NOTES ON THE HISTORY OF THE GENERAL EQUATIONS OF HYDRODYNAMICS*

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Introduction. The following article consists mainly in the descriptions of volumes in an exhibit in the Indiana University Mathematics Library commemorating the two-hundredth anniversary of the general equations of fluid dynamics.

1752–1952. In 1748 the Berlin Academy proposed as the subject for the prize competition of 1750 the theory of the resistance of fluids. The tentatives of Newton were known to be wrong, and nothing but empirical formulae had taken their place. The subject, besides being close to the unexplored foundations of the concept of inertia, was thus of great mathematical interest and practical moment.

In December, 1749, d'Alembert sent to Berlin his entry, a long *Essai d'une Nouvelle Théorie de la Résistance des Fluides*. The Academy, of which d'Alembert

* There are only two relatively extensive historical works on fluid dynamics:

O. Flachsbart, Geschichte der experimentellen Hydro- und Aeromechanik, insbesondere der Widerstandsforschung, pp. 1–61 of vol. 4_2 of *Handbuch der Experimentalphysik*, Leipzig, 1932.

R. Giacomelli & E. Pistolesi, Historical Sketch (of aerodynamics), pp. 305–394 in vol. 1 of *Aerodynamic Theory*, ed. W. F. Durand, Berlin, 1934.

Neither is complete nor entirely reliable; in both, the mathematical aspects of the subject are almost entirely neglected. There is some relevant material in

E. Hoppe, Geschichte der Physik, pp. 1–179 of vol. 1 *Handbuch der Phys.*, Berlin, 1926,

R. Dugas, *Histoire de la Mécanique*, Neuchâtel and Paris, 1950.

Both these works are scholarly; the former is entirely reliable, the latter, mainly so.

The few remarks concerning hydrodynamics in the standard histories of mathematics or of physics, when not entirely false, are so disconnected and incomplete as to be virtually meaningless. Many consist in more or less distorted selections from

Lagrange, *Méchanique Analitique*, Paris, 1788; see Seconde Partie, Septième Section. Slightly enlarged in the second and later editions (*e.g. Oeuvres* **11, 12**), where it appears as Section Dixième.

This brief and elegantly written history is scholarly, accurate in statement, and penetrating, but it omits most of the subject entirely. Lagrange dismisses all the early work with a phrase uniting Archimedes and Galilei: "for the interval which separated these two great geniuses disappears in the history of mechanics." It is only total ignorance which can produce such total contempt. The most serious omission of all, only partly corrected in the second edition, is the inexplicable neglect of Euler, from whose papers most of the portions of the *Mécanique Analytique* which concern fluids are taken over almost unchanged. And of course it stops in 1788.

My teacher, the late Dr. P. F. Neményi, left unfinished the MS of a remarkable history of the concepts of fluid mechanics, dealing mainly with the physical side of the subject. Within the next two years I hope to be able to fulfill our plan of balancing it by a corresponding study of the mathematical ideas and methods.

Meanwhile, however, I have thought it worthwhile to publish these remarks on the formation of the backbone of the whole subject. At the referee's suggestion, I have added this explanatory note and a few references to other historical works; in this connection it should perhaps be added that any statement for which no such reference is given represents my own opinion, formed entirely from study of the original sources.

was a non-resident member, judged that no contestant had earned the prize, sent back all manuscripts, and urged the authors to compare their predictions with the results of experiment. D'Alembert withdrew at once, and in 1752 he published his essay. It is described in §3 below. In the introduction d'Alembert details the circumstances in angry and sarcastic terms.

At the time of the contest Euler was director of the Mathematical Division of the Berlin Academy. It seems difficult to believe that he could have expected seriously that a theory of fluids could give results in accord with existing experiments on the resistance suffered by submerged bodies. The data itself was confused and sometimes contradictory. Nearly all the experiments concerned turbulent flow; they were not to be understood even qualitatively for a full century following; and to the present day the phenomenon of resistance remains, as far as any rigorous theory is concerned, a partially open question. Rather, it is likely that the reason given out was only a pretext, offered in place of the truth, much more difficult to substantiate to an author, that d'Alembert's reasoning was tortuous, incomprehensible, and endless, and that in illustration of his equations he had not succeeded in exhibiting a single two-dimensional flow.

This view is supported by the fact that by 1752 Euler had read to the Academy a paper *De motu fluidorum in genere*, treating this subject. Eneström* identified this paper with that described in §4 below. No resistances of specific bodies are calculated in it, nor in any of Euler's papers on the general theory, and there is no attempt to compare predictions with experimental measurements. But much that d'Alembert had tried to do is achieved in a direct and economical fashion, not only more general but also far more convincing—as Euler said in a later paper, "directly from the first axioms of mechanics."

It was the facile opinion of Euler's contemporaries, repeated in most of the present histories of mathematics, that† "M. Euler seemed sometimes to be occupied by the pure pleasure of mathematical calculation and of considering the questions of mechanics and physics only as occasions for exercising his genius and for abandoning himself to his predominant passion. So scientists have reproached him for having sometimes lavished his calculus on physical hypotheses, or even on metaphysical principles, of which he had not sufficiently examined the likelihood and solidity." In contrast, d'Alembert is represented as a physicist who "explained with his usual clearness and insight the necessary dependence of calculation upon experiment."‡ This estimate of d'Alembert seems to derive from Lagrange, who in various places in the *Mécanique Analytique* attributed to d'Alembert various things which it is extremely difficult to locate at all in the actual writings of d'Alembert himself, and extremely hard to follow even when located. I believe that any modern scientist who actually

* Verzeichnis der Schriften Leonhard Eulers, *Jahresbericht der deutschen Math. Verein.*, 4. Erganzungsband, 1910. See No. 258.

† Éloge de M. Euler (by Condorcet), *Hist. Acad. Sci. Paris*, 1783, (1786), quoted by Giacomelli and Pistolesi, *op. cit.*, p. 325.

‡ Giacomelli and Pistolesi, *loc. cit.*

reads the papers of d'Alembert and Euler will soon adopt a directly reversed view. In fact, in theories of nature Euler was so superior to all his contemporaries that parts of his writings on mechanics and physics were dismissed as too "mathematical," while to the modern reader they appear perfectly clear and in large part correct. It should be added also that Euler not only kept himself fully informed in experimental physics, but was always ready to make immediate use of the results of his calculations—and moreover, that those of his hydrodynamical writings which made the greatest permanent advances are those least influenced by experiment, while those in which he compromised with practicality have lost all but historical interest, since the experimental data available was obscure, incomplete, and sometimes faulty.*

The work of d'Alembert in hydrostatics was examined in detail by Todhunter, who found it "almost uniformly wrong" and containing "but little of importance." After careful search of his writings on hydrodynamics, I find the same words applicable, except for the fact that he did somehow arrive at the correct equations for plane and axially-symmetric irrotational flow. Philosophical meandering and doubts regarding the validity of the theory he is constructing fill so great a part of his work that there is little space left for development of the theory itself. It is quite possible that d'Alembert's discussion of experiments and repeated criticism of the mathematical method in physics may have appealed to some physicists of his day, as doubtless in ours; but no progress, either in mathematics or in physics, was achieved by these merely negative statements.

1. I. Newton, *Philosophiae naturalis principia mathematica*, London, 1687. Newton's views on fluids were heterodox. In the *Principia* we find both a molecular model of a "rare medium, consisting of equal particles freely disposed at equal distances from each other," and side by side with it a representation of a fluid as a smooth continuum without voids. The former view, in which impacts of the fluid particles upon each other and hence also those upon the back side of a submerged body are neglected, led Newton to the famous law of resistance proportional to the square of the velocity, and to his difficult and incorrect theory of sound. It has excited a great deal of notice ever since 1687, but has not led to any significant development in hydrodynamics.

One of the several passages where fluids are regarded as continuous media may be found in Lib. II, Sect. IX. Here Newton states a "Hypothesis," which may be translated as follows: "the resistance arising from the want of slipperiness in the parts of a fluid is, other things being equal, proportional to the velocity with which the parts of the fluid separate from one another." It is the earliest correct statement regarding the internal friction of fluids. When properly generalized by Stokes (see §10), it forms the basis of the modern theory of

* A detailed analysis of Euler's writings on hydrodynamics will appear shortly in the introduction to vols. 12 and 13 of *L. Euleri Opera Omnia*, ser. 2.

viscosity. Theorem 39, however, which follows immediately, is false, as is virtually every specific conclusion of Newton concerning fluids.

Historical research has destroyed much of the attribution of originality to Book I of the *Principia*, where occurs much of the now standard material given in elementary mechanics courses. This part of the treatise now appears mainly as an attempt at organization and rigorous, deductive treatment of known material. Earlier works contained errors freely intermingled with correct results, while Newton's presentation is correct in principle and in detail. Book II, however, which concerns the properties of fluids and of bodies moving in them, consists almost entirely in original work of Newton, who was the founder of hydrodynamics. The plan of deducing everything from the axioms, so distinctive a feature of Book I, is abandoned, and fresh hypotheses are put forward at every turn. We may gather from a remark in the preface that it is necessity, not choice, which has enforced this breakdown of the original scheme. Newton's might of intellect is revealed even more clearly in this largely erroneous but path-breaking part of the work, which gave rise to two hundred years of controversy in the attempt to correct and complete what Newton had begun. Finally, in Book III Newton shows his peerless ability to get specific, numerical conclusions from an analytical theory and to compare them with the results of experiment.

2. D. Bernoulli, "Theoria nova de motu aquarum per canales quoscunque fluentium," *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 2 (1727), 111–125 (1729). The work "hydrodynamics" was created by D. Bernoulli with the publication of his *Hydrodynamica* in 1738. The Indiana University Library does not possess a copy of this book; exhibited instead is a paper, published a few years earlier, which may serve as a specimen of the methods and results.

No general equations or general principles of fluid motion are to be found here. The *Hydrodynamica* is famous for "Bernoulli's theorem,"

$$p + \tfrac{1}{2}\rho v^2 = \text{const.},$$

which is taught in every freshman physics course. In this simple form, however, it is not to be found in the book, nor is its use given any prominence. In both the journal article (p. 112) and the book the analysis rests on the "principle of the conservation of living forces," which Daniel Bernoulli attributes to his father, John Bernoulli. At this time there was a vigorous general controversy regarding *vis viva* (kinetic energy). In the book, the conservation of energy is asserted in the form of the "principle of equality of potential ascent and actual descent." The terms are defined as follows:*

"We shall note that the *potential ascent* of a system whose several particles are moved with an arbitrary velocity signifies the vertical altitude which the

* *Hydrodynamica*, Sectio tertia, §1.

center of gravity of that system would reach if the several particles are conceived to turn their velocities upward and rise as far as they can; that the *actual descent* denotes the vertical altitude through which the center of gravity descends after the several particles had come to rest." The book was praised by Lagrange as "a work which shines with analysis as elegant in its ordering as simple in its results."

The "Bernoulli equation" in its now usual simplicity was first written down by Euler (see §4, 5), who derived it as an integral of his general hydrodynamic equations. In so doing, he generalized it in three ways, obtaining the forms valid in unsteady potential flow, in barotropic potential flow, and (with the constant generally assuming a value differing from one stream-line to another) for steady rotational flow.

That Daniel Bernoulli did not attain the simplicity of a modern engineering treatment, while Euler, beginning with a much more elaborate mathematical structure, by purely analytic manipulations attained the result in its now customary form, should not astonish us. Daniel Bernoulli wrote not only before Euler had made it customary to express general physical principles as mathematical equations, but also before the concept of internal fluid pressure had been created. His hydrodynamical works consist of long sequences of special problems ingeniously solved by means of the energy principle. Many of these problems are of the type which can be treated somewhat more simply nowadays by use of what is now called "Bernoulli's theorem." His father, the old John Bernoulli, in envy at his son's success* immediately wrote his *Hydraulica*, pre-dating it by seven years so as to make his son appear a plagiarizer. Nevertheless, the new work was enthusiastically praised by Euler as the first to use the "true and genuine method" and to be based on "the most certain principles of mechanics." By the former Euler meant the method of infinitesimal elements; by the latter, the momentum principle. In John Bernoulli's treatise we find also the first rather rough use of internal pressure, and the first actual equation equivalent to a fairly general case of the modern "Bernoulli equation." All this work concerned flow in tubes, and in Euler's early hydraulic papers the greater generality of the problems treated led to forms of the result gradually approaching the modern one.

The paper of Daniel Bernoulli which we were considering ends with a discussion of the validity of the two principles by whose application the foregoing special problems have been solved. Bernoulli states that the "conservation of living forces" remains valid as long as there is no friction or other exterior resistance, but that the "proportionality of the velocity to the inverse of the amplitude," while only an approximation, is valid for most vessels, provided there is no motion suddenly created or destroyed. The latter statement is the principle of incompressibility, expressed in an approximate form correct only for flow of incompressible substances in nearly straight pipes.

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* Cf. *e.g.* pp. 95–96 of O. Spiess, *Leonhard Euler*, Frauenfeld and Leipzig, 1929.

3. **J. L. d'Alembert,** *Essai d'une Nouvelle Theorie de la Rêsistance des Fluides*, Paris, 1752. Of this rare book only two copies, located at the Navy Department Library and at the Brown University Library, are known to be in the United States. The photostat on exhibit is loaned by the Naval Research Laboratory.

The circumstances of composition of this work are mentioned in the introductory section above. It contains the first publication of the equations governing irrotational flow of incompressible fluids in two dimensions. For axially-symmetric flow, these equations are stated in §45, the notations having been explained in §43. The equations for plane motion are given summarily in §73. The ideas and form of presentation are strongly influenced by the work of d'Alembert's lifelong rival, Clairaut, in his *Thêorie de la Figure de la Terre, tirêe des Principes de l'Hydrostatique*, Paris, 1743, where the principles of hydrostatics are similarly expressed. Like Clairaut, d'Alembert makes use of the principle of equivalent canals.

The *Essai*, despite the fact that it is so elaborate and so burdened with philosophy that very likely it has never been read in its entirety since 1755, is nevertheless a turning point both in physics and in mathematics. It contains the first field description of media in motion, a description which in other hands led quickly to partial differential equations and in particular to potential theory. It contains also what is generally described as d'Alembert's solution of the equation $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0$ in the form $\phi=f(x+iy)+g(x-iy)$, although in fact d'Alembert himself did not obtain the partial differential equation, deriving an equivalent result by manipulating differentials. The analysis is contained in §§57–60, which are famous for the first appearance of the "Cauchy-Riemann equations." The reasoning is essentially as follows: if

$$\frac{\partial p}{\partial z} = -\frac{\partial q}{\partial x}, \qquad \frac{\partial p}{\partial x} = \frac{\partial q}{\partial z},$$

then $qdx+pdz$ and $pdx-qdz$ are complete differentials. By purely formal steps it is shown that $p+iq=f(x-iz)$, $p-iq=g(x+iz)$.

In §70 appears the famous "d'Alembert paradox." Put in modern terms, this "paradox" asserts that a submerged smooth body moving at uniform speed in a continuous irrotational flow of an ideal fluid at rest at infinity experiences no resultant force. It had been stated and proved more simply and more generally by Euler in 1745*, and Oseen† has traced it back to Spinoza. All the early statements refer to symmetrical bodies and employ some form of reversibility argument, whereby the pressure on the back side of a body is shown to cancel precisely the pressure at a corresponding point of the front side. However, no such special reasoning is required, and the result holds for smooth bodies of any form. It is essential, however, that the velocity field fall off

* *Neue Grundsätze der Artillerie, aus dem Englischen des Herrn Benjamin Robins übersetzt und mit vielen Anmerkungen versehen*, Berlin, 1745, *Opera omnia* (2) **14.** See Ch. 2, Law 1, remark 3.

† *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Leipzig, 1927. See introd.

sufficiently rapidly at infinity. For incompressible fluids, the necessary order condition is available from potential theory; for subsonic flow of compressible fluids it has been established rigorously only in recent months.

Since he is writing about a new theory of the *resistance* of fluids, and that theory turns out to give no resistance at all in the most natural situations, d'Alembert appears to be rather dismayed. He does not make a great point of the matter, however, and passes on to treat of many other topics at great length but with few concrete results. Twelve years later he published the paradox again* in rather more sensational terms.

In the *Essai*, most of d'Alembert's dark suspicions regarding the theory are limited to the impossibility, rather than the uselessness, of the calculations. But time has shown that ideal hydrodynamics is far from useless. Even in the problem of resistance, d'Alembert turned out to be wrong. While he claimed to have shown that all motions are irrotational, three years later Euler proved (§5) that irrotational motion is only a very special case; and in rotational motion, the d'Alembert paradox no longer holds. Through proper development and exploitation of this idea, the modern theory of airplane lift has developed.

4. **L. Euler,** "Principia motus fluidorum" (1752–1755), *Acta Academiae Imperialis Scientiarum Petropolitensis*, **6** (1756–1757), 271–311 (1761). This paper contains Euler's first published treatment of the general problem of fluid motion. In 1752, shortly after the Berlin Academy's rejection of all pieces contesting for the prize on the resistance of fluids, Euler read before it a paper *De motu fluidorum in genere*, and Eneström regarded this fact, which was ascertained by Jacobi, as sufficient evidence to date the present paper 1752. Another paper of Euler, itself dated 1755 (see §5), refers to this paper by its actual title.

The first part of the paper calculates in detail the change in volume of an infinitesimal tetrahedron carried by the motion through an infinitesimal time. Setting this change equal to zero yields a general equation expressing incompressibility. The calculation is carried out first in two dimensions, then in three. On page 286 is the conclusion of the three dimensional treatment, with the resulting *equation of continuity* in a notation still used at the present day:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

where u, v, w are the velocity components.

The second part of the paper introduces rather hastily the concept of fluid pressure and derives the dynamical equations for ideal fluids.

The work is dense with ideas waiting to be worked out. Among its contents we may observe the first occurrence of the secular determinant and its expansion, the introduction of the velocity potential and the acceleration potential,

* Suite des recherches sur le mouvement des fluides, *Opuscules mathématiques*, 5 (1768), no. 34, 132–170. See §I.

derivation of "Laplace's equation" for the former (cf. §3), construction of the general polynomial harmonic of degree n, derivation of "Helmholtz's vorticity equation" as a condition of integrability for the pressure field, derivation of "Killing's equation" for rigid motion, derivation of "Bernoulli's equation" for unsteady irrotational motions, a theory of fluid friction, derivation of the theory of flow in narrow tubes. It is interesting to notice that in this paper Euler concurs with d'Alembert's decision that all fluid motions possess a velocity-potential (irrotational flow).

5. L. Euler, "Principes généraux du mouvement des fluides," *Mémoires de l'Académie Royale des Sciences*, Berlin, **11** (1755), 274–315 (1757). This paper is the second of three parts of a treatise on the mechanics of fluids, published in consecutive pages in the same volume of the journal. In the first part Euler established the partial differential equations of hydrostatics and their general solution. In order to do so, he introduced the concept of *pressure* as a field acting upon an imaginary closed boundary and mechanically equivalent to the action of the fluid exterior to the boundary upon that interior to it. Previous authors had regarded the pressure as the weight of a unit column of fluid, or the height to which water would rise if the vessel were punctured. The distinction is extremely important, and it is Euler's concept of pressure which is used in modern hydrodynamics. Neither Clairaut's hydrostatics (1743) nor d'Alembert's hydrodynamics used the pressure at all, except possibly by implication. Their general principles consist in statements that certain differentials shall be exact. Euler was the first to conceive any branch of mathematical physics as a field theory governed by partial differential equations. Lagrange* in comparing the work of d'Alembert and Euler wrote:

"But these equations did not yet possess all the generality and the simplicity of which they were susceptible. It is to Euler that we owe the first general formulae for the motion of fluids, founded on the laws of their equilibrium, and presented with the simple and luminous notation of partial differences. By this discovery, all the mechanics of fluids was reduced to a single point of analysis. . . ." The importance of the pressure, even for hydrostatics alone, was put so highly by Todhunter† that he wrote:

"Before Euler thus illustrated [*i.e.*, illuminated] the subject, there had been *demonstrations* in hydrostatics, but I cannot consider that these demonstrations were altogether intelligible."

The paper shown here clarifies and generalizes the pioneer work of 1752. Unlike the latter, it is smooth and pleasant reading, expressed in simple and easy notations. Both physically and mathematically it shows on every page what Fourier described as "that admirable clarity which is the principal character of all Euler's writings." The hydrodynamical equations are derived from what Euler called "the first axioms of mechanics," *viz.*, the indestructibility of

* *Loc. cit.*, but only in the second (1815) and later editions.

† *Op. cit.*, §301.

matter and momentum.

On p. 286 Euler obtains his general dynamical equations for ideal fluids:

$$f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \text{ etc.}$$

Previously he had derived the continuity equation for the compressible case:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0,$$

and he emphasizes the generality and the completeness of the set of four equations, complemented by a relation between pressure and density.

After a great many interesting tentatives and discoveries, Euler at the end of the paper concludes that mathematical analysis must be greatly improved before further progress in fluid dynamics can be made. "We see . . . how far distant we still are from a complete knowledge of the motion of fluids, and what I have just explained is only a feeble beginning. Nevertheless all that the theory of fluids includes is contained in the two equations presented above, so that it is not the principles of mechanics which we need for the pursuit of these researches, but analysis alone, which is not yet sufficiently cultivated for this end. One sees clearly what discoveries remain for us to make in this science before we can arrive at a more perfect theory of the motion of fluids."

6. L. Euler, "Recherches sur la propagation des ébranlemens dans une milieu élastique," *Mélanges de Turin*, **2**[2] (1760–1761), 1–10 (1762); *Opera* (2) **10**, 255–263; *Oeuvres de Lagrange* **14**, 178–188. Throughout the latter part of his life Euler several times returned to the theory of fluids, either to extend its applications or to strengthen its foundations. His final work on the subject is a definitive Latin treatise in four lengthy instalments, published partly after his death.

The incorrect value for the speed of sound obtained by Newton in the *Principia* gave rise to a century and a half of controversy. It occurred to Euler that acoustics should be a branch of hydrodynamics, and that a proper value for the speed of sound should be obtainable from the hydrodynamical field equations. The problem had been taken up meanwhile by Lagrange, then an unknown young man at the beginning of his mathematical career. Lagrange had written to Euler on this subject as well as on the calculus of variations, and Euler replied with his usual generosity and candor.

This paper is the 1762 printing of one of Euler's letters to Lagrange, dated January 1, 1760. In it he derives a new set of hydrodynamical equations, expressed in terms of the positions of the fluid particles at some arbitrary initial instant. These initial positions are what are now called the "Lagrangian coordinates." The misnomer arose from the fact that Lagrange in the *Méchanique Analitique* did not quote the specific sources of any of his hydrodynamical material, permitting the reader to gain the impression that much of it appeared there for the first time, while in fact all of it is contained in earlier papers by Euler or by Lagrange himself.*

Euler's second development of the field equations for fluids is in some ways more complete than the first, since it employs a more detailed kinematical description. All the principal equations, including the general formulae for inversion of partial differentiations, are given in the letter.

As is now easily understood, the proper speed of sound was not and could not be calculated from the hydrodynamical field equations alone. The result of Euler and Lagrange's researches was to establish the wave equation on a solid foundation, at the same time showing that the essence of the problem of the speed of sound lies in the form of the relation between pressure and density.

7. C.-L.-M.-H. Navier, "Mémoire sur les lois du mouvement des fluides" (1822), *Mémoires de l'Académie des Sciences de l'Institut de France,* (2) **6,** 389–440, (1827). Euler's equations when applied to flow in a long straight pipe possess an infinite number of solutions; in fact, any continuously differentiable flow with straight parallel stream-lines is such a solution. However, a physical fluid when forced down such a pipe at sufficiently low speed assumes a single, perfectly definite velocity distribution. It follows that in Euler's equations some essential physical phenomenon has been neglected. In fact, Euler's equations refer to fluids without dissipative mechanisms, sometimes called "perfect" or "ideal" fluids. While such fluids closely approximate the behavior of physical fluids in many circumstances, in cases where internal friction is important a more accurate mathematical model is required. For example, neglect of the adherence of fluids to solid boundaries is one of the reasons for the "d'Alembert paradox" (§3).

* Actually it is almost certain that Euler *began* his work on the general theory of fluids with the material description, but soon took up the simpler spatial description, which alone is used in his earliest publication. D'Alembert in the last section (§160) of the *Essai* refers sarcastically to "a manuscript theory on the current of rivers," which, he says, could not have fallen into his hands until August, 1750. He says further that "it would not even be impossible that the method presented in my work was unknown to the author of the memoir of which I speak, and did not help him in his researches on the current of rivers." Now there is a paper by Euler called *Recherches sur le mouvement de rivières*, which appears on pp. 101–118 of the Berlin Mémoires for 1760, not published until 1767. This paper presents internal evidence of being written before the *Principia motus* (1752); in particular, it employs a strange set of units used by Euler at this time, but abandoned in later papers; the formal structure is incomplete; and the specific flows considered as examples are trivial. According to Jacobi (quoted by Eneström, *Op. cit.*), a paper of this title was presented to the Berlin Academy on May 6, 1751. That it was *not* published sooner may be explained by the fact that in content it is a fragmentary and unsuccessful attempt, doubtless cast aside by Euler as he progressed in his researches. That it *was* published in 1767 may well be explained by Euler's angry departure from Berlin in 1766, depriving the academy of half its customary volume of MSS for publication.

If we can accept the date 1751 for this paper, we can set that year as the discovery of the material description. For it is this description alone which is employed in this paper.

By the year 1820 the foregoing facts were becoming rather dimly apprehended. In 1821 Navier, a French engineer, constructed imaginary models both for solid bodies and for fluids by regarding them as nearly static assemblages of "molecules," mass-points obeying certain intermolecular force laws. Forces of cohesion were regarded as arising from summation of the multitudinous intermolecular actions. Such models were not new, having occurred in philosophical or qualitative speculations for millennia past. Navier's magnificent achievement was to put these notions into sufficiently concrete form that he could derive equations of motion from them.

On p. 414 occur the basic dynamical equations, now called "the Navier-Stokes equations," which govern the motion of incompressible viscous fluids. In modern vector notation these read:

$$\mathbf{f} - \frac{1}{\rho}\,\mathrm{grad}\,p - \frac{\mu}{\rho}\,\nabla^2\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\mathrm{grad}\,\mathbf{v},$$

where $\mathbf{v}$ is the velocity vector and μ is the *shear viscosity*, a coefficient for which Navier gave an expression in terms of the intermolecular forces, but which is now usually regarded as an experimental function of temperature for each type of fluid. Navier's equation differs from Euler's in the presence of the term $(\mu/\rho)\nabla^2\mathbf{v}$, representing the effect of the internal friction, and, in so doing, raising the order of the differential equations from 1 to 2.

8. **A. Cauchy,** "Sur les équations qui expriment les conditions de l'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique," *Exercises de Mathêmatique*, **3** (1828); *Oeuvres* (2) **8**, 195–226. Fresnel's optical discoveries and the elastic theory of the ether which he proposed concurrently excited the intense interest of the entire scientific world. Immediately Cauchy took up the theoretical side of the subject, generalizing it profoundly and at the same time expressing the principles and results with the clarity and simplicity of the Eulerian tradition. The major results were announced in 1823, but proofs and explanations did not begin to appear until 1827, the second year of Cauchy's remarkable *Exercises*, annual volumes of short original papers by Cauchy alone, where jewels of number theory alternate with mechanics, complex function theory with algebra, pure geometry with the theory of infinite series.

Cauchy's first view regards matter as a continuous medium whose mechanical response is specified through *internal pressures*. These differ from Euler's hydrostatic pressure only in that their local resultant need not be normal to the surface element across which they act. This concept has the simplicity of genius. Its profound originality can be grasped only when one realizes that a whole century of brilliant geometers had treated very special elastic problems in very complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became the foundation of the mechanics of distributed matter.

Cauchy's main interest was in optical application, and he put great efforts into the elastic theory of light. §III of this paper contains a by-product, mentioned briefly in passing. If a continuous body is "altogether devoid of elasticity," the stress tensor t_{ij} assumes the form (Cauchy's eq. (95) (97), written in indicial notation)

$$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}),$$

where v_i is the velocity vector. When these are put into Cauchy's general form of the momentum principle,

$$t_{ij,j} + f_i = \rho\left(\frac{\partial v_i}{\partial t} + v_{i,j}v_j\right),$$

there result dynamical equations which differ from those for viscous compressible fluids only in lacking the hydrostatic pressure term. Cauchy apparently regarded them as suitable for very soft solid bodies. If $\lambda = 0$, while μ is regarded as a suitable function of stress, rather than of temperature, they become the equations of plasticity proposed by St. Venant (1870), Levy (1870), and v. Mises (1913), but no such interpretation was intended by Cauchy, who stated that for homogeneous bodies μ is a constant.

9. S.-D. Poisson, "Mémoire sur les équations générales de l'équilibre et du mouvement des corps solides élastiques et des fluides" (1829), *Journal de l'Ecole Polytechnique,* **13** (cahier 20), 1–174 (1831). There is no doubt that Poisson first learned of the general equations of elasticity and fluid dynamics from the memoirs of Navier, which were read at the French Academy in 1821 and 1822. However, Poisson contended that despite the correctness of the final results, Navier's analysis was wholly faulty. The lives of both these savants were embittered by the senseless public controversy which followed: Navier repeatedly claimed priority, Poisson repeatedly denied it, on the ground that a result derived by false reasoning is not derived at all. It is interesting to note the change of fashions in physics: Poisson, who was a physicist, based his attack mainly on Navier's having replaced summations over the individual particles by integrations over volumes.

Neither critic nor criticised argued soundly. Both viewed a liquid as a nearly static assembly of Newtonian mass-points; they differed one from the other only in details which now seem unimportant, since such a picture of the liquid state has been entirely abandoned, and both derivations must be rejected. Their models are of interest only as examples of how a wholly wrong molecular theory can lead to wholly right gross conclusions—a fact often overlooked in physical circles today.

Although Poisson's part in this controversy was not admirable, his physical sagacity and analytical skill did not fail to get new results. Just as it was Navier, and Navier alone, who discovered the equations for viscous incompressible fluids, so it was Poisson, and Poisson alone, who discovered those for compressible viscous fluids. In modern notations, Poisson's stress formula is

$$t_{ij} = - p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}).$$

Here occur both the pressure p, as in Euler's and Navier's equations, and the two viscosities λ and μ, as in Cauchy's equations. While the generalization is quite straightforward from the continuum viewpoint, it must be remembered that Poisson labored with an intricate molecular model, which he endeavored to manipulate both with physical propriety and with mathematical exactness.

10. G. G. Stokes, "On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solides" (1845), *Transactions of the Cambridge Philosophical Society*, **8** (1844–1849), 287–319; *Papers* **1**, 75–129. When Stokes wrote this paper, the general equations for elastic bodies and for viscous fluids had been in use in France for some twenty years, but they were not yet employed in England. It may be said that in England a result is not counted as known until an Englishman has rediscovered it. However, unlike some more recent compatriots, Stokes read, understood, and admired the continental literature, particularly the work of the great French mathematical physicsts, whose methods he brought to a Cambridge stultified by traditional adherence to the faults rather than the virtues of Newton's methods. In this paper as in his other works Stokes did not disdain to add to the results of deep physical insight and high analytical skill the indications of thorough scholarship, and in particular he discussed in detail the work of Cauchy and Poisson on the same subjects.

Stokes's contribution, as far as the general equations are concerned, is an extraordinarily natural and incisive discussion of the concepts. In §2 we find a "Principle" which expresses the notion of internal friction in its most general form:

"That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned." A hundred years later this same principle, expressed mathematically in tensor form, was shown by Reiner and Rivlin* to lead to much more general equations of fluid dynamics and to predict extremely interesting new physical phenomena. Stokes himself, content with a linear approximation, derived the

* M. Reiner, A mathematical theory of dilatancy, Am. J. Math. **67**, 350–362 (1945).

R. S. Rivlin, The hydrodynamics of non-Newtonian fluids, Proc. R. Soc. London (A) **193**, 260–281 (1948), Proc. Cambr. Phil. Soc. 45, 88–91, (1949).

See also C. Truesdell, A new definition of a fluid, J. Math. Pures Appl. (9) **29**, 215–244, (1950); **30**, 111–158 (1951).

same equations as had Poisson, but in so doing he put the theory on a sound and clear phenomenological basis. As far as the received theory of fluids with linear viscous response is concerned, this paper was final.

A part of Stokes's discussion concerns the "Stokes relation" $3\lambda+2\mu=0$ between the two viscosities. While put forward with reserve by Stokes, and later virtually retracted by him, unfortunately this specialization was universally accepted in the physical literature until very recently. Experimental results, however, show that it can be correct only for monatomic gases at most.

The remainder of the paper contains: a determination of the boundary condition for viscous fluids; solution of the problem of decay of free plane infinitesimal waves in viscous fluids, whence arose later the theory of ultrasonic absorption and dispersion; correction of Newton's erroneous solution of the problem of a cylinder rotating in a fluid (see §1); solution of the problem of flow in a straight pipe; a scholarly and penetrating discussion of the theorem of Lagrange in ideal hydrodynamics; development of elasticity theory; and a critique of the work of Poisson.

Postscript. It would certainly be wrong to conclude that analysis of the principles fundamental to the hydrodynamical equations stopped in 1845. True, such is the impression which one gains often enough from current textbooks, some of which produce the Navier-Stokes equations as empirical results established by experiment. As we have seen, while *experience* is indeed the basis of hydrodynamics, *experiment* played no direct part in the development of the general equations. The mathematical analysis of Navier, Cauchy, Poisson, and Stokes makes it plain that their results are only *first approximations* to those which would follow from a more general theory. It was natural enough that the main effort in hydrodynamics since 1845 should be directed toward obtaining solutions of the equations of Euler or Navier-Stokes and in comparing these solutions with experimentally measured values. As a result of this vigorous and thorough exploitation, we now have a fair idea both of the mathematics of classical hydrodynamics and of the range of its physical validity. But from time to time there have been attempts at precision of the more general ideas standing in the shadows behind Stokes's Principle. During the past seven years this question, whose theory might be named *Non-linear viscosity*, has become an active and vigorous field of modern mechanics.*

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* A skeleton history is given in §18 of my paper cited above. A treatise on the state of the theory up to January, 1952, is given in Chapter V of my paper, The mechanical foundations of elasticity and fluid dynamics, *J. Rational Mech. Anal.* **1**, 125–300 (1952); **2**, 593–616 (1953).