Example of Application of Morrison Equation

5. Horizontal velocity component of a wave propagating in x-direction in water of constant depth d is described by the equation

$$v_x = \frac{a\,g\,k}{\omega}\,\frac{\cosh(k\,(z+d))}{\cosh(k\,d)}\,\cos(k\,x - \omega\,t)\,,$$

where a is wave amplitude, g is gravity acceleration, $k = 2\,\pi/\lambda$ is wave number, λ is wave length, $\omega = \sqrt{k\,g\,\tanh(k\,d)}$ is frequency of the wave, $z = 0$ and $z = -d$ represent water surface and bottom respectively. A vertical cylindrical oil rig column of $10\,m$ in diameter is placed in $50\,m$ deep water. Calculate the maximal horizontal force and the moment about the bottom mounting applied to the column by a $200\,m$ long wave of $3\,m$ amplitude. The values of drag coefficient and inertial coefficient are $C_D = 1$ and $C_M = 2$. Discuss the applicability of your solution.

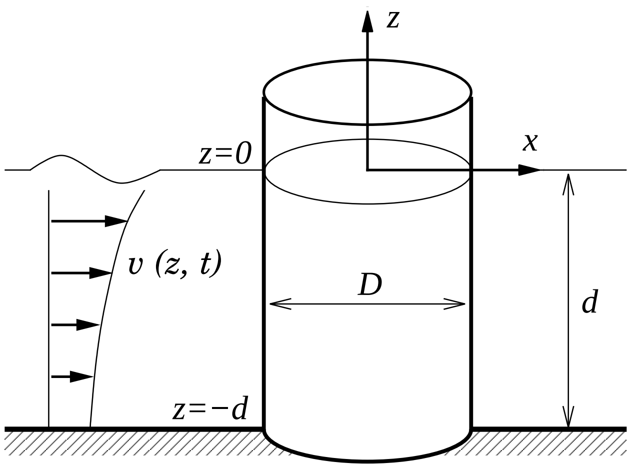

Fig. 5.1

Solution:
For a given position ($x = 0$) the horizontal velocity in the wave can be written as the product of a function of t, a function of z and a constant

$$v = U\,\cosh(\,k\,(z+d)\,)\,\cos(\omega\,t)\,, \tag{1}$$

where the constant U represents the maximal velocity at the bottom ($z = -d$) and can be calculated using the problem data

$$U = \frac{a\,g\,k}{\omega\,\cosh(k\,d)}\,.$$

Force dF on a small column element of thickness dz (Fig. 5.2) can be calculated by using Morrison's equation (*What assumptions justify its application?*)

$$dF = \frac{1}{2}\,C_D\,\rho\,D\,v\,|v|\,dz\;+\;C_M\,\rho\,\frac{\pi\,D^2}{4}\,\dot{v}\,dz\,. \tag{2}$$

Using (1) each term (2) can be represented as the product of a function of time, a function of z and a constant

$$dF = K_D\,\cos(\omega\,t)\,|\cos(\omega\,t)|\,\cosh^2(\,k\,(z+d)\,)\,dz$$
$$-\,K_I\,\sin(\omega\,t)\,\cosh(\,k\,(z+d)\,)\,dz\,,$$

where the constants

$$K_D = \frac{1}{2}\,C_D\,\rho\,D\,U^2; \quad K_I = C_M\,\rho\,\frac{\pi\,D^2}{4}\,U$$

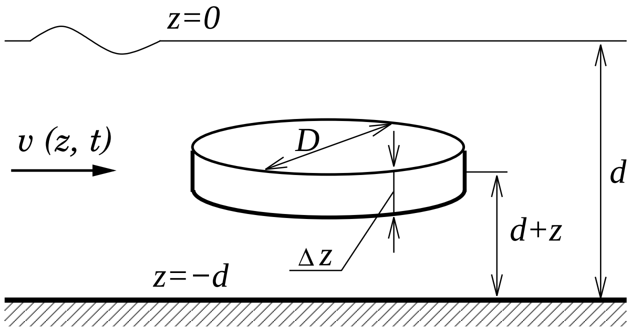

Fig. 5.2

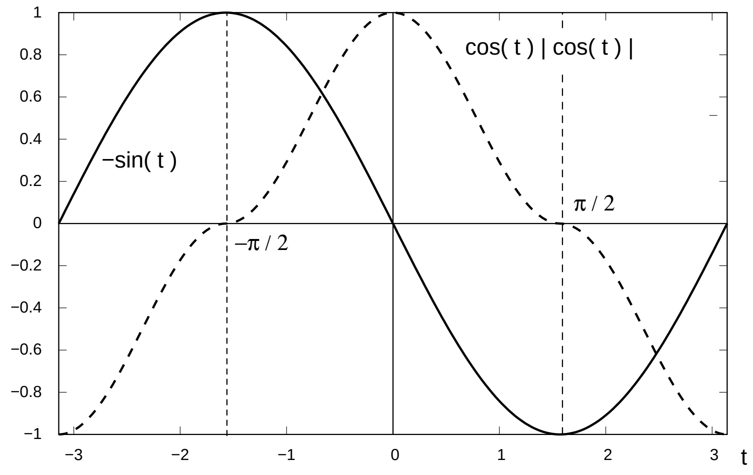

Fig. 5.3

can be calculated using the problem data. The corresponding overturning moment about the base can be obtained by multiplying dF by the distance to the base $z + d$

$$dM = K_D \cos(\omega t) \, |\cos(\omega t)| \, \cosh^2(\, k\,(z+d)\,)\,(z+d)\,dz$$
$$- K_I \sin(\omega t) \, \cosh(\, k\,(z+d)\,)\,(z+d)\,dz \,,$$

To calculate the total force and moment we have to integrate dF and dM from the bottom $z = -d$ to the surface $z = 0$. This implies taking the following integrals

$$I_1 = \int_{-d}^{0} \cosh^2(\, k\,(z+d)\,)\,dz; \quad I_2 = \int_{-d}^{0} \cosh(\, k\,(z+d)\,)\,dz\,;$$

$$I_3 = \int_{-d}^{0} \cosh^2(\, k\,(z+d)\,)\,(z+d)\,dz; \quad I_4 = \int_{-d}^{0} \cosh(\, k\,(z+d)\,)\,(z+d)\,dz\,.$$

The values of the integrals for the particular values of d and k can be estimated numerically, for example using Excel (e.g. see 1^{st} year Cylinder lab), or by using analytical methods and integral tables (refer to your math course). The following formulas are valid for the integrals I_1–I_4

$$I_1 = \frac{2\,k\,d + \sinh(2\,k\,d)}{4\,k}; \quad I_2 = \frac{\sinh(k\,d)}{k}\,;$$

$$I_3 = \frac{1 + 2\,(k\,d)^2 - \cosh(2\,k\,d) + 2\,k\,d\,\sinh(2\,k\,d)}{8\,k^2}; \quad I_4 = \frac{1 - \cosh k\,d + k\,d\,\sinh(k\,d)}{k^2}\,,$$

and their values can be calculated using the problem data.

Now we have the force and overturning moment as functions of time

$$F(t) = K_D\, I_1\, \cos(\omega\, t)\,|\cos(\omega\, t)| - K_I\, I_2\, \sin(\omega\, t)\,;$$

$$M(t) = K_D\, I_3\, \cos(\omega\, t)\,|\cos(\omega\, t)| - K_I\, I_4\, \sin(\omega\, t)$$

and our task is to find their maximal values. From Fig. 5.3 it is clear that the maximum occurs when both terms in expressions for $F(t)$ and $M(t)$ are positive, that is when $-\pi/2 < \omega\, t < 0$. It is convenient to apply a phase shift $\omega\, t \to \omega\, t + \pi/2$ and consider the interval $0 < \omega\, t < \pi/2$. We can write then

$$\begin{aligned} F(t) &= K_D\, I_1\, \sin^2(\omega\, t) + K_I\, I_2\, \cos(\omega\, t)\,; \\ M(t) &= K_D\, I_3\, \sin^2(\omega\, t) + K_I\, I_4\, \cos(\omega\, t) \end{aligned} \tag{3}$$

and maxima occur when time derivatives are 0.

Force:

$$\dot{F} = 2\,\omega\, K_D\, I_1 \sin(\omega\, t)\cos(\omega\, t) - \omega\, K_I\, I_2 \sin(\omega\, t) = \omega\, \sin(\omega\, t)\ \left(2\, K_D\, I_1 \cos(\omega\, t) - K_I\, I_2\right) = 0\,,$$

which happens when either

$$\omega\, t = 0 \qquad \text{or} \qquad \omega\, t = \arccos\left(\frac{K_I\, I_2}{2\, K_D\, I_1}\right)\,.$$

To calculate the value of the force maximum substitute these values of $\omega\, t$ to the first equation of (3) and select the maximal value. Note that when $K_I\, I_2 > 2\, K_D\, I_1$ arccos does not exist. The force maximum then occurs at $t = 0$ and equals to the inertial force component $K_I\, I_2$.

Similarly, the moment maximum occurs when

$$\omega\, t = 0 \qquad \text{or} \qquad \omega\, t = \arccos\left(\frac{K_I\, I_4}{2\, K_D\, I_3}\right)\,.$$