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CALCULATING A CAM PROFILE FOR A CONSTANT-FORCE MECHANISM

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A procedure is proposed for calculation of cam profiles and analysis of their geometric properties for mechanisms that convert forces in accordance with specified laws. The procedure is applied to the case of a mechanisms that exerts a constant force. Representation of the profile as a section of a bifurcation set of potential functions made it possible to obtain an expression for the profile in parametric form and to study its singularities as functions of two determining parameters, applying the mathematical methods of catastrophe theory. A graphical diagram showing the positions of the profile singularities is presented, as are plots of specific curves.

One area in which profiled cams are used encompasses devices for conversion of a certain force acting on a mechanism in accordance with a specified law. The cam mechanism discussed in the present paper functions to set up a constant force or torque by using a linear elastic element. Devices of this type are used in various fields of engineering. One example is the mechanism that equalizes the mainspring torque of a stationary chronometer as it unwinds [1,2]. It incorporates a profiled cam known as a fusee (snail or hump). The other end of a sprocket chain on the fusee drives a circular roller. The spiral spring is attached to the shaft of the cam. Similar mechanisms are used in self-balancing hoisting devices [3-5].

A schematic diagram of the device appears in Fig. 1. The mechanism includes cam 1, cable 2, roller 3, and elastic element 4. The cable is secured to the cam at point U and to the roller at point W. The generatrix profile and constant of the elastic element must be so chosen that the tension force of the cable will be constant and equal to Q at any cam rotation angle. The torque created by the elastic element is offset by the constant torque applied to the roller.

In this paper we determine the profile of the cam and analyze its geometric properties. The method proposed for solution of these problems can be used effectively to calculate other force-transmitting cam mechanisms.

1. To the cam we attach the moving polar coordinate system ρ, ϕ with its center at point O and polar axis OU (Fig. 1). The cam profile can be specified by an equation $\rho = \rho(\phi)$. The turn angle of the cam is denoted by γ. We appoint distant O'O as the unit distance, assuming that all linear quantities are normalized to it.

The points at which the cable leaves the cam and roller will be denoted by T and N. The thickness of the cable will be neglected. The angles formed by line OT with the polar axis OU and by line O'N with line O'O will be denoted by ϕ and α, respectively. We shall assume that the cable between points T and N is a straight-line segment. The projection OM of the polar radius OT = ρ onto the perpendicular to TN is

$$h=\rho\cos(\alpha-\varphi-\gamma). \qquad (1)$$

From inspection of trapezoid O'OMN we find

$$h=q-\cos\alpha. \qquad (2)$$

where q is the radius of the roller.

We note that the variables h and α can be found from (1) and (2) as functions of the angle γ and the quantities ρ, ϕ, q, i.e., $h=h(\gamma;\rho,\varphi,q)$, $\alpha=\alpha(\gamma;\rho,\varphi,q)$.

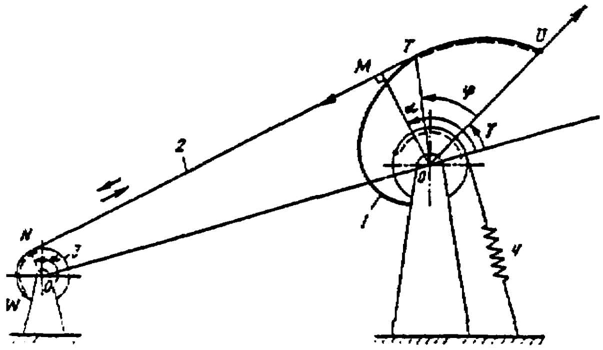

Fig. 1

The unit work done by the cable tension force Q on rotation of the cam through unit angle $d\gamma$ equals $dA_1 = Qh\,d\gamma$. We shall henceforth put Q = 1, thereby specifying the corresponding normalization. The elementary work of the linear elastic force $dA_2 = -p(\gamma-\gamma_0)d\gamma$, where p is the constant of the elastic element and γ_0 is the rotation angle at which the elastic element is undeformed.

The forces acting on the system admit the existence of a potential function whose differential dV is equal to the sum of the elementary works taken with the opposite sign: $dV = -(dA_1 + dA_2)$ [6]. The potential function depends on the angle γ (the sole variable of state that determines the position of the system) and on ρ, ϕ, p and q, which we shall call the controlling or control parameters. According to the virtual-displacement principle [6], the system equilibrium condition is

$$dV(\gamma;\ \rho,\ \varphi,\ p,\ q) = 0, \tag{3}$$

or

$$[-h(\gamma;\ \rho,\ \varphi,\ q) + p(\gamma-\gamma_0)]\,d\gamma = 0. \tag{4}$$

The behavior of the system under study is determined entirely by the potential function V. First of all, the condition of vanishing derivative $dV/d\gamma$ implicitly specifies a map $\gamma = \gamma(\rho, \varphi, p, q)$, the singularities of which remain to be studied, and, secondly, it determines a certain set point in γ, ρ, ϕ, p, q-space. With a view to use of the mathematical methods of catastrophe theory in the analysis that follows [7] we shall hold to its terminology and refer to this set as a catastrophic manifold.

By hypothesis, the cam profile must be such as to ensure an indifferent equilibrium position as the rotation angle γ varies. Hence follows the condition

$$d^2V/d\gamma^2 = 0. \tag{5}$$

Equation (5) specifies a singular set on the catastrophic manifold. The projection of the singular set into the space of control parameters is a bifurcation set. The section of the bifurcation set cut by the plane p = const, q = const gives the unknown plane cam profile. In the case of a fusee in the form of a variable-diameter drum, the three-dimensional form of the groove is determined by the section of the singular set cut by the same plane p = const, q = const.

In other words, the system of equations (3) and (5) can be used to find the curved cam profile for given constant values of the parameters p and q. Differentiating relation (1), we obtain $\frac{dh}{d\gamma} = -\rho\sin(\alpha-\varphi-\gamma)\left(\frac{d\alpha}{d\gamma}-1\right)$. Using this formula and differentiating (4), we expand (5):

$$k\left(\frac{d\alpha}{d\gamma}-1\right) + p = 0, \text{ where } k = \rho\sin(\alpha-\varphi-\gamma).$$

With (4), relation (2) yields

$$\cos\alpha = q - p(\gamma-\gamma_0), \tag{6}$$

whence

$$d\alpha/d\gamma=p/\sin\alpha. \tag{7}$$

It is now possible to find ρ and ϕ as functions of the angle α. We have $\rho=\sqrt{h^2+k^2}$, $\varphi=\arg(h,\ k)+\alpha-\gamma$. The function $\psi=\arg(x,\ y)$ is given by the relations $\cos\psi=x/\sqrt{x^2+y^2}$, $\sin\psi=y/\sqrt{x^2+y^2}$.

Finally, we obtain

$$\rho=\sqrt{(q-\cos\alpha)^2+\frac{p^2\sin^2\alpha}{(p-\sin\alpha)^2}},$$
$$\varphi=\arg\left[(q-\cos\alpha),\frac{p\sin\alpha}{p-\sin\alpha}\right]+\alpha+\frac{\cos\alpha-q}{p}-\gamma_0. \tag{8}$$

Formulas (8) specify a curved cam profile in the parametric form $\rho=\rho(\alpha),\ \varphi=\varphi(\alpha)$.

As we see from (6), the angle $(\gamma-\gamma_0)$ can vary only within the segment $[(q-1)/p,\ (q+1)/p]$ if $p > 0$ and the segment $[(q+1)/p,\ (q-1)/p]$ if $p < 0$. This constraint is of fundamental geometric nature, and it is now possible to obtain a wider range of variation of the cam rotation angle γ in this scheme of the mechanism. To judge from (4), we would conclude that when $\gamma=\gamma_0$ the tangent TN to the profile passes through the center of rotation O, at which point there is a change of sign of the arm h, i.e., of the sign of the moment of the cable tension force about the cam rotation axis O.

Since formulas (8) are invariant under the substitution $\alpha\to-\alpha,\ \gamma\to-\gamma+2\gamma_0,\ \rho\to\rho,\ \gamma\to-\gamma-2\gamma_0,\ p\to-p,\ q\to q$, the cam profile for $p < 0$ can easily be obtained by mirror reflection of the profile for $p > 0$ about the line $\phi=-\gamma_0$. Below we shall concentrate our attention basically on the case $p > 0$.

2. Let us investigate the properties of the cam-profile curve described by formulas (8). The radius ρ increases without limit at $\sin\alpha = p$. This can happen only for $|p|\leq 1$. It follows from (6) that the equality $\sin\alpha = p$ holds in two cases: $\gamma-\gamma_0=(q-\sqrt{1-p^2})/p$, if $\cos\alpha>0$; $\gamma-\gamma_0=(q+\sqrt{1-p^2})/p$, if $\cos\alpha < 0$. The corresponding cam-generatrix asymptotes:

$$\varphi_1=\pm\frac{\pi}{2}+\arcsin p-\frac{q-\sqrt{1-p^2}}{p}-\gamma_0,$$
$$\varphi_2=\pm\frac{\pi}{2}+\pi-\arcsin p-\frac{q+\sqrt{1-p^2}}{p}-\gamma_0.$$

Proceeding with determination of other singularities of the profile, we exclude the neighborhoods of points corresponding to $\sin\alpha = p$ from further consideration.

Points of the bifurcation set, i.e., of the cam profile, parametrize degenerate critical points of the potential function V. To establish the properties of the bifurcation set we first find

$$\frac{d^2V}{d\gamma^2}=h\left(\frac{d\alpha}{d\gamma}-1\right)^2+k\frac{d^2\alpha}{d\gamma^2}, \tag{9}$$

for which (7) gives $d^2\alpha/d\gamma^2=-p^2\cos\alpha/\sin^3\alpha$.

If the third derivative of (9) does not equal zero, there will be no singularities of the profile at this point. The derivative $d^3V/d\gamma^3$ vanishes where the profile curve has a cuspidal point of the first kind ($d\rho/d\alpha=0,\ d\varphi/d\alpha=0$). Let us examine the latter possibility more closely. We expand (9) and equate this expression to zero:

$$(q-\cos\alpha)(\sin\alpha-p)^3-p^2\cos\alpha=0,\ \sin\alpha\neq 0. \tag{10}$$

To investigate the left side of (10), we introduce the new variables $u=\cos\alpha/q,\ v=\sin\alpha/p$. Then (10) can be given the form

$$F(u,\ v)=(1-u)(v-1)^3-u=0. \tag{11}$$

Figure 2 shows a plot of curve (11) on the (u, v) plane. The curve consists of two

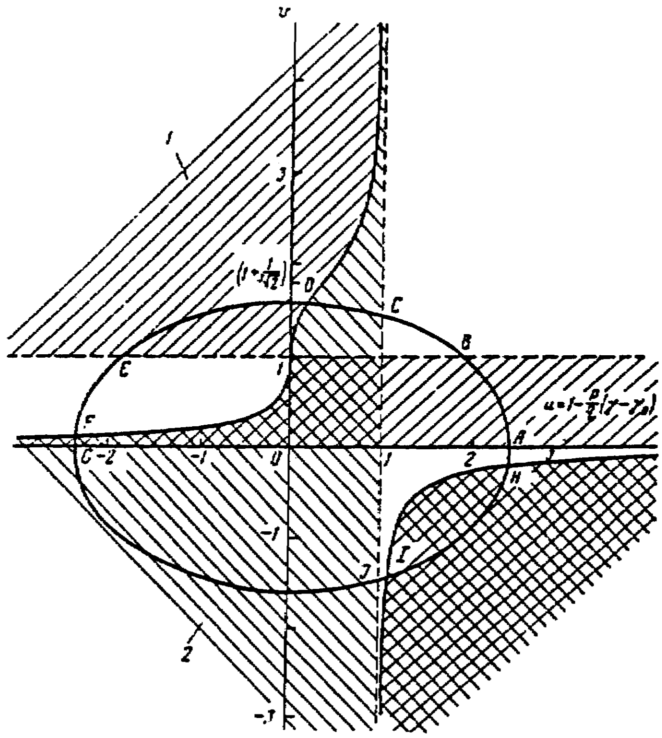

Fig. 2. Diagram of singularities of function specifying cam profile. Regions 1 and 2, where $\rho(\alpha)$ (when $p > 0$) and $\phi(\alpha)$ increase, respectively, are shaded separately.

branches and has two asymptotes: $v = 0$ and $u = 1$ and two inflection points: $u = 0$, $v = 1$ and $u = \ldots$, $v = 1 + (1/\sqrt[3]{2})$.

In the region bounded by the two branches of this curve, which contains the coordinate origin, the function $F(u, v)$ is negative. It is positive on the rest of the (u, v) plane.

The derivatives

$$\frac{d\rho}{d\alpha} = pq\frac{v|v-1|}{(v-1)^3}\frac{F(u,v)}{\sqrt{G(u,v)}}, \quad \frac{d\varphi}{d\alpha} = q^2(u-1)\frac{F(u,v)}{G(u,v)}, \qquad (12)$$

can be found from (8); here $G(u,v) = q^2(v-1)^2(1-u)^2 + p^2v^2 \geq 0$ ($G(u,v) = 0$ only when $\rho = 0$, i.e., at $u = 1$, $v = 0$).

If $p > 0$, the angle α can either increase from zero to π or decrease from zero to $-\pi$. But if $p < 0$, the angle α can either decrease from π to zero or increase from $-\pi$ to zero. The values of $(\gamma - \gamma_0)$ increases in all cases.

As the parameter α varies, the mapping point on the (u, v) plane moves along an arc of an ellipse: $u^2q^2 + v^2p^2 = 1$. One such ellipse for $q = 1/2.4$ and $p = 1/1.6$ is represented in Fig. 2. The regions in which the derivatives (12) have various combinations of signs are also indicated on the same figure.

Following the mapping point along the arc of the ellipse, we can easily track all singularities of the cam profile and obtain information on the decreases or increases of ρ and ϕ. Let us illustrate this for the specific case of the ellipse in Fig. 2. Let the initial value of the parameter α be zero, and then let it increase all the way to π, i.e., let us move along arc ABCDEFG. The corresponding $\rho(\alpha)$, $\phi(\alpha)$ and $(\gamma(\alpha)-\gamma_0)$ ($\gamma_0 = \pi/4$) curves appear in Fig. 3, and the curve of the profile itself, $\rho(\phi)$, in Fig. 4.

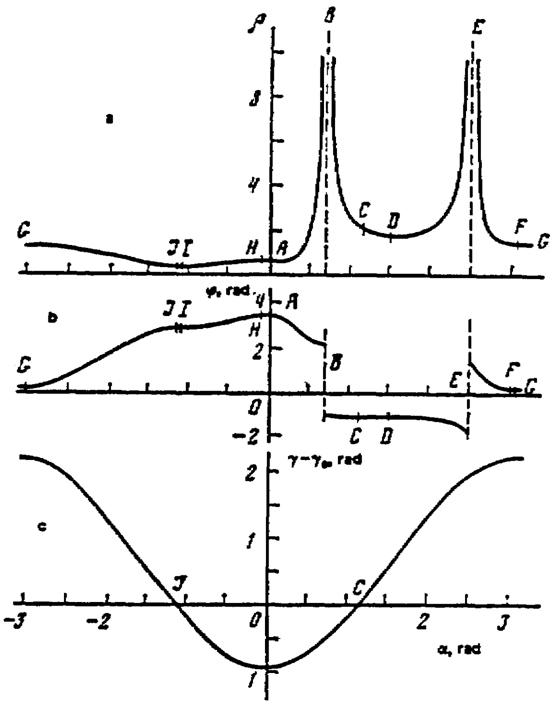

Fig. 3. Curves of polar coordinates ρ, ϕ and cam rotation angle $(\gamma - \gamma_0)$ vs. the parameter α.

We see from (12) that the tangent to the profile at point A is normal to the radius vector of this point. On segment AB, as the polar angle ϕ decreases, the radius ρ increases, going infinite at point B. The intersection of the line EB(v = 1) corresponds to a discontinuity of the profile. Both ϕ and ρ decrease simultaneously on arc BC. At the intersection of the line u = 1 at point C, the tangent to the profile crosses the rotation axis and the elastic element passes through its undeformed state, with a change in the sign of its deformation. Then the angle ϕ begins to increase, but ρ continues to decrease to the point D of intersection with the branch of the F(u, v) = 0 curve. This point corresponds to a cuspidal point of the profile curve, beyond which ρ again increases, while ϕ decreases. At point E, as at B, ρ increases without limit and the profile curve has a discontinuity. Both ρ and ϕ decrease simultaneously on segment EF. Point F is another cuspidal point of the profile curve; ρ and ϕ increase on arc FG. At the terminal point G, the tangent to the profile is normal to the radius vector.

A similar analysis can also be performed for the lower half-arc of the ellipse HIJG (Figs. 2-4) and for ellipses that correspond to other values of p and q, including negative p.

It is easily understood from Fig. 2 that the lengths of arcs CD, FG, AH and IJ tend to zero as $p \to 0$, $q \to 0$. It can be shown that the shapes of parts of the profile approach circle evolutes in this case outside of the neighborhood of the points of discontinuity ($\sin \alpha = p$). Since the ellipse does not intersect the line v = 1 when p > 1, the profile has no points of discontinuity. The set of points of intersection of the ellipse and the curve F(u, v) = 0 (i.e., the set where $\frac{d^2V}{d\gamma^2} = 0$) may contain from four points to one or be empty. Points D and F (or H and I) may merge with the ellipse tangent only to the branch of F(u, v) = 0. To establish the nature of the singularity that appears at such a point, we return again to the potential function V. Its fourth derivative

$$\frac{d^4V}{d\gamma^4} = 3h\left(\frac{d\alpha}{d\gamma} - 1\right)\frac{d^2\alpha}{d\gamma^2} + k\left(\frac{d^3\alpha}{d\gamma^3} - \left(\frac{d\alpha}{d\gamma} - 1\right)^3\right),$$

where $\frac{d^3\alpha}{d\gamma^3} = \frac{p^2(1+2\cos^2\alpha)}{\sin^3\alpha}$.

Equating $d^nV/d\gamma^n$ $(n = 1, 2, 3, 4)$ to zero, we obtain a system in which the equation equivalent to $d^4V/d\gamma^4 = 0$, is brought to the form

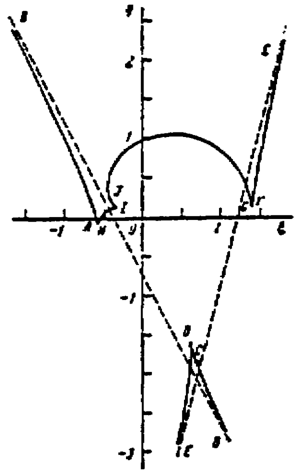

Fig. 4. Cam profile for $p=\frac{1}{1{,}6}$, $q=\frac{1}{2{,}4}$ and $\gamma_0=\pi/4$, Oξ is the polar axis.

$$v^5-4v^4+6v^3-6v^2+\frac{3}{p^2}=0. \qquad (13)$$

It can be interpreted as the condition of contact between the ellipse and the F(u, v) = 0 curve in Fig. 2. There is obviously a single-parameter family of solutions that cause all $d^nV/d\gamma^n$ up to and including n = 4 to vanish.

We find the next potential-function derivative:

$$\frac{d^5V}{d\gamma^5}=h\left[3\left(\frac{d^2\alpha}{d\gamma^2}\right)^2+\left(\frac{d\alpha}{d\gamma}-1\right)\left(4\frac{d^3\alpha}{d\gamma^3}-\left(\frac{d\alpha}{d\gamma}-1\right)^3\right)\right]+$$
$$+k\left[\frac{d^4\alpha}{d\gamma^4}-6\left(\frac{d\alpha}{d\gamma}-1\right)^2\frac{d^2\alpha}{d\gamma^2}\right],$$

where

$$\frac{d^4\alpha}{d\gamma^4}=\frac{-3p^4\cos\alpha(3+2\cos^2\alpha)}{\sin^7\alpha}.$$

Converting the system $d^nV/d\gamma^n=0$ $(n=1, 2, 3, 4, 5)$, we can arrive at an equation only for v:

$$(v-1)(v-2)(5v^2-6v+6)=0. \qquad (14)$$

It is easily shown that system (13), (14) has no satisfactory solution (it must be remembered that $|vp|<1$). This corresponds to the obvious fact that the ellipse and the curve F(u, v) = 0 in Fig. 2 cannot have second-order tangency for any combination of p and q.

Let us denote by M_m the point set in γ, ρ, ϕ, p, q-space that is specified by the system $d^nV/d\gamma^n=0$, $d^{m+1}V/d\gamma^{m+1}\neq 0$ $(n=1,\ldots,m)$. The potential function V is (m + 1)-definite on set M_m. At points belonging to a one-dimensional M_4, V can be brought by a smooth substitution of variables to the standard form x^5 - rostok katastrofy A_4 (swallowtail). It can be shown that on set M_4 the remaining (4 - 1) control parameters ensure universal deformation of the potential function, which is equivalent to the standard A_4 deformation $a_1x^3+a_2x^2+a_3x$. On the two-dimensional set M_3, the dovetail catastrophe splits into 2 contiguous $A_{\pm 3}$ catastrophes sborki i dvoistvennoi sborki. It is their characteristic semicubic klyuvi that represent the cam profile in the neighborhoods of cuspidal points (Fig. 4). For the purposes of our problem - study of the profile - the differences between sborka i dvoistvennaya sborka are of no importance. Set M_2 of folds (of type A_2 catastrophes) is three-dimensional and includes all remaining points of the bifurcation set

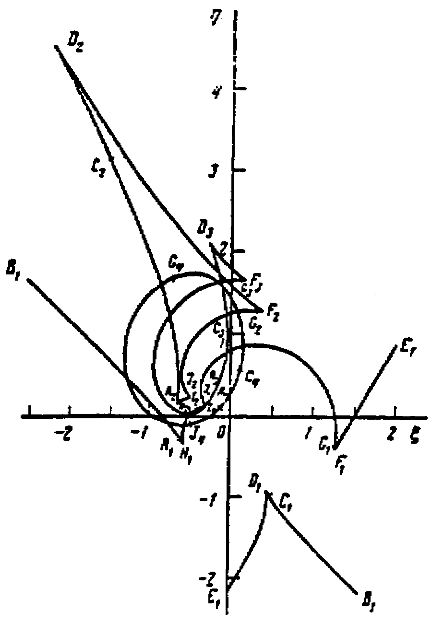

Fig. 5

(and, consequently, the cam profile), except for points that belong to M_3 and M_4. These singularities of the cam profile, which are associated with a sequence of cuspoid catastrophes, are structurally stable. The existence of four control parameters in the present problem led us to hope that we might also find a structurally stable point of a higher order cuspoid catastrophe, namely A_5 (butterfly), but the investigation showed that the structure of the potential function excluded this possibility.

Figure 5 shows four typical profile curves that correspond to the following values of the parameters: 1) $p=1/2$, $q=1/3$; 2) $p=3/4$, $q=1/3$; 3) $p=2$, $q=2/3$; 4) $p=5$, $q=3/4$ $(\gamma_0=\pi/4)$.

The evolution of the profile on variation of the determining parameters can be followed in Fig. 5. One dovetail point occurs on the transition from curve 2 to 3 with merging of points H_2 and I_2, and the other on transition from curve 3 to 4 with merging of D_3 and F_3.

The curves shown in Figs. 3-5 were generated on a graph plotter with a specially written computer program that computes the curve of the profile from formulas (8). The program can be used to construct the cam profile in a specified scale for any parameter values.

Thus, representation of the cam profile as the section cut by the plane p = const, u = const through the potential-function bifurcation set made possible quick acquisition of the profile function in parametric form and investigation of its singularities, which were found to be organized as a type A_4 (dovetail) catastrophe. The singularity diagram (Fig. 2) can be used in cam design. It permits immediate inferences as to all singularities of the profile and enables us to choose the initial and final points of the cam generatrix arc correctly. The sample profiles (Figs. 4 and 5) illustrate various cases that correspond to representative sets of parameters.

The proposed method for calculation of cam profiles and analysis of their geometric properties can be used for a broad class of cam mechanisms that convert forces in accordance with specified laws.

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