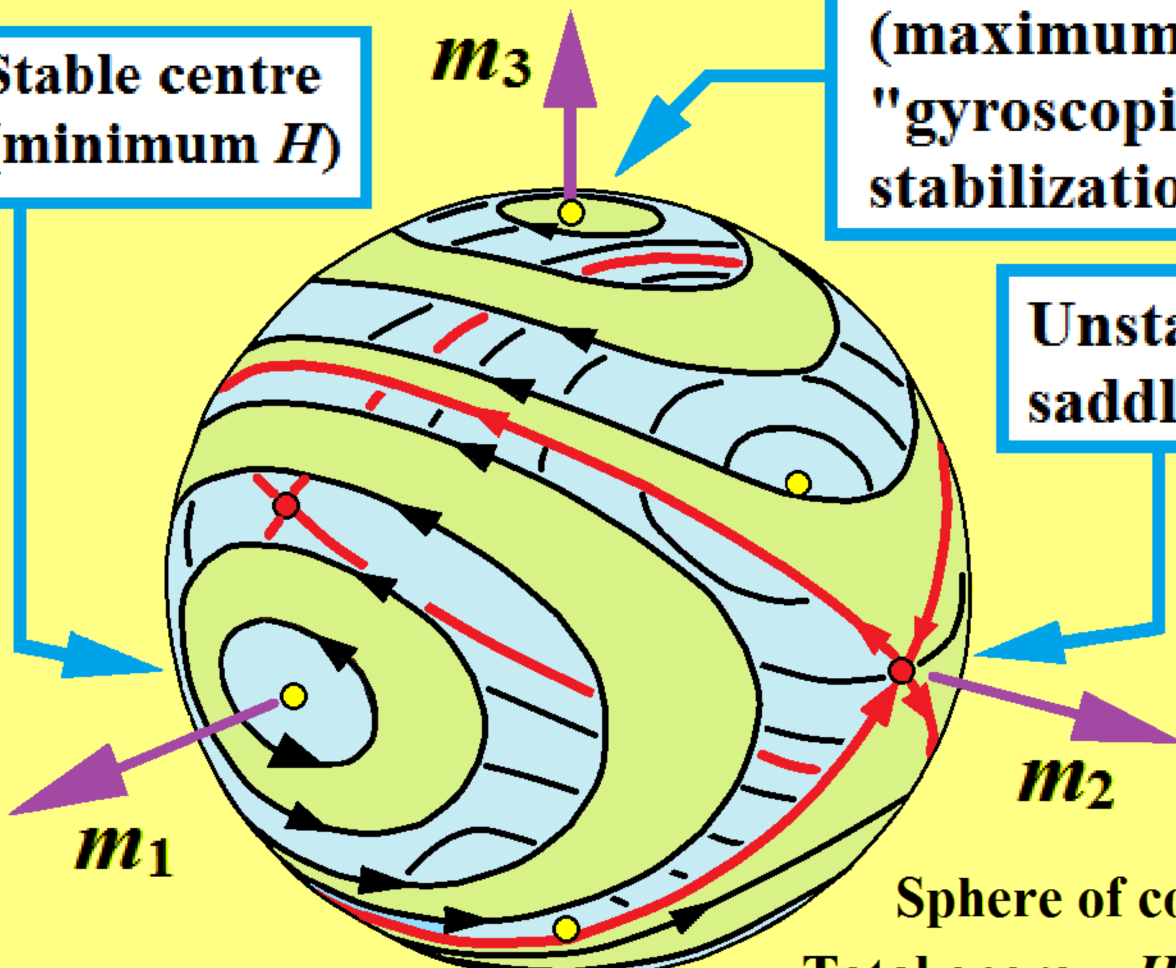
Moments of inertia: $I_1 > I_2 > I_3$
Angular velocities: $\omega_1, \omega_2, \omega_3$
Angular momenta: $m_i = I_i \, \omega_i$

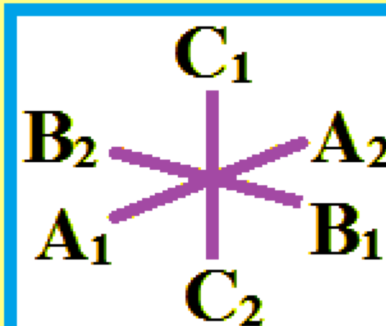
Stable centre (minimum H)

Stable centre (maximum H) "gyroscopic stabilization"

Saddle connections

Unstable saddle



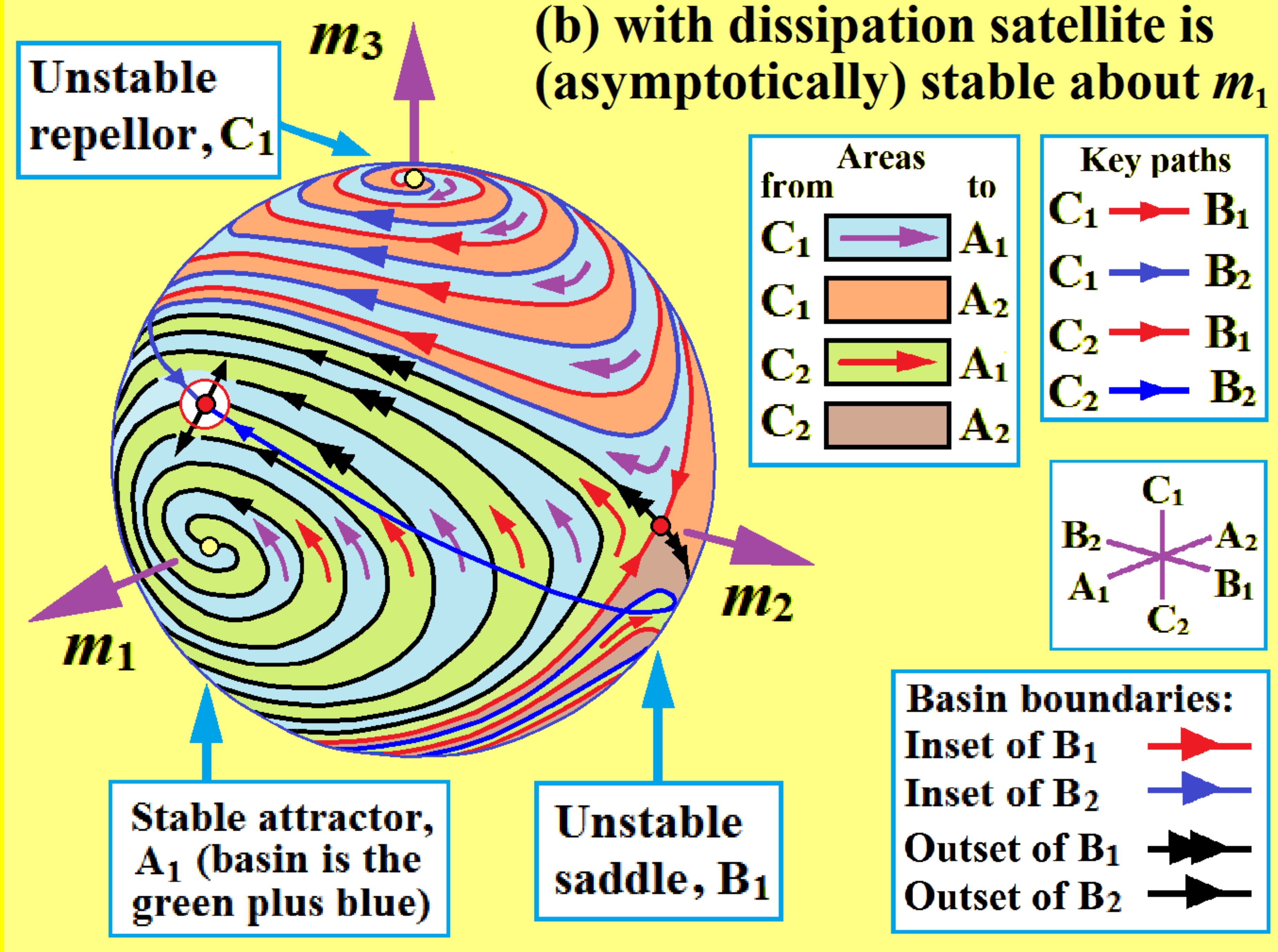


Fixed points

Racquet: B_1 to B_2 where m_2 is negative about flipped axis

Sphere of constant angular momentum.

Total energy, H, is constant on a trajectory.



(b) with dissipation satellite is (asymptotically) stable about m_1
Unstable repellor, C_1
m_3
m_2
m_1
Areas
from
to
C_1 A_1
C_1 A_2
C_2 A_1
C_2 A_2
Key paths
C_1 B_1
C_1 B_2
C_2 B_1
C_2 B_2
C_1
B_2
A_2
A_1
B_1
C_2
Basin boundaries:
Inset of B_1
Inset of B_2
Outset of B_1
Outset of B_2
Stable attractor, A_1 (basin is the green plus blue)
Unstable saddle, B_1

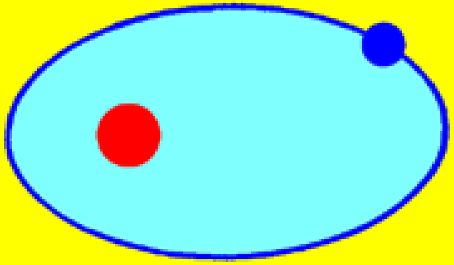
Illustrative experi
with 8D phase s

The Traffic Cop is a system of pendulums that illustrates in a dramatic and amusing way the surprises and complexities of chaotic motion

[m16 traffic]

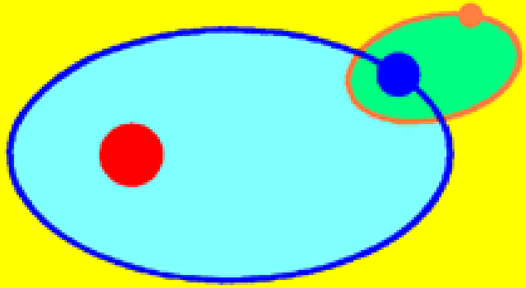
Henri Poincaré: BIRTH OF CHAOS

In 1887 the King of Sweden offered a prize to whoever could answer the question "Is the solar system stable?"



Newton solved 2-body problem

Poincaré won the prize with his work on the 3-body problem



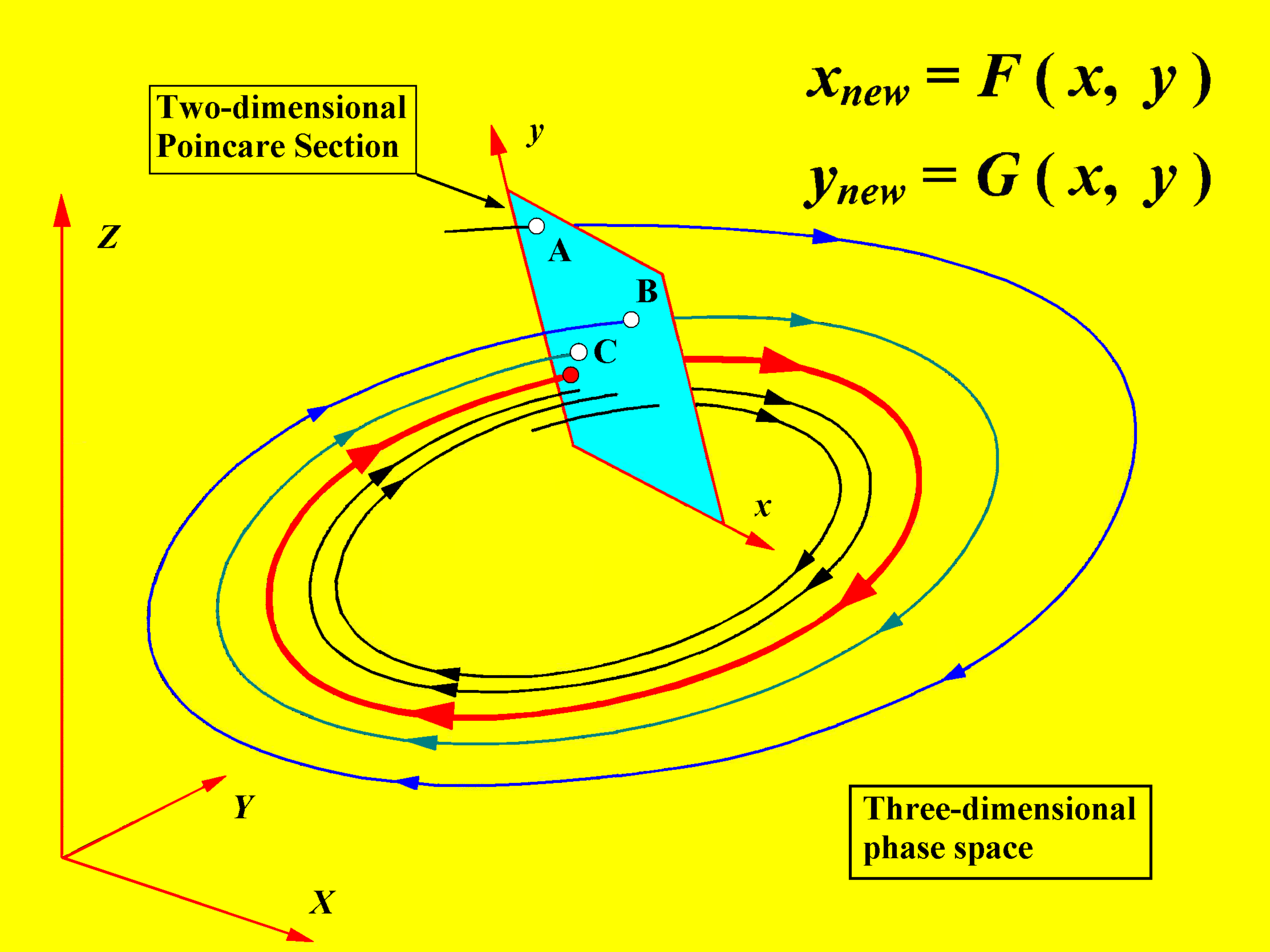
Poincare showed 3-body unsolvable

This has some unstable fixed-points

Introducing a Poincaré section, he saw that homoclinic tangles must occur

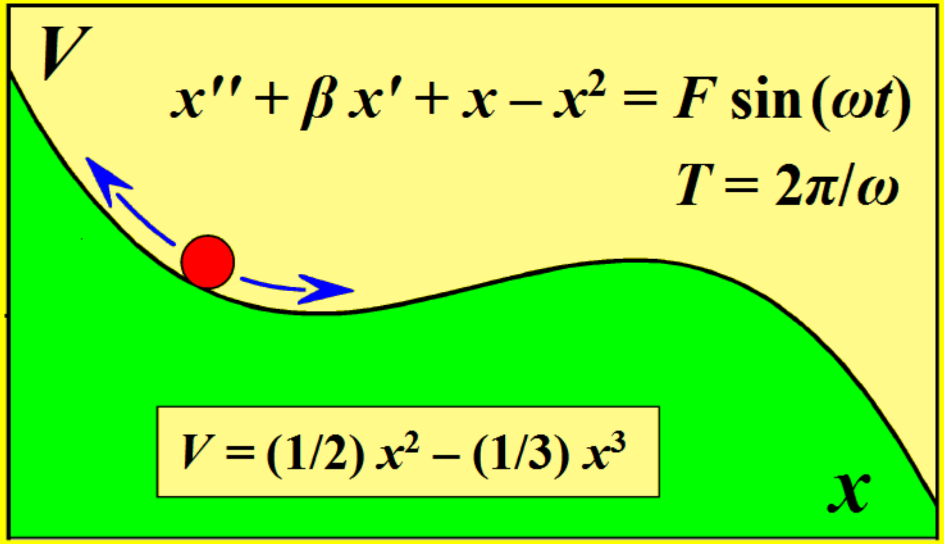
These must give rise to chaos and unpredictability

... and the hairy sphere theorem!

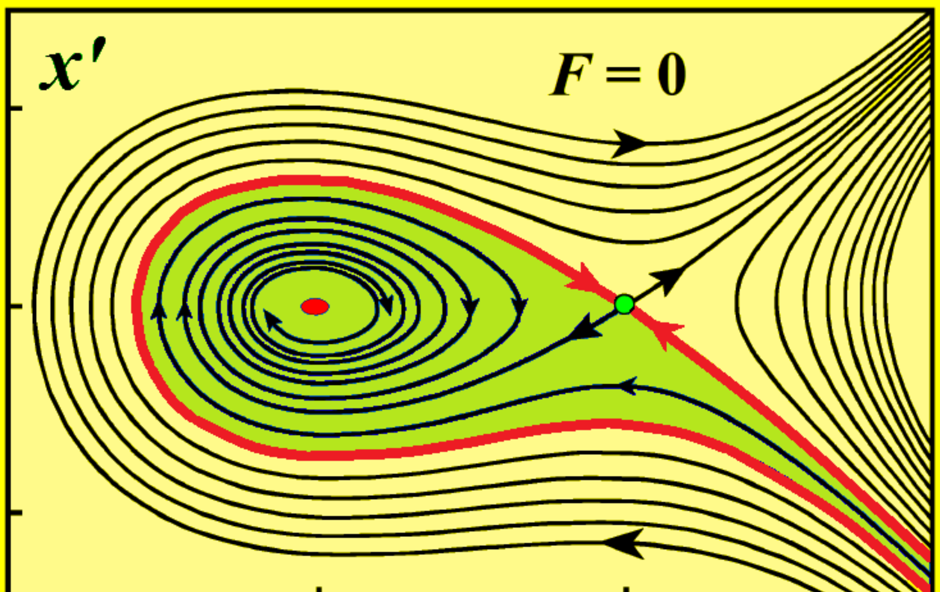


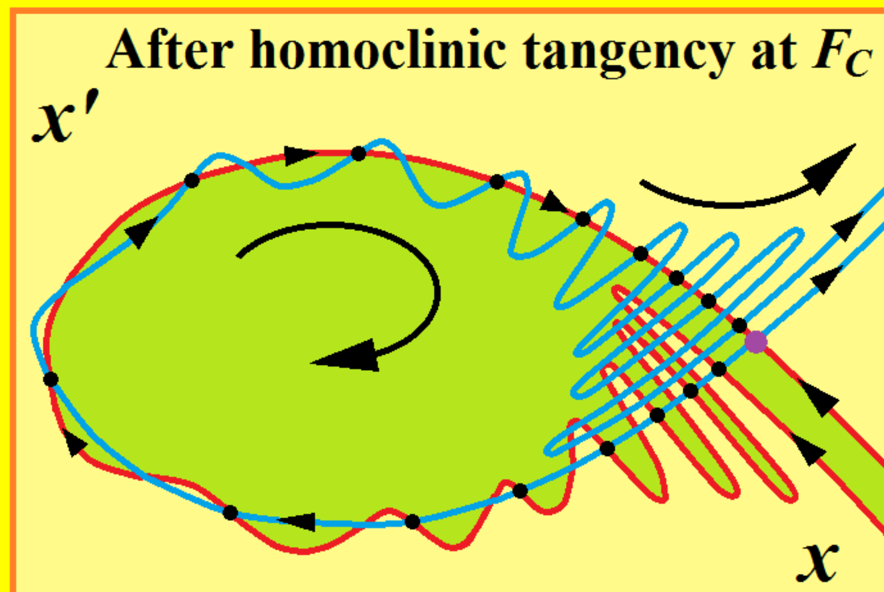
$x_{new} = F(x, y)$
$y_{new} = G(x, y)$
Two-dimensional Poincare Section
Three-dimensional phase space
A
B
C
x
y
X
Y
Z

Poincaré despaired on realizing that the 3-body problem contained a 'tangle'



When $F \neq 0$ phase-space is 3D, and we need a Poincaré section (stroboscopic sampling). This gives a dot-map and when $F > F_C$ the inset and outset intersect an infinite number of times. Near the homoclinic tangle will be chaos and infinitely many periodic orbits.





Chaotic Tumbling of Saturn's Moon

Saturn with its rings

In 2004 the Cassini spacecraft orbited and studied Saturn.

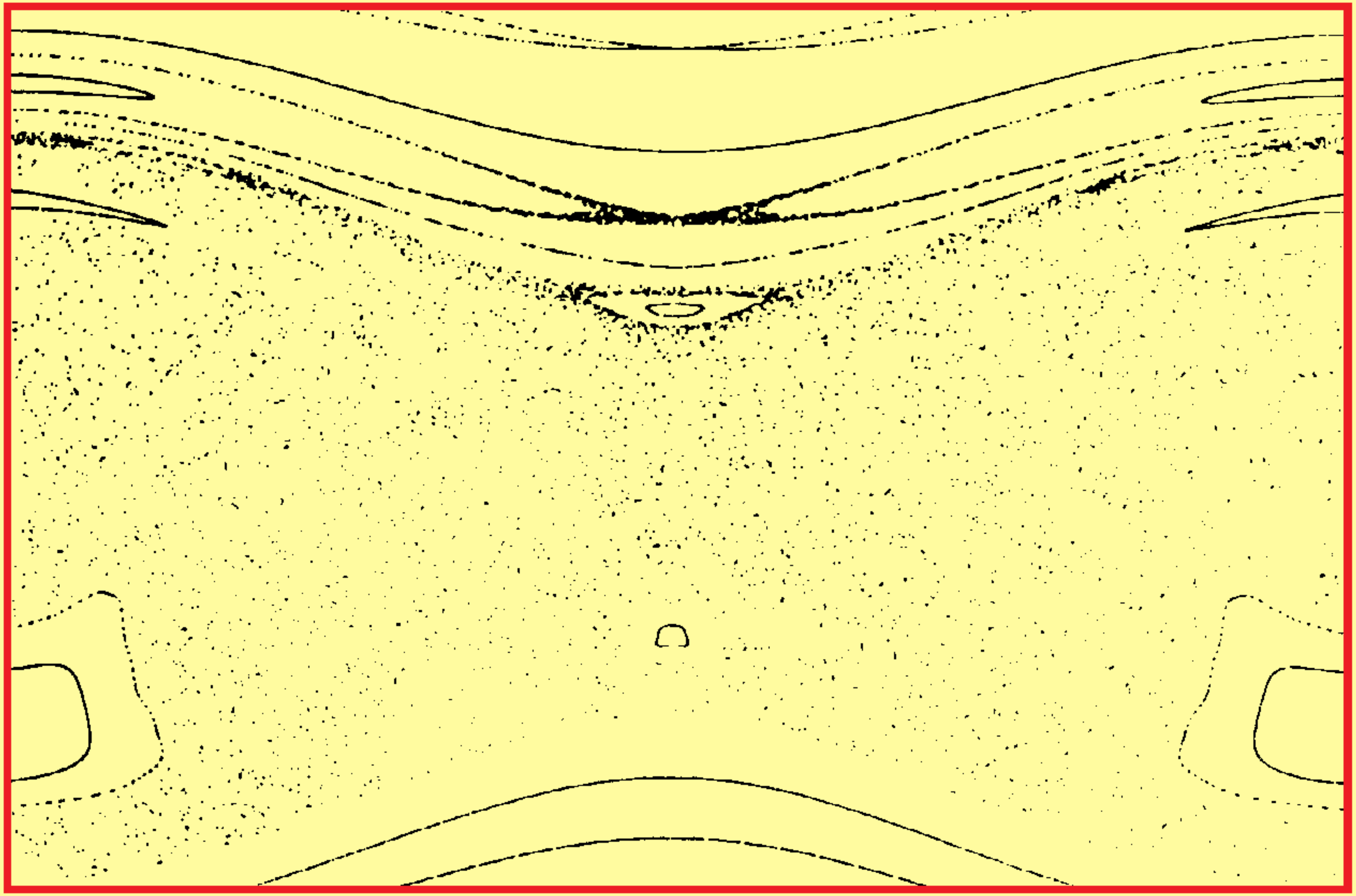
Particles in the rings range in size from dust to mountains.

Two moons orbit in the gaps.

Saturn's moon, Hyperion

Hyperion tumbles **chaotically and unpredictably** as it orbits Saturn. It appears to be sponge-like and porous with a low density. Irregular in shape, its average diameter is about 270 km.

Chaotic Tumbling of Hyperion (simulation)

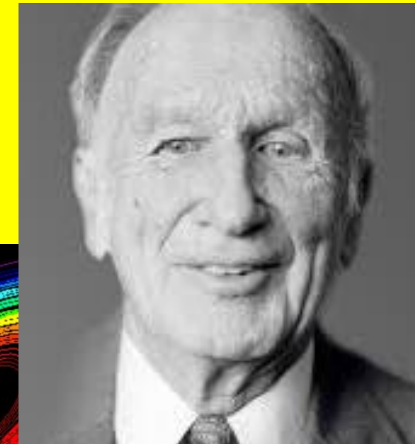


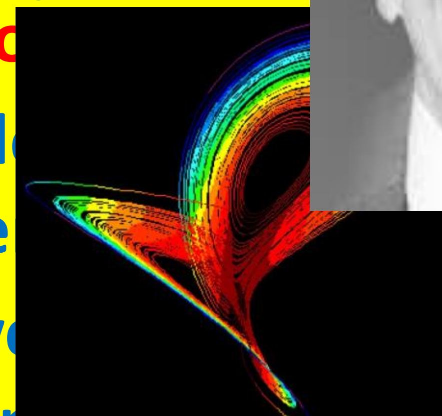
Poincaré section of spin-angle versus spin-rate.
Elliptical orbit generates the chaos (dots).
Curves are quasi-periodic motions.

Ed Lorenz, the Butterfly Effect

In 1963 Lorenz (1917–2008) was trying to improve weather forecasting

Using a recently available computer, he discovered the first chaotic attracto





(*x*, *y*, *z*) define convection of a mode atmosphere, with a thermal gradie

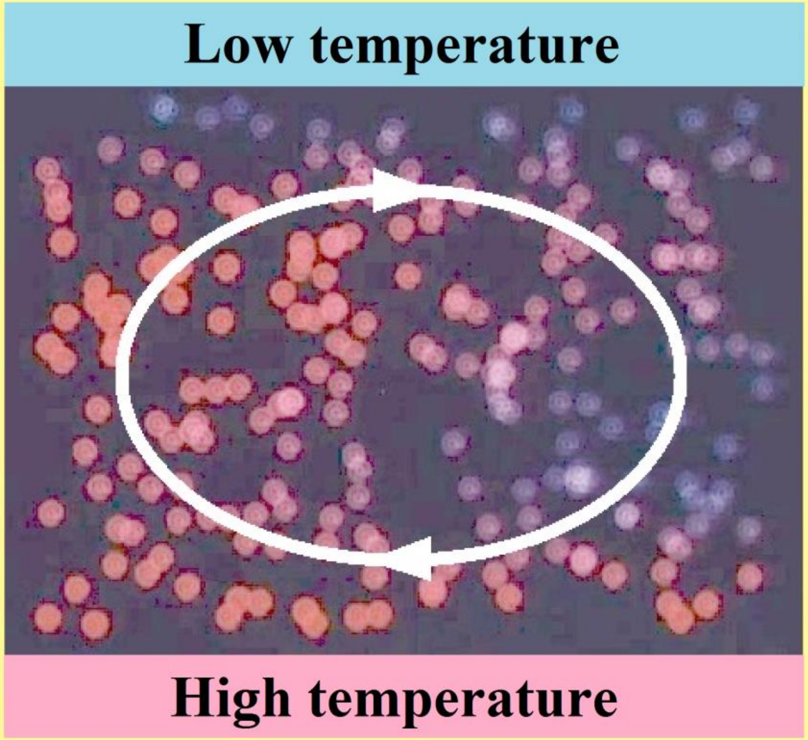
At fixed *R*, changing convection give trajectories in a 3D (*x*, *y*, *z*) phase space

At high *R*, trajectories from all starts settle onto a strange, chaotic attractor

Right and left flips occur as randomly as heads and tails. Prediction is impossible

s04 lorenz

Lorenz: atmospheric convection in a box (cf Rayleigh-Benard cells)



$$x' = -10\,(x - y)$$
$$y' = R\,x - y - x\,z$$
$$z' = x\,y - (8/3)\,z$$

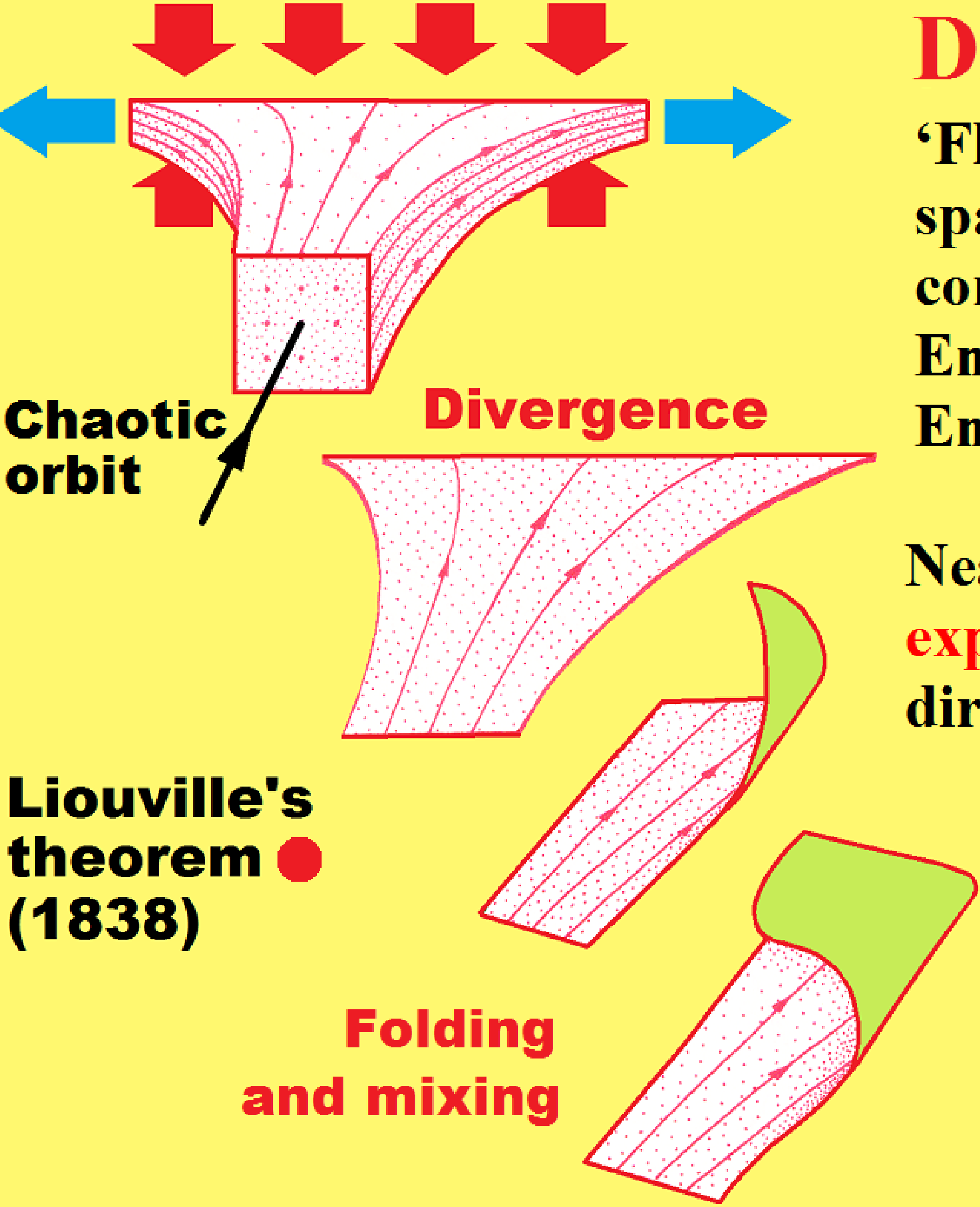
div = -10 -1 - (8/3)

Rayleigh number, *R*, is ratio of thermal driving to fluid damping. Convection starts at *R* = 1, and Lorenz identified chaos at *R* = 28.

***x* is rate of circulation, *y* and *z* describe the temperature distribution**

Since div < 0, phase volumes contract and all motions settle: at *R* = 28, to a chaotic attractor

m35 lorenzHbs



Divergence (div)

'Flow' of trajectories in phase space is like a fluid: if energy is conserved, it is incompressible.

Energy dissipation: div < 0

Energy input: div > 0

Near a chaotic motion we have exponential divergence in one direction, contraction in another

Sensitive Dependence on initial state

Exponential growth as in the divergence of chaotic trajectories

Exponential change is very rapid!

Positive div: 2^{x} gives 2, 4, 8, 16, 32, 64, 128, …

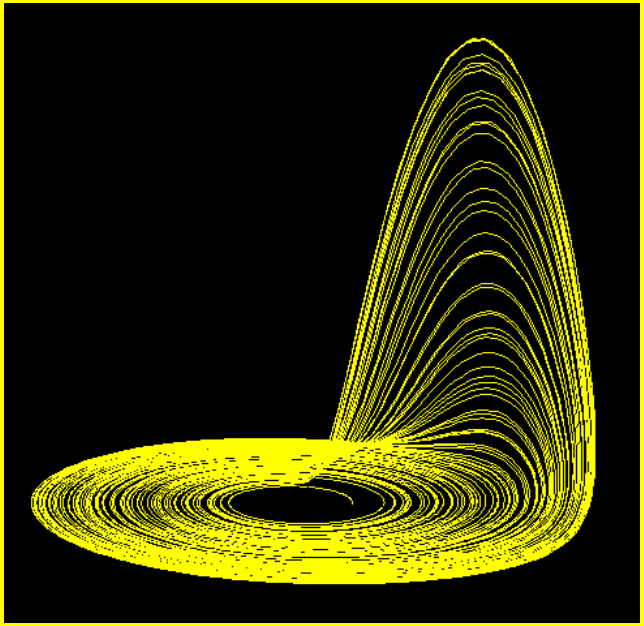
Negative div: 2^{-x} gives 1/2, 1/4, 1/8, 1/16, …

Roll out flaky pastry: reduce by 10 each time. After 8 rolls (10^{-8}) you are splitting an atom!

Bang!!

Rossler's equations (1976)

Rossler devised a system of 3 equations to display the folding and mixing of chaos.

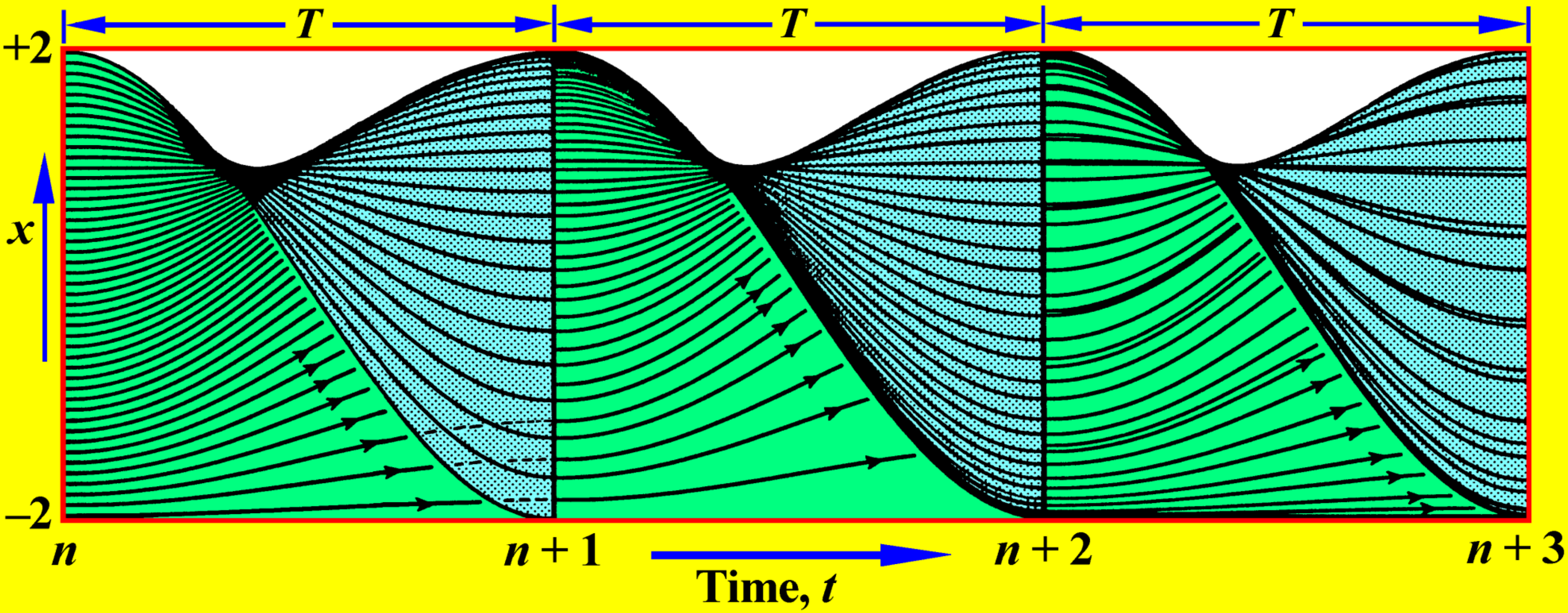


The repeated folding is like making flaky pastry.

This creates an infinite number of infinitely thin layers: a fractal structure.

s05 ross

Repeated Folding and Mixing



There is no crossing in phase space: so how do complex chaotic motions arise?

The answer is by **divergence, folding and mixing** (possible with **nonlinearity and 3D**)

Next lecture …