WEAK GRAVITATION DESTROYS NON-SYMMETRIC PERIODIC TRAJECTORIES OF A BILLIARD IN A CIRCLE

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ABSTRACT

A simple mechanical system consisting of a mass point moving in a circle is considered. The point reflects by the perfect elastic law when it encounters the circular boundary. In the absence of external forces, this system is the Birkhoff billiard in a circle, which allows for a continuum of periodic trajectories. Each of them may be rotated as a whole by an arbitrary angle around the centre of the circle giving birth to a new periodic trajectory. For a trajectory to be periodic (closed) its angle of reflection which is invariant must be commensurable with π. It turns out that this rotation symmetry property does not hold any more when a weak gravitational field is imposed.

An analytic proof is given that superimposing an infinitesimally weak homogeneous constant gravitational field causes disappearence of non-symmetric 2-, 3-, 4-, 5-, 6-, 7-, and 9-link periodic trajectories of all possible types, including polygons and star-like self-intersecting trajectories. The non-existence is proven for 12 different periodic non-symmetric motions.

INTRODUCTION

This pendulum-like system with a unilateral constraint was studied in (Beletsky *et al.* 1995), mostly for strong gravitation. The numerical implementation of the Poincaré mapping offered a means of describing the phase pictures with regular and chaotic regions as well as their evolution as the gravitation parameter changes.

At first glance it would seem natural if, after superposition of an arbitrarily weak force field, a countable set of periodic trajectories survived including both symmetric and non-symmetric with respect to the force vector. However, numerical experiments have revealed no periodic trajectories that are not symmetric about the vertical diameter of the circle. The sole exception is h-shaped motions with various number of links, but they exist in the case of strong gravitation. Thus, the gravitational field acts selectively and eliminates non-symmetric periodic trajectories.

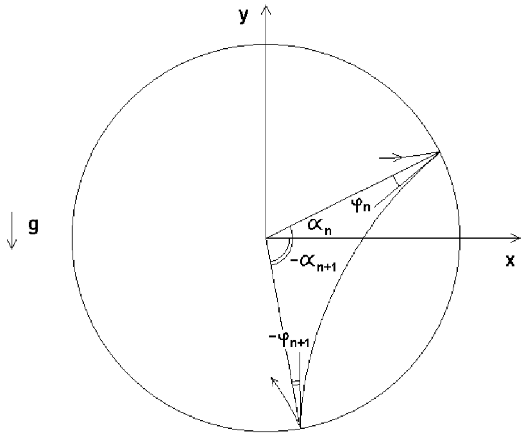

Figure 1: Configurational space of a billiard in a circle.

The problem is integrable between one collision and another. This made it possible to build an analytic proof of non-existence for 12 different periodic non-symmetric motions in the weak gravitational field.

The trajectories under study consist of near-straight links. Therefore, the proof is based on the expansions into power series in terms of the gravitation parameter. The terms of up to the third power were retained.

CONSTRUCTION OF THE MAPPING

In a weak gravitational field g, $\epsilon \stackrel{\triangle}{=} -\frac{1}{2}g, |\,\epsilon\,| \ll 1$, one link of a trajectory is described by the equation

$$y - \sin\alpha_n = \epsilon(x - \cos\alpha_n)^2 + \tan(\alpha_n + \varphi_n)(x - \cos\alpha_n), \quad (1)$$

where α_n is an angular coordinate of the n-th collision point, n=0, 1, 2,…, and φ_n is the reflection angle, $-\frac{\pi}{2} < \varphi_n < \frac{\pi}{2}$ (Fig. 1).

For the Birkhoff billiard, $\epsilon = 0$ and a trajectory link is a segment of a straight line, the reflection angle φ_n is an invariant and the mapping of the phase cylinder boundary is given by

$$\alpha_{n+1} = \alpha_n + 2\varphi_n - \pi, \quad \varphi_{n+1} = \varphi_n, \quad n = 0,\ 1,\ 2, \dots\ . \tag{2}$$

A necessary and sufficient condition of a trajectory periodicity is the commensurability of the invariant $\varphi \equiv \varphi_n$ with π, i. e.,

$$\varphi = \frac{p}{2q}\pi, \quad p - q = 2k, \quad |\, p \,|<|\, q \,|, \quad p,\ q,\ k \in \mathsf{Z}. \tag{3}$$

In a weak gravitational field ($|\ \epsilon\ | \ll 1$), we search for the mapping $(\alpha_n, \varphi_n) \to (\alpha_{n+1}, \varphi_{n+1})$ in the form

$$\alpha_{n+1} = \alpha_n + 2\varphi_n - \pi + \epsilon\delta_{n+1} + \epsilon^2\beta_{n+1} + \epsilon^3\chi_{n+1} + O(\epsilon^4),$$

$$\varphi_{n+1} = \varphi_n + \epsilon\gamma_{n+1} + \epsilon^2\kappa_{n+1} + \epsilon^3\eta_{n+1} + O(\epsilon^3).$$

By using (1) and the equation for the angle of incidence

$$\left.\frac{dy}{dx}\right|_{x=x_{n+1}} = \tan(\alpha_{n+1} - \varphi_{n+1}),$$

we come to

$$\begin{aligned} \delta_{n+1} &= -4\cos\varphi_n \cos^3(\alpha_n + \varphi_n), \qquad (4)\\ \beta_{n+1} &= -8\cos\varphi_n \cos^5(\alpha_n + \varphi_n) \times \\ &\times [2\sin(\alpha_n + \varphi_n)\cos\varphi_n + \sin\varphi_n \cos(\alpha_n + \varphi_n)],\\ \chi_{n+1} &= \frac{1}{24}\delta^3_{n+1} + \cos(\alpha_n + \varphi_n) \times \\ &\times [\delta^2_{n+1}(\frac{1}{2}\cos\varphi_n \cos^2(\alpha_n + \varphi_n) + \\ &+ \cos(2\alpha_n + 3\varphi_n)) + 2\beta_{n+1}\cos(\alpha_n + \varphi_n) \times \\ &\times (\sin(\alpha_n + 2\varphi_n) + \cos\varphi_n \sin(\alpha_n + \varphi_n))],\\ \gamma_{n+1} &= 0,\\ \kappa_{n+1} &= 4\cos^2\varphi_n \cos^4(\alpha_n + \varphi_n)\sin 2(\alpha_n + \varphi_n),\\ \eta_{n+1} &= \chi_{n+1} - \frac{1}{6}\delta^3_{n+1} + \\ &+ \cos(\alpha_n + \varphi_n)[2\kappa_{n+1}\cos\varphi_n \sin 2(\alpha_n + \varphi_n) - \\ &- \delta^2_{n+1}(2\cos(2\alpha_n + 3\varphi_n) + \\ &+ \cos\alpha_n \cos(\alpha_n + \varphi_n)) - \\ &- 2\beta_{n+1}(\sin(2\alpha_n + 3\varphi_n) + \cos\alpha_n \sin(\alpha_n + \varphi_n))]. \end{aligned}$$

The initial value of φ_0 may be written, for a periodic trajectory, in the form

$$\varphi_0 = \varphi + \sum_{\imath - 1}^{infty} \varphi_{0i}\epsilon^i,$$

where φ is determined by (3).

Taking this into account, the coefficient κ_{n+1} may be expressed from iterative formulas (4) as

$$\kappa_{n+1} = \kappa^{(0)}_{n+1} + \epsilon\kappa^{(1)}_{n+1} + O(\epsilon^2),$$

where

$$\begin{aligned} \kappa^{(0)}_{n+1} &= 4\cos^2\left(\tfrac{\pi p}{2q}\right) \times \qquad (5)\\ \times \cos^4\left(\alpha_0 + \tfrac{\pi p}{2q} + n\pi\tfrac{p}{q}\right) &\sin\left(2\alpha_0 + \pi\tfrac{p}{q} + 2n\pi\tfrac{p}{q}\right),\\ \kappa^{(1)}_{n+1} &= \frac{1}{4}(1 + \cos\frac{p}{q}\pi) \times \\ \times [\frac{\cos 3(\alpha_0 + n(\frac{p}{q} + 1)\pi) - \cos 3\alpha_0}{2\cos\frac{p}{q} - 1} &+ \\ + 3(\cos(\alpha_0 + n(\frac{p}{q} + 1)\pi) - \cos\alpha_0)]. \end{aligned}$$

We have after q mappings

$$\varphi_q = \varphi_0 + \epsilon^2 \sum_{n=0}^{q-1} \kappa^{(0)}_{n+1} + \epsilon^3 \sum_{n=0}^{q-1} (\kappa^{(1)}_{n+1} + \eta^{(0)}_{n+1}) + qO(\epsilon^3).$$

The periodicity conditions for the gravitating billiard are as follows:

$$\alpha_q = \alpha_0 + 2\pi k, \qquad \varphi_q = \varphi_0. \tag{6}$$

The second equation (6) yields the necessary conditions of periodicity:

$$\sum_{n=0}^{q-1} \kappa^{(0)}_{n+1} = 0; \tag{7}$$

$$\sum_{n=0}^{q-1} (\kappa^{(1)}_{n+1} + \eta^{(0)}_{n+1}) = 0, \tag{8}$$

where $\eta^{(0)}_{n+1}$ may be obtained from (4), where the unperturbed values $\alpha_0 + n\pi(\frac{p}{q} - 1)$ and φ_0 are substituted for α_n and φ_n, respectively, in the right-hand sides.

In view of (5), the first condition (7) may be expressed in the form

$$\begin{aligned} \sum_{n=0}^{q-1} [\frac{5}{4}\sin\left(c + 2n\pi\tfrac{p}{q}\right) + \sin 2(c + 2n\pi\frac{p}{q}) + \qquad (9)\\ + \frac{1}{4}\sin 3(c + 2n\pi\frac{p}{q})] = 0, \qquad c \triangleq 2\alpha_0 + \frac{p}{q}\pi. \end{aligned}$$

By developing the sines of the sums in (9) and taking into account (11)–(12) (see Appendix), we find that (9) is always met identically for any c, except in the following essentially different cases:

I. $p = 0, q = 2$. II. $p = \pm 1, q = 3$.
III. $p = \pm 2, q = 4$. IV. $p = \pm 4, q = 6$.

The negative values of p correspond to time-reversed motion.

2-, 3-, 4-, AND 6-LINK TRAJECTORIES

Now we consider the cases I–IV.

I. $p = 0, q = 2$. The unperturbed trajectory is a diameter of the circle. Inserting the values of p, q in (9) yields

$$\cos^4\alpha_0 \sin 2\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{2}i, \qquad i \in \mathsf{Z}.$$

Thus, among all possible two-link trajectories, the gravity selects only two symmetric trajectories, which are born from the vertical and horizontal diameters (recall that we are dealing with a weak field).

II. $p = \pm 1, q = 3$. The unperturbed trajectory is a equilateral triangle. Inserting the values of p, q in (9) yields

$$\sin 6\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{6} i, \qquad i \in \mathsf{Z}.$$

Only triangles, one vertex of which lies at the lowest or highest point of the circle (the first type) or at the intersection of the horizontal diameter with the circle (the second type), are selected. The numerical experiment revealed similar three-link trajectories of the first type only. These trajectories are symmetric about the vertical diameter. Recall that the condition (9) is just the necessary condition for periodicity of a trajectory. Note that the condition (8) also does not forbid the solutions of the second type.

III. $p = \pm 2, q = 4$. Squares in the unperturbed motion. Equation (9) yields

$$\sin 4\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{4} i, \qquad i \in \mathsf{Z}.$$

These are generating trajectories — either a square with its vertices at intersections of the horizontal and vertical diameters with the circle (the first type) or one with the horizontal and vertical sides (the second type). The numerical computation discovered the trajectories of both types.

IV. $p = \pm 4, q = 6$. Regular hexagons in the unperturbed motion. Equation (9) yields

$$\sin 6\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{6} i, \qquad i \in \mathsf{Z}.$$

This condition selects two types of hexagons: with a pair of vertical sides (the first type) and with a pair of horizontal sides (the second type). The trajectories of either type were found during the numerical computation.

5-, 7-, AND 9-LINK TRAJECTORIES

We now pass on to the third order equation (8). It can be put in the following form:

$$\sum_{n=0}^{q-1} \sum_{j=0}^{4} \sum_{\substack{0 \le l_j \le 2j+1 \\ 0 \le m_j \le 7}} (-1)^{l_j} A_{m_j} \times \tag{10}$$
$$\times \cos\left((2j+1)\alpha_0 + (l_j n + m_j)\pi \tfrac{p}{q} \right) = 0.$$

The values of the coefficients A_{m_j}, l_j, and m_j are not presented here because of a lack of space.

It can be seen that there exist only few new pairs of the parameters p and q such that the equation (10) is not an identity for an arbitrary α_0. Now we list these new essentially different cases:

V.I. $p = \pm 1, q = 5$.	V.II. $p = \pm 3, q = 5$.
VI.I. $p = \pm 1, q = 7$.	VI.II. $p = \pm 3, q = 7$.
VI.III. $p = \pm 5, q = 7$.	VII.I. $p = \pm 1, q = 9$.
VII.II. $p = \pm 5, q = 9$.	VII.III. $p = \pm 7, q = 9$.

In order to obtain the above and to derive the following equations from (10) the formulas (13) and (14) were used.

V.I. $p = \pm 1, q = 5$. The unperturbed trajectory is a regular five-pointed star. The angle of reflection is $\varphi_0 = \frac{\pi}{10}$. Inserting the values of p, q in (10) yields after all

$$\cos 5\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{10} + \frac{\pi}{5} i, \qquad i \in \mathsf{Z}.$$

This gives two different trajectories: stars with one their vertex in the highest or lowest point of the circle, respectively. Both trajectories were found during the numerical computation.

V.II. $p = \pm 3, q = 5$. The unperturbed trajectory is a regular pentagon. The angle of reflection is $\varphi_0 = \frac{3}{10}\pi$. Equation (10) is brought into the same form as in case V.I:

$$\cos 5\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{10} + \frac{\pi}{5} i, \qquad i \in \mathsf{Z}.$$

We have two different solutions: pentagons with one their vertex in the highest or lowest point of the circle, respectively. Both trajectories were computed numerically.

VI.I. $p = \pm 1, q = 7$. The unperturbed trajectory is a regular seven-pointed star with 14 self-intersections. The angle of reflection is $\varphi_0 = \frac{\pi}{14}$. Inserting the values of p, q in (10) yields finally

$$\cos 7\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{14} + \frac{\pi}{7} i, \qquad i \in \mathsf{Z}.$$

This gives two different trajectories: stars with one their vertex in the highest or lowest point of the circle, respectively. Numerical experiments confirmed the existence of both of them.

VI.II. $p = \pm 3, q = 7$. The unperturbed trajectory is a regular seven-pointed star with 7 self-intersections. The angle of reflection is $\varphi_0 = \frac{3}{14}\pi$. The set of reflection points is exactly the same as in the previous case:

$$\cos 7\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{14} + \frac{\pi}{7} i, \qquad i \in \mathsf{Z}.$$

All solutions fall into two groups: stars with one their vertex in the highest or lowest point of the circle, respectively. Numerical experiments revealed both of them.

VI.III. $p = \pm 5, q = 7$. The unperturbed trajectory is a regular heptagon with no self-intersections. The angle of reflection is $\varphi_0 = \frac{5}{14}\pi$. The set of reflection points is exactly the same as in the two previous cases VI.I and VI.II:

$$\cos 7\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{14} + \frac{\pi}{7}i, \qquad i \in \mathsf{Z}.$$

All solutions are classified into two groups: heptagons with one their vertex in the highest or lowest point of the circle, respectively. The existence of both of them was confirmed in the numerical experiment.

VII.I. $p = \pm 1, q = 9$. The unperturbed trajectory is a regular nine-pointed star with 27 self-intersections. The angle of reflection is $\varphi_0 = \frac{\pi}{18}$. Inserting the values of p, q in (10) yields finally

$$\cos 9\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{18} + \frac{\pi}{9}i, \qquad i \in \mathsf{Z}.$$

Two different trajectories are distinguished: stars with one their vertex in the highest or lowest point of the circle, respectively. Both of them were found numerically.

VII.II. $p = \pm 5, q = 9$. The unperturbed trajectory is a regular nine-pointed star with only 9 self-intersections. The angle of reflection is $\varphi_0 = \frac{5}{18}\pi$. The set of reflection points is exactly the same as in the case VII.I and equation (10) gives

$$\cos 9\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{18} + \frac{\pi}{9}i, \qquad i \in \mathsf{Z}.$$

There are two different trajectories: stars with one their vertex in the highest or lowest point of the circle, respectively. Both of them were computed numerically.

VII.III. $p = \pm 7, q = 9$. The unperturbed trajectory is a regular nonagon with no self-intersections. The angle of reflection is $\varphi_0 = \frac{7}{18}\pi$. The set of reflection points is exactly the same as in the cases VII.I and VII.II:

$$\cos 9\alpha_0 = 0 \qquad \Rightarrow \qquad \alpha_0 = \frac{\pi}{18} + \frac{\pi}{9}i, \qquad i \in \mathsf{Z}.$$

There are two different trajectories: nonagons with one their vertex in the highest or lowest point of the circle, respectively. Both of them were found numerically.

CONCLUISION

We have shown that imposing the weak gravitational field leads to the disappearence of non-symmetric two-, three-, four-, five-, six-, seven-, and nine-link trajectories of all types. The eight-link trajectories are not covered by the theory in the approximation considered and they are allowed to be arbitrarily orientated. There seems every reason to believe that "the selection" of the eight-links as well as the trajectories having ten and more links may be carried out by taking into account the expansion terms of the gravitation parameter powers higher than three.

The theory developed can be called quasi-linear in the sense that the trajectories under study consist of near-straight links. In the essentially non-linear cases for the strong gravitation, it is also sometimes possible to demonstrate the selective properties of the gravitational field (the selection of bilaterally symmetric trajectories) (Beletsky *et al.* 1995).

The phenomenon arises from interaction between two factors — the perfect symmetry of the circular billiard and the superposition of a homogeneous force field. By contrast, there exists, for example, an elliptic gravitating billiard with the boundary $x = A\cos\alpha,\ y = \sin\alpha,\ 0 < A < 1$, allows for the non-symmetric two-link trajectory (Beletsky *et al.* 1996).

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APPENDIX

Below are presented four formulas for trigonometrical sums. They were used for solution of the equations (9) and (10).

$$\sum_{i=0}^{q-1} \sin 2\pi i \frac{m}{q} = 0; \tag{11}$$

$$\sum_{i=0}^{q-1} \cos 2\pi i \frac{m}{q} = \begin{cases} q, & \text{if } \frac{m}{q} = j, \\ 0, & \text{if } \frac{m}{q} \neq j; \end{cases} \tag{12}$$

$$\sum_{i=0}^{q-1} (-1)^i \sin \pi i \frac{m}{q} = \tag{13}$$

$$= \begin{cases} -\tan \frac{\pi m}{2q}, & \text{if } m+q = 2l+1, \frac{m}{q} \neq j, \\ 0, & \text{if } m+q = 2l \text{ or } \frac{m}{q} = j; \end{cases}$$

$$\sum_{i=0}^{q-1} (-1)^i \cos \pi i \frac{m}{q} = \tag{14}$$

$$= \begin{cases} q, & \text{if } \frac{m}{q} = 2j+1, \\ 0, & \text{if } \frac{m}{q} = 2j, q = 2l \text{ or } m+q = 2j, \frac{m}{q} \neq l, \\ 1, & \text{if } \frac{m}{q} = 2j, q = 2l+1 \text{ or } m+q = 2j+1, \frac{m}{q} \neq l; \end{cases}$$

$$m,\ l,\ j \in \mathsf{Z}.$$