Chaos and Fractals: Understanding the Unpredictable

by

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A course of 5 lectures for the Cambridge U3A
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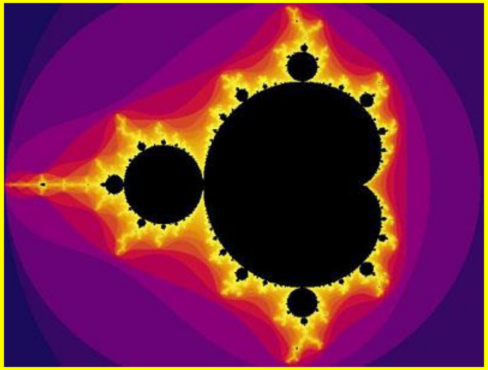
Topics covered in the Lecture Course

Newton's Clockwork Universe
Chaos and Fractals
From Mayflies to Butterflies
Chaos in Driven Oscillators
Fractal Basins, Chaotic Crises
Concluding Remarks



Uncontrollable shimmy

Lecture 1
The Clockwork Universe of Newton

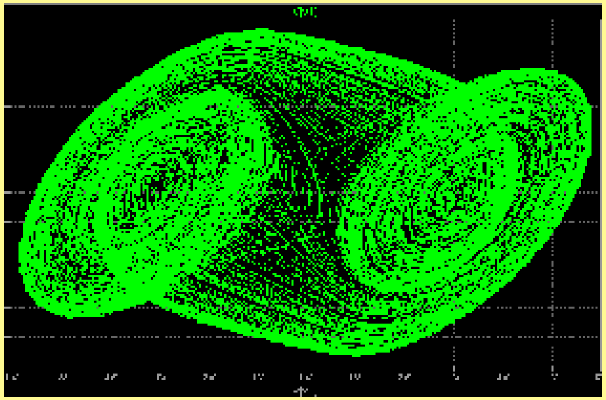
- The sound of chaos
- Double Pendulum: a taster
- Quest to Predict the Future
- Newton: *Principia,* mechanics
- In Poetry, Art and Philosophy
- Galileo, swinging lamp
- Introduction to phase space
- Homoclinic orbits create chaos
- English church bells

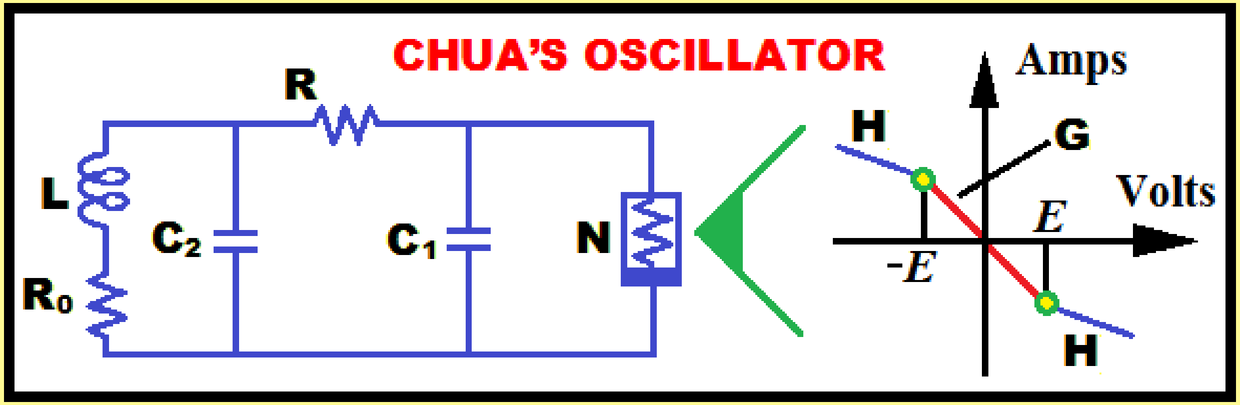
Chaos in Electronics and Telecommunications

An archetypal circuit (Chua, 1992) exhibits a double-scroll chaotic attractor.

We can listen to its chaos intended for the ambient sound to 'Marisol' by José Rivera.

Here an old and senile God is dying and taking the rest of the universe with him!





LINEAR: inductor L with resistance R_0 ; resistor R ; capacitors C_1, C_2

NONLINEAR: driving diode resistor N (slopes G, H)

Double Pendulum: a taste of chaos

The double pendulum gives us a first glimpse of chaos.

A real pendulum will have friction, air resistance, etc, which will dissipate energy: it will eventually stop.

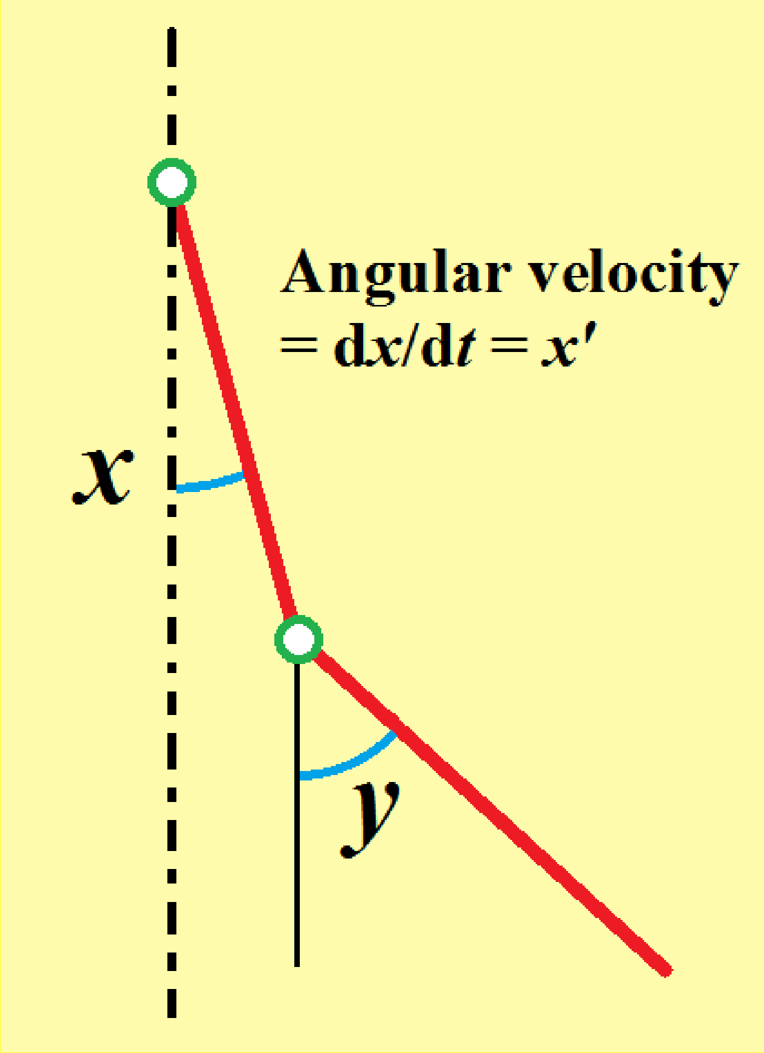
But often it is useful to imagine a non-dissipative pendulum which oscillates for ever.

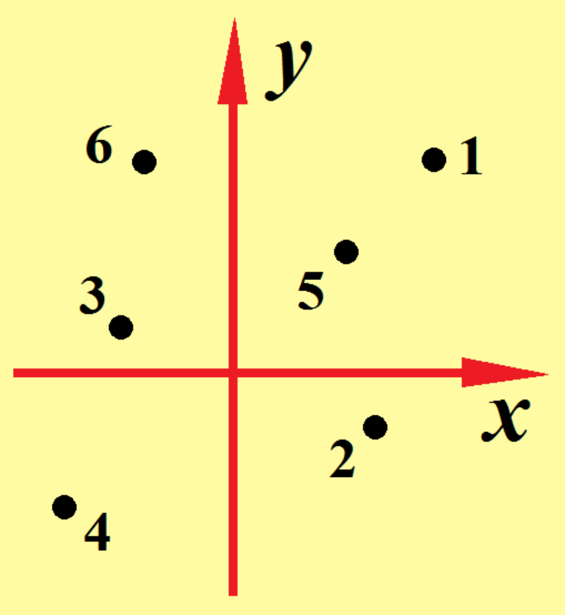
Experiment

Double Pendulum without Dissipation

In this conservative system, motions continue, and do not settle.

We employ a Poincaré Section by plotting the two linkage angles when the upper link has zero angular velocity.

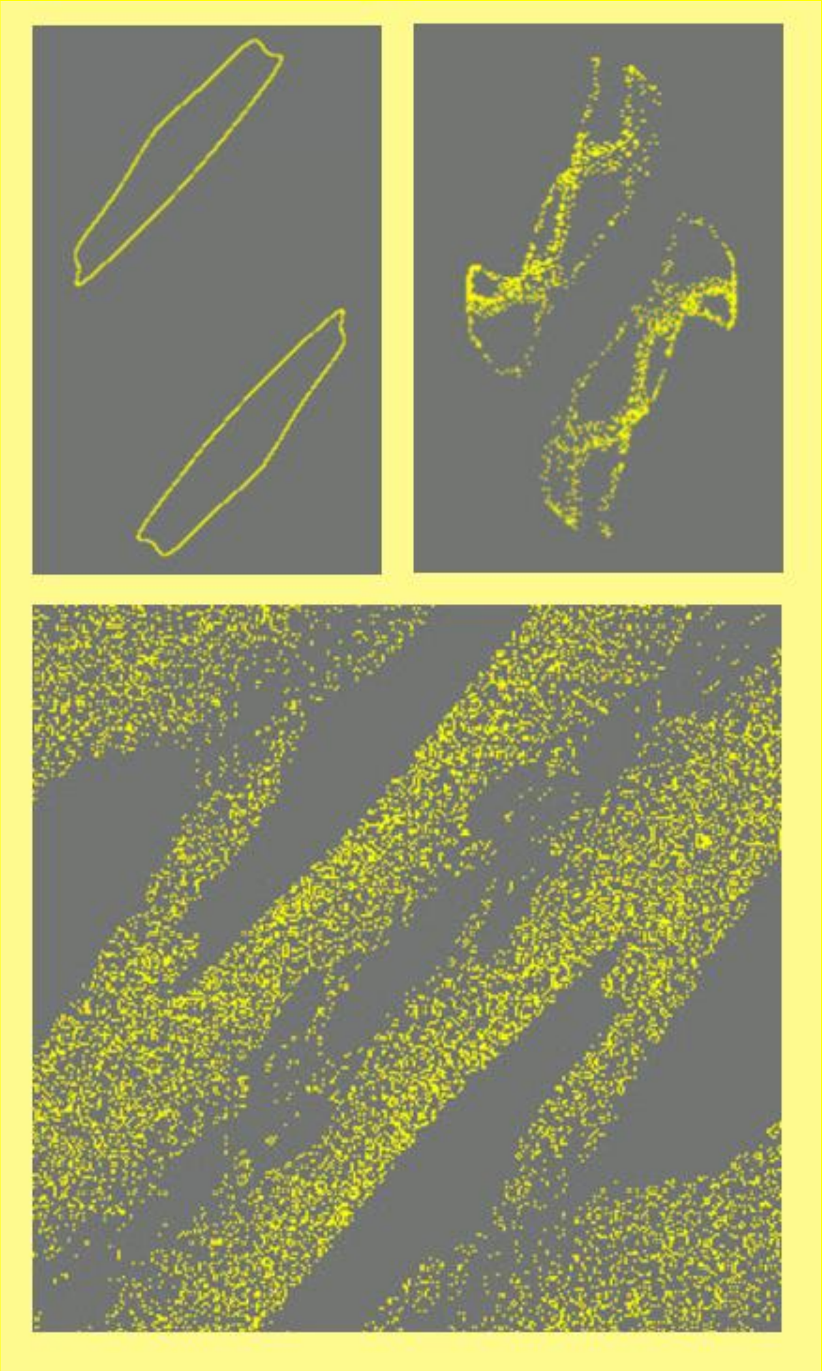


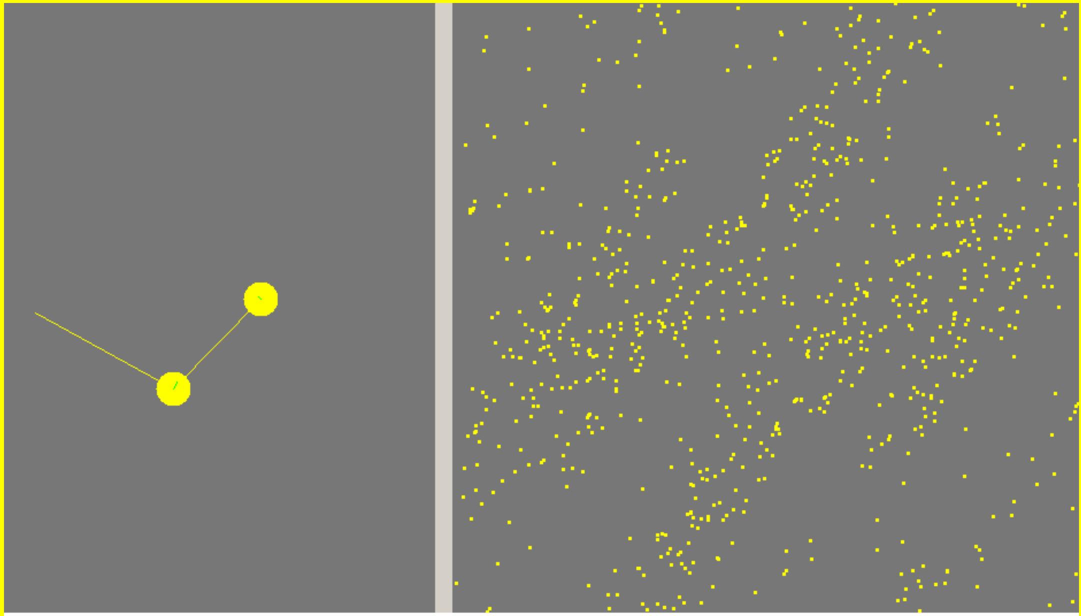


Depending on the start:

Regular oscillatory motion (upper left)

Irregular chaotic motion (others)





Undamped double pendulum and its Poincaré sampling

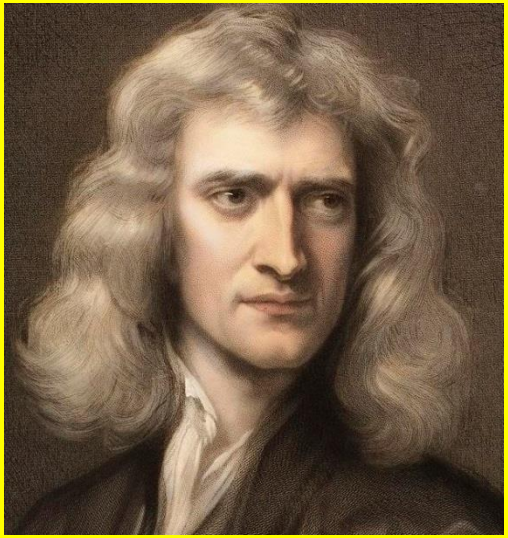
s02 d-pend   s06 henon

The Quest to Predict the Future

Human beings have always sought to understand the working of nature

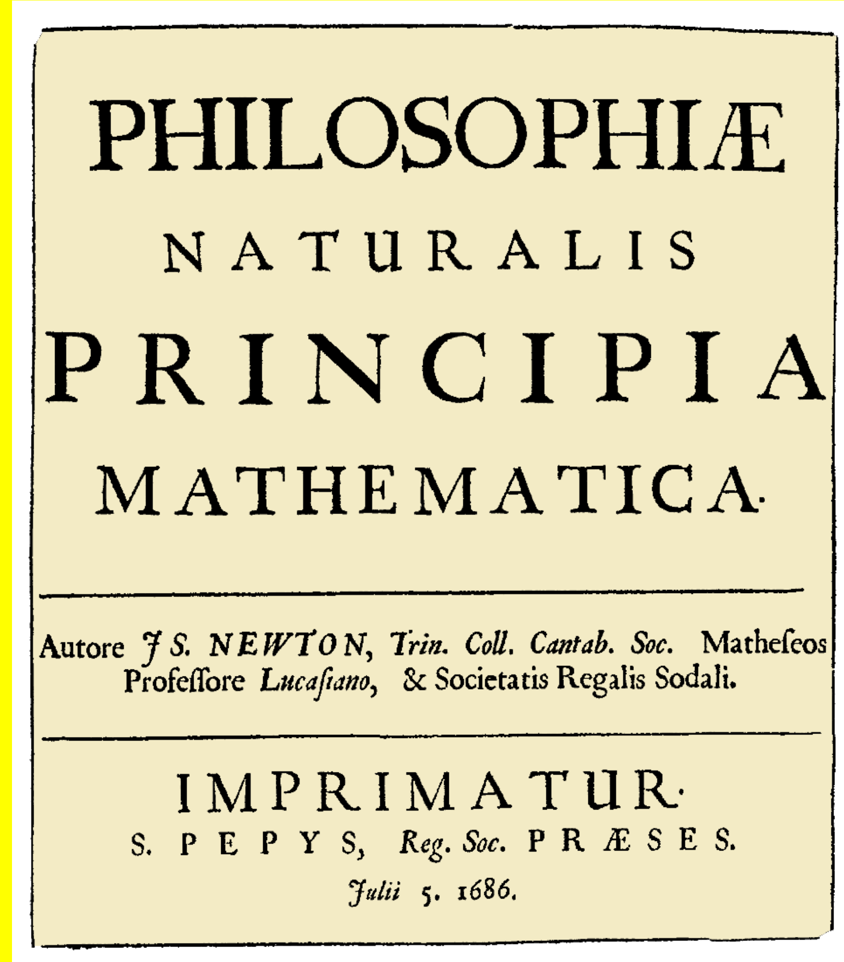
Early in this search, astronomy proved to be a fertile field

Early civilizations devised calendars to predict the seasons on Earth, and the eclipses of the sun and moon in the sky



Newton's *Principia* 1686

- **Newton's Laws of Motion revolutionised science**
- **Giving a dynamical system with a differential equation**
- **Still the ideal way to model a system evolving in time**
- **Starting conditions, define a unique future**



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J S.* NEWTON, *Trin. Coll. Cantab. Soc.* Matheſeos
Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

Stephen Hawking was Lucasian Professor at Cambridge, 1979-2009

From Newton to Chaos

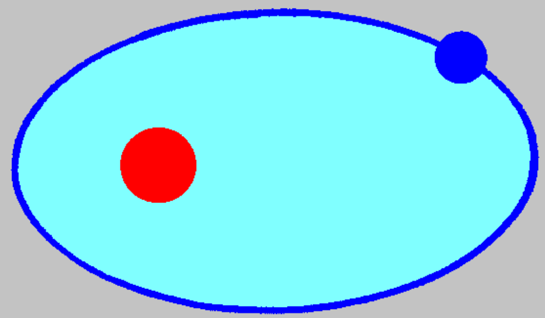
In mechanics, Newton's Laws allow precise analytical solutions to two problems:

The simple pendulum
The Sun-Earth (2 body) system

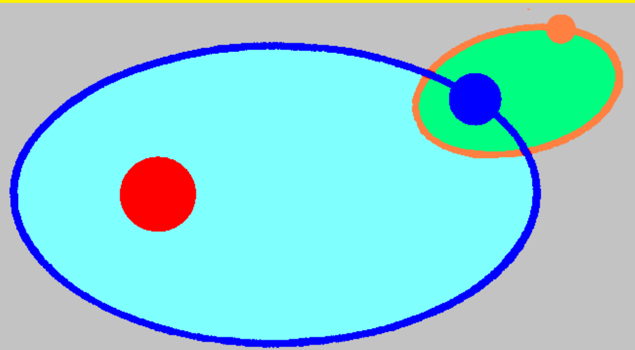
One small increase in complexity gives the following with chaos and no such solution:

The double pendulum
The Sun-Earth-Moon (3 body) system

Note that these astronomical problems relate to point masses interacting according to Newton's inverse-square law of gravitation



Newton solved the 2-body problem



Poincaré proved the 3-body problem is chaotic and essentially unsolvable

Status of Newtonian Mechanics

- Today, Newton's Laws are used throughout science and engineering
- In (for example) the design, all trajectory calculations, and the guidance of the Mars missions
- They are superseded at the sub-atomic level by quantum mechanics:
- and at speeds near that of light, or at cosmic distances, by relativity theory m39 ed white 1965
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Newton’s Impact in Poetry and Art

Blake depicted him

with divider and scroll at the bottom of the sea!

Blake actually opposed the Enlightenment.

Alexander Pope wrote:

Nature and Nature’s laws lay hid in night:

God said, *Let Newton be!* and all was light

Newton in Philosophy

Pierre Laplace (1749-1827)

Ball rolling on 4 shaken wells

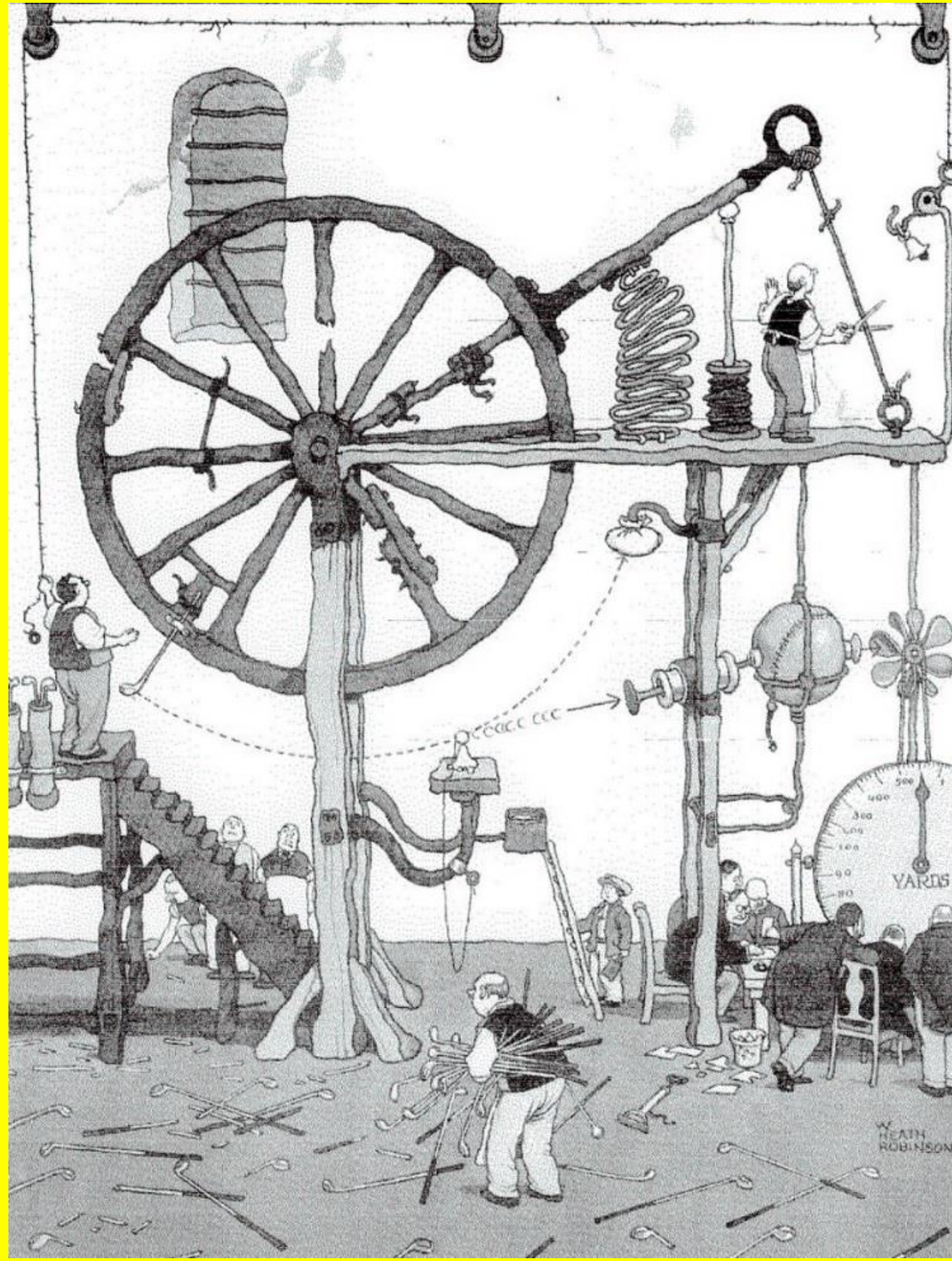
- Suppose a Universe (lifeless, say) is made up of particles interacting according to Newton's laws
- Then it is a dynamical system, governed by a (very large!) set of differential equations
- Given the starting positions and velocities of all particles, there is a unique outcome (true even for chaos!)
- This was a bombshell to scientists and philosophers
- Laplace, a French mathematician, wrote extensively about the clockwork universe
- However ...

m26 lv ball

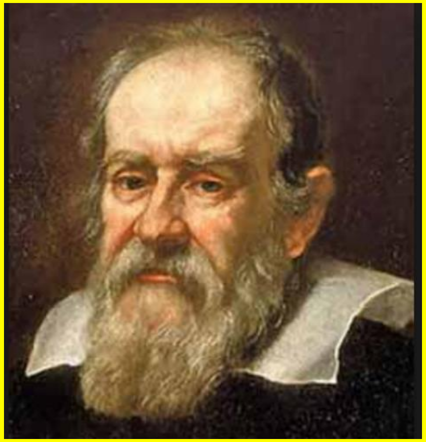
The Clockwork Universe

Newton’s Laws suggested that the Universe has the predictability of a machine

Golf club testing by Heath Robinson



The Simple Pendulum



Galileo Galilei

The pendulum is the classical example of a dynamical system

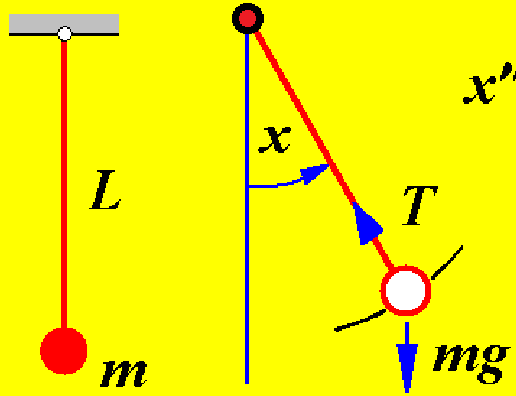
During a service in the cathedral of Pisa, Galileo (1564-1642) pondered the constant oscillation of a lamp swinging in the breeze

Used for centuries in clocks, it is the very essence of regularity and predictability

Pendulum: oscillations and phase space

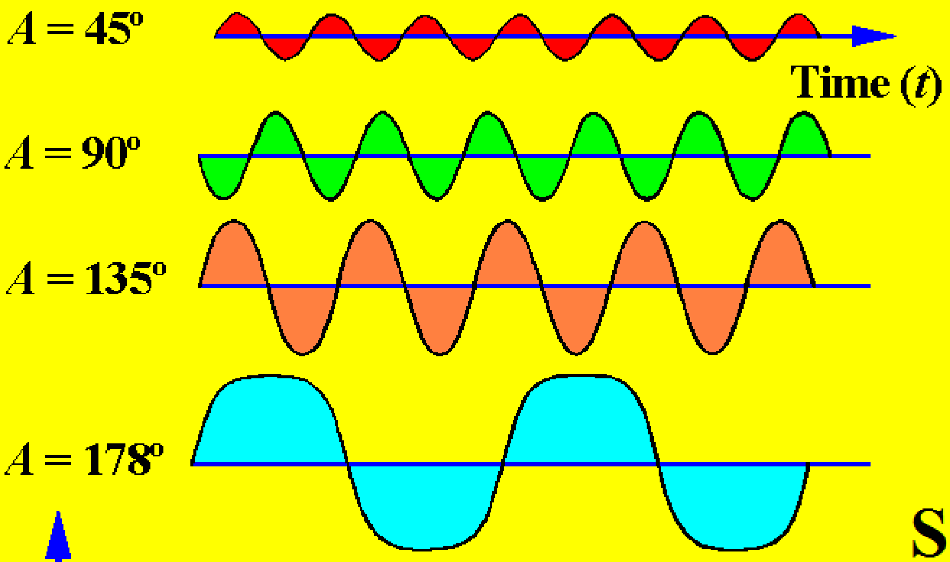
SIMPLE PENDULUM

Large-amplitude oscillations of an undamped pendulum



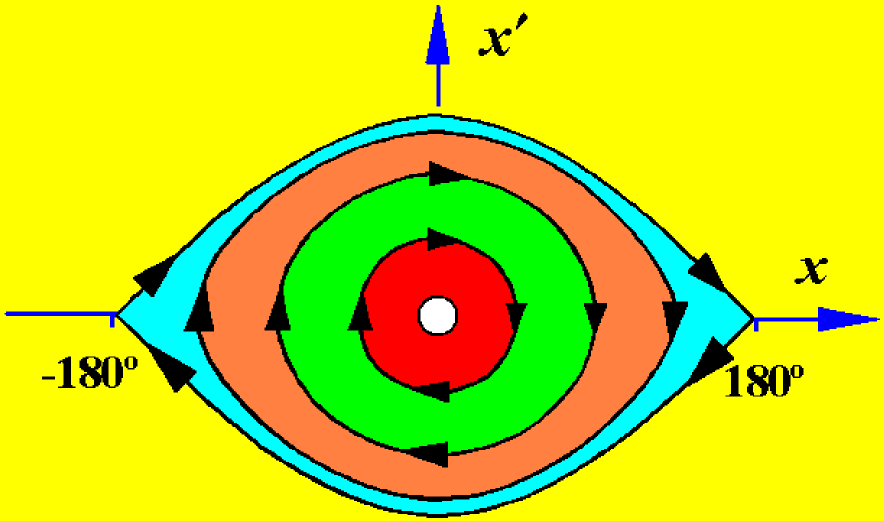
$$x'' + (g/L)\sin x = 0$$

Time histories of angle (x) versus time (t)



Amplitude

The phase space of angular velocity (x') versus angle (x)

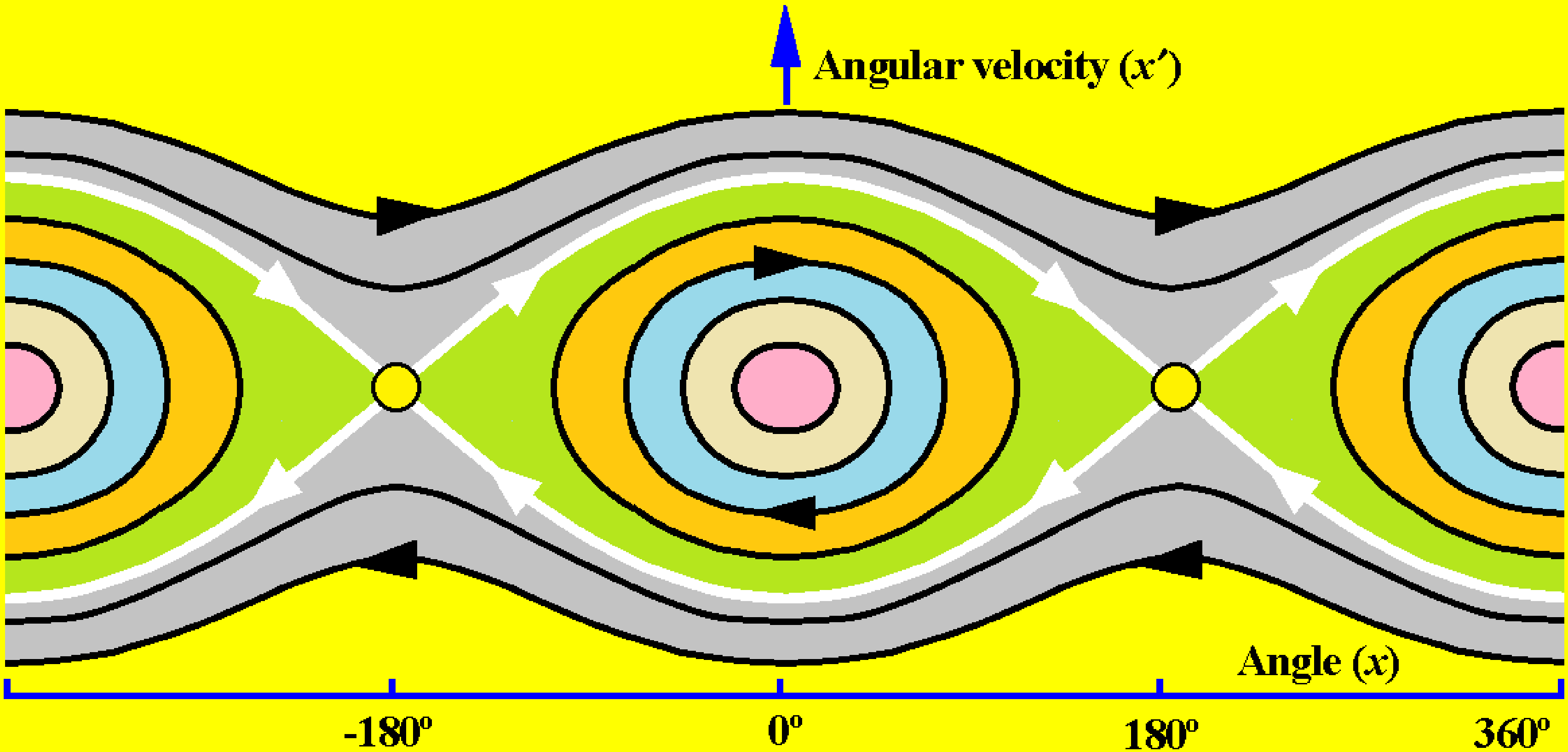


Space is full of non-crossing trajectories giving unique path

Phase space: of the starting conditions **Experiment**

The Homoclinic Trajectory

The white trajectory travels, in infinite time, from one saddle to the other. It looks like a heteroclinic connection between two saddles. But for the pendulum these are the same states, so we call it homoclinic.



The white homoclinic curves form the basin boundaries between the coloured oscillations and the grey whirling (propeller) motions.

Homoclinic orbit creates chaos

Imagine a very small number,

E (say 1/1000000000).

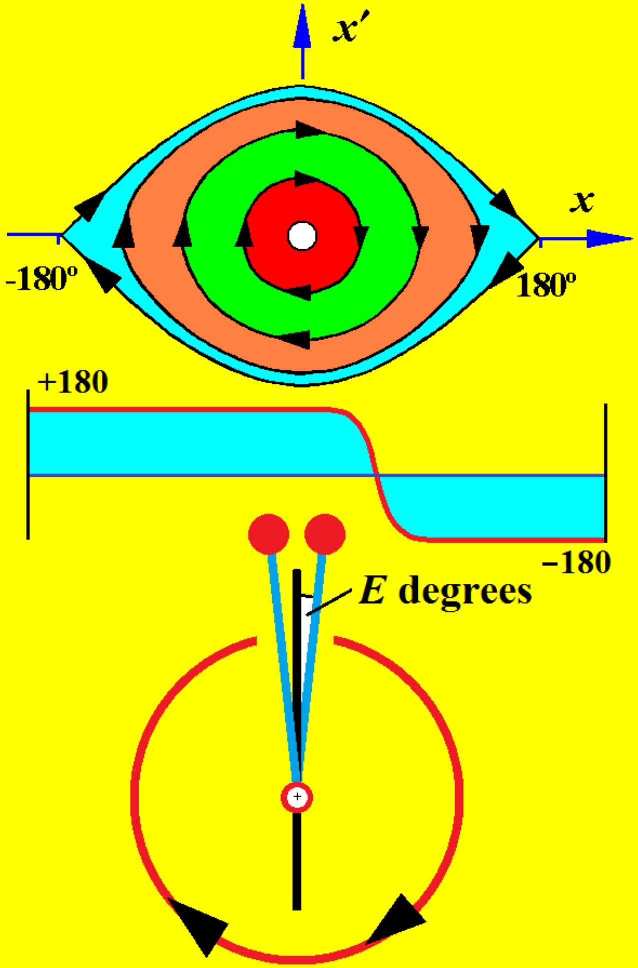
Suppose Galileo released a pendulum

at rest, E degrees off vertical in 1606.

For 410 years it slowly gathered speed.

Today (2016) it makes a rapid transit of

about 360º.



Then for 410 years it will slow down, coming to rest at E degrees on the other side of the vertical.

If disturbed, it may or may not get over the top: here lies a possible trigger for chaos.

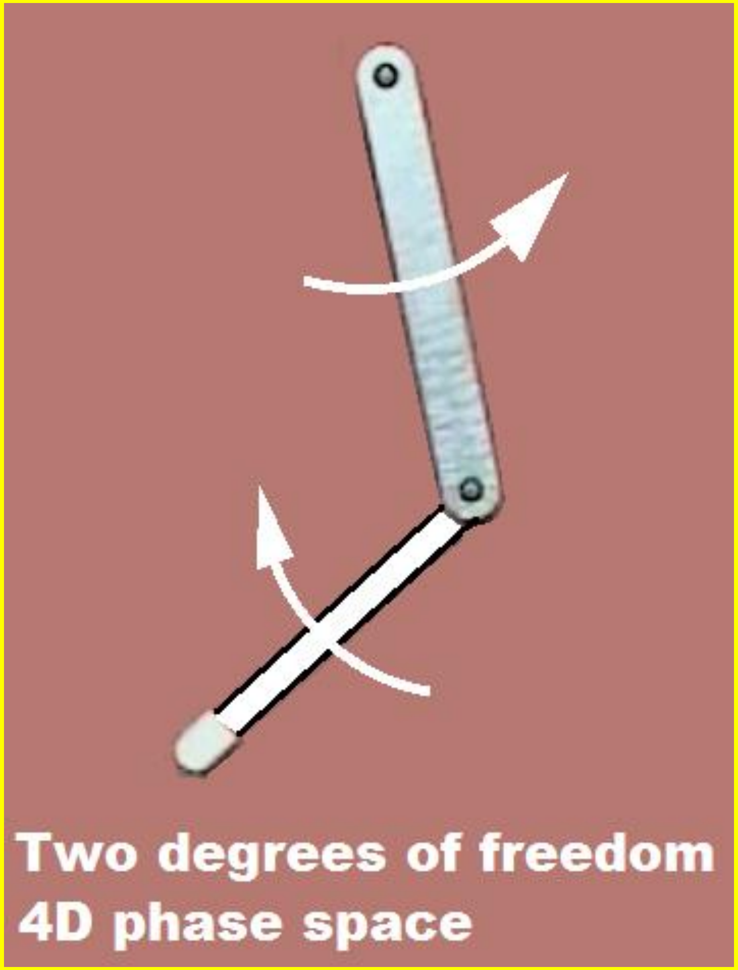
Experiments on a Double Pendulum

One pendulum effectively disturbs the other, giving chaos by a '*repeated passage near a homoclinic plus a regular disturbance*'

We shall see:

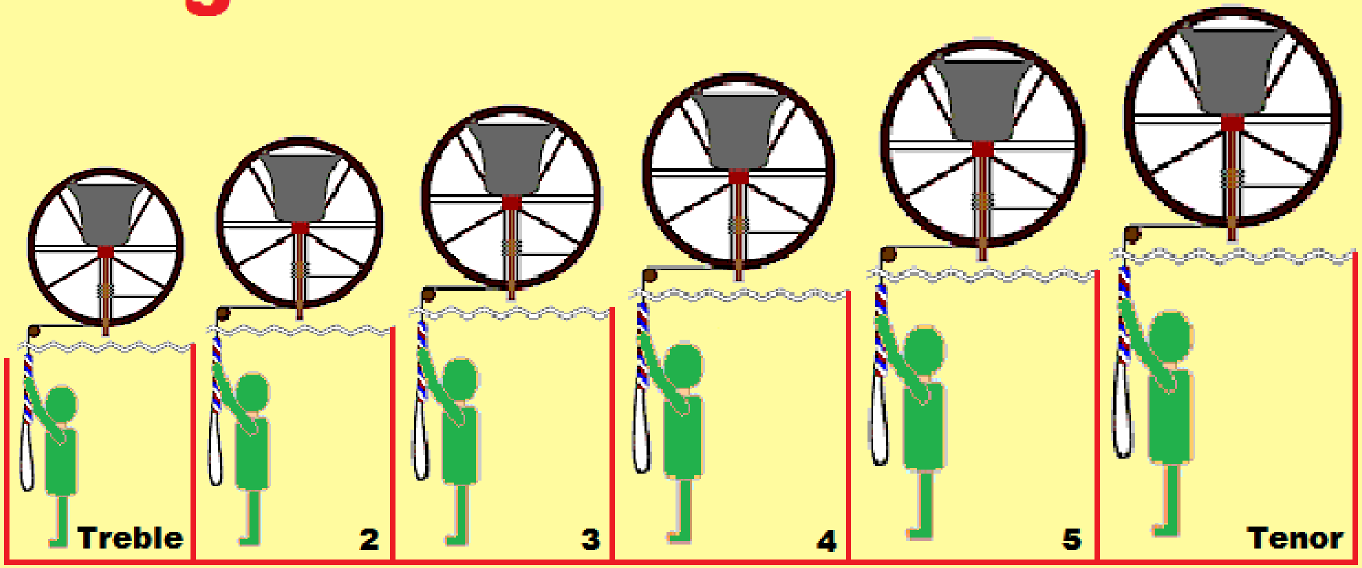
Normal modes of linear theory

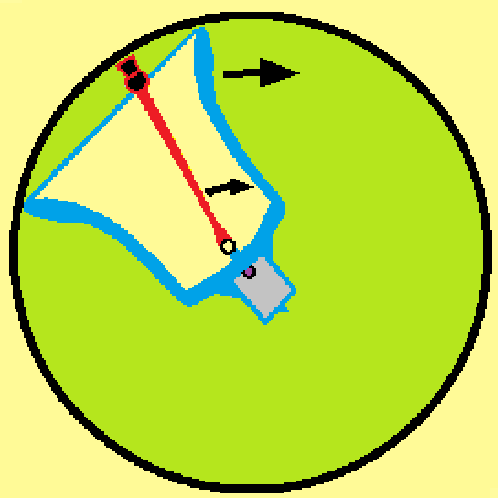
Nonlinearity and chaos



m15 d-pend

English Church Bells





Ringing is akin to controlling chaos.
A ringer rests the bell close to its inverted state.
Then with little effort, but considerable skill, it can be tugged into its homoclinic trajectory.
It swings through ~ 360° and back to 'rest'.

The clapper, rotating faster, is about to strike and ring the bell.

m30 bells (sound)

End of Lecture 1

s01 ex-pend

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Next time ... throwing a book!