Free manifolds of dynamical billiards

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A mechanical system with impacts is called a mathematical billiard if it may be reduced to a mass point moving in a plane and reflecting from a bounding curve [1]. A billiard is called dynamical if an external force field is imposed [2–6]. The objective of this work is to suggest a new method of representation of dynamical billiards as generalized mathematical billiards on curved surfaces, or free manifolds embedded in $\mathbb{R}^3$. *The trajectories of the mass point on a free manifold consist of segments of geodesics. The mass point moves on a free manifold with no external forces acting on it. The free manifolds of dynamical billiards in the constant and Newtonian force fields are shown to be surfaces of revolution in the three-dimensional Euclidean space. The parabolic case is of particular interest, because it is equivalent to the classical planar mathematical billiard.*

1 Free manifolds of dynamical billiards

Let $\mathbb{R}^2$ be the plane with Cartesian coordinates (x, y). We define a dynamical billiard B on this plane as a triple $\{G, U, h\}$, where G is a closed connected domain in $\mathbb{R}^2$ (an action domain of the billiard) and $U = U(x, y)$ a smooth function in this domain, which is a force function of the forces, acting on a billiard point, which moves inside G. The constant h is an energy level for which the motion of the billiard point proceeds. If $U \equiv 0$, then we have a classical mathematical billiard.

We assume that the billiard point has a unit mass. Its kinetic energy T in the Cartesian coordinates (x, y) is a homogeneous quadratic form $T = \dfrac{1}{2}(\dot{x}^2 + \dot{y}^2)$. Therefore, the system may be treated with the help of the least action principle in the Jacobi form. Now we formulate it.

Let w_0 be a collision-free interval of the point's trajectory, with its beginning in some point p_1 and its end in point p_2. For every point of trajectory, the energy integral is valid: $T = U + h$. Let $W = \{w\}$ be the family of smooth curves w with the beginning in the point p_1 and the end in the point p_2. We define the functional Φ_0 on the family W:

$$\Phi_0 = \int_{p_1}^{p_2} \sqrt{(U + h)}\,ds,$$

where the curve's arc differential ds is determined by the quadratic form of the kinetic energy T: $ds^2 = dx^2 + dy^2$. The least action principle in the Jacobi form suggests that the curve w_0 is an extremal of the functional Φ_0 on the set W and vice versa, if the curve $w_1 \in W$ is an extremal of the functional Φ_0

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on the set W, then it may be parameterized in such a way that it becomes a trajectory of the system under consideration.

Instead of the functional Φ_0, we further use the functional $\Phi = \sqrt{\lambda}\Phi_0$, where $\lambda > 0$ is a positive constant normalizing factor, the value of which will be adjusted below. The extremals of the functionals Φ and Φ_0 are the same.

Let M^2 be a two-dimensional surface (manifold), parameterized with (u, v) and such that its linear element takes the form

$$ds^2 = \lambda(U(u, v) + h)(du^2 + dv^2).$$

The geodesics on this surface are extremals of the functional Φ (expressed in coordinates (u, v)), and vice versa. We map the plane of the dynamical billiard onto M^2 by the simple rule

$$u = x, \quad v = y. \tag{1.1}$$

Therefore, the billiard domain G maps onto some domain $\tilde{G}$ in M^2, and the collision-free intervals of the billiard trajectories map in a one-to-one manner onto the geodesics in M^2, not intersecting the boundary of the region $\tilde{G}$. In addition, since the coordinates (u, v) on M^2 are isothermal (i.e., the metric tensor is spherical), then the above mapping preserves the angles between curves in the billiard's plane and, in particular, the "reflection law", i.e., the correspondence between the angles at which the trajectory approaches and leaves the boundary of the domain.

Consider now a mathematical billiard on M^2 in the domain $\tilde{G}$. The point moves freely in this system, without action of external forces. In this case the trajectories of motion coincide with the corresponding geodesics on M^2. This means that the initial dynamical billiard and the free billiard on M^2 coincide geometrically, i.e., the mapping (1.1) maps the trajectories of the first billiard onto the trajectories of the second one and vice versa. However, notice that the temporary law of motion along the trajectories may not retain.

The manifold M^2 is called the *free manifold* of the original dynamical billiard. In what follows, we consider the billiards with two particular potentials: constant gravitational and Newtonian. In each case, we shall show that a planar billiard with the imposed force field is trajectorically equivalent to the force-free billiard on the corresponding surface of revolution. The procedure of construction of that surface will be developed.

2 Free manifold of a billiard in the constant force field

As a first example, we examine the free manifold for a planar dynamical billiard in the constant force field $U = -gy$. According to the Jacobi principle, the linear element of the free manifold M^2 then takes the form

$$ds^2 = \lambda(h - gv)(du^2 + dv^2), \tag{2.1}$$

where h is the energy constant. We shall show that M^2 may be embedded in $\mathbb{R}^3$ isometrically as a many-sheeted surface of revolution. The image of the domain G may be located entirely within one sheet unless it is too large.

Let us introduce new coordinates (f, l) in M^2 such that

$$f = \sqrt{\lambda} u, \qquad l = \sqrt{\lambda} \int\limits_{v}^{h/g} \sqrt{(h - g\tau)} d\tau.$$

It is an easy matter to obtain

$$l = \sqrt{\lambda} \frac{2}{3g} (h - gv)^{3/2} \tag{2.2}$$

and

$$h - gv(l) = \left(\frac{3g}{2\sqrt{\lambda}} l \right)^{2/3}.$$

In the new coordinates, the linear element is

$$ds^2 = \left(\frac{3g}{2\sqrt{\lambda}} l \right)^{2/3} df^2 + dl^2. \tag{2.3}$$

Now introduce three variables (φ, η, ζ), related to the variables (f, l) as follows:

$$\varphi = f, \qquad \eta = \sqrt{h - gv(l)}, \qquad \left(\frac{d\eta}{dl} \right)^2 + \left(\frac{d\zeta}{dl} \right)^2 = 1. \tag{2.4}$$

By using Eq. (2.2), we obtain from the two last formulas in (2.4)

$$\frac{d\eta}{dl} = \frac{g}{2\eta^2 \sqrt{\lambda}}, \qquad \frac{d\zeta}{dl} = \sqrt{1 - \left(\frac{d\eta}{dl} \right)^2} = \frac{1}{\eta^2} \sqrt{\eta^4 - \frac{g^2}{4\lambda}}. \tag{2.5}$$

Eqs. (2.5) parametrically define a curve $\Gamma(l) = (\eta(l), \zeta(l))$ in the plane (η, ζ). Further, we obtain

$$\eta = 3^{1/3} 2^{-1/3} g^{1/3} l^{1/3} \lambda^{-1/6}, \qquad \frac{d\zeta}{dl} = \sqrt{1 - 3^{-4/3} 2^{-2/3} l^{-4/3} \lambda^{-1/3} g^{2/3}}.$$

By virtue of the last condition (2.4), the parameter l is an intrinsical parameter along this curve. We have from Eq. (2.3) $ds^2 = \eta^2 d\varphi^2 + dl^2$, or

$$ds^2 = \eta^2 d\varphi^2 + \left(\left(\frac{d\eta}{dl} \right)^2 + \left(\frac{d\zeta}{dl} \right)^2 \right) dl^2. \tag{2.6}$$

Let (x_1, x_2, x_3) be a three-dimensional Cartesian space. We introduce in it cylindrical coordinates (φ, η, ζ), such that (η, φ) are the radius and the polar angle in the plane (x_1, x_2), and ζ is a displacement along the axis x_3. Then the linear element of the manifold M^2, written in the form (2.6), coincides with the linear element of the surface of revolution M^2_ρ, formed by rotation of the curve $\Gamma(l)$ around the axis x_3 in the Cartesian space (x_1, x_2, x_3). Therefore, the map $(\varphi(u,v), \eta(u,v), \zeta(u,v)) : M^2 \to M_\rho^2$ is an isometric embedding of M^2 in $\boldsymbol{R}^3$.

From Eqs. (2.5), one can find $\dfrac{d\zeta}{d\eta}$ to give:

$$\varphi = \sqrt{\lambda} u, \qquad \eta = \sqrt{h - gv}, \qquad \frac{d\zeta}{d\eta} = \sqrt{\frac{4\lambda}{g^2}\eta^4 - 1}. \tag{2.7}$$

Having found from Eqs. (2.7) the variables (φ, η, ζ) as functions of (u,v), we obtain the following isometrical map of the space (u,v) with metric (2.1) into the space $\boldsymbol{R}^3 : (x_1, x_2, x_3)$ with metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$:

$$x_1 = \eta\cos\varphi, \qquad x_2 = \eta\sin\varphi, \qquad x_3 = \zeta. \tag{2.8}$$

Note that the right-hand side of the last equation (2.7) has the domain of definition $\eta^2 \geq \dfrac{g}{2\sqrt{\lambda}}$, or

$$v \leq \frac{h}{g} - \frac{1}{2\sqrt{\lambda}}. \tag{2.9}$$

This means that the map of the isometric embedding (2.8) exists only in the domain (2.9), where the coordinate ζ rises monotonically as η increases. It is interesting to note that the rest of the manifold M^2 may be embedded isometrically in the three-dimensional pseudoeuclidean space $\boldsymbol{R}_1^3$ with pseudometric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$.

The last equation (2.7) may be integrated to give $\zeta(\eta) = \frac{1}{3}\left[\eta\sqrt{\frac{4\lambda}{g^2}\eta^4 - 1} - \frac{\sqrt{g}}{\lambda^{1/4}} F\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + C$, where $\alpha = \arccos\sqrt{\dfrac{g}{2}}\dfrac{1}{\lambda^{1/4}\eta}$, and F is the elliptic integral of the first kind. If we put $\zeta = 0$ for $\eta = \sqrt{\dfrac{g}{2}}\dfrac{1}{\lambda^{1/4}}$, then the constant $C = 0$.

Now consider a sufficiently fast billiard, for which the inequality $h - gv > \varepsilon$ is valid everywhere in the billiard domain G for some $\varepsilon > 0$. This means that the trajectory of this billiard can reach every point of the boundary of its domain with a uniformly positive velocity ($\dot{x}^2 + \dot{y}^2 > 2\varepsilon$). Then, having taken $\lambda = \dfrac{g^2}{4\varepsilon^2}$, we have the situation when the whole billiard domain G may be isometrically embedded by the map (2.8) in the three-dimensional Euclidean space as a part of the surface of revolution.

3 Free manifold of a billiard in the Newtonian force field

Let us now consider the form of the free manifold for a planar dynamical billiard in the Newtonian force field $U = \gamma/r$, where $r = \sqrt{x^2 + y^2}$. We introduce in the plane of the billiard the polar coordinates (θ, r), such that $x = r\cos\theta$, $y = r\sin\theta$. (The angle θ is assumed to lie in the range

$(-\infty, \infty)$, i.e., the billiard plane is treated as many-sheeted.) In these coordinates, the metric takes the form:

$$ds^2 = r^2 d\theta^2 + dr^2, \tag{3.1}$$

and the kinetic energy of the unit mass point is $T = \frac{1}{2}(r^2\dot{\theta}^2 + \dot{r}^2)$. According to the Jacobi principle, the linear element of the free manifold M^2 will look like:

$$ds^2 = \lambda \left(h + \frac{\gamma}{v}\right)(v^2 du^2 + dv^2), \tag{3.2}$$

where h is the energy constant and $\lambda > 0$ the positive normalizing factor. Notice that, unlike the above sections where the variable u was introduced as a linear coordinate which then became angular, u is considered such in this section.

Since the metric (3.2) in the free manifold differs from the original one (3.1) by a factor only, the mapping $B : (\theta, r) \to M^2 : (u, v)$ preserves angles (i.e., it is conformal).

We shall show that, in this case, M^2 may be also embedded isometrically in $\boldsymbol{R}^3$ as a many-sheeted surface of revolution. We introduce new coordinates (f, l) in M^2, such that

$$f = \sqrt{\lambda}u, \qquad l = \sqrt{\lambda}\int\limits_0^v \sqrt{h + \frac{\gamma}{\tau}}d\tau. \tag{3.3}$$

It is easy to check that for $h > 0$

$$l(v) = \sqrt{\lambda}\left(\sqrt{v(hv+\gamma)} + \frac{\gamma}{\sqrt{h}}\ln(\sqrt{hv} + \sqrt{hv+\gamma}) - \frac{\gamma}{\sqrt{h}}\ln(\sqrt{\gamma})\right),$$

for $h = 0$

$$l(v) = 2\sqrt{\lambda\gamma v}$$

and

$$l(v) = 2\sqrt{-\frac{\lambda}{h}}\left(\arctan\sqrt{\frac{-hv}{hv+\gamma}} - \arctan\frac{1}{\sqrt{\gamma}}\right)$$

for $h < 0$.

Denote

$$\eta(l) = v(l)\sqrt{h + \frac{\gamma}{v(l)}} = \sqrt{hv^2(l) + \gamma v(l)}. \tag{3.4}$$

Then the linear element takes the form $ds^2 = \eta^2 df^2 + dl^2$.

Similarly to the previous section, we introduce the variable $\varphi = f$ and the variable ζ, related to the variables (f, l) by the following formula:

$$\left(\frac{d\eta}{dl}\right)^2 + \left(\frac{d\zeta}{dl}\right)^2 = 1. \tag{3.5}$$

Eqs. (3.3) – (3.5) define parametrically a curve $\Gamma(l) = (\eta(l), \zeta(l))$ in the plane (η, ζ), and Eqs. (3.3) – (3.4) imply

$$\frac{dv}{dl} = \frac{\sqrt{v}}{\sqrt{\lambda(hv+\gamma)}}, \qquad \frac{d\eta}{dl} = \frac{d\eta}{dv}\frac{dv}{dl} = \frac{1}{\sqrt{\lambda}}\left(1 - \frac{\gamma}{2(hv+\gamma)}\right),$$
$$\frac{d\zeta}{dl} = \sqrt{1 - \frac{1}{\lambda}\left(1 - \frac{\gamma}{2(hv+\gamma)}\right)^2}. \tag{3.6}$$

The derivative $\dfrac{d\zeta}{dv}$ may be expressed from Eq. (3.6) as a function of v and the general solution of the resulting differential equation may be shown to be obtained in terms of elliptic integrals.

In view of Eq. (3.5), the parameter l is an intrinsical parameter along this curve. We have in the new coordinates

$$ds^2 = \eta^2 d\varphi^2 + \left(\left(\frac{d\eta}{dl}\right)^2 + \left(\frac{d\zeta}{dl}\right)^2\right) dl^2. \tag{3.7}$$

Let us consider the three-dimensional Cartesian space (x_1, x_2, x_3) with the cylindrical coordinates (φ, η, ζ) in it, such that (η, φ) are the radius and the polar angle in the plane (x_1, x_2), and ζ is a displacement along the axis x_3. Then the linear element of the manifold M^2, written in the form (3.7), coincides with the linear element of the surface of revolution M^2_ρ, formed by rotation of the curve $\Gamma(l)$ around the axis x_3 in the Cartesian space (x_1, x_2, x_3). Therefore, the map $(\varphi(u,v), \eta(u,v), \zeta(u,v)) : M^2 \to M^2_\rho$ is an isometric embedding of M^2 in $\boldsymbol{R}^3$.

Having found from Eqs. (3.3), (3.4), (3.6) the variables (φ, η, ζ) as functions of (u, v), we obtain the following isometric map of the space (u, v) with metric (2.1) into the space $\boldsymbol{R}^3 : (x_1, x_2, x_3)$ with metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$:

$$x_1 = \eta\cos\varphi, \qquad x_2 = \eta\sin\varphi, \qquad x_3 = \zeta. \tag{3.8}$$

Note that the right-hand side of the last equation (3.6) has the domain of definition

$$2(1-\sqrt{\lambda}) \le \frac{\gamma}{(hv+\gamma)} \le 2(1+\sqrt{\lambda}). \tag{3.9}$$

This means that the map of the isometric embedding (3.8) exists only in the domain (3.9). It is interesting to note that the rest of the manifold M^2 may be embedded isometrically in the three-dimensional pseudoeuclidean space $\boldsymbol{R}^3_1$ with pseudometric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$.

Now consider a billiard, for which the following inequality is valid everywhere in the billiard domain G:

$$|\ hv + \gamma\ | > \varepsilon \tag{3.10}$$

for some $\varepsilon > 0$. Then, having taken $\lambda > \left(1 + \frac{|\gamma|}{2\varepsilon}\right)^2$, we come to the situation when the whole billiard domain G may be isometrically embedded by the map (3.8) in the three-dimensional Euclidean space as a part of the surface of revolution.

Let the potential be repulsive, i.e., $\gamma < 0$. Then $h = T - \frac{\gamma}{v} > 0$ (where T is the kinetic energy). The accessibility domain of this mechanical system is an exterior of the circle $v = -\frac{\gamma}{h-T} \geq -\frac{\gamma}{h}$. If the billiard domain lies strictly inside the accessibility domain, i.e., if there exists R such that $-\frac{\gamma}{h} < R \leq v$, then the condition (3.10) is valid for $\varepsilon = \frac{hR+\gamma}{2}$.

Let the potential be attracting, i.e., $\gamma > 0$. Three cases are possible depending on the sign of the energy integral constant h.

Elliptic motion for $h < 0$. All collision-free segments of the billiard trajectory are elliptic arcs. In order to satisfy the condition (3.10), it is necessary and sufficient that the whole billiard domain lies in the accessibility domain of this mechanical system, i.e., there exists R such that $v \leq R < -\frac{\gamma}{h}$. This means that the billiard is fast and that its trajectory may approach any point of its boundary with a uniformly positive velocity.

For $\lambda = 1$, the shape of the ruling curve in the plane (η, ζ) is determined by Eq. (3.4) for the coordinate η and by the following solution for ζ:

$$\zeta = \frac{\sqrt{3}\gamma}{\sqrt{-h}} E\left(\beta, \frac{2}{\sqrt{3}}\right) + const,$$

where $\beta = \arcsin\sqrt{-\frac{hv}{\gamma}}$, and E is the elliptic integral of the second kind.

Parabolic motion for $h = 0$. All collision-free segments of the billiard trajectory are parabolic arcs. The condition (3.10) is satisfied, e.g., for any billiard domain for $\lambda = 4$, i.e., $\varepsilon = \frac{\gamma}{2}$. The parabolic billiard is of particular interest, because, in some sense, it is equivalent to the classical mathematical billiard in a plane. This case is considered in the next section in more detail.

Hyperbolic motion for $h > 0$. All collision-free segments of the billiard trajectory are hyperbolic arcs. The condition (3.10) is satisfied, e.g., for any billiard domain for $\lambda = 4$, i.e., $\varepsilon = \frac{\gamma}{2}$.

To sum up all the listed cases, we conclude that both for the attracting and repulsive Newtonian potential, if the billiard domain belongs strictly to the interior of the accessibility domain of this mechanical system, then the free manifold of the billiard may be embedded isometrically in the three-dimensional Euclidean space as a part of a surface of revolution. The condition formulated means that the billiard is sufficiently "energetic" and the absolute value of the billiard point velocity on the boundary of the billiard domain is uniformly bounded below by a positive value.

4 The parabolic billiard

Let us consider the form of the free manifold of the planar dynamical billiard in the attracting Newtonian force field in the parabolic case. Then $h = 0$, hence, Eq. (3.6) implies

$$\frac{dv}{dl} = \sqrt{\frac{v}{\lambda\gamma}}, \qquad \frac{d\eta}{dl} = \frac{1}{2\sqrt{\lambda}}, \qquad \frac{d\zeta}{dl} = \sqrt{1 - \frac{1}{4\lambda}},$$

or, after integration,

$$l = 2\sqrt{\gamma\lambda v}, \qquad \eta = \frac{l}{2\sqrt{\lambda}}, \qquad \zeta = \sqrt{1 - \frac{1}{4\lambda}}\, l. \tag{4.1}$$

According to Eq. (3.8), these equations determine a many-sheeted conic surface of revolution. This surface may be developed into the plane $\tilde{R}^2(\tilde{x}, \tilde{y})$ with polar coordinates l and φ in it

$$\tilde{x} = l\cos\varphi, \qquad \tilde{y} = l\sin\varphi.$$

The original billiard lies in the plane $R^2(x, y)$ with the Newtonian force field. In this plane, the full circle $(r, \theta), \theta \in [0, 2\pi)$ maps into the arc $(\eta, f), f \in [0, 2\pi\sqrt{\lambda})$. n such arcs should make up one full circle of radius l: $n2\pi\sqrt{\lambda}\eta = 2\pi l$. Substituting η from (4.1) then immediately gives $n = 2$, i.e., a pair of sheets of the conic surface may be developed into the plane. Therefore, we take the two-sheeted part of the representation of the plane $\boldsymbol{R}^2(x, y)$, which is defined in the polar coordinates by $x = r\cos\theta, y = r\sin\theta, 0 \leq \theta \leq 4\pi$.

From the above it follows that the original plane $\mathbb{R}^2(r, \theta)$ is equivalent "isobilliardly" to the plane $\tilde{\boldsymbol{R}}^2(l, \varphi)$ with the following map:

$$l = 2\sqrt{\lambda\gamma r}, \qquad \varphi = \frac{\theta}{2}. \tag{4.2}$$

Now it is clear that in the parabolic case the map may be considered as the map $z \to \sqrt{z}$ from the two-fold Riemann surface rooted at zero to the ordinary complex plane. For this map, it is known that a parabola with focus in the origin becomes a straight line. More precisely, one can easily check that a parabola $r = \dfrac{p}{1 + \cos(\theta - \theta_0)}$, p, θ_0 parameters, in $R^2(r, \theta)$ maps into the straight line $l = \dfrac{\sqrt{2\lambda\gamma p}}{|\cos(\varphi - \theta_0/2)|}$ in $\tilde{\boldsymbol{R}}^2(l, \varphi)$.

By way of illustration let us consider the dynamical billiard in the unit circle D^2 in $\mathbb{R}^2$. Let the centre of the circle be offset by r_0 from the gravitating centre. Without loss of generality, we assume that the circle is shifted along the axis Ox. Then the billiard domain is given by the inequality $r^2-2rr_0\cos\theta\le 1-r_0^2$. In this case, the two-sheeted representation of the parabolic billiard is equivalent trajectorically to the one-sheeted planar mathematical billiard in the domain $\tilde{D}^2$, determined by the inequality

$$\frac{l^4}{16\gamma^2\lambda^2}-\frac{l^2 r_0\cos 2\varphi}{2\gamma\lambda}\le 1-r_0^2,$$

or

$$(\tilde{x}^2+\tilde{y}^2)^2-8\gamma r_0\lambda(\tilde{x}^2-\tilde{y}^2)\le 16\gamma^2\lambda^2(1-r_0^2). \tag{4.3}$$

According to Eq. (4.3), the bounding curve of the domain $\tilde{D}^2$ is a cassinoid.

If the centre of the billiard circle coincides with the gravitating centre, i.e., $r_0=0$, then the domain $\tilde{D}^2$ degenerates into a circle of radius $2\sqrt{\lambda\gamma}$. The Cassini oval is simply connected for $r_0<1$, when the billiard domain contains the gravitating centre. If the last one lies on the boundary ($r_0=1$), then the cassinoid becomes a lemniscate with an intersection point in the origin. If $r_0>1$ (the billiard domain does not contain the gravitating centre), then the whole circle D^2 is located in one sheet of $\mathbb{R}^2$. Therefore, the usual (one-sheeted) dynamical billiard is equivalent trajectorically to the planar mathematical billiard in one of the two simply connected domains (4.3).

Now look more closely at the special case $r_0=0$, when the parabolic billiard is equivalent to the classical mathematical billiard in the circle.

Let us construct periodic trajectories of the original parabolic billiard. Let them be centrally symmetrical and consist of n parabolic arcs ($n\ge 2$). Each arc should meet the bounding circle at points with the circular arc distance between them being equal to $2\pi\frac{m}{n}$; m,n coprime integers ($m\ge 1$). Then, in order for the parabola $r=\frac{p}{1+\cos\theta}$ to be a segment of the periodic trajectory, the following relation should be true: $1=\dfrac{p}{1+\cos\frac{m}{n}\pi}$, from which we get $p=1+\cos\frac{m}{n}\pi$. It is easy to compute the angle μ between the tangent to the parabolic arc and the radius direction in the collision point $\theta=\frac{m}{n}\pi$: $\tan\mu=\dfrac{r}{\frac{dr}{d\theta}}=\dfrac{1+\cos\frac{m}{n}\pi}{\sin\frac{m}{n}\pi}$, that implies $\mu=\frac{\pi}{2}-\frac{m}{2n}\pi$.

The mapping (4.2) is conformal, hence, the angle of reflection is the same for the image periodic polygonal trajectory of the mathematical billiard. If $m=1$, then we have $\mu=\frac{\pi}{2}-\frac{\pi}{2n}$, which corresponds to a regular $2n$-gon. Thus, the centrally symmetrical periodic n-segment trajectory of the parabolic billiard maps into the regular $2n$-gon. The simplest case is a trajectory consisting of two equal parabolic arcs. It should be passed twice to complete the corresponding square in the force-free billiard. Accordingly, the parabolic triangle maps into the regular hexagon and so on. When $m=1$, all trajectories are convex and various star-like trajectories arise, if $m>1$.

The constructed map allows us to extend the properties of the Birkhoff billiard to the parabolic one. For example, it is well known, that each periodic trajectory of the Birkhoff billiard in a circle consists of straight line segments of equal length. Therefore, we may conclude that the parabolic billiard in the circle with the non-shifted centre also possesses periodic trajectories containing only identical arcs. Of course, this particular statement may be also derived by means of direct considerations.

In conclusion, we consider the parabolic billiard in the ellipse, one focal point of which coincides with the attracting centre. The mapping (4.2) transforms such ellipses into the ellipses with the centre in the origin [7]. Therefore, the parabolic billiard in the ellipse $r = \dfrac{p_e}{1 + e_e \cos\theta}, 0 < e_e < 1$, is found to be equivalent to the Birkhoff (force-free) billiard in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, a = 2\sqrt{\dfrac{\lambda\gamma p_e}{1+e_e}}, b = 2\sqrt{\dfrac{\lambda\gamma p_e}{1-e_e}}$.

It is interesting to note that the inverse map $z \longrightarrow z^2$ of the complex plane was applied to analysis of the billiards in the ellipse with the imposed Newtonian potential. A trajectory equivalence of the billiard in the ellipse with the Newtonian attracting force directed to the focal point and the billiard in the ellipse with the Hooke linear elastic force attracting the particle to the centre of the ellipse was proved [8]. In this context, the parabolic motion may be treated as a limiting case, corresponding to zero elastic constant in the Hooke law [7].

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