SYNOPSIS E466 APPLIED MECHANICS (DYNAMICS)

Generalised coordinates $q = (q_1, ..., q_n)^T$ (n degrees of freedom); $q_i = q_i(\boldsymbol{r}_1, ..., \boldsymbol{r}_N)$; holonomic ($f(\boldsymbol{r}_i) = 0$) and nonholonomic ($f(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i) = 0$) constraints

Virtual displacement, virtual work: $\delta W = \sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^n Q_i \, \delta q_i$ (Q_i generalised forces)

Kinetic energy T, potential energy V, Lagrangian $L = T - V$ (where $L = L(q_i, \dot{q}_i)$)

Euler-Lagrange equations of motion (including non-conservative generalised forces Q_i for damping, forcing, etc.): $(\mathrm{d}/\mathrm{d}t)(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = Q_i \quad (i = 1, ..., n)$

Hamilton's principle of stationary action: $\delta S = 0$, where $S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i) \, \mathrm{d}t$, yields the Euler-Lagrange equations (in the conservative case)

Small displacements: quadratic forms $T = \frac{1}{2} \dot{q}^T M \dot{q}$, $V = \frac{1}{2} q^T K q$ (M mass matrix, K stiffness matrix, both symmetric: $M^T = M$, $K^T = K$)

Small oscillations are governed by the linear system of equations $M\ddot{q} + Kq = F$, with F the vector of generalised applied forces (this follows from the Euler-Lagrange equations)

Natural frequencies (eigenvalues) ω_i and natural mode shapes (eigenvectors) $v^{(i)}$ of free ($F = 0$) synchronous vibrations are given by solutions of the eigenvalue problem $(K - \omega^2 M)v = 0$

Orthogonality of eigenvectors with respect to M and K: $v^{(i)^T} M v^{(j)} = 0 = v^{(i)^T} K v^{(j)}$ $(i \neq j)$

Normalisation: $v^{(i)^T} M v^{(i)} = I$; then $v^{(i)^T} K v^{(i)} = \omega_i^2$ (normalised eigenvectors are called normal modes)

The $v^{(i)}$ form a basis: any q can be expressed as a linear combination of eigenvectors, i.e., $q = \sum_{i=1}^n c_i v^{(i)}$ (expansion theorem), where $c_i = v^{(i)^T} M q$ (this assumes **normalised** eigenvectors)

Modal matrix $V = (v^{(1)}, ..., v^{(n)})$ (columns of V are (usually normalised) eigenvectors); transformation to normal coordinates c according to $q = Vc$ leads to a fully uncoupled system of equations $\ddot{c}_i + \omega_i^2 c_i = Q_i$, where $Q = V^T F$, with solution $c_i(t) = A \cos \omega_i t + B \sin \omega_i t + (1/\omega_i) \int_0^t Q_i(\tau) \sin \omega_i (t - \tau) \, \mathrm{d}\tau$ (the expression $Q = V^T F$ assumes **normalised** eigenvectors)

Zero eigenvalues correspond to non-oscillatory rigid body modes (translation and/or rotation)

System with linear damping: $M\ddot{q} + C\dot{q} + Kq = F$; in case of proportional damping (i.e., $C = \alpha M + \beta K$, for some α and β) the modal expansion is preserved with decaying c_is

Forced systems ($F \neq 0$), harmonic or non-harmonic forcing; harmonic forcing at one of the natural frequencies ω_i leads to resonance (in the absence of damping this gives rise to unbounded oscillations)

Given an estimate v of the fundamental mode shape, Rayleigh's quotient $R(v) = v^T K v / v^T M v$ gives an estimate of the corresponding fundamental frequency, good to second order; in fact, $\omega_i^2 = v^{(i)^T} K v^{(i)} / v^{(i)^T} M v^{(i)}$ and $R(v) \geq \omega_1^2$ for all v

Small free vibrations of **uniform** continuous systems: lateral vibrations of a string, longitudinal vibrations of a bar and torsional vibrations of a shaft are all governed by the wave equation $c^2 \partial^2 u / \partial x^2 = \partial^2 u / \partial t^2$

Small free vibrations of a **uniform** Euler-Bernoulli beam are governed by the beam equation $c^2\partial^4 u/\partial x^4 + \partial^2 u/\partial t^2 = 0$

Looking for separable solutions (synchronous motion), $u(x,t) = Y(x)Z(t)$, leads to a boundary-value problem for Y requiring 2 boundary conditions (one on each side) for the string, bar or shaft, and 4 boundary conditions (two on each side) for the beam; there are infinitely many solutions (eigenvalues, natural frequencies) ω_n of the characteristic equation, each with corresponding eigenfunction (natural mode) $Y_n(x)$ $(n = 1, 2, ...)$

There is again an orthogonality principle for the mode shapes $Y_n(x)$; there is also an expansion theorem which allows one to reduce the problem to an (infinite) set of uncoupled harmonic oscillators c_i; hence the problem of finding the response of a beam to external lateral excitation is entirely analogous to that for a discrete system

The finite-element method offers a way of approximately solving the equations of motion for a continuous structure by dividing the structure into elements and expressing the displacement $u(x,t)$ at any point of a given element in terms of displacements u_i at the boundaries (the nodes) of that element; for the approximate mode shapes we take solutions of the corresponding statics problem (the shape functions $\phi_i(x)$); the mass matrix M and stiffness matrix K follow by performing the x-integration of the kinetic and potential (strain) energy; the vector of nodal forces F is found by using the virtual work expression; the result is a discrete system of equations in the form $M\ddot{q} + Kq = F$, where $q = (u_1, u_2, ...)^T$ is the vector of nodal displacements

Torsion element (2×2 matrices); beam element (4×4 matrices)

The global mass and stiffness matrices M and K for the whole structure (an assemblage of finite elements) are obtained by suitably integrating the individual elemental matrices M_i and K_i into matrices for the overall structure: $M = \sum_{i=1}^{N} A_i^T M_i A_i$, $K = \sum_{i=1}^{N} A_i^T K_i A_i$ (sum over N finite elements), where the matrices A_i express the nodal displacements of element i in terms of displacements relative to a global system of coordinates

Instead of consistent-mass matrices for an element one can use simpler (diagonal) lumped-mass matrices

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