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ON THE NUMBER OF TUBES TOUCHING A SPHERE OR A TUBE

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- Introduction
- Tubes touch a ball from the outside
- Tubes touch the surface of a ball from the inside
- On the number of tubes touching one tube
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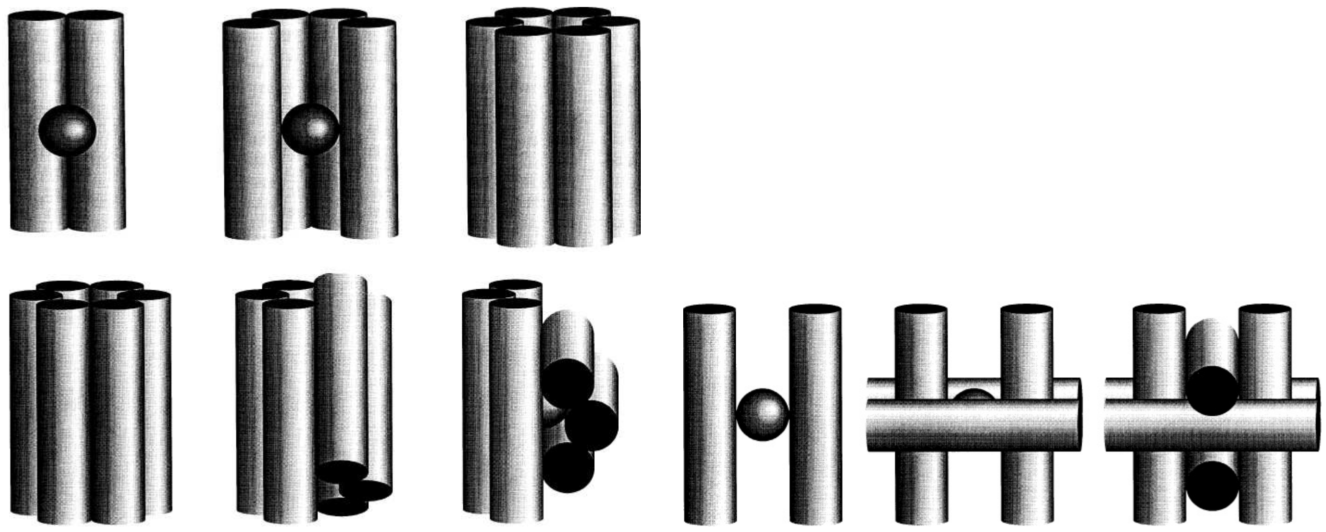
Motivation

The Kuperberg problem is to determine the maximum number N_{cyl} of unit-radius infinite cylinders touching a unit-radius ball.

$N_{cyl} < 8$ [P. Braß and C. Wenk]



Cylinders $\Longrightarrow$ perfect tubes and ask the same question.

Cylinders $\Longrightarrow$ perfect tubes

Space curve $\mathcal{C}$, piecewise of class C^2: $\mathcal{C} = \{\mathbf{r} : M \to \mathbb{R}^3\}$; s arc length, tangent $\mathbf{t} = \frac{d\mathbf{r}}{ds} \neq \mathbf{0}$. M is either $\mathbb{R}$ for an infinite curve or S^1 for a closed one.

Definition 1 *Global radius of curvature* is

$$\rho_G(\mathbf{x}) = \inf_{\mathbf{y},\mathbf{z}\in\mathcal{C},\mathbf{x}\neq\mathbf{y}\neq\mathbf{z}\neq\mathbf{x}} R_c(\mathbf{x},\mathbf{y},\mathbf{z}),$$

where $R_c(\mathbf{x},\mathbf{y},\mathbf{z}) \geq 0$ is the radius of the smallest circle containing $\mathbf{x},\mathbf{y},\mathbf{z}$ [O. Gonzalez and J. H. Maddocks].

Assume that $\rho_G(\mathbf{x}) \geq \rho > 0$, $\mathbf{x} \in \mathcal{C}$.

Cylinders $\Longrightarrow$ perfect tubes

Definition 2 *ρ-tube* $\mathcal{T}$ based on axis $\mathcal{C}$: $\mathcal{T} = \{\mathbf{x} \in \mathbb{R}^3, \|\mathbf{x} - \mathbf{y}\| < \rho, \mathbf{y} \in \mathcal{C}\}$, and the closed ρ-tube $\bar{\mathcal{T}} = \{\mathbf{x} \in \mathbb{R}^3, \|\mathbf{x} - \mathbf{y}\| \leq \rho, \mathbf{y} \in \mathcal{C}\}$.

ρ-tube is embedded in $\mathbb{R}^3$.

Fix the scale $\rho = 1$ and all but one tubes, we shall be dealing with, are 1-tubes. We shall call them either unit tubes or just tubes. The only exception is a P-tube.

Definition 3 *Ring* = tube with closed axis.

Definition 4 *Bialy* $\mathcal{Y} = \mathcal{T}(\mathcal{C})$ with $\mathcal{C}$ the circle of unit radius [Kusner],i.e. a bialy is a torus with no hole.

Definition 5 *Ball* of radius R with centre at $\mathbf{z}$ is $\mathcal{B}_R(\mathbf{z}) = \{\mathbf{x} \in \mathbb{R}^3, \|\mathbf{x} - \mathbf{z}\| \leq R\}$. In particular, $\mathcal{B}_P \equiv \mathcal{B}_P(\mathbf{0})$ is the central ball of radius P.

Any closed tube may be thought of as a union of balls:

$$\bar{\mathcal{T}} = \bigcup_{\mathbf{x} \in \mathcal{C}} \mathcal{B}_\rho(\mathbf{x}).$$

Sphere is $\mathcal{S}_R(\mathbf{z}) = \partial \mathcal{B}_R(\mathbf{z})$.

A unit *orthogonal cross-sectional disk* of a tube is

$\mathcal{D}(\mathbf{r}(s)) \equiv \mathcal{D}(s) = \{\mathbf{r}(s) + \nu(s), \nu(s) \in \mathbb{R}^3, \nu(s) \cdot \mathbf{t}(s) = 0, \|\nu(s)\| \leq 1\}.$

Tubes touch a ball from the outside

Lemma 1 Let $\mathcal{B}_P$ be the central ball and $\mathcal{S}_R$ a central sphere of greater radius $R > P$.

Let $\mathcal{T}$ be a tube with axis $\mathcal{C} = \{\mathbf{r}_C(s)\}$ touching $\mathcal{B}_P$ in point $Q = \bar{\mathcal{T}} \cap \mathcal{B}_P$.

Let $A(\mathcal{C})$ be an intersection of the tube and the sphere: $A(\mathcal{C}) = \bar{\mathcal{T}} \cap \mathcal{S}_R$.

Then, for every R in the range $P < R \leq P + 2$, the area of A reaches its minimum for $\mathcal{T} = \mathcal{Y}$.

Lemma 1: Sketch of proof

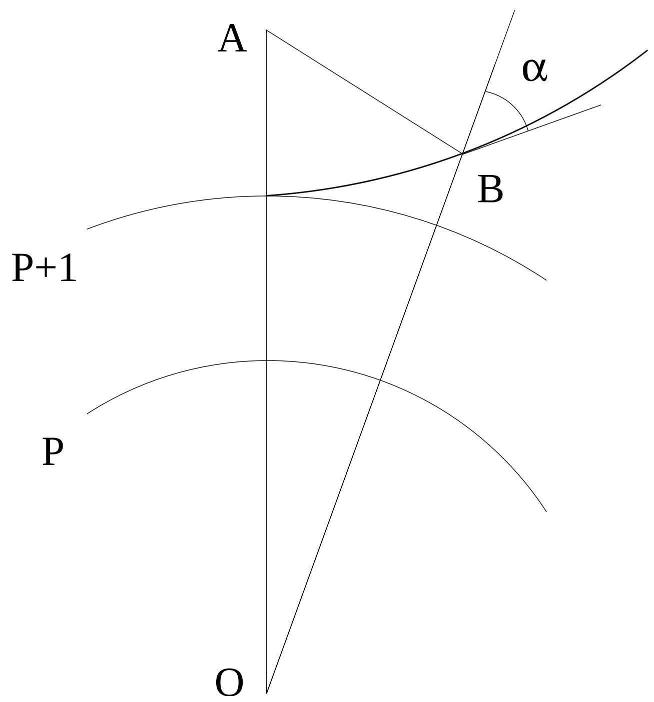
1. Show that for any tube with non-planar centreline $\mathcal{C}$, there exists a tube with planar axis having the same area of intersection with the R-ball.

Lemma 1: Sketch of proof

2. Consider planar axes for $s \in [0, L]$. Let $s = 0$ corresponds to the contact with the P-ball and $s = L$ to a section outside the $(P+2)$-ball. The function $r(s)$ starts at $r(0) = P + 1$. Then

$$\alpha(r) \geq \alpha_0(r) = \arcsin \frac{(P+2)^2 - 1 - r^2}{2r}$$

for any curve with constrained curvature.



Lemma 1: Sketch of proof

3. Show that the area A cannot be smaller then the area of intersection of bialy and $\mathcal{S}_R$.

Theorem 1 Let tubes $\mathcal{T}_i$, $i = 1, \ldots, n$ be such that

1. $\mathcal{T}_i \cap \mathcal{B}_P = \emptyset$.

2. $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset, i \neq j$.

3. $\bar{\mathcal{T}}_i \cap \mathcal{B}_P = Q_i, Q_i = \{\mathbf{q}_{ik} \in \mathbb{R}^3, k = 1, ..., m_i \geq 1\}$ ($\mathbf{q}_{ik}$ is k-th contact point of i-th tube with the central P-ball; i-th tube has m_i contact points).

4. $\forall i, k \ \exists \sigma_{ik} : \mathcal{D}(\sigma_{ik}) \cap \mathcal{B}_{P+2} = \emptyset$, $\mathcal{D}(s_{ik}) \cap \mathcal{B}_P = \mathbf{q}_{ik}$, $\infty < \sigma_{i0} < s_{i1} < \sigma_{i1} < s_{i2} < \sigma_{i2} < \ldots < s_{ik} < \sigma_{ik} < s_{i,k+1} < \ldots < \sigma_{i,m-1} < s_{im} < \sigma_{im} < \infty$.

Theorem 1: Claim

Then the total number of contacts is bounded: $\sum_{i=1}^{n} m_i \leq \tilde{N}(P)$, where the function $\tilde{N}(P)$ will be defined below.

If $m_i = 1, i = 1, \ldots, n$, then the alternating condition 4 is released and the theorem claims that $n \leq \tilde{N}(P)$.

Theorem 1: Sketch of proof

1. Note that the intersections of tubes with any sphere $\mathcal{S}_R$ concentric with the central ball are disjoint because the tubes are disjoint.

2. Consider only $\mathcal{S}_R$ such that $P \leq R \leq P+2$.

3. Lemma 1 implies that the minimal area of intersection of the tube with $\mathcal{S}_R$ is achieved when the part of the tube inside the sphere is a fragment of the bialy centred at the (maximally possible) distance $P+2$ from the centre of the ball.

Theorem 1: Sketch of proof

4. Fix the origin of the reference frame at the centre of the ball with the centre of the bialy lying on the z-axis and its circular axis being in the xz-plane. Then the bialy is described by

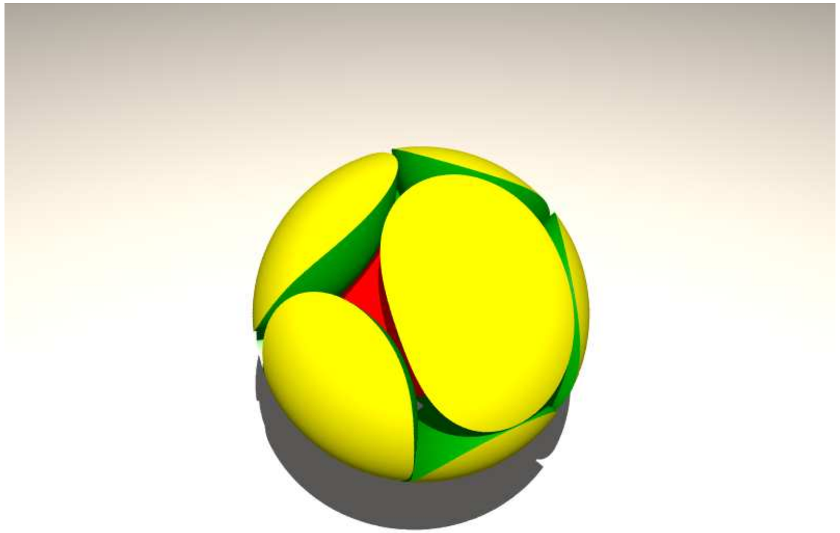
$$y^2 + \left(\sqrt{x^2 + (z - P - 2)^2} - 1\right)^2 = 1. \tag{1}$$

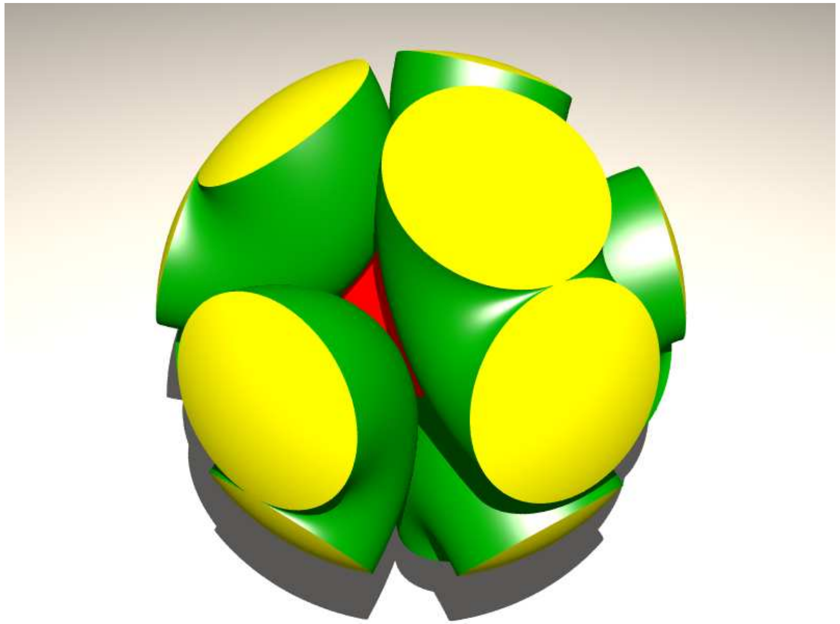
We are interested in its intersection with the sphere $\mathcal{S}_R$ given by

$$x^2 + y^2 + z^2 = R^2. \tag{2}$$

Theorem 1: Sketch of proof

5. The intersection domains: 9 unit bialies touch the unit ball and clipped with the sphere of radius $R = 2$ (left) and $R = 3$ (right).





Theorem 1: Sketch of proof

6. The boundary of the overlapping region on the surface of the sphere satisfies the equation

$$[R^2 - 2(P+2)z + (P+2)^2]^2 = 4[x^2 + (z-P-2)^2]. \qquad (3)$$

Introduce cylindrical coordinates by $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$. Then Eq. (3) may be represented as a quadratic equation $4Uz^2 - 4Vz + W = 0$ with coefficients depending on P, R and ϕ. Therefore, $z = z(\phi) = \frac{V \pm \sqrt{V^2 - UW}}{2U}$ (for $R \le P+2$, only the sign "$-$" is meaningful).

Theorem 1: Sketch of proof

7. Compute the intersection area

$$S = \int_0^{2\pi} \int_0^{\rho(\phi)} \frac{R}{\sqrt{R^2 - \rho^2}} \rho \, d\rho \, d\phi = R \int_0^{2\pi} \left(R - \sqrt{R^2 - \rho^2(\phi)} \right) \, d\phi =$$

$$= 4R \int_0^{\pi/2} (R - z(\phi)) \, d\phi = \frac{1}{2} S_R - 4RZ(P, R), (4)$$

where $S_R = 4\pi R^2$ is the entire area of $\mathcal{S}_R$ and $Z(P, R) \equiv \int_0^{\pi/2} z(\phi) \, d\phi$.

The last integral may be expressed in terms of elliptic integrals.

Theorem 1: Sketch of proof

8. Consider function $N = N(P, R) = \frac{S_R}{S}$,

$$N^{-1} = \frac{1}{2} - \frac{1}{\pi R} Z(P, R). \quad (5)$$

Show that, for fixed $P > \epsilon > 0$, N has a minimum in the interval $R \in (P, P + 2)$ for some $R^*(P)$. To this purpose, note that both function $Z(P, R)$ and its derivative with respect to R may be expressed in elementary functions for $R = P$ and $R = P + 2$. The derivative $\frac{\partial N^{-1}(P,R)}{\partial R}$ has different signs at the ends of the interval $[P, R + 2]$.

Theorem 1: Sketch of proof

9. Applying the Weierstrass intermediate value theorem to the function $\frac{\partial N^{-1}(P,R)}{\partial R}$, we conclude that there exists $R^* \in (P, P+2)$ such that $\left.\frac{\partial N^{-1}(P,R)}{\partial R}\right|_{R=R^*} = 0$, which implies that the function $N(P,R)$ has a local minimum on that interval. In order to find R^*, we solve numerically the equation

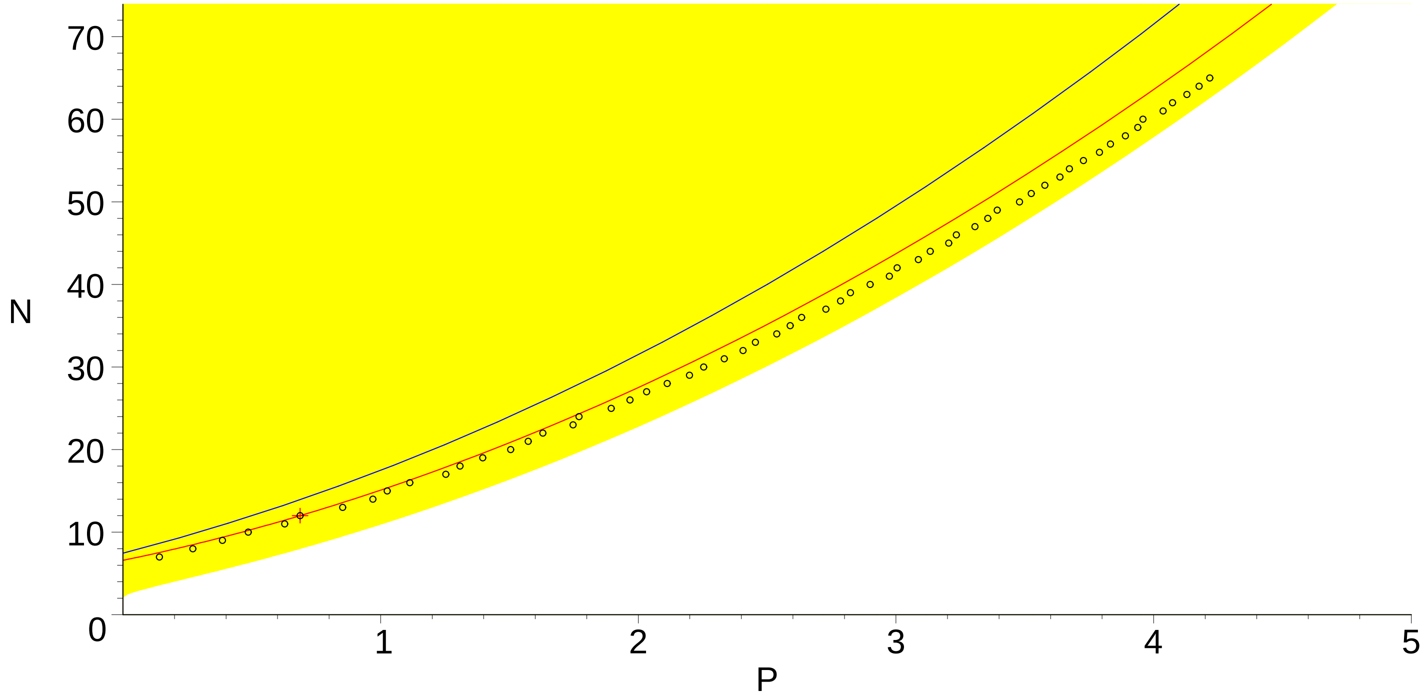
$$Z(P,R) - R\frac{\partial Z(P,R)}{\partial R} = 0. \tag{6}$$

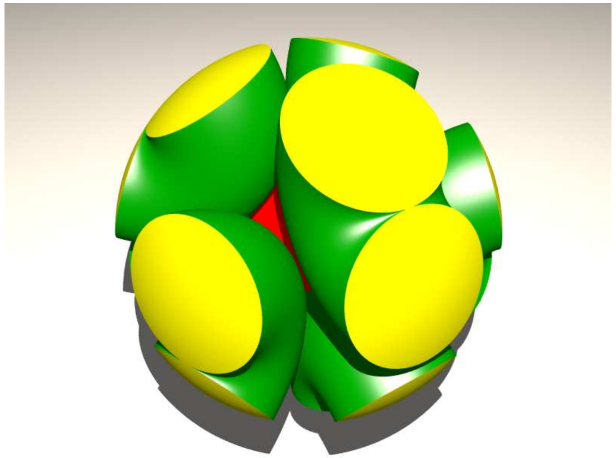
Theorem 1: Sketch of proof

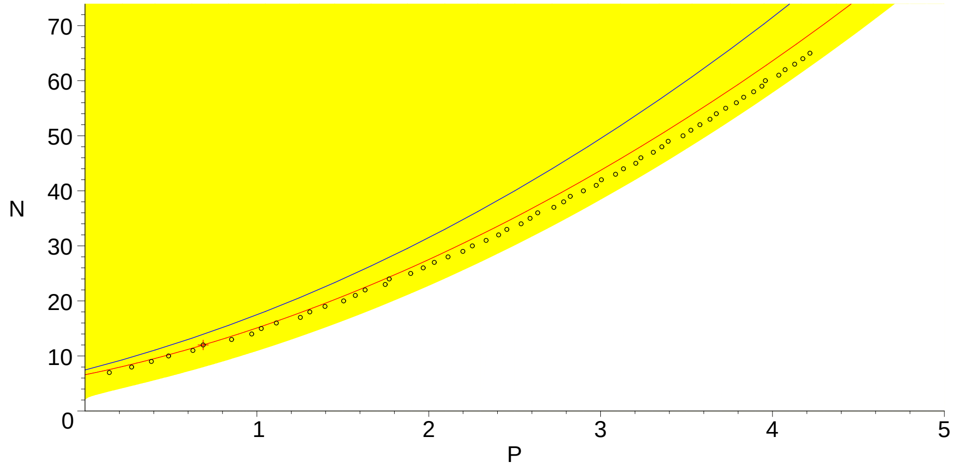
10. Thus, for given P, the number of contacts may not exceed $\tilde{N}(P) = N(P, R^*(P))$. The function $\tilde{N}(P)$ may be computed with arbitrary precision. Numerical computation suggests that there is only one local minimum of $N(P, R)$ for any fixed $P > 0$. However, the uniqueness property in no way affects the correctness of the above estimate because *any* R-sphere may be taken.

Theorem 1: Sketch of proof

11. Shaded region with the boundary $\tilde{N}(P)$ corresponds to forbidden numbers of contacts of unit-radius tubes with the ball of radius P from the outside.







Remark 1 If a unit bialy crossed with $\mathcal{S}_R$ which passes through the bialy's centre ($R = P + 2$), then the intersection domain is a pair of spherical caps, each bounded with a unit-radius circle, and having area $S = 4\pi R(R - \sqrt{R^2 - 1})$. Hence, the intersection area is bounded from below by S and we can obtain the estimate

$$N^- \equiv N(P, P+2) = \frac{P+2}{P+2-\sqrt{(P+1)(P+3)}},$$

shown as the upper (blue) curve on the right graph.

Remark 2 Assume that there are N^+ contacts. Then the tubes must cross the $(P+2)$-sphere at least $2N^+$ times. The minimal area of every crossing is that of a unit spherical cap. Therefore the number N^+ cannot exceed one half of the maximal number n_0 of free unit-radius circles packed on the sphere.

The latter number corresponds to the solution of the Tammes problem which is to find a configuration of given number n_T of points on the sphere such that it maximizes the minimum distance between any pair of points. The extremal configuration is called a spherical code.

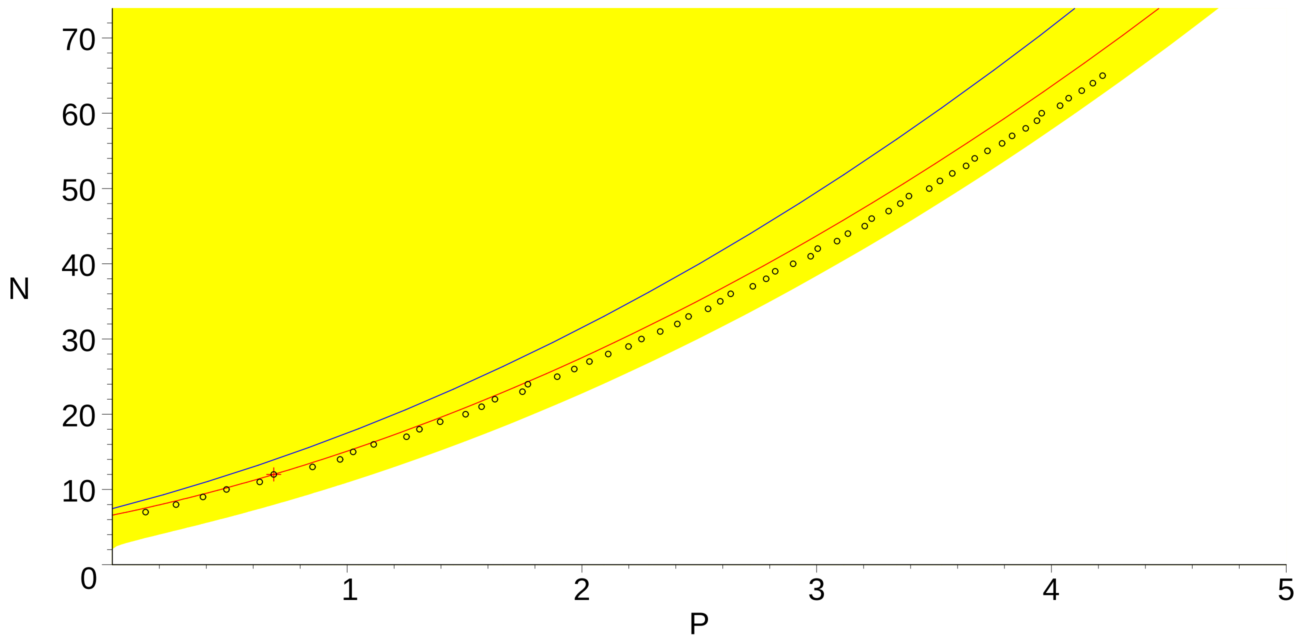
A number of upper bounds are known for the Tammes problem, e.g. two proved by R. M. Robinson:

$$n_0 < \frac{12(\pi R^2 + \Sigma_2)}{4\Sigma_1 + \Sigma_2 + \Sigma_3}, \qquad \text{for } 12 \leq n_0 \leq 24; \tag{7}$$

$$n_0 < \frac{6(\pi R^2 - \Sigma_2 + 2\Sigma_3)}{2\Sigma_1 + \Sigma_3}, \qquad \text{for } n_0 \geq 24, \tag{8}$$

where Σ_1 is a spherical area of a equilateral triangle of side 2γ (with angles β_1), Σ_2 is a halved spherical area of a regular quadrangle of side 2γ (with angles β_2) and Σ_3 is a spherical area of a triangle with two sides equal to 2γ and included angle $2\pi - 4\beta_1$.

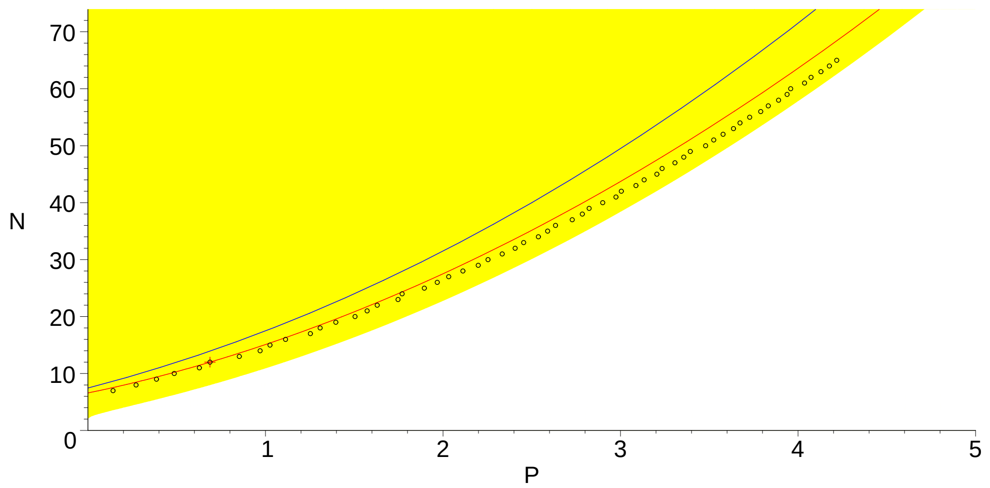
The right-hand sides of Eqs. 7, 8 give the upper bounds for a number of unit circles that can be packed on the sphere of radius $R = \csc\gamma$. This implies that the number N^+ of the bialies that touch the ball of radius $P = R - 2 = \csc\gamma - 2$ does not exceed $n_0/2$. The lower (red) curve on the graph represents the improved upper bounds based on Robinson's estimates.



Remark 3 The solutions of the Tammes problem are currently known with proofs only for all $n_T \leq 14$ and for $n_T = 24$.

A variety of existing numerical algorithms have produced presumably extremal configurations.

The best solutions currently known are collected and updated on the web site `www.research.att.com/~njas/packings/` for all $n_T \leq 130$.

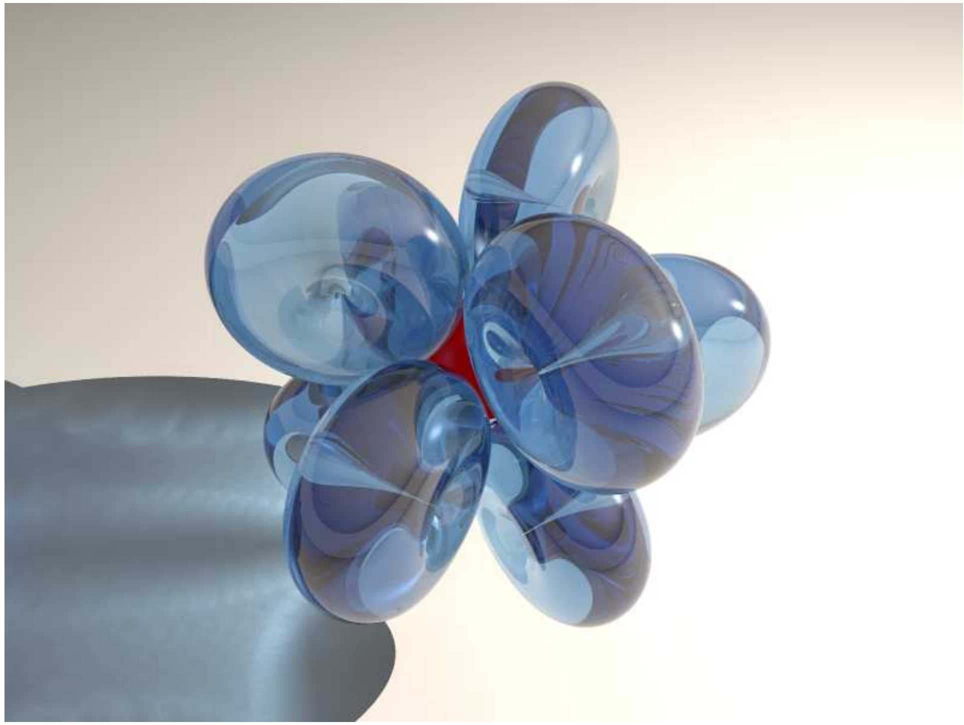


The discrete points on the figure were computed with the data from `www.research.att.com/~njas/packings/`.

The estimate based on spherical codes approaches $\tilde{N}$ as the radius P increases.

Two parts of Robinson's bound meet each other at a point for $N = 12$, which corresponds to the isolate proved solution to the Tammes problem.

Example: A configuration of 9 unit bialies touching a unit ball. $N(1,2) \approx 10.858914 < 11$ means that 11 tubes cannot be in contact with the same unit ball.



Does a configuration with 10 tubes exist or not?

Tubes touch the surface of a ball from the inside

Now all tubes are rings.

Lemma 2 Let $\mathcal{S}_P$, $P > 2$ be a central sphere and $\mathcal{S}_R$ another central sphere of a smaller radius $R < P$. Let $\mathcal{T}$ be a ring inside of $\mathcal{S}_P$ with axis $\mathcal{C}$ touching $\mathcal{S}_P$ in point $Q = \bar{\mathcal{T}} \cap \mathcal{S}_P$. Let $A(\mathcal{C})$ be an intersection of the ring and the R-sphere: $A(\mathcal{C}) = \bar{\mathcal{T}} \cap \mathcal{S}_R$.

Then the area of A reaches its minimum, if $\mathcal{T} = \mathcal{Y}$ for $P - 2 \leq R < P$.

Theorem 2 Let tubes $\mathcal{T}_i$, $i = 1, \ldots, n$ be such that

1. $\mathcal{T}_i \cap \mathcal{B}_P = \mathcal{T}_i$, $P \geq 3$.

2. $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset, i \neq j$.

3. $\bar{\mathcal{T}}_i \cap \mathcal{S}_P = Q_i, Q_i = \{\mathbf{q}_{ik} \in \mathbb{R}^3, k = 1, ..., m_i \geq 1\}$ ($\mathbf{q}_{ik}$ is k-th contact point of i-th tube with the central P-sphere; i-th tube has m_i contact points).

4. $\forall i, k \ \exists \sigma_{ik} : \mathcal{D}(\sigma_{ik}) \cap \mathcal{B}_{P-2} = \mathcal{D}(\sigma_{ik})$, $\mathcal{D}(s_{ik}) \cap \mathcal{S}_P = \mathbf{q}_{ik}$, $\infty < \sigma_{i0} < s_{i1} < \sigma_{i1} < s_{i2} < \sigma_{i2} < \ldots < s_{ik} < \sigma_{ik} < s_{i,k+1} < \ldots < \sigma_{i,m-1} < s_{im} < \sigma_{im} < \infty$.

Theorem 2: Claim

Then the total number of contacts is bounded: $\sum_{i=1}^{n} m_i \leq \tilde{N}_{in}(P)$, where the function $\tilde{N}_{in}(P)$ will be defined below.

If $m_i = 1, i = 1, \ldots, n$, then the alternating condition 4 is released and the theorem claims that $n \leq \tilde{N}_{in}(P)$.

Theorem 2: Sketch of proof

1. Condition 4 $\Longrightarrow$ each tube has to enter the $(P-2)$-ball between the contacts to the P-sphere

$\Longrightarrow$ there should exist a cross-sectional disk belonging to this small ball. This is always possible for $P \geq 3$.

2. Let $\mathbf{r}(s)$, $s \in M$, $s_{ik} < s < s_{i,k+1}$ be the centreline of the piece(s) of i-th ring inside the $(P-2)$-ball. Then $r^\star \equiv \min_{s \in M} r(s) \leq P-3$, because otherwise no cross-section would be immersed in the $(P-2)$-ball.

The plane of the section $\mathcal{D}(\mathbf{r}^\star)$ passes through the origin and the tube touches the surface of the ball of radius $\rho \leq |P-4|$ for the centreline point $\mathbf{r}^\star$. Note that the ρ-ball lies inside of the closed tube for $3 \leq P < 4$.

Then, since all the tubes are disjoint, it follows immediately that only one ring has room inside of the $(P-2)$-ball, i.e.

$$\tilde{N}(P) = 1, \; P \in [3, 4).$$

3. Now let $P \geq 4$. What is the minimal normalized area of the intersection of the ring with $\mathcal{S}_R$, $R \leq P-2$?

If $r^\star \geq R-1$ for the piece of the ring inside $\mathcal{S}_R$, then Lemma 1 $\Longrightarrow$ the minimal area is reached when the piece is a part of the bialy.

If the ring sinks deeper into the $(P-2)$-ball, i.e. if $r^\star < R-1$, then nothing prevents the ring from crossing $\mathcal{S}_R$ orthogonally and the intersection domain is simply a pair of spherical caps.

4. It is sufficient to consider only the bialies with centres at $\mathcal{S}_{P-2}$, they are in touch with the $(P-4)$-ball. Then the intersection area is given by the old Eq. (4)

$$S = \frac{1}{2} S_R - 4RZ(P,R),\ Z(P,R) \equiv \int\limits_0^{\pi/2} z(\phi)\, d\phi,\ z(\phi) = \frac{V \pm \sqrt{V^2 - UW}}{2U},$$

but we should formally substitute $P \to P-4$ in the expressions for the coefficients U, V, W.

5. Define $N_{in}(P, R) \equiv N(P-4, R)$ where N is given by Eq. (5)

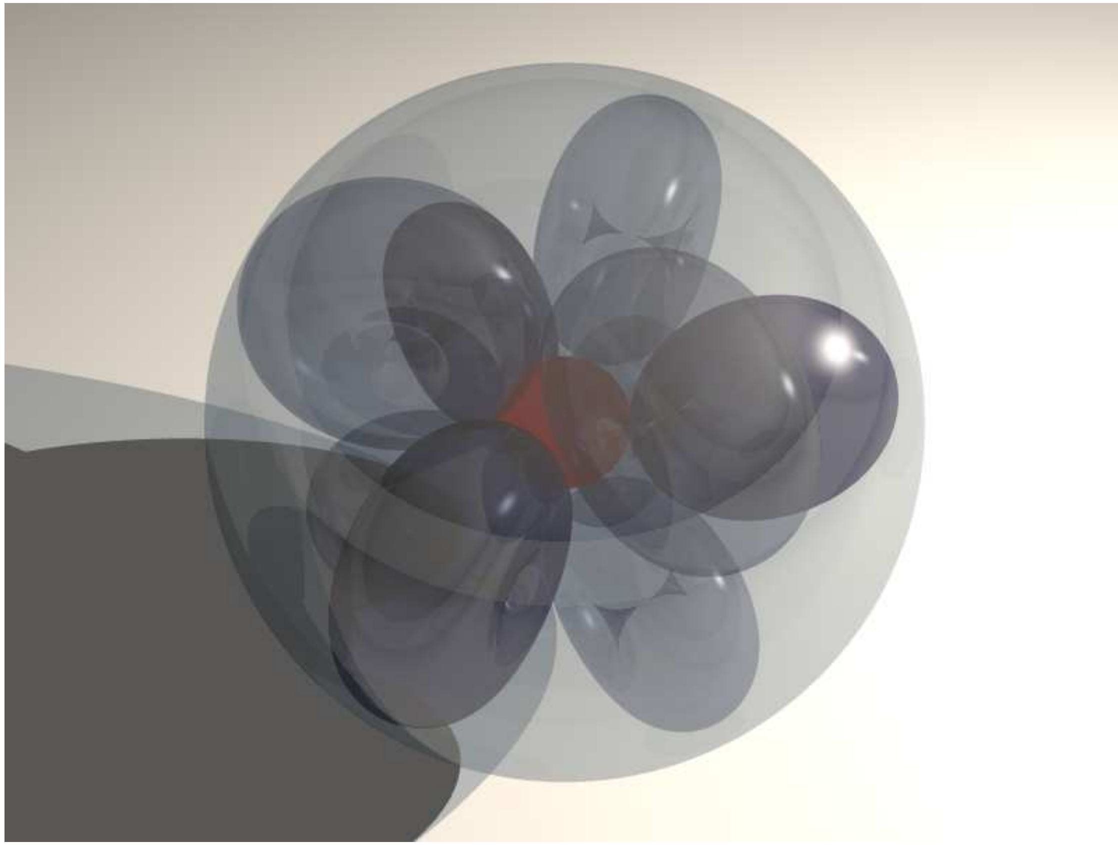
$$N = \left(\frac{1}{2} - \frac{1}{\pi R} Z(P, R)\right)^{-1}.$$

For fixed $P \geq 4$, N_{in} has a minimum in the interval $R \in (P-4, P-2)$: $\tilde{N}_{in}(P) = \min_{R \in (P-4, P-2)} N_{in}(P, R)$, which is an upper bound for the number of contacts of the rings with $\mathcal{S}_P$.

The estimate graph in the interior case is essentially the same as in the exterior one, only shifted along the P-axis by 4.

The duality property: the same value serves both to bound the number of unit tubes touching the surface of the small P-ball from the outside and of the larger $(P+4)$-ball from the inside.

Example: A configuration of 9 unit bialies touching a sphere of radius $R = 5$ from the inside. 11 tubes cannot be in contact with the same surface. What about 10?



On the number of tubes touching one tube

Theorem 3 Let $\mathcal{T}_0$ be a tube of radius P and $\mathcal{D}_0$ be its section. Let unit tubes $\mathcal{T}_i$, $i = 1, \ldots, n$ be such that

1. $\mathcal{T}_\lambda \cap \mathcal{T}_\mu = \emptyset, \lambda \neq \mu$; $\lambda, \mu = 0, \ldots, n$ and

2. $\bar{\mathcal{T}}_i \cap \mathcal{D}_0 = Q_i, Q_i = \{\mathbf{q}_{ik} \in \mathbb{R}^3, k = 1, ..., m_i \geq 1\}$ ($\mathbf{q}_{ik}$ is k-th contact point of i-th tube, $i = 1, \ldots, n$). It belongs to the cross-sectional disk $\mathcal{D}_{ik}$ with normal $\mathbf{n}_{ik}$.

Then the number of touching unit tubes n does not exceed $\pi \arcsin(P + 1)^{-1})^{-1}$. If each unit tube has exactly one contact with the central one ($m_i = 1$) and $P = \csc \frac{\pi}{n} - 1$, then all $n + 1$ discs $\mathcal{D}_0$ and $\mathcal{D}_{i1}$ lie in the same plane.

Theorem 3: Sketch of proof

1. Central disk $\mathcal{D}_0$, the other disks $\mathcal{D}_{ik}$ with centres $\mathbf{r}_{ik}$, $\|\mathbf{r}_{ik}\| = P + 1$, $i = 1, \ldots, n$, $k = 1, ..., m_i$. $\mathbf{n}_0$ is the tangent to the axis of the central tube and $\mathbf{n}_0 \cdot \mathbf{r}_{ik} = 0 \Longrightarrow$ the centres of all $n + 1$ disks lie in the same plane $\mathcal{P}_0 \supset \mathcal{D}_0$.

$\mathcal{D}_0 \cap \mathcal{D}_{ik} = \{\mathbf{q}_{ik}\} \neq \emptyset \Rightarrow \mathcal{B}_P \cap \mathcal{B}_1(\mathbf{r}_{ik}) = \{\mathbf{q}_{ik}\} \neq \emptyset$.

$\mathcal{B}_1(\mathbf{r}_{ik})$ and $\mathcal{B}_1(\mathbf{r}_{jl})$ $(i \neq j)$ have no common interior points either, because, for any i, the interior of $\mathcal{B}_1(\mathbf{r}_{ik})$ belongs to $\mathcal{T}_i$ and the tubes do not overlap.

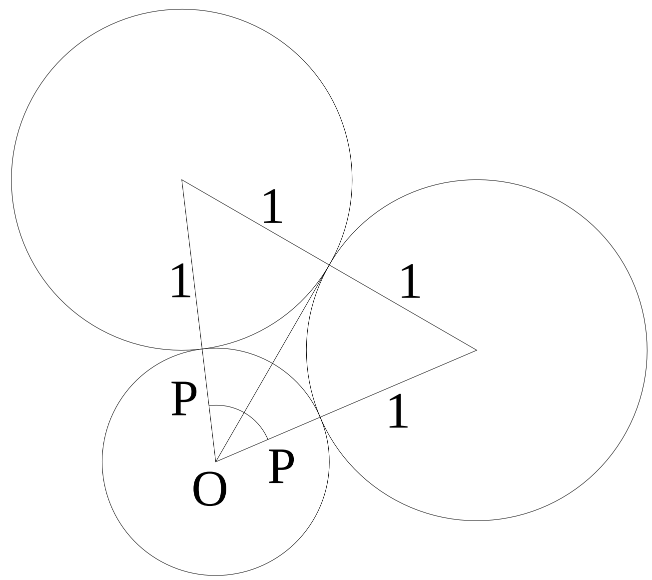
2. $n \leq$ number of the unit balls touching the central ball, all having their centres coplanar.

But that number, in turn, $\leq$ the kissing number in $\mathbb{R}^2$, which is

$$[\pi(\arcsin(P+1)^{-1})^{-1}]$$

for unit circles contacting the common central circle of radius P.

This proves the first claim of Theorem 3.



3. Suppose that all the balls belong to different tubes ($m_i = 1$) and that $P = \csc\frac{\pi}{n} - 1$. The coplanar points $\mathbf{r}_{i1} \in \mathcal{P}_0$ lie each at distance $P+1$ from the origin and the distance between each pair of them ≥ 2.

$\Longrightarrow$ they are vertices of a regular n-gon. Renumber them so that $\|\Delta\mathbf{r}_j\| = 2$, $\Delta\mathbf{r}_j \equiv \mathbf{r}_{j+1} - \mathbf{r}_j$, $j = 1, \ldots, n$ $(\mathbf{r}_{n+1} \equiv \mathbf{r}_1)$.

$\Longrightarrow$ the piece of tube, corresponding to $\mathbf{r}_j$ touches both the central tube and two other tubes, $\mathcal{T}_{j\pm 1}$.

The tangent to the centreline of j-th tube at $\mathbf{r}_j$ is $\mathbf{n}_j =$ the normal to $\mathcal{D}_j$.

$\Longrightarrow$ $\mathbf{n}_j \cdot \mathbf{r}_j = 0$ and $\mathbf{n}_j \cdot \Delta\mathbf{r}_j = 0$

$\Longrightarrow$ $\mathbf{n}_j$ is normal to the plane $\mathcal{P}_0$, i.e. all $\mathcal{D}_j \in \mathcal{P}_0$.

Corollary Let the maximal allowed number of tubes be in a *continuous* contact with the central tube whose centreline is $\mathbf{r}_0(s)$, s arc length.

Then the 2nd part of Theorem 3 $\Longrightarrow$

the vector field $\Delta\mathbf{r}_{j0}(s) \equiv \mathbf{r}_j(s) - \mathbf{r}_0(s)$ is relatively parallel.

Proof

$$\frac{d\mathbf{r}_j(s)}{ds} = \frac{d\mathbf{r}_0(s)}{ds} + \omega \times \Delta\mathbf{r}_{j0}(s), \qquad (9)$$

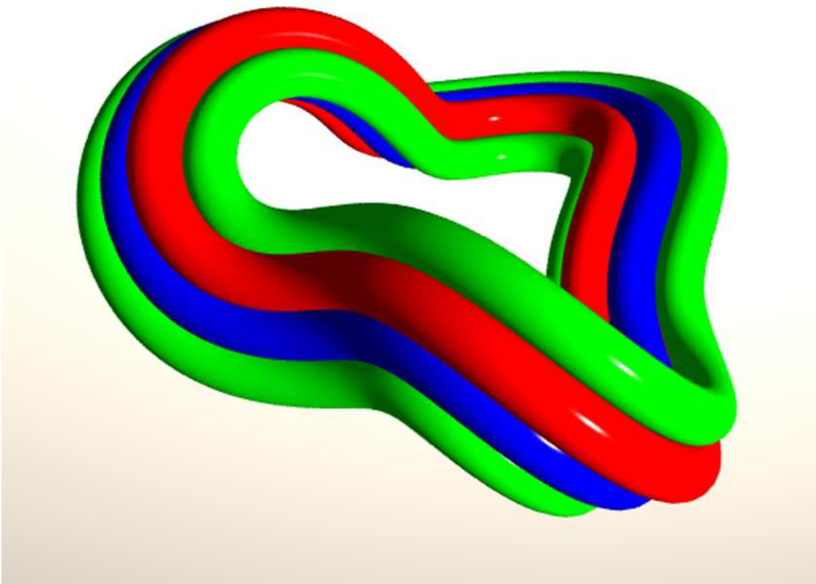
where $\omega = \omega_0 \mathbf{n}_0(s) + \omega_1 \Delta\mathbf{r}_{j0}(s) + \omega_2 \mathbf{n}_0(s) \times \Delta\mathbf{r}_{j0}(s)$ is an angular velocity of rotation of the orthogonal frame $\{\mathbf{n}_0(s), \Delta\mathbf{r}_{j0}(s), \mathbf{n}_0(s) \times \Delta\mathbf{r}_{j0}(s)\}$ as s varies.

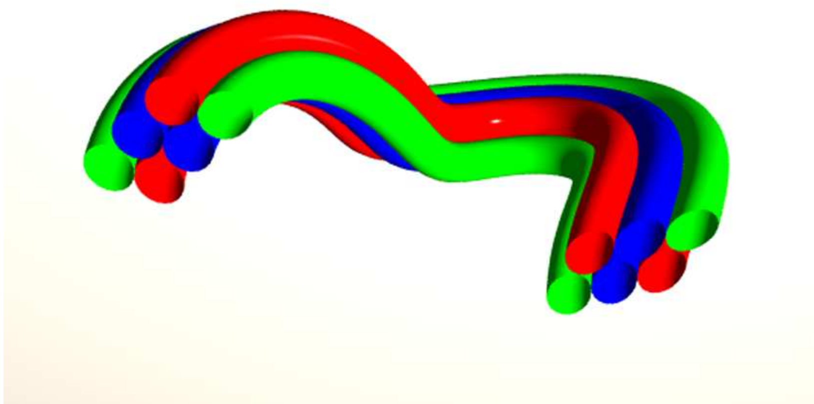
Cross-product of Eq. (9) and $\mathbf{n}_0(s) \Longrightarrow \omega_0 = 0 \Longrightarrow$

$$\frac{d\Delta\mathbf{r}_{j0}(s)}{ds} = -\omega_2 \Delta\mathbf{r}_{j0}^2(s)\mathbf{n}_0(s).$$

Example

Suppose that the centreline $\mathbf{r}_0(s)$ is smoothly closed. Let the other tubes be closed, too. We denote by $\mathcal{L}k_j$ the linking number of $\mathbf{r}_0(s)$ and $\mathbf{r}_j(s)$. Then the writhing number of each of the curves $\mathbf{r}_0(s)$ and $\mathbf{r}_j(s)$ equals $\mathcal{L}k_j$ and, consequently, it is an integer. This follows from the Călugăreanu-White-Fuller theorem $\mathcal{L}k = \mathcal{T}w + \mathcal{W}r$, because the twisting number $\mathcal{T}w = 0$.





Concluding remarks

- The approach may be readily extended to tubes of limited flexibility.
- H. Schiessel et al. deal with equilibria of a tubular polymer chain attracted to a spherical organizing centre. They estimate an upper bound for the number of contacts as of order $P^{3/2}$. This can be justified by taking into account the limited flexibility of the polymer.
- J. R. Banavar & A. Maritan have proposed a tube model to better understand the geometry of protein folding. In view of this approach, the estimates for the contact numbers may be useful when applied to globular proteins to count exposed fragments of the amino acids chain.
- The estimates may serve as sterical constraints for validation of a computed secondary structure of RNA.
- The property of the relative parallelism and an integer writhe (for closed configurations) may be applied to describe toroidal conformations produced as a result of the DNA condensation, characterized by hexagonal lattice packing (Preprint *mpi-pks/0410005*: E.Starostin, ON THE PERFECT HEXAGONAL PACKING OF TUBES)

The talk is based on the preprint *mpi-pks/0408004* and the paper is to appear in *Geometriae Dedicata*.

Martin Held. Lebensschleife. 2003